J. Math. Soc. Japan
Vol. 39, No. 2, 1987

On the inverse scattering problem for the acoustic equation

By Masaru IKEHATA

(Received July 12, 1985)

§1. Introduction.

The inverse problem for quantum and acoustic scatterings has been investigated extensively. Little attention, however, seems to have been paid to the inverse problem for the acoustic scattering. P. Lax and R. Phillips [**4**, p. 174], showed that the scattering operator associated with the wave equation in an exterior domain Ω (with $\partial\Omega$ smooth and bounded) whose solutions satisfy the boundary condition of being zero on $\partial\Omega$ uniquely determines the obstacle Ω^c. But in the case of a metric perturbation for the wave equation in $\boldsymbol{R}^n$ it is not known whether the scattering operator uniquely determines the metric or not. The purpose of this paper is to give an answer to an inverse problem related to this problem.

Let $g(x)$ be a C^∞-Riemannian metric on $\boldsymbol{R}^n$ $(n\geqq 2)$ satisfying $g(x)=I_n$ (the unit matrix of degree n) for $|x|\geqq r_0$ where r_0 is a positive number. Consider the scattering problem for the acoustic equation

$$(1.1)\qquad (\partial_t^2-\nabla\cdot g(x)^{-1}\nabla)\,u(t, x)=0 \qquad \text{in } \boldsymbol{R}^1\times\boldsymbol{R}^n,$$

where $\nabla={}^t(\partial_{x_1}, \cdots, \partial_{x_n})$. Let $S(s, \theta, \omega)$ be the scattering kernel for this problem. For each ω, $\theta\in S^{n-1}$, it is well known that $S(\cdot, \theta, \omega)$ is a distribution on $\boldsymbol{R}^1$ (see H. Soga [**2**], [**3**]). In what follows we adopt the following convention: $\sup \operatorname{sing\,supp} S(\cdot, \theta, \omega)=-\infty$ if $\operatorname{sing\,supp} S(\cdot, \theta, \omega)=\emptyset$. We consider the following

PROBLEM. Find an inhomogeneous media $g(x)$ from the known $\sup \operatorname{sing\,supp} S(\cdot, \theta, \omega)$.

Now let us prepare notations in order to give our answer to this problem. Let $g_e^n(x)$ be a surface of revolution on $\boldsymbol{R}^n$ with center 0 treated by H. Gluck and D. Singer [**1**] in the case that $n=2$, namely

$$(1.2)\qquad g_e^n(x)=I_n-\frac{e(|x|)}{|x|^2}x\,{}^tx, \qquad x={}^t(x_1, \cdots, x_n)\in\boldsymbol{R}^n\setminus 0,$$

where e is a smooth even function with support in $[-r_0, r_0]$, $e(0)=0$, and $E(r)=1-e(r)$ is positive. Then we see that $g_e^n(x)$ can be represented in polar coordinates by

$$ds^2 = E(r)dr^2 + r^2 d\omega^2$$

and $g_e^n(x)=I_n$ for $|x| \geqq r_0$. Let δ_e be the deflection function between $g_e^n(x)$ and I_n introduced in [1]. That is

$$\delta_e(c) = c\int_c^{r_0} \frac{r^{-1}(E(r)^{1/2}-1)}{(r^2-c^2)^{1/2}} dr, \qquad 0<c<r_0. \tag{1.3}$$

Then δ_e extends to a smooth odd function with support in $[-r_0, r_0]$. For $0<r_1<r_0$, we set

$$\begin{aligned} \mathrm{RO}_{r_0}^{\pm}(r_1) = \{g_e^n(x) \mid & \pm(d\delta_e/dc)(c)>0 \quad \text{for } 0<c<r_1, \\ & \pm(d\delta_e/dc)(c)<0 \text{ for } r_1<c<r_0 \text{ and } |\delta_e(r_1)|<\pi/2\}. \end{aligned} \tag{1.4}$$

$$\mathrm{RSO}_{r_0}^{\pm} = \{g_e^n(x) \in \mathrm{RO}_{r_0}^{\pm}(r_0/2) \mid \delta_e(c) = \delta_e(r_0-c) \text{ for } 0<c<r_0/2\}. \tag{1.5}$$

By definition, $\mathrm{RSO}_{r_0}^{\pm}$ is contained in $\mathrm{RO}_{r_0}^{\pm}$ $(r_0/2)$. Furthermore, by Theorem 4.3 in [**1**, p. 212], these sets are non-empty and uncountable.

An answer to the preceding problem in the case that $n=2$ is contained in

THEOREM 1.1. (i) *There exist infinitely many* $g_e^2(x) \in \mathrm{RO}_{r_0}^{\pm}(r_1)$ *with the same* sup sing supp $S(\cdot, \theta, \omega)$ $(\theta \neq \omega)$.
(ii) *Let* $g_{e_i}^2(x) \in \mathrm{RSO}_{r_0}^{\pm}$ *and* $g_{e_i}^2(x)-I_2$ *is sufficiently small in the* C^2*-topology* $(i=1, 2)$. *If both* $g_{e_1}^2(x)$ *and* $g_{e_2}^2(x)$ *have the same* sup sing supp $S(\cdot, \theta, \omega)$ $(\theta \neq \omega)$, *then* $g_{e_1}^2(x)=g_{e_2}^2(x)$.

Theorem 1.1 follows from a result obtained by H. Soga [**3**] and the following Theorems 1.3, 1.4 and 1.6, which may be of independent interest. Let $g(x)$ be a C^∞-Riemannian metric on $\boldsymbol{R}^n$ with $g(x)=I_n$ for $|x| \geqq r_0$, which satisfies the following assumption (G):

(G) For any $\omega \in S^{n-1}$ and y with $y \cdot \omega = -r_0$,

$$\lim_{t \to +\infty} |q(t; -r_0, y, \omega)| = +\infty,$$

where the dot $\cdot$ denotes the ordinary scalar product over $\boldsymbol{R}^n$, S^{n-1} is the $(n-1)$-dimensional unit sphere and where $(q(t; s, x, \xi), p(t; s, x, \xi))$ denotes the solution of the Hamilton equation

$$\begin{gathered} \frac{dq}{dt} = \nabla_\xi H(q, p), \qquad \frac{dp}{dt} = -\nabla_x H(q, p) \\ (q, p)|_{t=s} = (x, \xi) \in \boldsymbol{R}^n \times (\boldsymbol{R}^n \setminus 0), \qquad s \in \boldsymbol{R}^1, \end{gathered} \tag{1.6}$$

where $H(x, \xi)=\{g(x)^{-1}\xi\cdot\xi\}^{1/2}$. For any θ, $\omega\in S^{n-1}$ we set $\pi_\omega=\{y\in \boldsymbol{R}^n \mid y\cdot\omega=-r_0\}$ and let us put

$$(1.7)\qquad M_\omega(\theta)=\{y\in\pi_\omega \mid \lim_{t\to+\infty} p(t\,; -r_0, y, \omega)=\theta\},$$

$$(1.8)\qquad s_\omega(\theta)=\sup\{\lim_{t\to+\infty}\{q(t\,; -r_0, y, \omega)\cdot\theta-t\} \mid y\in M_\omega(\theta)\} \qquad (\text{if } M_\omega(\theta)\neq\emptyset),$$
$$= -\infty \qquad (\text{if } M_\omega(\theta)=\emptyset),$$

$$(1.9)\qquad \tilde{M}_\omega(\theta)=\{y\in M_\omega(\theta) \mid \lim_{t\to+\infty}\{q(t\,; -r_0, y, \omega)\cdot\theta-t\}=s_\omega(\theta)\}.$$

Then H. Soga [**3**] proved the following theorem.

THEOREM 1.2. *Let $n=2$ and $\omega\neq\theta$. Assume that*

$$(1.10)\qquad \det\partial_x q(t\,; -r_0, y, \omega)\neq 0,$$

$$(1.11)\qquad \det\partial_\xi p(t\,; -r_0, y, \omega)\neq 0$$

for any $(t, y)\in[-r_0, +\infty[\times\tilde{M}_\omega(\theta)$. Then

$$\sup \operatorname{sing}\operatorname{supp} S(\cdot, \theta, \omega)=s_\omega(\theta).$$

Concerning the hypotheses of Theorem 1.2 we have the following theorems.

THEOREM 1.3. *If $g_e^n(x)$ is a surface of revolution on $\boldsymbol{R}^n$ with center* 0, *the condition* (G) *is satisfied with $g(x)$ replaced by $g_e^n(x)$.*

THEOREM 1.4. *Let $n=2$ and $g_e^2(x)\in \mathrm{RO}_{r_0}^{\pm}(r_1)$. If $g_e^2(x)-I_2$ is sufficiently small in the C^2-topology, then $g_e^2(x)$ satisfies* (1.10) *and* (1.11) *for ω, $\theta\in S^1$ $(\omega\neq\theta)$ and $(t, y)\in[-r_0, +\infty[\times\tilde{M}_\omega(\theta)$.*

As for the case that $n\geqq 3$, we obtain

THEOREM 1.5. *Let $n\geqq 3$.* (i) *Let $g_e^n(x)\in\mathrm{RO}_{r_0}^{+}(r_1)$. Then for any ω, $\theta\in S^{n-1}$ $(\omega\neq\theta)$ and $y\in M_\omega(\theta)$, there exists $t\in]-r_0, +\infty[$ such that*

$$\det\partial_x q(t\,; -r_0, y, \omega)=0.$$

(ii) *Let $g_e^n(x)\in\mathrm{RO}_{r_0}^{-}(r_1)$. If $g_e^n(x)-I_n$ is sufficiently small in the C^2-topology, then* (1.10) *and* (1.11) *hold for any ω, $\theta\in S^{n-1}$ $(\omega\neq\theta)$ and $(t, y)\in[-r_0, +\infty[\times\tilde{M}_\omega(\theta)$.*

The following theorem yields the relation between $g_e^n(x)$ and $s_\omega(\theta)$.

THEOREM 1.6. *Let $n\geqq 2$.* (i) *There exist infinitely many $g_e^n(x)\in\mathrm{RO}_{r_0}^{\pm}(r_1)$ such that $g_e^n(x)-I_n$ is sufficiently small uniformly in the C^2-topology and $g_e^n(x)$ has the same $s_\omega(\theta)$.*
(ii) *Let $g_{e_i}^n(x)\in\mathrm{RSO}_{r_0}^{\pm}$ $(i=1, 2)$. If both $g_{e_1}^n(x)$ and $g_{e_2}^n(x)$ have the same $s_\omega(\theta)$, then $g_{e_1}^n(x)=g_{e_2}^n(x)$ and the following inversion formulas hold:*

(a) *If* $g_e^n(x)\in \mathrm{RSO}_{r_0}^+$, *then*

$$E(r)=\left(1+\frac{1}{\pi}\int_r^{r_0}\frac{c\,dc}{s^{(2)}((s^{(1)})^{-1}(-c))(c^2-r^2)^{1/2}}\right)^2 \qquad (if\ r_0/2\leqq r\leqq r_0), \tag{1.12}$$

$$=\left(1+\frac{1}{\pi}\int_{r_0/2}^{r_0}\frac{c\,dc}{s^{(2)}((s^{(1)})^{-1}(-c))(c^2-r_0^2/4)^{1/2}}\right.$$
$$\left.-\frac{1}{\pi}\int_r^{r_0/2}\frac{c\,dc}{s^{(2)}((s^{(1)})^{-1}(c-r_0))(c^2-r_0^2)^{1/2}}\right)^2 \qquad (if\ 0\leqq r\leqq r_0/2).$$

(b) *If* $g_e^n(x)\in \mathrm{RSO}_{r_0}^-$, *then*

$$E(r)=\left(1-\frac{1}{\pi}\int_r^{r_0}\frac{c\,dc}{s^{(2)}((s^{(1)})^{-1}(r_0-c))(c^2-r^2)^{1/2}}\right)^2 \qquad (if\ r_0/2\leqq r\leqq r_0), \tag{1.13}$$

$$=\left(1-\frac{1}{\pi}\int_{r_0/2}^{r_0}\frac{c\,dc}{s^{(2)}((s^{(1)})^{-1}(r_0-c))(c^2-r_0^2/4)^{1/2}}\right.$$
$$\left.+\frac{1}{\pi}\int_r^{r_0/2}\frac{c\,dc}{s^{(2)}((s^{(1)})^{-1}(c))(c^2-r^2)^{1/2}}\right)^2 \qquad (if\ 0\leqq r\leqq r_0/2).$$

Here $s(\eta)=s^e_{(1,0,0,\cdots,0)}(\cos\eta,\ \sin\eta,\ 0,\ \cdots,\ 0)$, $\eta\in\boldsymbol{R}^1$, $s^{(1)}=ds/d\eta$, $s^{(2)}=d^2s/d\eta^2$, *and* $(s^{(1)})^{-1}$ *is the inverse function of* $s^{(1)}$.

The rest of this paper is organized as follows. In Section 2 we give some lemmas and prove Theorem 1.3. Using the results in Section 2, we prove Theorem 1.6 in Section 3 and we give further properties of $s_\omega(\theta)$. We shall prove Theorems 1.4 and 1.5 in Section 4.

The author would like to thank Professor M. Murata for his encouragements and the referee for some comments.

§ 2. Proof of Theorem 1.3.

First we introduce some notations. Let $n\geqq 2$. Set

$$\Sigma_-(r_0)=\{(z,\ \omega)\in\boldsymbol{R}^n\times S^{n-1}\mid |z|=r_0 \text{ and } z\cdot\omega<0\}.$$

Let $\Gamma^i_{jk}(x)$ be the Christoffel's symbol associated with a C^∞-Riemannian metric $g(x)$ on $\boldsymbol{R}^n$ and put

$$\Gamma(x)(u,\ v)={}^t\Big(\sum_{j,k}\Gamma^1_{jk}(x)u_jv_k,\ \cdots,\ \sum_{j,k}\Gamma^n_{jk}(x)u_jv_k\Big)$$

for u, v and $x\in\boldsymbol{R}^n$. Let $(q_\omega(t\,;x),\ v_\omega(t\,;x))$ be the solution of the following ordinary differential equation (i. e. the geodesic flow for $g(x)$)

$$(2.1) \qquad \frac{dq}{dt} = v, \qquad \frac{dv}{dt} = -\Gamma(q)(v, v),$$
$$(q, v)|_{t=0} = (x, \omega) \in \boldsymbol{R}^n \times S^{n-1}.$$

Hereafter we assume that $g(x)=I_n$ for $|x| \geqq r_0$. Since $g(x)$ is geodesically complete, $(q_\omega(t; x), v_\omega(t, x))$ exists for all $t \in \boldsymbol{R}^1$. Let $T(z, \omega)$ be the sojourn time for $(z, \omega) \in \Sigma_-(r_0)$ in the ball of radius r_0 with center $0 \in \boldsymbol{R}^n$, that is

$$(2.2) \qquad T(z, \omega) = \sup\{t>0 \mid |q_\omega(\tau; z)| < r_0 \text{ for } 0<\tau<t\}.$$

Then it is easy to see that $T(z, \omega)$ is well defined and the value lies in $]0, +\infty]$. For $\omega \in S^{n-1}$, put $\pi_\omega^1 = \{y \in \pi_\omega \mid |y+r_0\omega| \geqq r_0\}$ and $\pi_\omega^2 = \{y \in \pi_\omega \mid |y+r_0\omega| < r_0\}$. Furthermore, put

$$(2.3) \qquad T^*(y, \omega) = -(2r_0^2 - |y|^2)^{1/2} + T(A_\omega(y), \omega),$$

for $y \in \pi_\omega^2$, where $A_\omega(y) = y + (r_0^2 - (2r_0^2 - |y|^2)^{1/2})\omega$.

We start with the following elementary lemma which is obtained by elementary calculations and we omit the proof.

LEMMA 2.1. *$g(x)$ satisfies* (G) *if and only if $T(z, \omega)$ has a finite value for any $(z, \omega) \in \Sigma_-(r_0)$ and the following equalities hold:*

(2.4) *For any $y \in \pi_\omega^2$ and $t \geqq T^*(y, \omega)$,*

$$q(t; -r_0, y, \omega) = Q(A_\omega(y), \omega) + (t - T^*(y, \omega))V(A_\omega(y), \omega),$$
$$p(t; -r_0, y, \omega) = V(A_\omega(y), \omega),$$

where $(Q(z, \omega), V(z, \omega)) = (q_\omega(t; z), v_\omega(t; z))|_{t=T(z,\omega)}$.

(2.5) *For any $y \in \pi_\omega^1$ and $t \in \boldsymbol{R}$,*

$$q(t; -r_0, y, \omega) = y + (t+r_0)\omega,$$
$$p(t; -r_0, y, \omega) = \omega.$$

Let $R(\alpha)$ $(\alpha \in \boldsymbol{R}^1)$ be the rotation of angle α around $0 \in \boldsymbol{R}^2$ and put $\omega^\perp = R(\pi/2)\omega$ for $\omega \in S^1$. For $z_0, z \in \boldsymbol{R}^2$, define

$$\begin{aligned} B(z_0, z) &= 2(z \cdot z_0/|z_0|^2)z_0 - z \qquad &(\text{if } z_0 \neq 0), \\ &= -z &(\text{if } z_0 = 0). \end{aligned}$$

In the following lemma sub/superscript e stands for dependence on $g_e^n(x)$.

LEMMA 2.2. *Let $n=2$. Suppose that $g(x)$ is equal to a surface of revolution $g_e^2(x)$ on $\boldsymbol{R}^2$ with center* 0. *Then $T_e(z, \omega) < +\infty$ for $(z, \omega) \in \Sigma_-(r_0)$, and*

(2.6) $$T_e(z, \omega) = T_0(z, \omega) - 2\int_{-c}^{r_0} r\frac{d\delta_e}{dc}(r)dr,$$

(2.7) $$q_\omega^e(T_e(z, \omega)/2\,; z) = R(-\delta_e(c))q_\omega^0(T_0(z, \omega)/2\,; z),$$

(2.8) $$Q^e(z, \omega) = B(q_\omega^e(T_e(z, \omega)/2\,; z), z),$$

(2.9) $$V^e(z, \omega) = -B(q_\omega^e(T_e(z, \omega)/2\,; z), \omega),$$

where $c=z\cdot\omega^\perp$.

REMARK 1. The fact corresponding to (2.7) is described in [**1**].

REMARK 2. The fact that $g_e^2(x)$ is invariant with respect to linear transformations $B(z_0, \cdot)$ and $R(\alpha)$ of $\boldsymbol{R}^2$ onto itself concludes (2.8) and (2.9).

REMARK 3. (2.6) is an important equality in this paper since the perturbation term of $T_e(z, \omega)$ from $T_0(z, \omega)$ is written by using only δ_e in a simple form.

PROOF OF LEMMA 2.2. By the same argument in [**1**, p. 208], we obtain

$$T_e(z, \omega) = 2\int_{|c|}^{r_0} r\left(\frac{E(r)}{r^2-c^2}\right)^{1/2} dr.$$

Compare this with the case that $e=0$, then

$$T_e(z, \omega) - T_0(z, \omega) = 2\int_{|c|}^{r_0} r((E(r))^{1/2}-1)(r^2-c^2)^{-1/2}dr$$

holds. By Theorems 4.1 and 4.2 in [**1**],

$$(E(r))^{1/2}-1 = -\frac{2}{\pi}\int_r^{r_0}(c^2-r^2)^{-1/2}c\frac{d\delta_e}{dc}(c)dc.$$

Using Fubini's Theorem and the evenness of the function

$$-2\int_r^{r_0} c\frac{d\delta_e}{dc}dc,$$

we obtain (2.6). Q.E.D.

Theorem 1.3 in the case that $n=2$ immediately follows from Lemmas 2.1 and 2.2. It remains to prove the theorem for $n\geqq 3$. To this end we prepare a lemma. Suppose that $n\geqq 2$ and W is a two dimensional linear subspace of $\boldsymbol{R}^n$. Then W becomes a two dimensional submanifold of $\boldsymbol{R}^n$. Let i_W be the imbedding of W into $\boldsymbol{R}^n$ and $i_W^* g_e^n(x)$ be the induced metric of $g_e^n(x)$ by i_W. Then, the following lemma holds for C^∞-Riemannian manifolds $(W, i_W^* g_e^n(x))$, $(\boldsymbol{R}^2, g_e^2(x))$ and $(\boldsymbol{R}^n, g_e^n(x))$.

LEMMA 2.3. (i) *Let* $[w_1, w_2]$ *be a basis for* W *satisfying* $w_i \cdot w_j = \delta_{ij}$ $(i, j=1, 2)$, *and* $\lambda=(w_1\ w_2)$ *be the* $n\times 2$ *matrix. Then* $(\boldsymbol{R}^2, g_e^2(x))$ *is isometrically imbedded onto* $(W, i_W^* g_e^n(x))$ *through the map* $\boldsymbol{R}^2 \ni z \mapsto \lambda z \in W$.
(ii) *The* C^∞*-submanifold* W *of* $(\boldsymbol{R}^n, g_e^n(x))$ *is totally geodesic: every geodesic for* $(W, i_W^* g_e^n(x))$ *is also a geodesic for* $(\boldsymbol{R}^n, g_e^n(x))$.

PROOF. (i) clearly holds. It is sufficient for the proof of (ii) to show that $\Gamma(x)(u, u)$ belongs to W for x and $u \in W$. By the definition of $\Gamma(x)(u, u)$ we find

$$\Gamma(x)(u, u) = -\Big((x\cdot u)^2 \frac{dG}{dr}(|x|^2) + |u|^2 G(|x|^2)\Big)\frac{x}{E(|x|)}$$

where $G(r)=e(r^{1/2})/r$ is a smooth function on $[0, +\infty[$ by the assumption for e. This completes the proof of Lemma 2.3. Q. E. D.

Using Lemmas 2.1~2.3 and the uniqueness of the solution for the ordinary differential equation, we get the following lemma, which together with Lemmas 2.1~2.3 shows Theorem 1.3 for $n\geqq 3$.

LEMMA 2.4. *Let* $n\geqq 2$ *and* $g_e^n(x)$ *be a surface of revolution on* $\boldsymbol{R}^n$ *with center* 0. *Let* $\lambda=(w_1\ w_2)$ *be an* $n\times 2$ *matrix with* $w_i\cdot w_j=\delta_{ij}$ *for* $i, j=1, 2$. *Then* $T^{(n)}(\lambda z, \lambda\omega) < +\infty$ *for* $(z, \omega)\in \Sigma_-^{(2)}(r_0)$, *and*

$$T^{(n)}(\lambda z, \lambda\omega) = T^{(2)}(z, \omega), \tag{2.10}$$

$$q^{(n)}(t; -r_0, \lambda y, \lambda\omega) = \lambda q^{(2)}(t; -r_0, y, \omega), \tag{2.11}$$

$$p^{(n)}(t; -r_0, \lambda y, \lambda\omega) = \lambda p^{(2)}(t; -r_0, y, \omega), \tag{2.12}$$

where $\omega\in S^1$, $y\in\pi_\omega^2$, $t\geqq T^{*(2)}(y, \omega)$. *Here the superscript* n *denotes the dependence on the dimension of* $\boldsymbol{R}^n$.

§3. Proof of Theorem 1.6.

Let O_{r_0} denote the set of all smooth odd real functions on $\boldsymbol{R}$ with support in $[-r_0, r_0]$. Let $\delta\in\mathrm{O}_{r_0}$, $\eta\in[0, 2\pi]$, and $c\in\boldsymbol{R}$. Put

$$\Lambda_\delta(\eta) = \{c\in\boldsymbol{R} \mid 2\delta(c)\equiv\eta \pmod{2\pi\boldsymbol{Z}}\}, \tag{3.1}$$

$$\Delta T_\delta(c) = 2\int_c^{r_0} r\frac{d\delta}{dc}(r)dr, \tag{3.2}$$

$$\begin{aligned}\sigma_\delta(\eta) &= \sup\{\Delta T_\delta(c) \mid c\in\Lambda_\delta(\eta)\} && (\text{if } \Lambda_\delta(\eta)\neq\emptyset),\\ &= -\infty && (\text{if } \Lambda_\delta(\eta)=\emptyset).\end{aligned} \tag{3.3}$$

Let $\omega\in S^1$. Define $\iota_\omega: \boldsymbol{R}\to\pi_\omega$ by

$$\iota_\omega(c) = c\omega^\perp - r_0\omega. \tag{3.4}$$

c is a so-called impact parameter.

We obtain

PROPOSITION 3.1. *Let $n=2$ and let $g_e^2(x)$ be a surface of revolution on $\boldsymbol{R}^n$ with center* 0. *Then*

$$\iota_\omega^{-1} M_\omega(\theta) = \{c \in \boldsymbol{R} \mid R(-2\delta_e(c))\omega = \theta\} \tag{3.5}$$

and

$$s_\omega(\theta) = \sup\left\{2\int_c^{r_0} r\frac{d\delta_e}{dc}(r)dr \;\middle|\; c \in \iota_\omega^{-1} M_\omega(\theta)\right\}, \tag{3.6}$$

where ω, $\theta \in S^1$, and ι_ω^{-1} is the inverse map of ι_ω.

PROOF. Use Lemmas 2.1 and 2.2. Q. E. D.

COROLLARY 3.2. *Let $n \geqq 2$ and let $g_e^n(x)$ be a surface of revolution on $\boldsymbol{R}^n$ with center* 0. *Then*

$$s_\omega(\theta) = \sigma_{\delta_e}(\mathrm{Arc}\cos \omega \cdot \theta), \tag{3.7}$$

where ω, $\theta \in S^{n-1}$.

PROOF. If $n=2$, (3.7) follows from (3.5) and (3.6). For the case that $n \geqq 3$, use Lemma 2.4. Q. E. D.

For $0 < r_1 < r_0$, define

$$\begin{aligned} \mathrm{O}_{r_0}^{\pm}(r_1) = \Big\{\delta \in \mathrm{O}_{r_0} \;\Big|\; & \pm\frac{d\delta}{dc}(c) > 0 \text{ for } 0 < c < r_1, \\ & \pm\frac{d\delta}{dc}(c) < 0 \text{ for } r_1 < c < r_0 \text{ and } |\delta(r_1)| < \frac{\pi}{2}\Big\}. \end{aligned} \tag{3.8}$$

We note that $\mathrm{O}_{r_0}^{\pm}(r_1)$ is contained in O_{r_0} by the definition.

PROPOSITION 3.3. (i) *Suppose that $\delta \in \mathrm{O}_{r_0}$. Then σ_δ is symmetric with respect to $\eta = \pi$.*
(ii) *Suppose that $\delta \in \mathrm{O}_{r_0}^{+}(r_1)$. Then*

$$\begin{aligned} \sigma_\delta(\eta) &= -2\int_0^{\eta/2} (\delta|_{[r_1, r_0]})^{-1}(\tau)d\tau && (\textit{if } 0 \leqq \eta \leqq 2|\delta(r_1)|), \\ &= -\infty && (\textit{if } 2|\delta(r_1)| < \eta \leqq \pi). \end{aligned} \tag{3.9}$$

Therefore, $d\sigma_\delta/d\eta$ is a strictly monotone increasing function on $[0, 2|\delta(r_1)|]$ onto $[-r_0, -r_1]$ which satisfies

$$\delta(c)=\frac{1}{2}\left(\frac{d\sigma_\delta}{d\eta}\right)^{-1}(-c) \qquad for\ r_1\leqq c\leqq r_0. \tag{3.10}$$

(iii) *Suppose that* $\delta\in O_{r_0}^{-}(r_1)$. *Then*

$$\begin{aligned}\sigma_\delta(\eta) &= -2\int_0^{-\eta/2}(\delta|_{[0,r_1]})^{-1}(\tau)d\tau-2\int_0^{r_0}\delta(r)dr \qquad (if\ 0\leqq\eta\leqq 2|\delta(r_1)|),\\ &= -\infty \qquad (if\ 2|\delta(r_1)|<\eta\leqq\pi).\end{aligned} \tag{3.11}$$

Therefore, $d\sigma_\delta/d\eta$ *is a strictly monotone increasing function on* $[0, 2|\delta(r_1)|]$ *onto* $[0, r_1]$ *which satisfies*

$$\delta(c)=-\frac{1}{2}\left(\frac{d\sigma_\delta}{d\eta}\right)^{-1}(c) \qquad for\ 0\leqq c\leqq r_1. \tag{3.12}$$

PROOF. For the proof of (i), use the fact that ΔT_δ is even and a real number c belongs to $\Lambda_\delta(\eta)$ if and only if $-c$ belongs to $\Lambda_\delta(2\pi-\eta)$. Except for (3.9) and (3.11), the statements (ii) and (iii) are easily obtained. Let $\delta\in O_{r_0}^{\pm}(r_1)$ and $\eta\in[0, 2\pi]$. Since range $\delta=[-|\delta(r_1)|, |\delta(r_1)|]$ and $|\delta(r_1)|<\pi/2$,

$$\Lambda_\delta(\eta)=\delta^{-1}(\eta/2). \tag{3.13}$$

Put

$$c_1=(\delta|_{[0,r_1]})^{-1}(\eta/2), \tag{3.14}$$

$$c_2=(\delta|_{[r_1,r_0]})^{-1}(\eta/2). \tag{3.15}$$

Then

$$\Lambda_\delta(\eta)=\{c_1, c_2\}, \tag{3.16}$$

$$\sigma_\delta(\eta)=\max\{\Delta T_\delta(c_1), \Delta T_\delta(c_2)\}. \tag{3.17}$$

By integration by parts, we obtain

$$\Delta T_\delta(c_2)-\Delta T_\delta(c_1)=2\int_{c_1}^{c_2}(\delta(r)-\eta/2)dr. \tag{3.18}$$

Therefore,

$$\begin{aligned}\sigma_\delta(\eta) &= \Delta T_\delta(c_2) \qquad (\text{if } \delta\in O_{r_0}^{+}(r_1)),\\ &= \Delta T_\delta(c_1) \qquad (\text{if } \delta\in O_{r_0}^{-}(r_1)).\end{aligned} \tag{3.19}$$

Using this, (3.9) and (3.11) are easily obtained. Q. E. D.

We prepare the following deformation lemmas, whose proof is straightforward and so omitted.

LEMMA 3.4. *Let $\delta \in O_{r_0}^{+}(r_1)$ and $0<2\beta<r_1$. Set*

$$c_\beta^1(\delta)=\min\left\{\frac{d\delta}{dc}(c)\,(>0)\;\middle|\;\beta\leqq c\leqq r_1-\beta\right\}.$$

Suppose that ε is a smooth even function with support in $[-(r_1-\beta), -\beta]\cup[\beta, r_1-\beta]$ which satisfies

$$\int_0^{r_1}\varepsilon(\tau)d\tau=0 \tag{3.20}$$

and

$$\sup_c|\varepsilon(c)|<c_\beta^1(\delta). \tag{3.21}$$

Set $\varepsilon^*(c)=\int_c^{r_1}\varepsilon(\tau)d\tau$ for $c\in \boldsymbol{R}^1$. Then $\delta+\varepsilon^*$ belongs to $O_{r_0}^{+}(r_1)$ and

$$\delta+\varepsilon^*|_{[r_1,r_0]}=\delta|_{[r_1,r_0]}. \tag{3.22}$$

LEMMA 3.5. *Let $\delta\in O_{r_0}^{-}(r_1)$ and let $0<2\beta<r_0-r_1$. Set*

$$c_\beta^2(\delta)=\min\left\{\frac{d\delta}{dc}(c)\,(>0)\;\middle|\;r_1+\beta\leqq c\leqq r_0-\beta\right\}.$$

Suppose that ε is a smooth even function with support in $[-r_0, -r_1]\cup[r_1, r_0]$ which satisfies

$$\varepsilon(c)=\varepsilon(r_0+r_1-c)\qquad for\ r_1\leqq c\leqq r_0, \tag{3.23}$$

$$\varepsilon(c)\geqq 0\qquad for\ r_1\leqq c\leqq r_1+\beta, \tag{3.24}$$

$$\varepsilon(c)\leqq 0\qquad for\ r_1+\beta\leqq c\leqq (r_0+r_1)/2, \tag{3.25}$$

$$\int_{r_1}^{r_0}\varepsilon(\tau)d\tau=0, \tag{3.26}$$

and

$$\sup_c|\varepsilon(c)|<c_\beta^2(\delta). \tag{3.27}$$

Set $\varepsilon^*(c)=\int_0^c\varepsilon(\tau)d\tau$ for $c\in\boldsymbol{R}$. Then $\delta+\varepsilon^*$ belongs to $O_{r_0}^{-}(r_1)$,

$$\int_0^{r_0}(\delta+\varepsilon^*)(\tau)d\tau=\int_0^{r_0}\delta(\tau)d\tau, \tag{3.28}$$

and

$$\delta+\varepsilon^*|_{[0,r_1]}=\delta|_{[0,r_1]}. \tag{3.29}$$

THEOREM 3.6 (H. Gluck and D. Singer [1]). *Let* $n \geqq 2$ *and let* k *be a non negative integer.* (i) (Existence) *For any positive number* γ, *choose a positive number* κ *sufficiently small. Then for any* $\delta \in O_{r_0}$ *which satisfies*

$$\max_{0 \leq j \leq 2k+1} \sup_c |\delta^{(j)}(c)| < \kappa,$$

there exists only one surface of revolution $g_e^n(x)$ *on* $\boldsymbol{R}^n$ *with center* 0 *such that*

$$\max_{|\alpha| \leq k} \sup_x |\partial_x^\alpha (g_e^n(x) - I_n)| < \gamma,$$

and $\delta_e = \delta$.
(ii) (Uniqueness) *Let* $g_{e_i}^n(x)$ $(i=1, 2)$ *be surfaces of revolution on* $\boldsymbol{R}^n$ *with center* 0. *Suppose that* $\delta_{e_1} = \delta_{e_2}$. *Then*

$$g_{e_1}^n(x) = g_{e_2}^n(x).$$

(iii) (Inversion formula) *Let* $g_e^n(x)$ *be a surface of revolution on* $\boldsymbol{R}^n$. *Then*

$$E(r) = \left(1 - \frac{2}{\pi} \int_r^{r_0} \frac{c d\delta_e}{dc}(c)(c^2 - r^2)^{-1/2} dc\right)^2, \qquad 0 < r < r_0.$$

Theorem 3.6 is a special case of Theorems 4.1 and 4.3 in [1] and we omit the proof.

PROPOSITION 3.7. *Let* κ *be a positive number and let* k *be a non negative integer. Suppose that* $\delta \in O_{r_0}^{\pm}(r_1)$ *and*

$$\max_{0 \leq j \leq 2k+1} \sup_c |\delta^{(j)}(c)| < \kappa/2.$$

Then there exist infinitely many $\delta_i \in O_{r_0}^{\pm}(r_1)$, $i=1, 2, \cdots$ *such that*

$$\max_{0 \leq j \leq 2k+1} \sup_c |\delta_i^{(j)}(c)| < \kappa$$

and

$$\sigma_{\delta_i} = \sigma_\delta.$$

PROOF. Use Proposition 3.3, Lemmas 3.4 and 3.5. Q.E.D.

Proof of (i) in Theorem 1.6 follows from Corollary 3.2, (i) in Theorem 3.6, and Proposition 3.7. Proof of (ii) in Theorem 1.6 follows from Corollary 3.2, Proposition 3.3 and (ii), (iii) in Theorem 3.6. In particular, (1.12) and (1.13) follows from (3.7), (3.10), (3.12), and (iii) in Theorem 3.6. This completes the proof of Theorem 1.6.

§ 4. Proof of Theorems 1.4 and 1.5.

Let $n\geqq 2$, and let $g(x)$ be a C^{∞}-Riemannian metric on $\boldsymbol{R}^n$ with $g(x)=I_n$ for $|x|\geqq r_0$, which satisfies (G). Recalling (2.2) and (2.3) we have

PROPOSITION 4.1. *Let $\omega\in S^{n-1}$ and $y\in\pi_{\omega}^2$.*
(i) *If $t\in[-r_0, -(2r_0^2-|y|^2)^{1/2}]$, then*

$$\partial_x q(t;-r_0, y, \omega)=I_n, \tag{4.1}$$

$$\partial_x p(t;-r_0, y, \omega)=0, \tag{4.2}$$

$$\partial_\xi q(t;-r_0, y, \omega)=(t+r_0)(I_n-\omega{}^t\omega), \tag{4.3}$$

$$\partial_\xi p(t;-r_0, y, \omega)=I_n. \tag{4.4}$$

(ii) *If $t\in[-(2r_0^2-|y|^2)^{1/2}, T^*(y, \omega)]$, then*

$$|\partial_x q(t;-r_0, y, \omega)-I_n|, \qquad |\partial_x p(t;-r_0, y, \omega)|,$$
$$|\partial_\xi q(t;-r_0, y, \omega)| \quad \text{and} \quad |\partial_\xi p(t;-r_0, y, \omega)-I_n|$$

are bounded by

$$CT(A_\omega(y), \omega)|g^{-1}|_\infty\|g\|\exp\{CT(A_\omega(y), \omega)|g^{-1}|_\infty(\|g\|+1)\},$$

where C is an absolute constant and $\|g\|=|g^{-1}|_\infty|\partial_x g|_\infty^2+|\partial_x^2 g|_\infty$.
(iii) *If $t\in[T^*(y, \omega), +\infty[$, then*

$$\begin{aligned}\partial_x q(t;-r_0, y, \omega)&=\partial_x q(T^*(y, \omega);-r_0, y, \omega)\\&\quad+(t-T^*(y, \omega))\{I_n-V(A_\omega(y), \omega){}^tV(A_\omega(y), \omega)\}\partial_x p(T^*(y, \omega);-r_0, y, \omega),\end{aligned} \tag{4.5}$$

$$\partial_\xi p(t;-r_0, y, \omega)=\partial_\xi p(T^*(y, \omega);-r_0, y, \omega). \tag{4.6}$$

REMARK. If $y\in\pi_\omega^1$, then (4.1), (4.2), (4.3) and (4.4) hold for any $t\in\boldsymbol{R}$.

PROOF. Differentiate the integral equation for $(q(t), p(t))=(q(t;-r_0, x, \xi), p(t;-r_0, x, \xi))$:

$$q(t)=x+\int_{-r_0}^t \nabla_\xi H(q(\tau), p(\tau))d\tau, \tag{4.7}$$

$$p(t)=\xi-\int_{-r_0}^t \nabla_x H(q(\tau), p(\tau))d\tau, \tag{4.8}$$

and apply Lemma 2.1 and Gronwall's inequality. Then we get the proposition.
Q. E. D.

PROPOSITION 4.2. *Let $\omega\in S^{n-1}$ and $y\in\pi_\omega^2\setminus\{-r_0\omega\}$. Choose $\eta\in S^{n-1}$ such that $y\cdot\eta<0$ and $\eta\cdot\omega=0$. If $t\in[T^*(y, \omega), +\infty[$, then*

$$\begin{aligned}(4.9)\qquad &\partial_x q(t\,;-r_0,\,y,\,\omega)\eta\\ &=\partial_h(Q(A_\omega(y+h\eta),\,\omega)-T^*(y+h\eta,\,\omega)V(A_\omega(y),\,\omega)\\ &\quad+(t-T^*(y,\,\omega))\{I_n-V(A_\omega(y),\,\omega)^t V(A_\omega(y),\,\omega)\}V(A_\omega(y+h\eta),\,\omega))|_{h\to+0},\end{aligned}$$

$$(4.10)\qquad \partial_x q(t\,;-r_0,\,y,\,\omega)\omega=V(A_\omega(y),\,\omega).$$

PROOF. Since $y+h\eta\in\pi_\omega^2$ for $0<h<2|y\cdot\eta|$, (2.4) in Lemma 2.1 yields

$$(4.11)\qquad q(T^*(y+h\eta,\,\omega)\,;-r_0,\,y+h\eta,\,\omega)=Q(A_\omega(y+h\eta),\,\omega).$$

Differentiating (4.11) with respect to h, we obtain

$$\begin{aligned}(4.12)\qquad &\partial_x q(T^*(y+h\eta,\,\omega)\,;-r_0,\,y+h\eta,\,\omega)\eta\\ &=\partial_h\{Q(A_\omega(y+h\eta),\,\omega)\}\\ &\quad-\partial_h\{T^*(y+h\eta,\,\omega)\}\times\partial_t q(T^*(y+h\eta,\,\omega)\,;-r_0,\,y+h\eta,\,\omega).\end{aligned}$$

On the other hand, using (1.6), (2.4) and $|q(T^*(y,\,\omega)\,;-r_0,\,y,\,\omega)|=r_0$, we see

$$\begin{aligned}(4.13)\qquad \partial_t q(T^*(y,\,\omega)\,;-r_0,\,y,\,\omega)&=p(T^*(y,\,\omega)\,;-r_0,\,y,\,\omega)\\ &=V(A_\omega(y),\,\omega).\end{aligned}$$

Put (4.13) into (4.12). In view of (4.5) we thus obtain (4.9).

Next we shall prove (4.10). Since $|y-h\omega|\geqq r_0$,

$$\begin{aligned}(4.14)\qquad &(q(T^*(y,\,\omega)\,;-r_0,\,y,\,\omega),\,p(T^*(y,\,\omega)\,;-r_0,\,y,\,\omega))\\ &=(q(T^*(y,\,\omega)+h\,;-r_0,\,y-h\omega,\,\omega),\,p(T^*(y,\,\omega)+h\,;-r_0,\,y-h\omega,\,\omega)).\end{aligned}$$

Therefore,

$$(4.15)\qquad \partial_h\{q(T^*(y,\,\omega)+h\,;-r_0,\,y-h\omega,\,\omega)\}|_{h\to+0}=0.$$

On the other hand,

$$\begin{aligned}(4.16)\qquad &\partial_h\{q(T^*(y,\,\omega)+h\,;-r_0,\,y-h\omega,\,\omega)\}\\ &=\partial_t q(T^*(y,\,\omega)+h\,;-r_0,\,y-h\omega,\,\omega)-\partial_x q(T^*(y,\,\omega)+h\,;-r_0,\,y-h\omega,\,\omega)\omega.\end{aligned}$$

Then we obtain (4.10) by (4.13), (4.15) and (4.16). Q. E. D.

We now compute the following quantities:

$$(4.17)\qquad \partial_h\{Q(A_\omega(y+h\eta),\,\omega)\}|_{h\to+0},$$

$$(4.18)\qquad \partial_h\{V(A_\omega(y+h\eta),\,\omega)\}|_{h\to+0},$$

$$(4.19)\qquad \partial_h\{T^*(y+h\eta,\,\omega)\}|_{h\to+0}.$$

LEMMA 4.3. *Let* $n=2$, $\omega\in S^1$ *and* $y\in\pi_\omega^2\setminus\{-r_0\omega\}$. *Put* $\eta=-(\mathrm{sgn}\,y\cdot\omega^\perp)\omega^\perp$. *Then*

$$\partial_h\{Q(A_\omega(y+h\eta),\ \omega)\}|_{h\to+0} = -\partial_c\{Q(A_\omega(\iota_\omega(c)),\ \omega)\}|_{c=y\cdot\omega^\perp}\operatorname{sgn} y\cdot\omega^\perp, \tag{4.20}$$

$$\partial_h\{V(A_\omega(y+h\eta),\ \omega)\}|_{h\to+0} = -\partial_c\{V(A_\omega(\iota_\omega(c)),\ \omega)\}|_{c=y\cdot\omega^\perp}\operatorname{sgn} y\cdot\omega^\perp, \tag{4.21}$$

$$\partial_h\{T^*(y+h\eta,\ \omega)\}|_{h\to+0} = -\partial_c\{T^*(\iota_\omega(c),\ \omega)\}|_{c=y\cdot\omega^\perp}\operatorname{sgn} y\cdot\omega^\perp. \tag{4.22}$$

PROOF. Put $c=y\cdot\omega^\perp-h\operatorname{sgn} y\cdot\omega^\perp$. Then $y+h\eta$ is equal to $\iota_\omega(c)$. Therefore, Lemma 4.3 is easily shown. Q.E.D.

From now until after the completion of the proof of Theorem 1.4 we assume that $n=2$ and $g(x)=g_e^2(x)$ is a surface of revolution on $\boldsymbol{R}^2$ with center 0. For simplicity, we write δ, δ' for δ_e, $d\delta_e/dc$, respectively. Suppose that $|c|<r_0$ and put $z=A_\omega(\iota_\omega(c))$. Then $A_\omega(\iota_\omega(c))=-(r^2-c^2)^{1/2}\omega+c\omega^\perp$ and $q_\omega^0(T_0(z,\omega)/2\,;z)=c\omega^\perp$. We note $-B(R(\alpha)\omega^\perp,\omega)=R(2\alpha)\omega$ for $\omega\in S^1$ and $\alpha\in\boldsymbol{R}$. Using Lemma 2.2, we obtain that

$$Q(A_\omega(\iota_\omega(c)),\ \omega) = B(cR(-\delta(c))\omega^\perp,\ -(r^2-c^2)^{1/2}\omega+c\omega^\perp), \tag{4.23}$$

$$V(A_\omega(\iota_\omega(c)),\ \omega) = R(-2\delta(c))\omega, \tag{4.24}$$

$$T^*(\iota_\omega(c),\ \omega) = (r_0^2-c^2)^{1/2}-\Delta T_\delta(c). \tag{4.25}$$

Differentiating these, we obtain that

$$\frac{d}{dc}(T^*(\iota_\omega(c),\ \omega)) = -c(r_0^2-c^2)^{-1/2}+2c\delta'(c). \tag{4.26}$$

$$\frac{d}{dc}(V(A_\omega(\iota_\omega(c)),\ \omega)) = -2\delta'(c)R(-2\delta(c))\omega^\perp, \tag{4.27}$$

$$\begin{aligned}\frac{d}{dc}&(Q(A_\omega(\iota_\omega(c)),\ \omega))\\ &= 2\{c(r_0^2-c^2)^{-1/2}R(-\delta(c))\omega^\perp\cdot\omega-(r_0^2-c^2)^{1/2}\delta'(c)\omega\cdot R(\delta(c))\omega\\ &\qquad+\omega\cdot R(\delta(c))\omega+c\delta'(c)\omega\cdot R(\delta(c))\omega^\perp\}R(-\delta(c))\omega^\perp\\ &\quad+2\delta'(c)\{-(r_0^2-c^2)^{1/2}R(-\delta(c))\omega^\perp\cdot\omega+cR(-\delta(c))\omega\cdot\omega\}R(-\delta(c))\omega\\ &\qquad-c(r_0^2-c^2)^{-1/2}\omega-\omega^\perp.\end{aligned} \tag{4.28}$$

Put $u=\omega\cdot R(\delta(c))\omega$ and $v=\omega^\perp\cdot R(\delta(c))\omega$. Let $\omega\in S^1$ and let $y\in\pi_\omega^2\setminus\{-r_0\omega\}$. Put $c=y\cdot\omega^\perp$. Then, $0<|c|<r_0$. By Lemma 4.3,

$$\begin{aligned}\partial_h&\{Q(A_\omega(y+h\eta),\ \omega)\}|_{h\to+0}\\ &= \{\Omega_1(c)^t(u\ \ v)\cdot{}^t(u\ \ v)+|c|(r_0^2-c^2)^{-1/2}\}\omega\\ &\quad+\{\Omega_2(c)^t(u\ \ v)\cdot{}^t(u\ \ v)-1\}\eta,\end{aligned} \tag{4.29}$$

$$\begin{aligned}\{I_n&-V(A_\omega(y),\ \omega)^tV(A_\omega(y),\ \omega)\}\partial_h\{V(A_\omega(y+h\eta),\ \omega)\}|_{h\to+0}\\ &= \{2\delta'(c)(\operatorname{sgn} c)(R(-2\delta(c))(I_n-\omega^t\omega)\omega^\perp)\cdot\omega\}\omega\\ &\quad+\{-2\delta'(c)(R(-2\delta(c))(I_n-\omega^t\omega)\omega^\perp)\cdot\omega^\perp\}\eta,\end{aligned} \tag{4.30}$$

$$V(A_\omega(y),\ \omega) = \{R(-2\delta(c))\omega\cdot\omega\}\omega + \{-\operatorname{sgn} c\ R(-2\delta(c))\omega\cdot\omega^\perp\}\eta\,, \tag{4.31}$$

$$\partial_h\{T^*(y+h\eta,\ \omega)\}\,|_{h\to+0} = |c|\{(r_0^2-c^2)^{-1/2}-2\delta'(c)\}\,, \tag{4.32}$$

where $\Omega_1(c)$ and $\Omega_2(c)$ are 2×2 matrices satisfying

$$\Omega_1(c)_{11} = -2|c|\delta'(c), \tag{4.33}$$

$$\Omega_1(c)_{12} = \Omega_1(c)_{21} = \operatorname{sgn} c\ (r_0^2-c^2)^{1/2}\{2\delta'(c)-(r_0^2-c^2)^{-1/2}\}\,, \tag{4.34}$$

$$\Omega_1(c)_{22} = -2|c|\{(r_0^2-c^2)^{-1/2}-\delta'(c)\}\,, \tag{4.35}$$

$$\Omega_2(c)_{11} = 2\{1-(r_0^2-c^2)^{1/2}\delta'(c)\}\,, \tag{4.36}$$

$$\Omega_2(c)_{12} = \Omega_2(c)_{21} = c\{(r_0^2-c^2)^{-1/2}-2\delta'(c)\}\,, \tag{4.37}$$

$$\Omega_2(c)_{22} = 2\delta'(c)(r_0^2-c^2)^{1/2}. \tag{4.38}$$

The preceding calculation leads to the following proposition.

PROPOSITION 4.4. *Let $n=2$ and $g_e^2(x)$ be a surface of revolution on $\boldsymbol{R}^2$ with center 0. Suppose that $\omega\in S^1$ and $y\in\pi_\omega^2\setminus\{-r_0\omega\}$. Put $\eta=-(\operatorname{sgn} y\cdot\omega^\perp)\omega^\perp$. Then the matrix representation for $\partial_x q(t\,;-r_0,\ y,\ \omega)$ with the basis $[\omega,\ \eta]$ of $\boldsymbol{R}^2$ is equal to $K(t\,;c,\ \omega)|_{c=y\cdot\omega^\perp}$, where $K(t\,;c,\ \omega)$, $t\geqq T^*(y,\ \omega)$ and $|c|<r_0$, is a 2×2 matrix satisfying*

$$K(t\,;c,\ \omega)_{11} = R(-2\delta(c))\omega\cdot\omega, \tag{4.39}$$

$$\begin{aligned} K(t\,;c,\ \omega)_{12} = {} & \Omega_1(c)^t(u\ \ v)\cdot{}^t(u\ \ v)+|c|(r_0^2-c^2)^{-1/2} \\ & -|c|(R(-2\delta(c))\omega\cdot\omega)\{(r_0^2-c^2)^{-1/2}-2\delta'(c)\} \\ & +2\{t+\Delta T_\delta(c)-(r_0^2-c^2)^{1/2}\}\delta'(c)(\operatorname{sgn} c)\{R(-2\delta(c))(I_n-\omega^t\omega)\omega^\perp\}\cdot\omega, \end{aligned} \tag{4.40}$$

$$K(t\,;c,\ \omega)_{21} = -\operatorname{sgn} c\ R(-2\delta(c))\omega\cdot\omega^\perp, \tag{4.41}$$

$$\begin{aligned} K(t\,;c,\ \omega)_{22} = {} & \Omega_2(c)^t(u\ \ v)\cdot{}^t(u\ \ v)-1 \\ & +|c|\operatorname{sgn} c\ (R(-2\delta(c))\omega\cdot\omega^\perp)\{(r_0^2-c^2)^{-1/2}-2\delta'(c)\} \\ & -2\{t+\Delta T_\delta(c)-(r_0^2-c^2)^{1/2}\}\delta'(c)\{R(-2\delta(c))(I_n-\omega^t\omega)\omega^\perp\}\cdot\omega^\perp. \end{aligned} \tag{4.42}$$

From this proposition we have immediately

PROPOSITION 4.5. *Let $n=2$ and $g_e^2(x)$ be a surface of revolution on $\boldsymbol{R}^2$ with center 0. Suppose that $\omega\in S^1$ and $y\in\pi_\omega^2\setminus\{-r_0\omega\}$. If $t\in[T^*(y,\ \omega),\ +\infty[$, then*

$$\begin{aligned} & \det\partial_x q(t\,;-r_0,\ y,\ \omega) \\ & = -2(t-T^*(y,\ \omega))\delta'(y\cdot\omega^\perp)+\det\partial_x q(T^*(y,\ \omega)\,;-r_0,\ y,\ \omega). \end{aligned} \tag{4.43}$$

PROOF. By Proposition 4.4, we see that $\det\partial_x q(t\,;-r_0,\ y,\ \omega)=a(t-T^*(y,\ \omega))+b$

for some $a, b \in \boldsymbol{R}$. From this $b=\det \partial_x q(T^*(y, \omega); -r_0, y, \omega)$, and a direct computation shows $a=-2\delta'(y\cdot\omega^{\perp})$. Q. E. D.

Now we can give the proof of Theorem 1.4.

PROOF OF THEOREM 1.4. Suppose that $y \in \pi_\omega^2$. By Lemma 2.2,

$$T(A_\omega(y), \omega) \leqq 2r_0(|\delta_e|_\infty+1). \tag{4.44}$$

We note that if $g_e^n(x)-I_n$ is sufficiently small in the C^2-topology, then $\|g_e^n\|$ is arbitrarily small. Therefore, we see by Proposition 4.1 that (1.10) holds and (1.11) holds for $t \in [-r_0, +\infty[$. Corollary 3.2 and the proof of Proposition 3.3 show that if $y \in \tilde{M}_\omega(\theta)$, then $y \in \pi_\omega^2$ and $\delta_e'(y\cdot\omega^{\perp}) \leqq 0$. Consequently, Proposition 4.5 shows that (1.10) holds for any $y \in \tilde{M}_\omega(\theta)$ and $t \in [T^*(y, \omega), +\infty[$. This completes the proof of Theorem 1.4. Q. E. D.

The rest of this section is devoted to the proof of Theorem 1.5. Let $n \geqq 3$, $\omega \in S^{n-1}$ and $y \in \pi_\omega$ with $0<|y+r_0\omega|$. Put $\omega^{\perp}(y)=(y+r_0\omega)/|y+r_0\omega|$, then $\omega^{\perp}(y)$ is perpendicular to ω and belongs to S^{n-1}. Choose $\eta_1, \cdots, \eta_{n-2} \in \boldsymbol{R}^n$ such that $[\omega, \omega^{\perp}(y), \eta_1, \cdots, \eta_{n-2}]$ becomes an orthonormal basis of $\boldsymbol{R}^n$. Fix $1 \leqq i \leqq n-2$ and put $\eta=\eta_i$. Let P be the orthogonal projection of $\boldsymbol{R}^n$ onto the orthogonal complement $(L[\omega, \omega^{\perp}(y), \eta])^{\perp}$ of the linear hull $L[\omega, \omega^{\perp}(y), \eta]$ of $\omega, \omega^{\perp}(y), \eta$ and put

$$\begin{aligned} R_h^{\omega, y}(x) = {} & (x\cdot\omega)\omega+\{R(h)^t(x\cdot\omega^{\perp}(y)\ \ x\cdot\eta)\cdot{}^t(1\ \ 0)\}\omega^{\perp}(y) \\ & +\{R(h)^t(x\cdot\omega^{\perp}(y)\ \ x\cdot\eta)\cdot{}^t(0\ \ 1)\}\eta+Px \end{aligned} \tag{4.45}$$

for $h \in \boldsymbol{R}$ and $x \in \boldsymbol{R}^n$. Since ${}^tR_h^{\omega, y}R_h^{\omega, y}=I_n$, $g_e^n(x)$ is invariant with respect to $R_h^{\omega, y}$. Therefore,

$$R_h^{\omega, y}(q_\omega(t; y)) = q_{\tilde{\omega}}(t; \tilde{y})|_{(\tilde{y}, \tilde{\omega})=(R_h^{\omega, y}(y), R_h^{\omega, y}(\omega))} \tag{4.46}$$

for any $t \in \boldsymbol{R}$. Using $R_h^{\omega, y}\omega=\omega$, we see that

$$q_\omega(t; R_h^{\omega, y}(y)) = R_h^{\omega, y}(q_\omega(t; y)). \tag{4.47}$$

Since $y=-r_0\omega+|y+r_0\omega|\omega^{\perp}(y)$,

$$R_h^{\omega, y}(y) = -r_0\omega+|y+r_0\omega|\{(\cos h)\omega^{\perp}(y)+(\sin h)\eta\} \tag{4.48}$$

and $\partial_h(R_h^{\omega, y}(y))=|y+r_0\omega|\{(-\sin h)\omega^{\perp}(y)+(\cos h)\eta\}$. Hence,

$$\partial_h(R_h^{\omega, y}(y))|_{h\to 0} = |y+r_0\omega|\eta. \tag{4.49}$$

On the other hand, the equation

$$q(t; -r_0, R_h^{\omega, y}(y), \omega) = R_h^{\omega, y}(q(t; -r_0, y, \omega)) \tag{4.50}$$

is easily shown by (4.47). Since $q(t;-r_0, y, \omega)\in L[\omega, \omega^{\perp}(y)]$, $q(t;-r_0, y, \omega)\cdot\eta=0$ and $Pq(t;-r_0, y, \omega)=0$. We obtain that

$$\begin{aligned}(4.51)\qquad & R_h^{\omega, y}(q(t;-r_0, y, \omega)) \\ & = (q(t;-r_0, y, \omega)\cdot\omega)\omega+(q(t;-r_0, y, \omega)\cdot\omega^{\perp}(y))\{(\cos h)\omega^{\perp}(y)+(\sin h)\eta\},\end{aligned}$$

$$\begin{aligned}(4.52)\qquad & \partial_h\{R_h^{\omega, y}(q(t;-r_0, \omega, y))\} \\ & = (q(t;-r_0, y, \omega)\cdot\omega^{\perp}(y))\{(-\sin h)\omega^{\perp}(y)+(\cos h)\eta\}.\end{aligned}$$

As $h\to 0$, we obtain that

$$(4.53)\qquad \partial_h\{R_h^{\omega, y}(q(t;-r_0, y, \omega))\}|_{h\to 0} = (q(t;-r_0, y, \omega)\cdot\omega^{\perp}(y))\eta.$$

Differentiate (4.50) and use (4.49) and (4.53). Then we obtain

$$\partial_x q(t;-r_0, y, \omega)\eta = |y+r_0\omega|^{-1}(q(t;-r_0, y, \omega)\cdot\omega^{\perp}(y))\eta.$$

Therefore, $L[\eta_1, \cdots, \eta_{n-2}]$ reduces $\partial_x q(t;-r_0, y, \omega): \boldsymbol{R}^n\to\boldsymbol{R}^n$ and the matrix representation of $\partial_x q(t;-r_0, y, \omega)|_{L[\eta_1,\cdots,\eta_{n-2}]}$ with the basis $[\eta_1, \cdots, \eta_{n-2}]$ of $L[\eta_1, \cdots, \eta_{n-2}]$ is given by

$$(4.54)\qquad |y+r_0\omega|^{-1}(q(t;-r_0, y, \omega)\cdot\omega^{\perp}(y))I_{n-2}.$$

In the case that $n=2$, we note that $\omega^{\perp}(y)=(\operatorname{sgn} y\cdot\omega^{\perp})\omega^{\perp}$ for $\omega\in S^1$, $y\in\pi_{\omega}\setminus\{-r_0\omega\}$. Therefore, we can easily rewrite (4.54) as below. Let $\omega\in S^1$, $y\in\pi_{\omega}\setminus\{-r_0\omega\}$, and $[w_1, w_2, \cdots, w_n]$ be an orthonormal basis of $\boldsymbol{R}^n$. Then, for $3\leqq j\leqq n$,

$$\begin{aligned}(4.55)\qquad & \partial_x q^{(n)}(t;-r_0, (w_1\ w_2)y, (w_1\ w_2)\omega)w_j \\ & = (\operatorname{sgn} y\cdot\omega^{\perp})|y+r_0\omega|^{-1}(q^{(2)}(t;-r_0, y, \omega)\cdot\omega^{\perp})w_j.\end{aligned}$$

Now, we compute the quantity $\det\partial_x q^{(n)}(t;-r_0, y, \omega)$ for $\omega\in S^{n-1}$, $y\in\pi_{\omega}^2\setminus\{-r_0\omega\}$, and $t\in[T^{*(n)}(y, \omega), +\infty[$.

PROPOSITION 4.6. *Let $n\geqq 3$, $\omega\in S^1$ and $y\in\pi_{\omega}^2\setminus\{-r_0\omega\}$. Let $\lambda=(w_1\ w_2)$ be an $n\times 2$ matrix with $w_i\cdot w_j=\delta_{ij}$ for $i, j=1, 2$. If $t\in[T^{*(2)}(y, \omega), +\infty[$, then*

$$\begin{aligned}(4.56)\qquad & \det\partial_x q^{(n)}(t;-r_0, \lambda y, \lambda\omega) \\ & = |y+r_0\omega|^{-(n-2)}(q^{(2)}(t;-r_0, y, \omega)\cdot(\operatorname{sgn} y\cdot\omega^{\perp})\omega^{\perp})^{n-2} \\ & \quad\times\det\partial_x q^{(2)}(t;-r_0, y, \omega).\end{aligned}$$

PROOF. Put $\eta=-(\operatorname{sgn} y\cdot\omega^{\perp})\omega^{\perp}$, then $y\cdot\eta<0$, $\omega\cdot\eta=0$, and $\eta\in S^1$. By Lemma 2.3 and Proposition 4.2,

$$(4.57)\qquad \partial_x q^{(n)}(t;-r_0, \lambda y, \lambda\omega)\lambda\eta = \lambda\partial_x q^{(2)}(t;-r_0, y, \omega)\eta,$$

$$(4.58)\qquad \partial_x q^{(n)}(t;-r_0, \lambda y, \lambda\omega)\lambda\omega = \lambda\partial_x q^{(2)}(t;-r_0, y, \omega)\omega.$$

Here we have used the fact $T^{*(n)}(\lambda y, \lambda\omega)=T^{*(2)}(y, \omega)$ by Lemma 2.4. Therefore, we obtain (4.56) by (4.55), (4.57), and (4.58). Q. E. D.

PROPOSITION 4.7. *Let* $n\geqq 3$, $\omega\in S^{n-1}$ *and* $y\in\pi_{\omega}^{2}\setminus\{-r_0\omega\}$. *If*

$$0<\delta(|y\cdot\omega^{\perp}|)<2^{-1}\pi, \tag{4.59}$$

then there exists a $t\in]-r_0, +\infty[$ *such that*

$$\det\partial_x q^{(n)}(t;-r_0, \lambda y, \eta\omega)=0, \tag{4.60}$$

where $\lambda=(w_1\ w_2)$, $w_i\cdot w_j=\delta_{ij}$ *and* $w_i\in\boldsymbol{R}^n$ $(i=1, 2)$.

PROOF. We note that

$$q^{(2)}(t;-r_0, y, \omega)\cdot(\operatorname{sgn} y\cdot\omega^{\perp})\omega^{\perp}|_{t=-r_0}=|y\cdot\omega^{\perp}|>0. \tag{4.61}$$

Furthermore, we see that

$$\lim_{t\to+\infty} q^{(2)}(t;-r_0, y, \omega)\cdot(\operatorname{sgn} y\cdot\omega^{\perp})\omega^{\perp}=-\infty.$$

By (2.4) in Lemma 2.1, it is sufficient for the proof of this fact to show that $V(A_{\omega}(y), \omega)\cdot(\operatorname{sgn} y\cdot\omega^{\perp})\omega^{\perp}<0$. Put $y=\iota_{\omega}(c)$ $(0<|c|<r_0)$. Using (4.24), we obtain the left hand side of the preceding inequality is equal to

$$-2(\operatorname{sgn} c)\{R(\pi/2-(\operatorname{sgn} c)\delta(|c|))\omega\cdot\omega\}\times\{R(-(\operatorname{sgn} c)\delta(|c|))\omega\cdot\omega\}.$$

Since $0<\delta(|c|)<\pi/2$, we see that $(\operatorname{sgn} c)\{R(\pi/2-(\operatorname{sgn} c)\delta(|c|))\}\omega\cdot\omega$ and $R(-(\operatorname{sgn} c)\delta(|c|))\omega\cdot\omega$ are positive. Therefore the preceding inequality is valid and there exists a $T\in]-r_0, +\infty[$ such that

$$q^{(2)}(t;-r_0, y, \omega)\cdot(\operatorname{sgn} y\cdot\omega^{\perp})\omega^{\perp}<0 \qquad \text{for } t\in[T, +\infty[. \tag{4.62}$$

Hence there exists a $t\in]-r_0, +\infty[$ such that

$$q^{(2)}(t;-r_0, y, \omega)\cdot(\operatorname{sgn} y\cdot\omega^{\perp})\omega^{\perp}=0. \tag{4.63}$$

Therefore, the preceding statement is obtained by (4.56) in Proposition 4.6.

Q. E. D.

PROPOSITION 4.8. *Let* $\omega\in S^1$ *and* $y\in\pi_{\omega}^{2}\setminus\{-r_0\omega\}$. *If*

$$q^{(2)}(t;-r_0, y, \omega)\cdot(\operatorname{sgn} y\cdot\omega^{\perp})\omega^{\perp}|_{t=T^{*(2)}(y,\omega)}>0, \tag{4.64}$$

$$-2^{-1}\pi<\delta(|y\cdot\omega^{\perp}|)<0, \tag{4.65}$$

then for any $t\in[T^{*(2)}(y, \omega), +\infty[$

$$q^{(2)}(t;-r_0, y, \omega)\cdot(\operatorname{sgn} y\cdot\omega^{\perp})\omega^{\perp}>0. \tag{4.66}$$

PROOF. Use Lemmas 2.1 and 2.2. Then the proposition is elementarily shown. Q. E. D.

PROOF OF (i) IN THEOREM 1.5. Since $2^{-1}\pi>\delta(c)>0$ for $0<c<r_0$, Propositions 4.6 and 4.7 imply (i). Q. E. D.

PROOF OF (ii) IN THEOREM 1.5. Since $g_\varepsilon^2(x)-I_2$ and $g_\varepsilon^3(x)-I_3$ are sufficiently small in the C^2-topology we obtain that

$$q^{(2)}(t\,;-r_0,\ y,\ \omega)\cdot(\operatorname{sgn} y\cdot\omega^{\perp})\omega^{\perp}|_{t=T^{*(2)}(y,\omega)}>0$$

for $\omega\in S^1$ and $y\in\pi_\omega^2\setminus\{-r_0\omega\}$. Since $-2^{-1}\pi<\delta(c)<0$ for $0<c<r_0$, Proposition 4.8 shows that $q^{(2)}(t\,;-r_0,\ y,\ \omega)\cdot(\operatorname{sgn} y\cdot\omega^{\perp})\omega^{\perp}>0$ for any $t\geqq T^{*(2)}(y,\ \omega)$. Therefore, Proposition 4.6 and Theorem 1.4 show that

$$\det\partial_x q^{(n)}(t\,;-r_0,\ y,\ \omega)>0$$

for $t\geqq T^{*(n)}(y,\ \omega)$ and $y\in\tilde{M}_\omega(\theta)$ $(\omega\neq\theta)$. For $-r_0\leqq t\leqq T^{*(n)}(y,\ \omega)$, we can use (i) and (ii) in Proposition 4.1. Q. E. D.

References

[1] H. Gluck and D. Singer, Scattering of geodesic fields, II, Ann. of Math., **110** (1979), 205-225.

[2] H. Soga, Singular support of the scattering kernel for the wave equation perturbed in a bounded domain, Proc. Japan Acad. Ser. A, **59** (1983), 13-16.

[3] H. Soga, Singular support of the scattering kernel for the acoustic equation in inhomogeneous media, Comm. Partial Differential Equations, **9** (1984), 467-502.

[4] P. Lax and R. Phillips, Scattering Theory, Academic Press, New York, 1967.

Masaru IKEHATA
Department of Mathematics
Tokyo Metropolitan University
Fukasawa, Setagaya-ku
Tokyo 158, Japan