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Spectral synthesis on the algebra of Hankel transforms

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1. Introduction.

Let $\nu \geqq -1/2$. The Hankel transform of order ν is given by

$$\hat{g}(y)=\int_{0}^{\infty} g(x) J_{\nu}(xy) x^{\nu+1} y^{-\nu} dx, \quad y \geqq 0$$

for a function $g(x)$ on $[0, \infty)$, where $J_{\nu}(t)$ is the Bessel function of the first kind of order ν. Let

$$A^{(\nu)}=\left\{\hat{g}\ ;\ \int_{0}^{\infty}|g(x)| x^{2\nu+1} dx<\infty\right\},$$

and introduce a norm to $A^{(\nu)}$ by

$$\|\hat{g}\|=\frac{1}{2^{\nu} \Gamma(\nu+1)} \int_{0}^{\infty}|g(x)| x^{2\nu+1} dx .$$

Then the followings are known (cf. [**10**], [7]):

(i) $A^{(\nu)}$ consists of continuous functions on $[0, \infty)$ vanishing at infinity.

(ii) $A^{(\nu)}$ is a semisimple regular Banach algebra with the product of pointwise multiplication, and the maximal ideal space of $A^{(\nu)}$ is identified with the interval $[0, \infty)$.

Let $A(\boldsymbol{R}^{n})$ be the Fourier algebra given by $A(\boldsymbol{R}^{n})=\{\tilde{g}\,;\, g \in L^{1}(\boldsymbol{R}^{n})\}$, $\|\tilde{g}\|=\|g\|_{L^{1}(\boldsymbol{R}^{n})}$, where $\tilde{g}$ is the Fourier transform of g, that is, $\tilde{g}(\xi)=(2\pi)^{-n/2}\int_{\boldsymbol{R}^{n}} g(v) e^{-v\xi i} dv$, $\xi \in \boldsymbol{R}^{n}$. Denote by $A_{r}(\boldsymbol{R}^{n})$ the Fourier transforms of the radial functions g, $g(v)=g(|v|)$ a.e. $v \in \boldsymbol{R}^{n}$, in $L^{1}(\boldsymbol{R}^{n})$. From the well-known formula $\tilde{g}(\xi)=\hat{g}(|\xi|)$ for a radial function g, it follows that $A_{r}(\boldsymbol{R}^{n})$ is isomorphic and isometric to $A^{(\nu)}$ if $\nu=(n-2)/2$, $n=1, 2, 3, \cdots$.

L. Schwartz [**11**] showed that the unit sphere S^{n-1} in $\boldsymbol{R}^{n}$ is not a set of spectral synthesis for $A(\boldsymbol{R}^{n})$, $n \geqq 3$. Reiter [7] proved that, if $n \geqq 3$, then the singleton $\{y_{0}\}$, $y_{0}>0$ is not a set of spectral synthesis for $A_{r}(\boldsymbol{R}^{n})$, that is, for $A^{(\nu)}$, $\nu=(n-2)/2$. This implies L. Schwartz's result. For, if $\tilde{g}_{j} \rightarrow \tilde{g}$ in $A(\boldsymbol{R}^{n})$, then the Fourier transforms of the means of g_{j} on S^{n-1} converge to the Fourier transform of the mean of g on S^{n-1} in $A_{r}(\boldsymbol{R}^{n})$. A. Schwartz [**10**] showed that

Reiter's result holds good for all $\nu \geqq 1/2$.

On the other hand, Herz [4] proved that S^1 is a set of spectral synthesis for $A(\boldsymbol{R}^2)$, which implies that $\{y_0\}$, $y_0>0$ is a set of spectral synthesis for $A^{(0)}$. Since S^0 is a set of spectral synthesis for $A(\boldsymbol{R}^1)$, the set $\{y_0\}$, $y_0>0$ is a set of spectral synthesis for $A^{(-1/2)}$.

For $y_0=0$, Reiter [7] proved that, for every $n \geqq 1$, the set $\{0\}$ is a set of spectral synthesis for $A_r(\boldsymbol{R}^n)$, that is, for $A^{(\nu)}$, $\nu=(n-2)/2$.

The purpose of this paper is to show the following:

THEOREM. *If* $-1/2 \leqq \nu < 1/2$ *and* $y_0>0$, *then* $\{y_0\}$ *is a set of spectral synthesis for* $A^{(\nu)}$. *The set* $\{0\}$ *is a set of spectral synthesis for* $A^{(\nu)}$ *for every* $\nu \geqq -1/2$.

Related results will be found in Igari and Uno [5], Cazzaniga and Meaney [1], and Wolfenstetter [12]. They are concerned with spectral synthesis for the algebra of absolutely convergent Jacobi polynomial series.

2. A lemma.

First we will prove a lemma.

LEMMA. *Let* $\nu>-1$ *and let* m *be the least integer exceeding* $\nu+3/2$. *Let* f *be an infinitely differentiable function with compact support contained in* $[-K, K]$. *Then*

$$\int_0^\infty |\hat{f}(u)|\, u^{2\nu+1} du \leqq C \sum_{j=0}^{m} K^j \sup_{0 \leqq y} |f^{(j)}(y)|,$$

where C *is a constant depending only on* ν.

PROOF. Let

$$\begin{aligned}\int_0^\infty |\hat{f}(u)|\, u^{2\nu+1} du &= \left(\int_0^{1/K} + \int_{1/K}^\infty\right) |\hat{f}(u)|\, u^{2\nu+1} du \\ &= I_1 + I_2 .\end{aligned}$$

Since $|J_\nu(t)| \leqq Ct^\nu$ for $0 \leqq t \leqq 1$, we have

$$\begin{aligned} I_1 &= \int_0^{1/K} \left| \int_0^K f(y) J_\nu(yu) y^{\nu+1} u^{-\nu} dy \right| u^{2\nu+1} du \\ &\leqq C \int_0^{1/K} \int_0^K |f(y)| (yu)^\nu y^{\nu+1} dy\, u^{\nu+1} du \\ &\leqq C(2\nu+1)^{-2} \sup_{0 \leqq y} |f(y)| . \end{aligned}$$

Here and below, the letter C means positive constants depending only on ν, and it may be different in each occasion.

Next we will estimate I_2. By the formula $(d/tdt)^n[t^\nu J_\nu(t)]=t^{\nu-n}J_{\nu-n}(t)$ (cf. [2, 7.2.8 (52)]), we have

$$\hat{f}(u)=u^{-2(\nu+m)}\int_0^\infty yf(y)\Big(\frac{d}{ydy}\Big)^m\{(yu)^{\nu+m}J_{\nu+m}(yu)\}dy\,.$$

Noting the fact $(d/ydy)^{m-j}\{(yu)^{\nu+m}J_{\nu+m}(yu)\}\,|_{y=0}=0$, $j=1, 2, \cdots, m$, and repeating integration by parts, we have

$$\hat{f}(u)=(-1)^m u^{-2(\nu+m)}\int_0^\infty\Big\{\frac{d}{dy}\Big(\frac{d}{ydy}\Big)^{m-1}f(y)\Big\}(yu)^{\nu+m}J_{\nu+m}(yu)dy\,,$$

and thus we have

$$I_2\leqq\int_{1/K}^\infty\int_0^K\Big|\frac{d}{dy}\Big(\frac{d}{ydy}\Big)^{m-1}f(y)\Big|(yu)^{\nu+m}|J_{\nu+m}(yu)|\,dy\,u^{-(2m-1)}du\,.$$

By a simple calculation, we have

$$\frac{d}{dy}\Big(\frac{d}{ydy}\Big)^{m-1}f(y)=\sum_{j=1}^m c_j y^{j-2m+1}f^{(j)}(y),$$

where every c_j is a constant depending only on j. Thus we have

$$I_2\leqq C\sum_{j=1}^m(\sup_{0\leq y}|f^{(j)}(y)|)\int_{1/K}^\infty\int_0^K y^{j-m+\nu+1}|J_{\nu+m}(yu)|\,dy\,u^{-m+\nu+1}du\,.$$

Let

$$\begin{aligned}&\int_{1/K}^\infty\int_0^K y^{j-m+\nu+1}|J_{\nu+m}(yu)|\,dy\,u^{-m+\nu+1}du\\&=\int_{1/K}^\infty\int_0^{1/u}+\int_{1/K}^\infty\int_{1/u}^K=S_{1j}+S_{2j}\,.\end{aligned}$$

Then we have

$$\begin{aligned}S_{1j}&\leqq C\int_{1/K}^\infty\int_0^K y^{j-m+\nu+1}(yu)^{\nu+m}dy\,u^{-m+\nu+1}du\\&=C\{(j+2\nu+2)j\}^{-1}K^j\,.\end{aligned}$$

By the inequality $|J_{\nu+m}(t)|\leqq Ct^{-1/2}$ for $t\geqq 1$, we have

$$\begin{aligned}S_{2j}&\leqq C\int_{1/K}^\infty\int_{1/u}^K y^{j-m+\nu+1}(yu)^{-1/2}dy\,u^{-m+\nu+1}du\\&=C\int_0^K\int_{1/y}^\infty u^{-m+\nu+(1/2)}du\,y^{j-m+\nu+1}dy\\&=C\{(m-\nu-(3/2))j\}^{-1}K^j\,.\end{aligned}$$

Thus we have

$$I_2\leqq C\sum_{j=1}^m(S_{1j}+S_{2j})\sup_{0\leq y}|f^{(j)}(y)|\leqq C\sum_{j=1}^m K^j\sup_{0\leq y}|f^{(j)}(y)|\,,$$

and therefore the proof is complete.

3. Proof of Theorem.

Let I be a closed ideal in $A^{(\nu)}$ and let $Z(I)=\{y\in[0, \infty)\ ;\ f(y)=0$ for all $f\in I\}$. Theorem is an immediate consequence of the following proposition.

PROPOSITION. *Let* $\nu\geqq-1/2$ *and let* I *be a closed ideal in* $A^{(\nu)}$ *such that* $Z(I)=\{y_0\}$, $y_0\geqq 0$. *If* $y_0>0$, *then* $I=\{f\in A^{(\nu)}; f^{(j)}(y_0)=0, j=0, 1, 2, \cdots, M\}$ *for some* $M\leqq\nu+1/2$. *If* $y_0=0$, *then* $I=\{f\in A^{(\nu)}; f(0)=0\}$.

PROOF. Let I be a closed ideal in $A^{(\nu)}$ such that $Z(I)=\{y_0\}$, $y_0\geqq 0$, and let ϕ be a continuous linear functional on $A^{(\nu)}$ such that $\phi(f)=0$ for all $f\in I$. Let $\mathcal{D}(-\infty, \infty)$ be the test function space on $(-\infty, \infty)$ with usual topology. For $f\in\mathcal{D}(-\infty, \infty)$, put $f_P(y)=f(y)$, $y\geqq 0$ and $f_N(y)=f(-y)$, $y\geqq 0$. Then, by the inversion formula of the Hankel transform and Lemma, we have that f_P and f_N are in $A^{(\nu)}$. We define $\Phi_+(f)=\phi(f_P)+\phi(f_N)$ and $\Phi_-(f)=\phi(f_P)-\phi(f_N)$ for $f\in\mathcal{D}(-\infty, \infty)$. By Lemma we have

$$|\Phi_\pm(f)|\leqq\|\phi\|(\|f_P\|+\|f_N\|)\leqq C\|\phi\|\sum_{j=0}^{m}K^j\sup_{-\infty<y<\infty}|f^{(j)}(y)|,$$

where K is a positive number such that $\operatorname{supp} f\subset[-K, K]$, and m is the least integer exceeding $\nu+3/2$. Thus $\Phi_\pm$ are continuous linear functionals on $\mathcal{D}(-\infty, \infty)$ with order not exceeding m. Since $A^{(\nu)}$ is semisimple and regular, the ideal I contains the ideal of functions in $A^{(\nu)}$ which vanish on a neighborhood of y_0 (cf. [**6**, Chapter VIII, 5.7]). This implies that the supports of $\Phi_\pm$ are the singleton $\{y_0\}$. Thus $\Phi_\pm$ have the forms

$$\Phi_+=\sum_{j=0}^{m}a_j^+\delta_{y_0}^{(j)}, \qquad \Phi_-=\sum_{j=0}^{m}a_j^-\delta_{y_0}^{(j)},$$

where $a_j^\pm$ are constants, and δ_{y_0} is the Dirac measure with mass at $\{y_0\}$ (cf. [**8**, 6.25]).

Now we will show that $a_j^\pm=0$ for j exceeding $\nu+1/2$ if $y_0>0$. Let ν, $\mu>-1$ and put

$$g_{s,\mu}(x)=\frac{2^{\nu+1}\Gamma(\nu+\mu+2)}{\Gamma(\nu+1)s^{2(\nu+\mu+1)}}(s^2-x^2)^\mu\chi_{[0,s)}(x),$$

where $\chi_{[0,s)}(x)$ is the characteristic function of $[0, s)$. Then $\|\hat{g}_{s,\mu}\|=1$ and

$$\hat{g}_{s,\mu}(y)=\frac{2^{\nu+\mu+1}\Gamma(\nu+\mu+1)}{(sy)^{(\nu+\mu+1)}}J_{\nu+\mu+1}(sy) \qquad [\mathbf{3},\ 8.5(33)].$$

Let $q(y)$ be a function in $\mathcal{D}(-\infty, \infty)$ such that $q(y)=1$ on a neighborhood of y_0 and $\operatorname{supp} q\subset(0, \infty)$. Then $q\hat{g}_{s,\mu}$ is in $\mathcal{D}(-\infty, \infty)$ and $|\Phi_\pm(q\hat{g}_{s,\mu})|\leqq|\phi(q\hat{g}_{s,\mu})|\leqq$

$\|\phi\|\|q\|$. On the other hand, by the formula $(d/dt)(t^{-\nu}J_\nu(at))=-at^{-\nu}J_{\nu+1}(at)$ and the asymptotic formula

$$J_\nu(t)=\left(\frac{2}{\pi t}\right)^{1/2}\cos(t-\nu\pi/2-\pi/4)+O(t^{-3/2})\quad(t\to\infty),$$

we have $\delta_{y_0}^{(j)}(q\hat{g}_{s,\mu})=O(s^{j-(\nu+1/2)-(\mu+1)})$ $(s\to\infty)$ and

$$\limsup_{s\to\infty}|\delta_{y_0}^{(j)}(q\hat{g}_{s,\mu})|\,s^{-\{j-(\nu+1/2)-(\mu+1)\}}>0$$

for $j=0, 1, 2, \cdots$. This implies that $\limsup_{s\to\infty}|\Phi_\pm(q\hat{g}_{s,\mu})|=\infty$ if $a_j^\pm\neq 0$ for some $j>(\nu+1/2)+(\mu+1)$. Since $\mu>-1$ is arbitrary, we have $a_j^\pm=0$ for $j>\nu+1/2$.

Next we will show that $a_j^\pm=0$ for $j>0$ if $y_0=0$. First we note that $\Phi_\pm$ have the forms

$$\Phi_+=\sum_k a_{2k}^+\delta_0^{(2k)}\qquad\text{and}\qquad\Phi_-=\sum_k a_{2k-1}^-\delta_0^{(2k-1)}.$$

Let $q_0(y)$ be an even function in $\mathcal{D}(-\infty,\infty)$ such that $q_0(y)=1$ for $y\in[-1, 1]$ and $q_0(y)=0$ for $y\notin(-2, 2)$. Put $q_s(y)=(sy/2)q_0(sy/2)$. Then we have $\|(q_s)_P\|=O(1)$ as $s\to\infty$ by Lemma. Let $g_{s,0}$ be the function $g_{s,\mu}$ with $\mu=0$. Then $|\Phi_+(q_0\hat{g}_{s,0})|=O(1)$ and $|\Phi_-(q_s\hat{g}_{s,0})|=O(1)$ as $s\to\infty$. On the other hand, it follows from the power series expansion of the Bessel function that

$$\delta_0^{(2k)}(q_0\hat{g}_{s,0})=\frac{(-1)^k(2k)!\,\Gamma(\nu+1)}{2^{2k}k!\,\Gamma(\nu+\mu+2)}s^{2k},$$

$$\delta_0^{(2k-1)}(q_s\hat{g}_{s,0})=\frac{(-1)^{k-1}(2k-1)!\,\Gamma(\nu+1)}{2^{2k-1}(k-1)!\,\Gamma(\nu+\mu+1)}s^{2k-1}.$$

This implies that $|\Phi_+(q_0\hat{g}_{s,0})|\to\infty$ $(s\to\infty)$ if $a_{2k}^+\neq 0$ for some $k>0$, and $|\Phi_-(q_s\hat{g}_{s,0})|\to\infty\,(s\to\infty)$ if $a_{2k-1}^-\neq 0$ for some $k>0$. Thus we have $a_j^\pm=0$ for $j>0$, and therefore we have that $\phi(f_P)$ is of the form

$$\phi(f_P)=(\Phi_+(f)-\Phi_-(f))/2$$
$$=\begin{cases}\sum_{j=1}^{N}a_j\delta_{y_0}^{(j)}(f),\ N\leqq\nu+1/2 & (y_0>0),\\ a_0\delta_0(f) & (y_0=0)\end{cases}$$

for $f\in\mathcal{D}(-\infty,\infty)$. A. Schwartz [**9**] showed that if f is in $A^{(\nu)}$, then f has p continuous derivatives and $|f^{(r)}(y)|\leqq C\|f\|$, $r=0, 1, 2, \cdots, p$, where p is the greatest integer not exceeding $\nu+1/2$. From this and the fact that $\{f_P;\ f\in\mathcal{D}(-\infty,\infty)\}$ is dense in $A^{(\nu)}$, it follows that ϕ is of the form

$$\phi(f)=\begin{cases}\sum_{j=1}^{N}a_j\delta_{y_0}^{(j)}(f),\ N\leqq\nu+1/2 & (y_0>0),\\ a_0\delta_0(f) & (y_0=0)\end{cases}$$

for $f\in A^{(\nu)}$. We define $N(\phi)=\max\{j\,;\,a_j\neq 0\}$. Let L_I be the space of continuous linear functionals ϕ on $A^{(\nu)}$ such that $\phi(f)=0$ for all $f\in I$. Put $M=\max\{N(\phi)\,;\,\phi\in L_I\}$. Then we note that $M=0$ for $y_0=0$ and $0\leqq M\leqq\nu+1/2$ for $y_0>0$. Let f be in I and let ϕ_0 be a functional in L_I such that $M=N(\phi_0)$. For $h\in A^{(\nu)}$, we have

$$0=\phi_0(fh)=\sum_{k=0}^{M}\left\{\sum_{j=k}^{M}\binom{j}{k}a_j f^{(j-k)}(y_0)\right\}h^{(k)}(y_0)\,.$$

Since there exist functions $h_m\in A^{(\nu)}$ such that $h_m^{(k)}(y_0)=\delta_{mk}$, $k, m=0, 1, 2, \cdots, M$, we have $\sum_{j=k}^{M}\binom{j}{k}a_j f^{(j-k)}(y_0)=0$ for $k=0, 1, 2, \cdots, M$. Thus $f^{(k)}(y_0)=0$ for $k=0, 1, 2, \cdots, M$. This implies that $I=\{f\in A^{(\nu)}\,;\,f^{(k)}(y_0)=0,\ k=0, 1, 2, \cdots, M\}$ since I is a space of $f\in A^{(\nu)}$ such that $\phi(f)=0$ for all $\phi\in L_I$. Therefore the proof is complete.

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