J. Math. Soc. Japan
Vol. 39, No. 4, 1987

Stationary 2-type surfaces in a hypersphere

In Memory of Professor Yozô Matsushima

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(Received April 7, 1986)

1. Introduction.

In terms of finite-type submanifolds [**3**], a well-known theorem of Takahashi [**9**] says that an n-dimensional, compact submanifold M of E^{m+1} is of 1-type if and only if M is a minimal submanifold of a hypersphere S^m of E^{m+1}. Such a submanifold is always *mass-symmetric* in S^m, i.e., the center of mass of M is the center of S^m in E^{m+1}. Thus, if one chooses the center of S^m as the origin of E^{m+1}, then the position vector x of M has the following form:

$$x = x_p\,, \qquad \Delta x_p = \lambda_p x_p\,, \tag{1.1}$$

where $\lambda_p = n/r^2$, r is the radius of S^m and Δ is the Laplacian of M. Submanifolds of S^m satisfying (1.1) are the simplest finite-type submanifolds. The study of such submanifolds has attracted many mathematicians for many years.

On the other hand, it was shown in [**4**] (see, also [**3**, p. 274]) that if M is a compact hypersurface of S^m such that M is not a small hypersphere, then M has constant mean curvature $\alpha' \neq 0$ and constant scalar curvature τ if and only if M is mass-symmetric and of 2-type. In this case, the position vector x of M in E^{m+1} has the following form:

$$x = x_p + x_q, \quad \Delta x_p = \lambda_p x_p \quad \text{and} \quad \Delta x_q = \lambda_q x_q\,. \tag{1.2}$$

Furthermore, α' and τ are completely determined by $\{\lambda_p, \lambda_q\}$. Applying this result, we see that all non-minimal, isoparametric hypersurfaces of S^m are mass-symmetric and of 2-type.

Mass-symmetric, 2-type submanifolds of S^m are the "simplest" submanifolds of E^{m+1} next to minimal submanifolds of S^m. Many important submanifolds are known to be of 2-type and are mass-symmetric (cf. [**1**, **3**, **4**, **7**, 8]). For instance, it was shown in [8] that any compact, non-totally geodesic, parallel, Einstein, complex submanifold of complex projective space CP^N is of 2-type if

*) Partially supported by a grant from the Comite Conjunto Hispano-Norteamericano para la Cooperacion Cientifica y Tecnologica, Spain.

we regard CP^N as a submanifold of a Euclidean space by its first standard imbedding. The complete classification of mass-symmetric, 2-type submanifolds of S^m is formidably difficult. However, the case of surfaces in S^3 was done by the second author (cf. [**3**, p. 279]). In Section 4, we will solve this problem for surfaces in S^4.

Given an isometric immersion $f: M \to M'$ of a surface M into a Riemannian manifold M', one has the conformal total mean curvature $\tau(f)$ (cf. Section 5). Surfaces which are critical points of $\tau(f)$ are called stationary. Related to the Chen-Willmore problem, Weiner asked in [**10**] whether minimal surfaces of S^m are the only stationary, mass-symmetric surfaces of S^m? N. Ejiri constructed in [**5**] a counter-example to Weiner's problem. It is easy to see that Ejiri's example is of 2-type.

In this paper, we will study stationary, mass-symmetric, 2-type surfaces of S^m in detail. In particular, we will prove that such surfaces are in fact flat surfaces which lie fully in S^5 or in S^7. By completely determining the connection form of such surfaces, we show that such surfaces are obtained by some doubly-periodic isometric immersions of the Euclidean plane $\boldsymbol{R}^2$ into S^5 or S^7. In the case of S^5, a surprising phenomenon occurs. The connection form depends only on the eigenvalue λ_p which satisfies $2/3 < \lambda_p < 2$. Furthermore, for each $\lambda_p \in (2/3, 4/3]$, there is only one possibility for the connection form, while for each $\lambda_p \in (4/3, 2)$ there are two possibilities. Moreover, for each such connection form, we can construct a stationary, mass-symmetric, 2-type, flat torus in S^5. Although such a torus is not unique, it comes from a "unique" doubly-periodic immersion of $\boldsymbol{R}^2$ into S^5. We also show that the estimate on λ_p is best possible.

In the case of S^7, the connection form depends on both λ_p and λ_q (and depends only on them). Such λ_p and λ_q must satisfy $0 < \lambda_p < 2 < \lambda_q < \infty$. Furthermore, we can give some concrete examples for this case. More precisely, for any real number $d \in (2, \infty)$, there are a real number $c \in (0, 2)$ and a stationary, mass-symmetric, 2-type flat torus in S^7 with $(\lambda_p, \lambda_q) = (c, d)$. For each such pair (c, d), the flat torus in S^7 is obtained from a "unique" doubly-periodic immersion of $\boldsymbol{R}^2$ into S^7.

This work was done while the first author was a visiting professor at the Department of Mathematics, Michigan State University. He wishes to thank the Department for providing him with nice facilities and his colleagues there for their hospitality.

The authors would like to express their thanks to the referee for his valuable suggestions.

2. Preliminaries.

Let M be a compact Riemannian manifold and Δ the Laplacian of M acting on differentiable functions in $C^{\infty}(M)$. Then Δ is an elliptic differential operator and it has an infinite sequence of eigenvalues: $0=\lambda_0<\lambda_1<\lambda_2<\cdots<\lambda_k<\cdots\uparrow\infty$. Let $V_k=\{f\in C^{\infty}(M)\mid \Delta f=\lambda_k f\}$ be the eigenspace of Δ with eigenvalue λ_k. Then each V_k is finite-dimensional. If we define an inner product on $C^{\infty}(M)$ by $(f, g)=\int fg\,dV$, then the decomposition $\sum_{k\geq 0}V_k$ is orthogonal and dense in $C^{\infty}(M)$ (in L^2-sense). For each $f\in C^{\infty}(M)$, let f_t be the projection of f into V_t. Then we have the decomposition: $f=\sum_{t\geq 0}f_t$ (in L^2-sense).

For an isometric immersion $x: M\to E^{m+1}$ of a compact Riemannian manifold M into the Euclidean $(m+1)$-space E^{m+1}, we put $x=(x_1, \cdots, x_{m+1})$, where x_A is the A-th Euclidean coordinate function of M. Thus, we may write $x=\sum_{t\geq 0}x_t$ (in L^2-sense) so that $\Delta x_t=\lambda_t x_t$ for each t. Since M is compact, x_0 is a constant vector in E^{m+1} and, moreover, there is a natural number p such that $x_p\neq 0$ and $x=x_0+\sum_{t\geq p}x_t$. If there are infinitely many x_t's which are nonzero, we put $q=\infty$. Otherwise, there is an integer $q\geqq p$ such that

$$x=x_0+\sum_{t=p}^{q}x_t, \qquad x_q\neq 0.$$

In both cases, we have the following decomposition:

$$x=x_0+\sum_{t=p}^{q}x_t \qquad \text{(in } L^2\text{-sense)}. \tag{2.1}$$

The submanifold M is said to be *of finite type* if q is finite. Otherwise, M is said to be *of infinite type*. The submanifold M is said to be *of k-type* if there is exactly k nonzero x_t's in the decomposition (2.1). The pair $[p, q]$ is called the *order* of the submanifold M [**3**].

Let M be an n-dimensional submanifold of an m-dimensional Riemannian manifold M'. We choose a local field of orthonormal frames $e_1, \cdots, e_n, \xi_{n+1}, \cdots, \xi_m$ in M' such that $e_1, \cdots, e_n$ are tangent to M. Let $\omega^1, \cdots, \omega^n$ be the dual frame of $e_1, \cdots, e_n$. Denote by ∇' the Riemannian connection of M'. We put $\nabla' e_i=\sum\omega_i{}^j e_j+\sum\omega_i{}^r\xi_r$ and $\nabla'\xi_r=\sum\omega_r{}^i\xi_i+\sum\omega_r{}^t\xi_t$, $i, j, k=1, \cdots, n$; $r, s, t=n+1, \cdots, m$. By Cartan's Lemma, we have

$$\omega_i{}^r=\sum h_{ij}{}^r\omega^j, \quad h_{ij}{}^r=h_{ji}{}^r, \tag{2.2}$$

where $h_{ij}{}^r$ are coefficients of the second fundamental form. The connection form of M in M' is given by $(\omega_A{}^B)$, $A, B=1, \cdots, m$.

Throughout this paper, we shall assume that the submanifold M is compact unless mentioned otherwise.

3. 2-type submanifolds of hyperspheres.

Let $x: M \to E^{m+1}$ be an isometric immersion of a compact, n-dimensional Riemannian manifold M into E^{m+1}. Denote by ∇ and $\tilde{\nabla}$ the Riemannian connections of M and E^{m+1}, respectively. And by h, A and D the second fundamental form, the Weingarten map and the normal connection of M in E^{m+1}, respectively.

For a fixed vector a in E^{m+1} and vector fields X, Y tangent to M, the formulas of Gauss and Weingarten give

$$YX\langle H, a\rangle = \langle D_Y D_X H, a\rangle - \langle \nabla_Y(A_H X), a\rangle - \langle A_{D_X H} Y, a\rangle - \langle h(Y, A_H X), a\rangle, \tag{3.1}$$

where H is the mean curvature vector of M in E^{m+1} and $\langle\ ,\ \rangle$ the inner product of E^{m+1}. Let $e_1, \cdots, e_n$ be an orthonormal local frame field tangent to M. Equation (3.1) implies

$$\Delta H = \Delta^D H + \sum_{i=1}^{n} \{h(e_i, A_H e_i) + A_{D_{e_i} H} e_i + (\nabla_{e_i} A_H) e_i\}, \tag{3.2}$$

where

$$\Delta^D H = \sum_{i=1}^{n} \{D_{\nabla_{e_i} e_i} H - D_{e_i} D_{e_i} H\} \tag{3.3}$$

is the Laplacian of H with respect to D. Regard ∇A_H and A_{DH} as (1, 2)-tensors on M and we set $\bar{\nabla} A_H = \nabla A_H + A_{DH}$. Then we have

$$\operatorname{tr}(\bar{\nabla} A_H) = \sum_{i=1}^{n} \{(\nabla_{e_i} A_H) e_i + A_{D_{e_i} H} e_i\}. \tag{3.4}$$

Let $U = \{u \in M \mid H \neq 0 \text{ at } u\}$. Then U is an open subset of M. On U we choose an orthonormal local frame $\xi_{n+1}, \cdots, \xi_{m+1}$ normal to M in E^{m+1} so that ξ_{n+1} is parallel to H. Then we have

$$\sum_{i=1}^{n} h(e_i, A_H e_i) = \|A_{n+1}\|^2 H + \mathfrak{A}(H), \tag{3.5}$$

where

$$A_r = A_{\xi_r}, \quad \|A_{n+1}\|^2 = \operatorname{tr} A_{n+1}^2 \tag{3.6}$$

and

$$\mathfrak{A}(H) = \sum_{r=n+2}^{m+1} (\operatorname{tr} A_H A_r) \xi_r \tag{3.7}$$

on U. If $H=0$ at a point u, $\mathfrak{A}(H)$ is defined to be zero. It is clear that (3.5) and (3.6) hold trivially on $M-U$. Therefore, we have (3.5) and (3.7) on the whole submanifold M. The vector field $\mathfrak{A}(H)$ is a well-defined vector field perpendicular to H, which is called the allied mean curvature vector of M in E^{m+1}. From (3.2), (3.4) and (3.5) we get

$$\Delta H=\Delta^{D}H+\|A_{n+1}\|^{2}H+\mathfrak{A}(H)+\mathrm{tr}(\bar{\nabla}A_{H}). \tag{3.8}$$

Now, assume that M is a submanifold of the unit hypersphere $S_0^m(1)$ of E^{m+1} centered at the origin 0. Denote by H', A' and D' the mean curvature vector, the Weingarten map and the normal connection of M in $S_0^m(1)$, respectively. Then we have

$$H=H'-x\,,\qquad \Delta^{D}H=\Delta^{D'}H'\,,\qquad Dx=0\,. \tag{3.9}$$

Moreover, for any vector η normal to M in $S_0^m(1)$, we have $A_\eta=A_\eta'$. Let ξ be a unit vector parallel to H' with $H'=\alpha'\xi$, $\alpha'=\|H'\|$. (If $H'=0$ at u, ξ can be chosen to be an arbitrary unit normal vector of M in $S_0^m(1)$.) We have the following.

LEMMA 1 ([**3**, p. 273]). *Let M be an n-dimensional submanifold of $S_0^m(1)$ in E^{m+1}. Then we have*

$$\Delta H=\Delta^{D'}H'+\mathfrak{A}'(H')+\mathrm{tr}(\bar{\nabla}A_{H})+(\|A_\xi\|^2+n)H'-n(\alpha)^2x\,, \tag{3.10}$$

where $\mathfrak{A}'(H')$ is the allied mean curvature vector of M in $S_0^m(1)$ (which is zero on $\{u\in M\,|\,H'=0$ at $u\}$).

We also need the following.

LEMMA 2 ([**3**, p. 274]). *If M is a mass-symmetric, 2-type submanifold of $S_0^m(1)$, then we have*

(1) *the mean curvature α' is constant which is given by*

$$(\alpha')^2=\left(1-\frac{\lambda_p}{n}\right)\left(\frac{\lambda_q}{n}-1\right)\neq 0 \tag{3.11}$$

and

(2) $\mathrm{tr}(\bar{\nabla}A_H)=0$, $0<\lambda_p<n<\lambda_q<\infty$.

LEMMA 3 ([**4**]). *Let M be an n-dimensional submanifold of $S_0^m(1)$ in E^{m+1}. Then $\mathrm{tr}(\bar{\nabla}A_H)=0$ if and only if*

$$n\,\mathrm{grad}(\alpha')^2+4\,\mathrm{tr}\,A_{DH'}=0\,. \tag{3.12}$$

Lemma 2 implies that the mean curvature $\alpha'=|H'|$ of M in $S_0^m(1)$ is determined by the order. In the following, let dH' denote the E^{m+1}-valued 1-form defined by $(dH')(X)=\tilde{\nabla}_XH'$, for X tangent to M. Then we have $\|dH'\|^2=\|D'H'\|^2+\|A_{H'}\|^2$. The following lemma shows that the length of dH' is also determined by the order of M.

LEMMA 4. *Let M be a mass-symmetric, 2-type submanifold of $S_0^m(1)$ in E^{m+1}. Then we have*

(3.13) $$\|dH'\|^2 = \{\lambda_p+\lambda_q-n\}\{n(\lambda_p+\lambda_q)-\lambda_p\lambda_q-n^2\}/n^2\,,$$

(3.14) $$\mathfrak{A}'(H') = |H'| \sum_{r=n+2}^{m} \{\mathrm{tr}(\nabla\omega_{n+1}{}^r)-\langle D'\xi, D'\xi_r\rangle\}\xi_r\,.$$

PROOF. Let $\xi_{n+1}, \cdots, \xi_m$ be a local orthonormal normal basis of M in $S_0{}^m(1)$ such that $\xi_{n+1}=\xi$ is parallel to H' (this condition holds automatically on $\{u\in M \mid H'=0 \text{ at } u\}$). By using Lemma 2 and (3.3), we may find

(3.15) $$\Delta^D H = \Delta^{D'} H' = \alpha' \sum_{r=n+2}^{m} \{\langle D'\xi, D'\xi_r\rangle-\mathrm{tr}(\nabla\omega_{n+1}{}^r)\}\xi_r+\langle D'\xi, D'\xi\rangle H'\,,$$

where

(3.16) $$\mathrm{tr}(\nabla\omega_{n+1}{}^r) = \sum_{i=1}^{n} (\nabla_{e_i}\omega_{n+1}{}^r)(e_i)\,.$$

Since M is mass-symmetric and of 2-type in $S_0{}^m(1)$, we have (cf. [**3**, p. 256])

(3.17) $$\Delta H = (\lambda_p+\lambda_q)H+(\lambda_p\lambda_q/n)x\,.$$

Combining Lemma 1, Lemma 2, (3.15) and (3.17), we may obtain (3.13) and (3.14). (Q. E. D.)

REMARK 1. By using Lemma 1, we may prove that there exist no mass-symmetric, 3-type hypersurfaces with constant mean curvature in a hypersphere of E^{n+2}.

4. A non-existence theorem.

First, we mention the following [**3**, p. 279].

THEOREM 1. *Let M be a mass-symmetric surface of $S_0{}^3(1)$. Then M is of 2-type if and only if M is the product of two plane circles of different radii.*

The Veronese surface in $S_0{}^4(1)$ is a nice example of (mass-symmetric) 1-type surface which lies fully in $S_0{}^4(1)$. In contrast with this, we give the following *Non-existence Theorem.*

THEOREM 2. *There exist no mass-symmetric, 2-type surfaces which lie fully in $S_0{}^4(1)$.*

PROOF. Assume that M is a mass-symmetric, 2-type surface which lies fully in $S_0{}^4(1)$. Then we have $\langle D\xi, D\xi_4\rangle=0$. Thus, (3.14) reduces to

(4.1) $$\mathfrak{A}'(H') = \alpha' \mathrm{tr}(\nabla\omega_3{}^4)\xi_4\,.$$

Combining (3.7) and (4.1), we obtain

(4.2) $$\mathrm{tr}(A_3A_4) = \mathrm{tr}(\nabla\omega_3{}^4)\,.$$

On the other hand, by using constancy of α' (Lemma 2), Lemmas 2 and 3 imply $\mathrm{tr}A_{D\xi}=0$. Let e_1, e_2 be eigenvectors of A_4. Since $\mathrm{tr}A_4=0$, we may assume that $A_4e_1=\mu e_1$ and $A_4e_2=-\mu e_2$. Thus by using $\mathrm{tr}A_{D\xi}=0$, we find $\mu\omega_3{}^4=0$, i.e., $A_4\omega_3{}^4=0$. Combining this with (4.2), we obtain $\mathrm{tr}(A_3A_4)=0$.

Let $W=\{u\in M|A_4\neq 0 \text{ at } u\}$. Assume that $W\neq\emptyset$ and U is a connected component of W. Then U is open and $D'\xi_3=D'\xi_4=0$. Let e_1, e_2 be an orthonormal tangent basis on U such that, with respect to e_1, e_2, A_3 and A_4 are given by

$$A_3=\begin{pmatrix}\beta & 0\\ 0 & \gamma\end{pmatrix},\quad A_4=\begin{pmatrix}c & b\\ b & -c\end{pmatrix}. \tag{4.3}$$

Since $\mathrm{tr}(A_3A_4)=0$, (3.7) and (4.3) give $(\beta-\gamma)c=0$. On the other hand, Lemmas 2 and 4 imply $\|A_3\|^2\neq 2(\alpha')^2$. Thus U is not pseudo-umbilical, i.e., $\beta\neq\gamma$. Consequently, we have $c=0$. Moreover, since $D'\xi_3=D'\xi_4=0$ on U, Ricci's equation gives $[A_3, A_4]=0$. Therefore, $b=0$ too. Hence, $W=\emptyset$. Thus, we have $A_4=0$ on M. This gives $\omega_1{}^4=\omega_2{}^4=0$. By taking exterior differentiation of these we obtain

$$\beta\omega^1\wedge\omega_3{}^4=\gamma\omega^2\wedge\omega_3{}^4=0. \tag{4.4}$$

Let G denote the Gauss curvature of M. Then we have $G=1+\beta\gamma$. Let $V=\{u\in M|G(u)\neq 1\}$. Then, on V, (4.4) implies $\omega_3{}^4=0$, i.e., $D\xi_3=0$. Thus, Lemmas 2 and 4 imply that both β and γ are constant. Hence, by taking the exterior differentiation of $\omega_1{}^3=\beta\omega^1$ and $\omega_2{}^3=\gamma\omega^2$, we obtain $\omega_1{}^2=0$. Thus, $G=0$. So, by the continuity of G on M, we obtain $G\equiv 0$ or $G\equiv 1$. If $G\equiv 0$, then by $A_4=\omega_3{}^4=0$, we conclude that M is in fact a flat surface in a great hypersphere of $S_0{}^4(1)$. This is a contradiction. Therefore, $G\equiv 1$ on M. Hence, $\beta\gamma=0$. Since $\beta+\gamma$ is constant, β and γ are both constant. Without loss of generality, we may assume that $\beta=0$. Since M is of 2-type, $\gamma\neq 0$. Thus, we have $\omega_1{}^3=0$ and $\omega_2{}^3=\gamma\omega^2$, $\gamma\neq 0$. By taking exterior differentiation of these equations, we obtain $\omega_2{}^1=0$ which implies $G=0$. This is a contradiction. (Q.E.D.)

5. Stationary, 2-type surfaces.

Let $f: M\to M'$ be an isometric immersion of a surface M into an m-dimensional Riemannian manifold M'. We denote by α' and R' the mean curvature of f and the sectional curvature of M' with respect to the tangent space of M and define $\tau(f)$ by

$$\tau(f)=\int_M((\alpha')^2+R')dV. \tag{5.1}$$

It was proved in [2] that $\tau(f)$ is an invariant under conformal changes of the metric of M' (cf. also [**3**, p. 207]). We call $\tau(f)$ the *conformal total mean curvature.* The variation of $\tau(f)$ was calculated in [**10**] (cf. also [**3**, pp. 213-225]). When M' is the unit hypersphere $S_0^m(1)$ of E^{m+1}, f is a stationary point of τ if and only if

$$\Delta^{D'} H' = -2(\alpha')^2 H' + \|A_\xi\|^2 H' + \mathfrak{A}'(H'), \tag{5.2}$$

where H' is the mean-curvature vector of M in $S_0^m(1)$ and $H'=\alpha'\xi$, $\alpha'=|H'|$.

In [**5**], Ejiri showed that the isometric immersion f from the flat torus $S^1(1)\times S^1(\sqrt{1/3})$ into S^5 defined by

$$f((x, y), (z, w)) = (\sqrt{2/3}\,x,\ xz,\ xw,\ \sqrt{2/3}\,y,\ yz,\ yw)$$

is a mass-symmetric, stationary, non-minimal surface in S^5.

It is easy to see that Ejiri's example is a 2-type surface in S^5. In the following, we want to classify stationary, 2-type, mass-symmetric surfaces in S^m. In particular, we shall obtain the following.

THEOREM 3. *Let M be a stationary, mass-symmetric, 2-type surface in $S_0^m(1)$. Then M is a flat surface which lies fully in a totally geodesic $S_0^5(1)$ or in a totally geodesic $S_0^7(1)$ in $S_0^m(1)$.*

PROOF. We need some lemmas.

LEMMA 5. *Let M be a stationary, mass-symmetric, 2-type surface in $S_0^m(1)$ in E^{m+1}. Then we have*

(1) *M is an $\mathfrak{A}$-surface,*

(2) $|H'|^2=(2-\lambda_p)(\lambda_q-2)/4\neq 0$,

(3) $\|A_\xi\|^2=\lambda_p+\lambda_q-2-\lambda_p\lambda_q/4$,

(4) $\|D'\xi\|^2=\lambda_p\lambda_q/4$,

(5) *M is not pseudo-umbilical,*

(6) $\mathrm{tr}(\bar{\nabla} A_{H'})=0$,

(7) $\mathrm{tr}(\nabla\omega_3^r)=\langle D'\xi, D'\xi_r\rangle$, $r=4, \cdots, m$.

Conversely, if M is an $\mathfrak{A}$-surface of $S_0^m(1)$ satisfying (2), (3), (4), (6), *and* (7), *then M is a stationary, mass-symmetric* 2*-type surface in $S_0^m(1)$.*

PROOF. If M is mass-symmetric and of 2-type in $S_0^m(1)\subset E^{m+1}$, α' is a nonzero constant. So, there is a unit normal vector field ξ on M which is parallel to H'. From Lemma 2 we obtain (2) and (6). Moreover, from Lemma 4 we find

$$\|A_\xi\|^2+\|D'\xi\|^2 = \lambda_p+\lambda_q-2. \tag{5.3}$$

Since M is stationary, (5.2) holds. Thus, by Lemmas 1 and 2, we find

(5.4) $$\Delta H = 2\mathfrak{A}'(H') + 2(\|A_\xi\|^2 - (\alpha')^2 + 1)H' - 2(\alpha')^2 x\,.$$

On the other hand, since M is mass-symmetric and of 2-type, we also have

(5.5) $$\Delta H = (\lambda_p + \lambda_q)H' + (\lambda_p\lambda_q/2 - (\lambda_p + \lambda_q))x\,.$$

Thus, by combining (5.4) and (5.5), we obtain (1), (2) and (3). Statement (4) follows from statement (3) and (5.3). Statement (5) follows from (2) and (3). Moreover, by applying Lemma 4 and (1), we obtain (7). The converse of this can be easily verified.

We choose $\{e_1, e_2\}$ which diagonalizes A_3. Then we have $h_{11}^4 = \cdots = h_{11}^m = 0$ because M is not pseudo-umbilical and it is an $\mathfrak{A}$-surface. From Lemma 5, we also have $D'\xi \neq 0$. Because M is 2-dimensional, we may assume that $D'\xi$ lies in the normal subspace spanned by ξ_4 and ξ_5. So, by a suitable choice of $\xi_4, \cdots, \xi_m$, we have

(5.6) $$A_3 = \begin{pmatrix} \beta & 0 \\ 0 & \gamma \end{pmatrix}, \quad A_4 = \begin{pmatrix} 0 & b \\ b & 0 \end{pmatrix}, \quad A_6 = \begin{pmatrix} 0 & c \\ c & 0 \end{pmatrix}, \\ A_5 = A_7 = \cdots = A_m = 0\,, \qquad D'\xi_3 = \omega_3^4\xi_4 + \omega_3^5\xi_5\,,$$

where β and γ are unequal constants.

LEMMA 6. *Under the hypothesis, M is flat and $\omega_1^2 = b\omega_3^4 = 0$.*

PROOF. Lemmas 2 and 3 imply $\operatorname{tr} A_{D'\xi} = 0$. Thus (5.6) gives $b\omega_3^4 = 0$. So, by taking differentiation of $\omega_1^3 = \beta\omega^1$ and $\omega_2^3 = \gamma\omega^2$, and by using (5.6) and structure equations, we obtain $\omega_1^2 = 0$. From $\omega_1^2 = 0$, we see that M is flat.

If $b \neq 0$, then Lemma 6 gives $\omega_3^4 = 0$ and $D'\xi_3$ being perpendicular to the first normal space. Thus, by choosing $\xi_3, \cdots, \xi_m$ such that the first normal space is spanned by ξ_3 and ξ_4, we obtain the following case (1). Otherwise, we have case (2):

Case (1). With respect to the frame field we have

(5.7) $$A_3 = \begin{pmatrix} \beta & 0 \\ 0 & \gamma \end{pmatrix}, \quad A_4 = \begin{pmatrix} 0 & b \\ b & 0 \end{pmatrix}, \quad A_5 = \cdots = A_m = 0\,, \\ D'\xi_3 = \omega_3^5\xi_5\,, \quad \omega_1^2 = 0\,, \quad b \neq 0\,,$$

or

Case (2). With respect to the frame field, we have

(5.8) $$A_3 = \begin{pmatrix} \beta & 0 \\ 0 & \gamma \end{pmatrix}, \quad A_4 = A_5 = 0, \quad A_6 = \begin{pmatrix} 0 & b \\ b & 0 \end{pmatrix}, \quad A_7 = \cdots = A_m = 0\,, \\ D'\xi_3 = \omega_3^4\xi_4 + \omega_3^5\xi_5\,, \qquad \omega_1^2 = 0\,.$$

In both cases, β, γ and b are constants with $b^2=1+\beta\gamma$.

We consider cases (1) and (2) separately.

If Case (1) holds, we have $\omega_1^3=\beta\omega^1$, $\omega_2^3=\gamma\omega^2$, $\omega_1^4=b\omega^2$, $\omega_2^4=b\omega^1$, $\omega_i^r=0$ for $i=1, 2$; $r=5, \cdots, m$. Taking differentiation of $\omega_i^r=0$, we obtain $\omega_4^r=0$, for $r=6, \cdots, m$. Thus,

$$D'\xi_4 = \omega_4^5\xi_5 . \tag{5.9}$$

We put

$$\omega_3^5 = \mu_1\omega^1+\mu_2\omega^2 , \qquad \omega_4^5 = \eta_1\omega^1+\eta_2\omega^2 . \tag{5.10}$$

Taking exterior differentiation of $\omega_i^5=0$, $i=1, 2$, we obtain $\omega_i^3\wedge\omega_3^5+\omega_i^4\wedge\omega_4^5=0$. Thus, by applying (5.7) and (5.10), we may obtain

$$\eta_1 = \beta\mu_2/b, \qquad \eta_2 = \gamma\mu_1/b . \tag{5.11}$$

On the other hand, since $\omega_3^4=0$, Lemma 5 and (5.10) imply

$$0 = \mathrm{tr}(\nabla\omega_3^4) = \mu_1\eta_1+\mu_2\eta_2 . \tag{5.12}$$

Combining (5.11) and (5.12) we find $\mu_1\mu_2=0$. Since $(\mu_1)^2+(\mu_2)^2=\lambda_p\lambda_q/4$ is a constant, we obtain $\mu_1\equiv 0$ or $\mu_2\equiv 0$. Without loss of generality, we may assume that $\mu_2\equiv 0$. Thus, we get

$$\omega_3^5 = \mu\omega^1 \neq 0, \qquad \mu^2 = \lambda_p\lambda_q/4, \qquad \omega_4^5 = (\lambda\mu/b)\omega^2 . \tag{5.13}$$

Now, since $\omega_3^r=0$ for $r=6, \cdots, m$, Lemma 6 implies $\nabla\omega_3^r=0$ where we use the definition of $\nabla\omega_3^r$. Thus, by (7) of Lemma 5, (5.7) and (5.13), we obtain

$$\omega_5^r(e_1) = 0, \qquad r = 6, \cdots, m . \tag{5.14}$$

Moreover, by taking exterior differentiation of $\omega_3^r=0$ and using (5.13), we may find $\omega^1\wedge\omega_5^r=0$. Combining this with (5.14) we find $\omega_5^r=0$ for $r=6, \cdots, m$. Since we already know that $\omega_3^r=\omega_4^r=0$ for $r=6, \cdots, m$, the normal subspace spanned by $\{\xi_3, \xi_4, \xi_5\}$ is parallel with respect to the normal connection D'. Since the first normal subspace is spanned by $\{\xi_3, \xi_4\}$, Therefore, by a reduction theorem of submanifold [6], we may conclude that in fact M lies in a totally geodesic $S_0^5(1)$ of $S_0^m(1)$. We summarize these as the following.

LEMMA 7. *Let M be a stationary, mass-symmetric, 2-type surface in $S_0^m(1)$. If Case (1) holds, then M lies fully in a totally geodesic 5-sphere $S_0^5(1)$ of $S_0^m(1)$. Moreover, with respect to a suitable orthonormal frame $\{e_1, e_2, \xi_3, \xi_4, \xi_5\}$ of M in $S_0^5(1)$, we have*

$$(5.15)\quad \begin{aligned} &A_3 = \begin{pmatrix} \beta & 0 \\ 0 & \gamma \end{pmatrix}, \quad A_4 = \begin{pmatrix} 0 & b \\ b & 0 \end{pmatrix}, \quad A_5 = 0, \quad \omega_1{}^2 = 0, \quad D'\xi_3 = \mu\omega^1\xi_5, \\ &D'\xi_4 = (\gamma\mu/b)\omega^2\xi_5, \quad \beta\gamma+1 = b^2, \quad \mu^2 = \lambda_p\lambda_q/4, \quad b^2(\beta-\gamma)+\gamma\mu^2 = 0. \end{aligned}$$

The last equation in (5.15) follows from the exterior differentiation of $\omega_3{}^4=0$. Now, we shall consider Case (2). In this case, we have

$$(5.16)\quad \begin{aligned} &\omega_1{}^2 = 0, \quad \omega_1{}^3 = \beta\omega^1, \quad \omega_2{}^3 = \gamma\omega^2, \quad \omega_i{}^4 = \omega_i{}^5 = 0, \quad \omega_1{}^6 = b\omega^2, \\ &\omega_2{}^6 = b\omega^1, \quad \omega_i{}^r = 0, \quad \omega_3{}^6 = \cdots = \omega_3{}^m = 0, \quad i = 1, 2, \quad r = 7, \cdots, m. \end{aligned}$$

Moreover, we have

$$(5.17)\qquad \beta\gamma+1 = b^2 \quad \text{and} \quad D'\xi_3 = \omega_3{}^4\xi_4+\omega_3{}^5\xi_5 \neq 0 .$$

Taking exterior differentiation of $\omega_i{}^4=\omega_i{}^5=0$ and applying (5.17), we obtain

$$(5.18)\qquad \beta\omega^1\wedge\omega_3{}^5+b\omega^2\wedge\omega_6{}^5 = \gamma\omega^2\wedge\omega_3{}^5+b\omega^1\wedge\omega_6{}^5 = 0 ,$$

$$(5.19)\qquad \beta\omega^1\wedge\omega_3{}^4+b\omega^2\wedge\omega_6{}^4 = \gamma\omega^2\wedge\omega_3{}^4+b\omega^1\wedge\omega_6{}^4 = 0 .$$

If $b=0$, (5.17) gives $\beta\gamma\neq 0$. Moreover, (5.18) and (5.19) imply $\omega_3{}^4=\omega_3{}^5=0$. This contradicts (5.17). Thus, we see that b is a nonzero constant.

By taking exterior differentiation of $\omega_i{}^r=0$, $r=7, \cdots, m$, and applying (5.16), we get

$$(5.20)\qquad \omega_6{}^r = 0, \qquad r = 7, \cdots, m .$$

We put

$$(5.21)\qquad \omega_3{}^4 = \alpha_1\omega^1+\alpha_2\omega^2, \qquad \omega_3{}^5 = \delta_1\omega^1+\delta_2\omega^2 .$$

Then by (5.18) and (5.19) we find

$$(5.22)\qquad \omega_6{}^4 = (\beta\alpha_2/b)\omega^1+(\gamma\alpha_1/b)\omega^2 ,$$

$$(5.23)\qquad \omega_6{}^5 = (\beta\delta_2/b)\omega^1+(\gamma\delta_1/b)\omega^2 .$$

Because $\omega_3{}^6=0$, Lemma 5 implies $0=\mathrm{tr}(\nabla\omega_3{}^6)=\langle D'\xi_3, D'\xi_6\rangle=0$. Therefore, (5.21), (5.22) and (5.23) give

$$(5.24)\qquad \alpha_1\alpha_2+\delta_1\delta_2 = 0 .$$

In the following, we may choose ξ_4 in such a way that

$$(5.25)\qquad D'_{e_1}\xi_3 = \omega_3{}^4(e_1)\xi_4, \qquad \delta_1 = 0 .$$

Therefore, we obtain

$$(5.26)\qquad \omega_3{}^5 = \delta\omega^2, \qquad \omega_6{}^5 = (\beta\delta/b)\omega^1 ,$$

where $\delta=\delta_2$. Since $\delta_1=0$, (5.24) gives

(5.27) $$\alpha_1=0 \quad \text{or} \quad \alpha_2=0.$$

If $\alpha_1=0$, we have

(5.28) $$D'_{e_1}\xi_3=0.$$

In this case, we may choose ξ_4 in such a way that

(5.29) $$D'\xi_3=\omega_3{}^4\xi_4, \qquad \omega_3{}^4=\alpha_2\omega^2.$$

Then we have $\omega_3{}^5=0$. Therefore, by interchanging ξ_4 and ξ_6, we obtain Case (1). If $\delta=0$, the same argument holds. Consequently, we obtain the following.

LEMMA 8. *If M is not the flat surface in a $S_0^5(1)$ mentioned in Lemma 7, then, with respect to a suitable orthonormal frame $\{e_1, e_2, \xi_3, \cdots, \xi_m\}$, we have*

$$A_3=\begin{pmatrix}\beta & 0\\ 0 & \gamma\end{pmatrix}, \quad A_4=A_5=0, \quad A_6=\begin{pmatrix}0 & b\\ b & 0\end{pmatrix}, \quad A_7=\cdots=A_m=0,$$

(5.30) $$\omega_1{}^2=0,\quad \omega_3{}^4=\alpha_1\omega^1,\quad \omega_3{}^5=\delta\omega^2,\quad \omega_3{}^6=\cdots=\omega_3{}^m=0,\quad \omega_4{}^6=-(\gamma\alpha_1/b)\omega^2,$$

$$\omega_5{}^6=-(\beta\delta/b)\omega^1,\quad \omega_6{}^7=\cdots=\omega_6{}^m=0,\quad \beta\gamma+1=b^2\neq 0,\quad \alpha_1\delta\neq 0.$$

Now, we also need the following.

LEMMA 9. *Under the hypothesis of Lemma 8, we may choose the frame $\{e_1, e_2, \xi_3, \cdots, \xi_m\}$ in such a way that, in addition to (5.30), we also have*

(5.31) $$\omega_4{}^5=0,\quad \omega_4{}^r=\omega_5{}^r=0 \qquad \textit{for } r=8, \cdots, m,$$

(5.32) $$\alpha_1 \textit{ and } \delta \textit{ are constant.}$$

PROOF. Since $A_4=0$, equation of Ricci implies

$$0=\langle D'_{e_1}D'_{e_2}\xi_6, \xi_4\rangle-\langle D'_{e_2}D'_{e_1}\xi_6, \xi_4\rangle.$$

Thus, by (5.30) and constancy of β, γ and b, we obtain

(5.33) $$\gamma e_1(\alpha_1)=\beta\delta\omega_5{}^4(e_2).$$

Similarly, by using $[A_5, A_6]=0$ and equation of Ricci, we also have

(5.34) $$\beta e_2(\delta)=\gamma\alpha_1\omega_4{}^5(e_1).$$

On the other hand, by (5.30), we have

(5.35) $$\mathrm{tr}(\nabla\omega_3{}^4)=e_1(\alpha_1),\quad \mathrm{tr}(\nabla\omega_3{}^5)=e_2(\delta).$$

Thus, by applying statement (7) of Lemma 5 and (5.30), we find

(5.36) $$e_1(\alpha_1)=\delta\omega_4{}^5(e_2),\quad e_2(\delta)=\alpha_1\omega_4{}^5(e_1).$$

Since M is not pseudo-umbilical (Lemma 5) and $\alpha_1\delta\neq 0$ (Lemma 8), (5.33), (5.34) and (5.36) imply

(5.37) $$\omega_4{}^5=0,$$

(5.38) $$e_1(\alpha_1)=e_2(\delta)=0.$$

On the other hand, since $[A_3, A_4]=[A_3, A_5]=0$, the equation of Ricci, (5.30) and (5.37) imply

(5.39) $$e_2(\alpha_1)=e_1(\delta)=0.$$

Combining (5.38) and (5.39) and using the constancy of β and γ, we see that $\gamma\alpha_1$ and $\beta\delta$ are constant.

Now, we want to prove that $\omega_4{}^r=\omega_5{}^r=0$ for $r=8, \cdots, m$. Since $\omega_3{}^r=0$, statement (7) of Lemma 5 and (5.30) give

(5.40) $$\alpha_1\omega_4{}^r(e_1)+\delta\omega_5{}^r(e_2)=0,\qquad r=7, \cdots, m.$$

On the other hand, from $\omega_6{}^r=0$, $r=7, \cdots, m$, we find

(5.41) $$-\gamma\alpha_1\omega_4{}^r(e_1)+\beta\delta\omega_5{}^r(e_2)=0,\qquad r=7, \cdots, m.$$

Since $(\beta+\gamma)\alpha_1\delta\neq 0$, (5.40) and (5.41) imply

(5.42) $$\omega_4{}^r(e_1)=\omega_5{}^r(e_2)=0,\qquad r=7, \cdots, m.$$

Thus, we have

(5.43) $$D'_{e_1}\xi_4=-\alpha_1\xi_3,\qquad D'_{e_2}\xi_5=-\delta\xi_3.$$

Now, since $D_{e_2}\xi_4$ has no component in span$\{\xi_3, \xi_4, \xi_5\}$, we may choose ξ_7 in such a way that we have

(5.44) $$D'_{e_2}\xi_4=\omega_4{}^6(e_2)\xi_6+\omega_4{}^7(e_2)\xi_7.$$

In this way, we have $\omega_4{}^8(e_2)=\cdots=\omega_4{}^m(e_2)=0$. Combining this with (5.42), we obtain $\omega_4{}^8=\cdots=\omega_4{}^m=0$.

Taking exterior differentiation of $\omega_3{}^r=0$, $r=7, \cdots, m$ and applying (5.30), we obtain

(5.45) $$\alpha_1\omega_4{}^r(e_2)=\delta\omega_5{}^r(e_1),\qquad r=7, \cdots, m.$$

Combining this with $\omega_4{}^r=0$ for $r=8, \cdots, m$, and (5.40), we have $\omega_5{}^r=0$ for $r=8, \cdots, m$. This proves the lemma.

From (5.42), we may put

(5.46) $$\omega_4{}^7=\mu_1\omega^2,\qquad \omega_5{}^7=\mu_2\omega^1\,.$$

Taking exterior differentiation of (5.46) we may obtain

(5.47) $$e_1(\mu_1)=e_2(\mu_2)=0$$

From (5.45) and (5.46) we get

(5.48) $$\alpha_1\mu_1=\delta\mu_2\,.$$

Since α_1 and δ are nonzero constants, (5.47) and (5.48) show that μ_1 and μ_2 are constants, too. Since $\alpha_1\delta\neq 0$, (5.48) implies that either $\mu_1=\mu_2=0$ or μ_1 and μ_2 are nonzero constants satisfying $\alpha_1\mu_1=\delta\mu_2$. If $\mu_1=\mu_2=0$, we obtain $\omega_4{}^7=\omega_5{}^7=0$. Thus, by applying Lemmas 8 and 9, and equation of Ricci, we find

$$\beta\gamma\alpha_1\delta=b^2\langle D'_{e_1}D'_{e_2}\xi_4,\,\xi_5\rangle=b^2\langle D'_{e_2}D'_{e_1}\xi_4,\,\xi_5\rangle=-\alpha_1\delta b^2\,.$$

Since $\beta\gamma+b^2=1$, we have $\alpha_1\delta=0$. This is a contradiction. Consequently, we conclude that μ_1 and μ_2 are nonzero constant. Now, taking exterior differentiation of $\omega_4{}^r=0$ for $r=8,\cdots,m$ and applying Lemmas 11 and 12 and (5.46), we may obtain $\omega^2\wedge\omega_7{}^r=0$ for $r=8,\cdots,m$. Similarly, by taking exterior differentiation of $\omega_5{}^r=0$, $r=8,\cdots,m$, we may conclude that $\omega^1\wedge\omega_7{}^r=0$. Consequently, we have $\omega_7{}^8=\cdots=\omega_7{}^m=0$. Combining these with (5.30) and (5.31), we see that the normal subspace $\nu=\text{span}\{\xi_3,\cdots,\xi_7\}$ is parallel with respect to the normal connection D'. Moreover, ν contains the first normal space span$\{\xi_3,\xi_6\}$. Thus, by a reduction theorem of submanifolds [**6**], we conclude that M is in fact contained in a totally geodesic 7-sphere $S_0^7(1)$ of $S_0^m(1)$. Furthermore, from the connection form $(\omega_A{}^B)$, $A, B=1,\cdots,8$, of M in $S_0^m(1)$, we may also conclude that M lies fully in $S_0^7(1)$. This completes the proof of the theorem.

6. Connection form.

Theorem 3 says that if M is a stationary, mass-symmetric, 2-type surface in $S_0^m(1)$, then M is flat and it lies fully in a $S_0^5(1)$ or $S_0^7(1)$. In this section, we shall determine the connection form of such surfaces.

THEOREM 4. *If M is a stationary, mass-symmetric, 2-type surface in $S_0^5(1)$, then M is flat and $2/3<\lambda_p<2$. Moreover, with respect to an adapted orthonormal frame field, the connection form is given by* (6.1) *if $2/3<\lambda_p\leqq 4/3$ and given by* (6.1) *or* (6.2) *if $4/3<\lambda_p<2$:*

$$(6.1)\qquad \left(\begin{array}{cc|ccc} 0 & 0 & \frac{\sqrt{2}-\sqrt{2c}}{\sqrt{3c-2}}\omega^1 & \frac{\sqrt{c}}{\sqrt{3c-2}}\omega^2 & 0 \\ 0 & 0 & \frac{\sqrt{c}}{\sqrt{3c-2}}\omega^2 & \frac{\sqrt{c}}{\sqrt{3c-2}}\omega^1 & 0 \\ \hline \frac{\sqrt{2c}-\sqrt{2}}{\sqrt{3c-2}}\omega^1 & \frac{-\sqrt{2}}{\sqrt{3c-2}}\omega^2 & 0 & 0 & \frac{-c}{\sqrt{3c-2}}\omega^1 \\ \frac{-\sqrt{c}}{\sqrt{3c-2}}\omega^2 & \frac{-\sqrt{c}}{\sqrt{3c-2}}\omega^1 & 0 & 0 & \frac{-\sqrt{2c}}{\sqrt{3c-2}}\omega^2 \\ 0 & 0 & \frac{c}{\sqrt{3c-2}}\omega^1 & \frac{\sqrt{2c}}{\sqrt{3c-2}}\omega^2 & 0 \end{array}\right)$$

where $c=\lambda_p$ *is a real number satisfying* $2/3<c<2$, *or*

$$(6.2)\qquad \left(\begin{array}{cc|ccc} 0 & 0 & \frac{c-4}{2\sqrt{3c-4}}\omega^1 & \frac{1}{2}\sqrt{c}\,\omega^2 & 0 \\ 0 & 0 & \frac{1}{2}\sqrt{3c-4}\omega^2 & \frac{1}{2}\sqrt{c}\,\omega^1 & 0 \\ \hline \frac{4-c}{2\sqrt{3c-4}}\omega^1 & -\frac{1}{2}\sqrt{3c-4}\,\omega^2 & 0 & 0 & \frac{-c}{\sqrt{6c-8}}\omega^1 \\ -\frac{1}{2}\sqrt{c}\,\omega^2 & -\frac{1}{2}\sqrt{c}\,\omega^1 & 0 & 0 & -\frac{\sqrt{c}}{\sqrt{2}}\omega^2 \\ 0 & 0 & \frac{c}{\sqrt{6c-8}}\omega^1 & \frac{\sqrt{c}}{\sqrt{2}}\omega^2 & 0 \end{array}\right)$$

where $c=\lambda_p$ *is a real number satisfying* $4/3<c<2$.

PROOF. Under the hypothesis, Lemma 7 implies

$$(6.3)\qquad \omega_1^2=\omega_1^5=\omega_2^5=\omega_3^4=0\,,\quad \omega_1^3=\beta\omega^1\,,\quad \omega_2^3=\gamma\omega^2\,,$$

$$(6.4)\qquad \omega_1^4=b\omega^2\,,\quad \omega_2^4=b\omega^1\,,\quad \omega_3^5=\mu\omega^1\,,\quad \omega_4^5=\frac{\gamma\mu}{b}\omega^2\,,$$

$$(6.5)\qquad \beta\gamma+1=b^2\,,\quad \mu^2=\lambda_p\lambda_q/4\,,\quad b^2(\beta-\gamma)+\gamma\mu^2=0\,.$$

Moreover, Lemmas 5 and 7 also imply

$$(6.6)\qquad (\beta+\gamma)^2=(2-\lambda_p)(\lambda_q-2)\,,$$

$$(6.7)\qquad \beta^2+\gamma^2=\lambda_p+\lambda_q-2-\lambda_p\lambda_q/4\,.$$

By using the second equation of (6.5), (6.6) and (6.7) we obtain

$$(6.8)\qquad 2\mu^2=(\beta-\gamma)^2\,.$$

Replacing ξ_5 by $-\xi_5$ if necessary, we may assume that

(6.9) $$\sqrt{2}\mu = \beta - \gamma .$$

By using (6.5) and (6.9), we may obtain

(6.10) $$\beta = (\gamma^2 - 2)/3\gamma, \quad \mu = -\sqrt{2}(1+\gamma^2)/3\gamma, \quad b^2 = (1+\gamma^2)/3 .$$

Replacing ξ_4 by $-\xi_4$ if necessary, we may assume that b is positive, so we have

(6.11) $$b = (1+\gamma^2)^{1/2}/\sqrt{3} .$$

Substituting the first equation of (6.10) into (6.6) and (6.7) and then solving λ_q in terms of λ_p, we may obtain

(6.12) $$\lambda_q = 2\lambda_p/(3\lambda_p - 4) \quad \text{or} \quad \lambda_q = 4\lambda_p/(3\lambda_p - 2) .$$

On the other hand, since M is mass-symmetric in $S_0^5(1)$ and of 2-type, Theorem 9.1 of Chen [3, p. 307] gives $0 < \lambda_p < 2 < \lambda_q$. Thus, by (6.12), we see that the first equation of (6.12) holds only when $4/3 < \lambda_p < 2$ and the second equation holds only when $2/3 < \gamma_p < 2$. Combining this with (6.3)-(6.7) and (6.9)-(6.11), we may obtain the theorem. (Q. E. D.)

REMARK 2. Theorem 4 shows that the immersion is rigid.

REMARK 3. In the next section, we will show that both cases of (6.1) and (6.2) occur and the estimates on λ_p are best possible.

For a stationary, mass-symmetric, 2-type surface which lies fully in $S_0^7(1)$, Lemmas 5, 8 and 9 and (5.46), (5.47) give

(6.13) $$\omega_1^2 = \omega_1^4 = \omega_2^4 = \omega_1^5 = \omega_2^5 = \omega_1^7 = \omega_2^7 = \omega_3^6 = \omega_3^7 = \omega_4^5 = \omega_6^7 = 0 ,$$

(6.14) $$\omega_1^3 = \beta\omega^1, \quad \omega_2^3 = \gamma\omega^2, \quad \omega_1^6 = b\omega^2, \quad \omega_2^6 = b\omega^1 ,$$

(6.15) $$\omega_3^4 = \alpha_1\omega^1, \quad \omega_3^5 = \delta\omega^2, \quad \omega_4^6 = -\frac{\gamma\alpha_1}{b}\omega^2, \quad \omega_5^6 = -\frac{\beta\delta}{b}\omega^1 ,$$

(6.16) $$\omega_4^7 = \mu_1\omega^2, \quad \omega_5^7 = \mu_2\omega^1 ,$$

(6.17) $$\beta\gamma + 1 = b^2, \quad \alpha_1\mu_1 = \delta\mu_2, \quad \beta \neq \gamma, \quad \beta + \gamma \neq 0 ,$$

(6.18) $$(\beta+\gamma)^2 = (2-\lambda_p)(\lambda_q - 2), \quad \beta^2 + \gamma^2 = \lambda_p + \lambda_q - \frac{1}{4}\lambda_p\lambda_q ,$$

(6.19) $$\alpha_1^2 + \delta^2 = \frac{1}{4}\lambda_p\lambda_q, \quad \alpha_1\delta b\mu_1\mu_2 \neq 0 ,$$

where β, γ, b, α_1, δ, μ_1 and μ_2 are constants. Moreover, by taking differentiation of $\omega_3^6 = 0$ and $\omega_4^5 = 0$, we may also obtain

(6.20) $$(\beta - \lambda)b^2 = \beta\delta^2 - \gamma\alpha_1^2, \quad \alpha_1\delta b^2 = \beta\gamma\delta\alpha_1 + b^2\mu_1\mu_2 .$$

Combining (6.17) and (6.20), we have

$$(6.21)\qquad (\mu_1)^2=\left(\frac{\delta}{b}\right)^2,\quad (\mu_2)^2=\left(\frac{\alpha_1}{b}\right)^2.$$

Without loss of generality, we may choose ξ_4, ξ_5, ξ_7 in such a way that α_1, δ and μ_1 are positive. Then we have $\mu_1=\delta/b$ and $\mu_2=\alpha_1/b$. From (6.18) we get $(\beta-\gamma)^2=\lambda_p\lambda_q/2$. Furthermore, we have $0<\lambda_p<2<\lambda_q<\infty$ by Theorem 9.1 of [**3**, p. 307]. By interchanging e_1 and e_2 and replacing ξ_3 by $-\xi_3$ if necessary, we may assume that $\beta<\gamma$ and $\beta+\gamma>0$. In this case, we have $\beta+\gamma=[(2-\lambda_p)(\lambda_q-2)]^{1/2}$. From these we have the following.

THEOREM 5. *If M is a stationary, mass-symmetric, 2-type surface which lies fully in $S_0^7(1)$, then M is a flat surface; moreover, with respect to an adapted orthonormal frame field, the connection form $(\omega_A{}^B)$ is given by*

$$(6.22)\qquad \left(\begin{array}{cc|ccccc} 0 & 0 & \beta\omega^1 & 0 & 0 & b\omega^2 & 0 \\ 0 & 0 & \gamma\omega^2 & 0 & 0 & b\omega^1 & 0 \\ \hline -\beta\omega^1 & -\gamma\omega^2 & 0 & \alpha_1\omega^1 & \delta\omega^2 & 0 & 0 \\ 0 & 0 & -\alpha_1\omega^1 & 0 & 0 & -\frac{\gamma\alpha_1}{b}\omega^2 & \frac{\delta}{b}\omega^2 \\ 0 & 0 & -\delta\omega^2 & 0 & 0 & -\frac{\beta\delta}{b}\omega^1 & \frac{\alpha_1}{b}\omega^1 \\ -b\omega^2 & -b\omega^1 & 0 & \frac{\gamma\alpha_1}{b}\omega^2 & \frac{\beta\delta}{b}\omega^1 & 0 & 0 \\ 0 & 0 & 0 & -\frac{\delta}{b}\omega^2 & -\frac{\alpha_1}{b}\omega^1 & 0 & 0 \end{array}\right)$$

where b, β, γ, α_1, δ are constants satisfying

$$\beta=(1/2)\sqrt{cd/2}+(1/2)\sqrt{(2-c)(d-2)},\quad \gamma=-(1/2)\sqrt{cd/2}+(1/2)\sqrt{(2-c)(d-2)},$$
$$\alpha_1=(\gamma-\beta)(2+3\beta\gamma-\beta^2)/2(\beta+\gamma),\quad \delta=(\beta-\gamma)(2+3\beta\gamma-\gamma^2)/2(\beta+\gamma),$$
$$b=\sqrt{1+\beta\gamma},$$

for some constants $c=\lambda_p$ and $d=\lambda_q$ so that $0<c<2<d<\infty$.

REMARK 4. For any real numbers c and d with $0<c<2<d<\infty$, the connection form given in Theorem 5 satisfies the structure equations (or integrability condition). Thus, by Fundamental Theorem of Submanifolds, we see that there is a "unique" isometric immersion y from $\boldsymbol{R}^2$ into $S_0^7(1)$ whose connection form is given by (6.22). When y is doubly-periodic, y yields many stationary, mass-symmetric, 2-type, flat surfaces in $S_0^7(1)$ with $\lambda_p=c$ and $\lambda_q=d$. Theorem 5 also implies that all stationary, mass-symmetric, 2-type surfaces which lie fully in

$S_0^7(1)$ are obtained in this way.

7. Examples.

Let $\boldsymbol{R}^2$ be the Euclidean plane with the Euclidean metric. Let u, v and w be real numbers with $u, v>0$. We define the lattice

$$\Lambda=\{(2n\pi u, 2m\pi v+2n\pi w) \mid n, m\in \boldsymbol{Z}\}. \tag{7.1}$$

The dual lattice of Λ is given by

$$\Lambda^* = \left\{\left(\frac{h}{2\pi u}-\frac{kw}{2\pi uv}, \frac{k}{2\pi v}\right) \,\middle|\, h, k\in \boldsymbol{Z}\right\}. \tag{7.2}$$

Let T_{uvw} be the flat torus given by $\boldsymbol{R}^2/\Lambda$. Then the spectrum of T_{uvw} is given by

$$\left\{\left(\frac{h}{u}-\frac{kw}{uv}\right)^2+\frac{k^2}{v^2} \,\middle|\, h, k\in \boldsymbol{Z}\right\}. \tag{7.3}$$

For any nonzero real number ε and two natural numbers h and $\bar{\varepsilon}$ satisfying

$$\varepsilon \neq 2h\bar{\varepsilon}^2/(\bar{\varepsilon}^2-2h^2), \tag{7.4}$$

we put

$$\begin{aligned} u &= \sqrt{3}\,\varepsilon\bar{\varepsilon}/(2\varepsilon^2+\bar{\varepsilon}^2)^{1/2}, \quad v = \bar{\varepsilon}/(2\varepsilon^2+\bar{\varepsilon}^2)^{1/2}, \\ w &= (h-\varepsilon)\bar{\varepsilon}/(2\varepsilon^2+\bar{\varepsilon}^2)^{1/2}, \quad e = \sqrt{2}\,\varepsilon/(2\varepsilon^2+\bar{\varepsilon}^2)^{1/2} \end{aligned} \tag{7.5}$$

and we define an isometric immersion y from $\boldsymbol{R}^2$ into $S_0^5(1)\subset E^6$ by

$$\begin{aligned} y(s, t) = \Big(&v\cos\frac{\varepsilon s}{u}\cos\frac{t}{v}, v\cos\frac{\varepsilon s}{u}\sin\frac{t}{v}, e\cos\frac{\bar{\varepsilon}s}{u}, \\ &v\sin\frac{\varepsilon s}{u}\cos\frac{t}{v}, v\sin\frac{\varepsilon s}{u}\sin\frac{t}{v}, e\sin\frac{\bar{\varepsilon}s}{u}\Big). \end{aligned} \tag{7.6}$$

The immersion y induces an isometric immersion from T_{uvw} into $S_0^5(1)$ which is denoted by x. Thus we have

$$x : T_{uvw} \longrightarrow S_0^5(1)\subset E^6. \tag{7.7}$$

It is easy to see that if $\varepsilon=\bar{\varepsilon}=h=1$, then (7.7) gives Ejiri's example mentioned in section 5.

PROPOSITION 1. *For any nonzero real number ε and two natural numbers h and $\bar{\varepsilon}$ satisfying* (7.4), *the immersion $x: T_{uvw}\to S_0^5(1)$ is a stationary, mass-symmetric, isometric immersion, where u, v and w are defined by* (7.5). *Furthermore, we have*

(a) *x is of 1-type if and only if $\bar{\varepsilon}^2=4\varepsilon^2$, in this case, $\lambda_p=2$.*

(b) *Otherwise, x is of 2-type with λ_p and λ_q given by*

$$\{\lambda_p, \lambda_q\} = \left\{\left(\frac{\bar{\varepsilon}}{u}\right)^2, \left(\frac{\varepsilon}{u}\right)^2+\left(\frac{1}{v}\right)^2\right\}. \tag{7.8}$$

PROOF (Outlined). From (7.6) we see that x is an isometric immersion. The Laplacian of T_{uvw} is given by $\Delta=-\partial^2/\partial s^2-\partial^2/\partial t^2$. Therefore, the coordinate functions of x are eigenfunctions of Δ with eigenvalues given by (7.7). From (7.5) and (7.6) we know that $\lambda_p=\lambda_q$ if and only if $\bar{\varepsilon}^2=4\varepsilon^2$. In this case, x is of 1-type. Otherwise, x is of 2-type.

By direct, long computation, we may prove that T_{uvw} is an $\mathfrak{A}$-surface of $S_0^5(1)$. Moreover, we may also prove that a mass-symmetric, 2-type, $\mathfrak{A}$-surface of $S_0^m(1)$ is stationary if and only if $\Delta H=2(\|A_\xi\|^2-2(\alpha')^2)H'+2\alpha^2 H$. So, by a long, straight-forward computation, we may in fact prove that the immersion x satisfies this equation.

REMARK 5. It is easy to check that w satisfies $w^2\leqq(2h^2+\bar{\varepsilon}^2)/2$. If one chooses $w^2\in(0, (2h^2+\bar{\varepsilon}^2)/2)$, then one obtains two non-isometric tori. Otherwise, if $w=0$ or $w^2=(2h^2+\bar{\varepsilon}^2)/2$, one obtains only one torus. Moreover, if $w=0$, the torus is defined by a rectangular lattice.

THEOREM 6. *We have the following two statements.*

(a) *For each real number c with $2/3<c\leqq 4/3$, there is a stationary, mass-symmetric, 2-type, flat torus in $S_0^5(1)$ whose connection form is given by* (6.1).

(b) *For each real number c with $4/3<c<2$, there are two stationary, mass-symmetric, 2-type, flat tori in $S_0^5(1)$ whose connection forms are given by* (6.1) *and* (6.2) *respectively.*

PROOF. Consider the stationary, mass-symmetric, 2-type, flat torus in $S_0^5(1)$ (with $\bar{\varepsilon}^2\neq 4\varepsilon^2$) given by (7.7). According to (7.5) and (7.8), we have

$$\{\lambda_p, \lambda_q\} = \left\{\frac{2}{3}+\frac{1}{3\varepsilon^2}, \frac{4}{3}+\frac{8}{3}\varepsilon^2\right\}. \tag{7.9}$$

Given a real number c with $2/3<c<2$, we consider the following equation

$$c = \lambda_p = \frac{2}{3}+\frac{1}{3\varepsilon^2}. \tag{7.10}$$

The only thing we need to prove is that the range of ε^2 is $(1/4, \infty)$. One notices that for any fixed natural numbers h and $\bar{\varepsilon}$, ε depends on w continuously over the domain. For instance, consider the case $h=\bar{\varepsilon}=1$, we have

$$\varepsilon = \{1+w(3-2w^2)^{1/2}\}/(1-2w^2), \quad \text{or} \quad \varepsilon = \{1-w(3-2w^2)^{1/2}\}/(1-2w^2). \tag{7.11}$$

In this case, the range of w is $\left(-\frac{\sqrt{6}}{2}, \frac{\sqrt{6}}{2}\right)-\{1\}$ (with $w^2=3/2$ corresponding to the 1-type case). Now, it is not difficult to see that the range of ε^2 is $(1/4, \infty)$.

If $4/3<c<2$, we may also consider the equation

$$c=\lambda_p=\frac{4}{3}+\frac{8}{3}\varepsilon^2. \tag{7.12}$$

By a similar argument we may see that the range of ε^2 is $(0, 1/4)$. (Q. E. D.)

REMARK 6. From the Fundamental Theorem of Submanifolds, we see that for any $c\in(2/3, 4/3]$, there is an isometric immersion y from $\boldsymbol{R}^2$ into $S_0^5(1)$ whose connection form is given by (6.1). Such immersions are unique up to rigid motions. According to Theorem 6, for such c, there is a flat torus T_c in $S_0^5(1)$ whose connection form is also given by (6.1). Thus, if we lift the immersion x of T_c up to its universal covering $\boldsymbol{R}^2$, we obtain an isometric immersion $\tilde{x}$ from $\boldsymbol{R}^2$ into $S_0^5(1)$. Since y and $\tilde{x}$ have the same connection form, they only differ by a rigid motion. Consequently, the immersion y is doubly-periodic. For $c\in(4/3, 2)$, we have two isometric immersions y_1, y_2 from $\boldsymbol{R}^2$ into $S_0^5(1)$ whose connection forms are given by (6.1) and (6.2), respectively. Theorem 6 implies that y_1 and y_2 are both doubly-periodic. From these, we conclude that all stationary, mass-symmetric, 2-type surfaces in $S_0^5(1)$ are always obtained in this way.

In Remark 4, we know that a stationary, mass-symmetric, 2-type surface which lies fully in $S_0^7(1)$ is obtained by a doubly-periodic isometric immersion of $\boldsymbol{R}^2$ into $S_0^7(1)$ whose connection form is given by (6.22). In the following, we give some concrete examples of such surfaces in $S_0^7(1)$.

Recall that for any real numbers u, v and w with $u, v>0$, we have a flat torus T_{uvw}. Given four natural numbers $(n, m, \bar{n}, \bar{m})$, we put

$$\varepsilon=n-\frac{mw}{v}, \quad \bar{\varepsilon}=\bar{n}-\frac{\bar{m}w}{v}. \tag{7.13}$$

We define an isometric immersion y from $\boldsymbol{R}^2$ into $S_0^7(1)\subset E^8$ by

$$\begin{aligned} y(s, t)=\Big(&c_1\cos\frac{\varepsilon s}{u}\cos\frac{mt}{v}, \quad c_1\cos\frac{\varepsilon s}{u}\sin\frac{mt}{v}, \\ &c_1\sin\frac{\varepsilon s}{u}\cos\frac{mt}{v}, \quad c_1\sin\frac{\varepsilon s}{u}\sin\frac{mt}{v}, \\ &c_2\cos\frac{\bar{\varepsilon} s}{u}\cos\frac{\bar{m}t}{v}, \quad c_2\cos\frac{\bar{\varepsilon} s}{u}\sin\frac{\bar{m}t}{v}, \\ &c_2\sin\frac{\bar{\varepsilon} s}{u}\cos\frac{\bar{m}t}{v}, \quad c_2\sin\frac{\bar{\varepsilon} s}{u}\sin\frac{\bar{m}t}{v}\Big), \end{aligned} \tag{7.14}$$

where c_1 and c_2 are two real numbers satisfying

$$c_1^2+c_2^2=1,\quad c_1^2\bar{\varepsilon}^2+c_2^2\varepsilon^2=u^2,\quad c_1^2m^2+c_2^2\bar{m}^2=v^2. \tag{7.15}$$

The immersion y induces an isometric immersion from T_{uvw} into $S_0^7(1)\subset E^8$ which is denoted by x. Thus we have

$$x : T_{uvw} \longrightarrow S_0^7(1)\subset E^8. \tag{7.16}$$

By using an argument similar to that of Proposition 1, we may prove the following.

PROPOSITION 2. *If* $v^2(\bar{\varepsilon}^2-\varepsilon^2)\neq u^2(m^2-\bar{m}^2)$, *then the immersion* $x: T_{uvw}\to S_0^7(1)$ *is a mass-symmetric, 2-type, isometric immersion with*

$$\{\lambda_p, \lambda_q\}=\left\{\left(\frac{\varepsilon}{u}\right)^2+\left(\frac{m}{v}\right)^2,\ \left(\frac{\bar{\varepsilon}}{u}\right)^2+\left(\frac{\bar{m}}{v}\right)^2\right\}. \tag{7.17}$$

Moreover, T_{uvw} *is an* $\mathfrak{A}$*-surface of* $S_0^7(1)$. *Furthermore, the immersion* x *is stationary if and only if the following equation holds:*

$$2c_1^2\left[\left(\frac{\varepsilon}{u}\right)^2-\left(\frac{m}{v}\right)^2\right]^2+2c_2^2\left[\left(\frac{\bar{\varepsilon}}{u}\right)^2-\left(\frac{\bar{m}}{v}\right)^2\right]^2=\lambda_p\lambda_q. \tag{7.18}$$

We also need the following.

PROPOSITION 3. *Let* $[p, q]$ *be the order of a stationary immersion given by* (7.16). *Then we have*

(a) $2/3<\lambda_p<2$,

(b) *if* $2/3<\lambda_p\leqq 4/3$, *then* $\lambda_q>4\lambda_p/(3\lambda_p-2)$, *and*

(c) *if* $4/3<\lambda_p<2$, *then* $2\lambda_p/(3\lambda_p-4)>\lambda_q>4\lambda_p/(3\lambda_p-2)$.

PROOF. Follows from (7.15), (7.18) and the fact $0<\lambda_p<2<\lambda_q$ of [**3**, p. 307].

Given two real numbers c and d with $0<c<2<d<\infty$, we put

$$\begin{aligned} F(c, d) &= c-\{cd(c-2)/2(2-d)\}^{1/2},\\ G(c, d) &= d+\{cd(c-2)/2(2-c)\}^{1/2}. \end{aligned} \tag{7.19}$$

LEMMA 10. *For any* $d\in(2, \infty)$ *and any rational number* $r\neq 0$, *there is a* $c\in(2/3, 2)$ *such that* $G(c, d)=r^2F(c, d)$.

PROOF. Under the hypothesis, it is easy to see that there is a $c\in(0, 2)$ satisfying $G=r^2F$. Because G is positive, F is also positive. Thus, we obtain $c\in(2/3, 2)$.

By using Lemma 10, (7.13), (7.16) and (7.20), we obtain the following.

THEOREM 7. *For any* $d\in(2, \infty)$, *there is a stationary, mass-symmetric,* *2-type, flat torus in* $S_0^7(1)$ *such that* $\lambda_q = d$ *and whose connection form is given by* (6.22).

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