J. Math. Soc. Japan
Vol. 40, No. 1, 1988

A theorem on the outradii of Teichmüller spaces

Dedicated to Professor Tadashi Kuroda on his sixtieth birthday

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(Received June 2, 1986)

1. Introduction.

The purpose of this paper is to present some results related to the Teichmüller spaces. Let Γ be a Fuchsian group acting on the upper half plane $\boldsymbol{U}=\{\operatorname{Im} z>0\}$. Then the Teichmüller space $\boldsymbol{T}(\Gamma)$ is represented as a bounded domain in the Banach space $\boldsymbol{B}(\boldsymbol{U}^*, \Gamma)$ of bounded quadratic differentials for Γ in the lower half plane $\boldsymbol{U}^*$ (Bers [1]). We consider the function $\varphi_\alpha(z)=\alpha z^{-2}$, $\alpha\in\boldsymbol{C}$, defined in $\boldsymbol{U}^*$. Let F_α be a solution of the differential equation $\{f, z\}=\varphi_\alpha(z)$, where $\{f, z\}=(f''/f')'-(1/2)(f''/f')^2$ denotes the Schwarzian derivative of f. Then it is known that F_α is univalent in $\boldsymbol{U}^*$ if and only if α belongs to the set $\boldsymbol{V}=\{\alpha=(1-re^{2i\theta})/2\,;\ r\leqq 4\cos^2\theta,\ 0\leqq\theta<\pi\}$ ([4, 5]). Since it has such a simple form, the function φ_α, $\alpha\in\boldsymbol{V}$, cannot belong to $\boldsymbol{T}(\Gamma)$ unless Γ is one of some elementary groups (see Section 4). However if we are allowed to vary Γ in its quasiconformal equivalence class, we obtain the following result:

THEOREM A. *Let $Q_U(\Gamma)$ be the set of all quasiconformal automorphisms of U compatible with Γ. If Γ contains a hyperbolic element, then for each $\alpha\in\boldsymbol{V}$ there exists a sequence w_n, $n=1, 2, \cdots$, in $Q_U(\Gamma)$ with an element $\varphi_n\in\boldsymbol{T}(w_n\circ\Gamma\circ w_n^{-1})$ such that φ_n converges normally (uniformly on every compact subsets of $\boldsymbol{U}^*$) to φ_α in $\boldsymbol{U}^*$.*

The motivation of this theorem originates from a problem related to the outradii of Teichmüller spaces. By a theorem of Nehari [8] the outradius $\boldsymbol{o}(\Gamma)$ of $\boldsymbol{T}(\Gamma)$ does not exceed 6. The following theorem shows that this value 6 is sharp within the range of the quasiconformal equivalence class.

THEOREM B. *Set $\mathcal{O}(\Gamma)=\sup\{\boldsymbol{o}(w\circ\Gamma\circ w^{-1});\ w\in Q_U(\Gamma)\}$. Then the equality $\mathcal{O}(\Gamma)=6$ holds if $0<\dim\boldsymbol{T}(\Gamma)$.*

Actually if Γ is of the second kind, Theorem B is trivially deduced from

This research was partially supported by Grant-in-Aid for Scientific Research (No. 60740108), Ministry of Education, Science and Culture.

the equality $o(\Gamma)=6$, which is established by Sekigawa and Yamamoto [**12, 13**]. However for any finitely generated Fuchsian group Γ of the first kind, $o(\Gamma)$ is strictly less than 6 (Sekigawa [**11**]). Chu showed in [**2**] an example of a sequence Γ_n, $n=1, 2, \cdots$, of finitely generated Fuchsian groups of the first kind such that $o(\Gamma_n)$ converges to 6, but the topological structure of the surface U/Γ_n becomes more and more complicated as n increases. We remark that Theorem B gives an amelioration of Chu's result, namely,

COROLLARY. *Let* $\sigma=(g; \nu_1, \cdots, \nu_n)$ $(g\geqq 0,\ 2\leqq\nu_1\leqq\cdots\leqq\nu_n\leqq\infty)$ *be a signature* (*for the definition, see e.g.* [**6**, p. 57]) *satisfying* (i) $2g-2+\sum_{j=1}^{n}(1-1/\nu_j)>0$, *and* (ii) $3g-3+n>0$, *then*

$$\sup\{o(\Gamma);\ \Gamma \text{ is a Fuchsian group with the signature } \sigma\}=6.$$

Note that the condition (ii) implies that $T(\Gamma)$ is not a single point. Since two Fuchsian groups with the same signature are quasiconformally equivalent to each other, thus this corollary follows.

The author owes much to Prof. H. Yamamoto with whom he has discussed the subject matter during his preparation of this paper. The author also express his deepest gratitude to Prof. Y. Kusunoki for his continual encouragement and useful suggestions.

2. Preliminaries.

In the following D denotes the unit disk $\Delta=\{|z|<1\}$ or the upper half plane U, and D^* denotes the exterior of $\bar{D}$ in the Riemann sphere $\hat{C}$. Let Γ be a Fuchsian group acting discontinuously on D and hence also on D^*. We denote by $B(D^*, \Gamma)$ the space of bounded quadratic differentials for Γ in D^*. In other words a holomorphic function φ in D^* belongs to $B(D^*, \Gamma)$ if and only if (i) $\varphi(\gamma z)\gamma'(z)^2=\varphi(z)$ *for all* $\gamma\in\Gamma$ *and all* $z\in D^*$, *and* (ii) *the norm is finite, i.e.*, $\|\varphi\|_{D^*}=\sup_{z\in D^*}\lambda(z)^{-2}|\varphi(z)|<\infty$, *where* $\lambda(z)$ *is the density of the hyperbolic metric on* D^* *which has constant* (*Gaussian*) *curvature* -4. Then $\lambda(z)=(|z|^2-1)^{-1}$ for $D^*=\Delta^*$, and $\lambda(z)=(-2\,\mathrm{Im}\,z)^{-1}$ for $D^*=U^*$. A quasiconformal automorphism w of $\hat{C}$ is said to be compatible with Γ if $w\circ\gamma\circ w^{-1}$ is a Möbius transformation for each $\gamma\in\Gamma$. Then the Teichmüller space $T_{D^*}(\Gamma)$ is the set of all φ of $B(D^*, \Gamma)$ with the following property: There is a quasiconformal automorphism w_φ of $\hat{C}$ compatible with Γ such that w_φ is conformal in D^* and its Schwarzian derivative $\{w_\varphi|_{D^*}, z\}$ coincides with φ. If Γ is the trivial group $\{\mathrm{id}\}$, we abbreviate $T_{D^*}(\{\mathrm{id}\})$ to $T_{D^*}(1)$ and call it the universal Teichmüller space. Then for any Γ, $T_{D^*}(\Gamma)$ is included in $T_{D^*}(1)$. Suppose that Γ acts on U. By using the Möbius transformation $h(z)=-i(z-1)/(z+1)$ we define the mapping h^* which takes φ of $B(U^*, \Gamma)$ into $(\varphi\circ h)h'(z)^2$ of $B(\Delta^*, h^{-1}\circ\Gamma\circ h)$. Then we can see that

h^* is an isometry of $\boldsymbol{B}(\boldsymbol{U}^*, \Gamma)$ onto $\boldsymbol{B}(\boldsymbol{\Delta}^*, h^{-1}\circ\Gamma\circ h)$ and that $h^*\boldsymbol{T}_{U^*}(\Gamma)= \boldsymbol{T}_{\Delta^*}(h^{-1}\circ\Gamma\circ h)$. By this mapping h^* we may identify these two Teichmüller spaces and in the following argument we shall replace the notations $\boldsymbol{T}_{D^*}(\Gamma)$ and $\|\varphi\|_{D^*}$ by $\boldsymbol{T}(\Gamma)$ and $\|\varphi\|$ respectively, when the domain $\boldsymbol{D}$ is not specified and no confusion will arise. The outradius $\boldsymbol{o}(\Gamma)$ of $\boldsymbol{T}(\Gamma)$ is defined to be $\sup\{\|\varphi\|;\ \varphi\in\boldsymbol{T}(\Gamma)\}$. By a theorem of Nehari [8] the inequality $\boldsymbol{o}(\Gamma)\leqq 6$ holds.

3. Behaviour of quadratic differentials.

Let Γ be a Fuchsian group acting on $\boldsymbol{D}$ $(=\boldsymbol{U}$ or $\boldsymbol{\Delta})$. We denote by $Q_D(\Gamma)$ the set of all quasiconformal automorphisms w of D compatible with Γ, that is, $w\circ\Gamma\circ w^{-1}$ is also a Fuchsian group acting on $\boldsymbol{D}$. The quotient space $R_\Gamma=\boldsymbol{D}/\Gamma$ is a Riemann surface with the hyperbolic metric induced by that on $\boldsymbol{D}$.

In the following we set $\boldsymbol{D}=\boldsymbol{U}$ and assume that Γ contains at least one hyperbolic element γ. We consider a sequence w_n, $n=1, 2, \cdots$, in $Q_U(\Gamma)$ with the following property:

(3.1) $\gamma_n=w_n\circ\gamma\circ w_n^{-1}$ *is of the form* $z\to\lambda_n z$, *where* $\lambda_n>1$, *and* $\lambda_n\to 1$ *as* $n\to\infty$.

An example of such a sequence is obtained by the method described in the proof of Theorem 11 in Bers's paper [**1**], namely by squeezing a simple closed curve on R_Γ. In that paper Bers considered only finitely generated groups, but the extremal length method which he employed there is applicable to infinitely generated ones. (See also the proof of Theorem 3 in [**1**].)

Let w_n, $n=1, 2, \cdots$, be a sequence in $Q_U(\Gamma)$ with the property (3.1). We choose an element φ_n from each $\boldsymbol{T}(\Gamma_n)$, where $\Gamma_n=w_n\circ\Gamma\circ w_n^{-1}$. The Nehari theorem yields the inequality $|\varphi_n(z)|\leqq 3/2(\operatorname{Im} z)^2$ in $\boldsymbol{U}^*$. Hence the φ_n's are locally uniformly bounded in $\boldsymbol{D}^*$ and then form a normal family. Let $\boldsymbol{V}$ be the set $\{\alpha=(1-re^{2i\theta})/2;\ r\leqq 4\cos^2\theta,\ 0\leqq\theta<\pi\}$.

PROPOSITION 3.1. *Let* $\{\varphi_{n_\nu}\}_{\nu=1}^\infty$ *be a subsequence of* $\{\varphi_n\}_{n=1}^\infty$ *which converges normally to a function* φ_∞ *in* $\boldsymbol{U}^*$. *Then* $\varphi_\infty(z)=\alpha z^{-2}$ *for some* $\alpha\in\boldsymbol{V}$.

PROOF. For convenience we replace the notations Γ_{n_ν}, $\{\varphi_{n_\nu}\}$ by Γ_n, $\{\varphi_n\}$ respectively. We set $\varphi_n(z)=z^{-2}P_n(z)$ for $n=1, 2, \cdots$; and $n=\infty$. Then obviously P_n, $n=1, 2, \cdots$, converges normally to P_∞ in $\boldsymbol{U}^*$. To show that P_∞ is constant in $\boldsymbol{U}^*$, we have only to show that P_∞ is constant along the negative imaginary axis I. Recall that under the condition (3.1) Γ_n contains an element of the form $\gamma_n(z)=\lambda_n z$. By substituting γ_n for γ in the equality $\varphi_n(\gamma z)\gamma'(z)^2=\varphi_n(z)$, which holds for all $\gamma\in\Gamma_n$, we obtain that $P_n(\lambda_n z)=P_n(z)$. Set $z=-i$ and take an arbitrary point $w=-ri$ on I. Let $\varepsilon>0$ be given. By the equicontinuity of the family $\{P_n\}$, we can choose $\delta>0$ so that $|P_n(\zeta)-P_n(w)|<\varepsilon$ holds for each n whenever $|\zeta-w|<\delta$. Since λ_n converges to 1, if N is taken to be sufficiently large, then every set $\{\lambda_n^\nu z\,;\,\nu=0,\ \pm1,\ \pm2,\ \cdots\}$ for $n>N$ intersects the δ-neighbourhood of w, and so $|P_n(w)-P_n(z)|=|P_n(w)-P_n(\lambda_n^{\nu_n}z)|<\varepsilon$ holds for a suitable choice of integer ν_n. Hence the inequality

$$|P_\infty(w)-P_\infty(z)|<|P_\infty(w)-P_n(w)|+|P_n(z)-P_\infty(z)|+\varepsilon$$

holds and by letting $n\to\infty$ it follows that $|P_\infty(w)-P_\infty(z)|<\varepsilon$. Since ε is arbitrary, the equality $P_\infty(w)=P_\infty(z)$ follows. Thus P_∞ is constant along I, and hence so is it in $\boldsymbol{U}^*$. Set $P_\infty\equiv\alpha$. Then φ_n converges normally to αz^{-2} in $\boldsymbol{U}^*$.

Next, to show that α belongs to $\boldsymbol{V}$, we use the mapping h^* defined in the previous section, which maps $\boldsymbol{T}_{U^*}(\Gamma_n)$ onto $\boldsymbol{T}_{\Delta^*}(h^{-1}\circ\Gamma_n\circ h)$ for each n. Set $\phi_n=h^*\varphi_n$. Then ϕ_n converges normally to $\phi_\infty(z)=4\alpha(z^2-1)^{-2}$ in $\boldsymbol{\Delta}^*$ (with respect to the spherical metric of $\hat{\boldsymbol{C}}$). Due to the fact which is proved in [**4**], we need only to show that a solution of the differential equation $\{f, z\}=\phi_\infty(z)$ is univalent in $\boldsymbol{\Delta}^*$. Let F_n be the solution of the equation $\{F_n, z\}=\phi_n(z)$ such that $F_n(z)=z+O(|z|^{-1})$ as $|z|\to\infty$. Since ϕ_n belongs to $\boldsymbol{T}_{\Delta^*}(h^{-1}\circ\Gamma_n\circ h)$, F_n is univalent in $\boldsymbol{\Delta}^*$ for all n. Then by taking a subsequence if necessary we may assume that F_n converges to a univalent function F_∞ in $\boldsymbol{\Delta}^*$ ([**9**, Theorem 1.7]). By the classical Cauchy's integral formula, the k-th derivative $F_n^{(k)}$ of F_n (in particular for $k=1, 2, 3$) converges normally to $F_\infty^{(k)}$, and so $\{F_n, z\}$ converges normally to $\{F_\infty, z\}$. Hence the univalent function F_∞ satisfies the equation $\{F_\infty, z\}=\phi_\infty(z)$. Now we complete the proof of the proposition.

PROOF OF THEOREM A. First let α be an interior point of the set $\boldsymbol{V}$. Let $\{w_n\}$ be a sequence in $Q_U(\Gamma)$ with the property (3.1). Then the Fuchsian group $\Gamma_n=w_n\circ\Gamma\circ w_n^{-1}$ contains the element $\gamma_n(z)=\lambda_n z$ with $\lambda_n\to1$ as $n\to\infty$. Here we may assume that γ_n is primitive in Γ_n, i. e., if $\gamma_n=\gamma^\nu$ for an element γ of Γ_n, then $\nu=\pm1$. Let K_n be the subgroup of Γ_n which consists of all elements keeping the imaginary axis invariant. Then K_n is either the cyclic group $\langle\gamma_n\rangle$ generated by γ_n or an extension of $\langle\gamma_n\rangle$ of index 2. For the latter case, by a conjugation of Γ_n by a Möbius transformation of the form $z\to\tau z$ ($\tau>0$) we may assume that the elliptic transformation $\eta(z)=1/z$ belongs to K_n. We may assume that γ_n represents a simple closed geodesic on $R_{\Gamma_n}=\boldsymbol{U}/\Gamma_n$ or on a two sheeted covering of R_{Γ_n}. Then the collar lemma (see e. g. [**3**]) provides the sector $S_n=\{z\in\boldsymbol{U}\,;\,\theta_n<\arg z<\pi-\theta_n\}$, where $\log(\operatorname{cosec}\theta_n+\cot\theta_n)=(2\sinh(\log\sqrt{\lambda_n}))^{-1}$, with the following property: $\gamma S_n=S_n$ for $\gamma\in K_n$ and $\gamma S_n\cap S_n=\emptyset$ otherwise. Note that $\theta_n\to0$ as $n\to\infty$.

For a technical reason we change the context of our argument to the unit disk $\boldsymbol{\Delta}$. To this end, we use the Möbius transformation $h(z)=i(1-z)/(1+z)$ and set $G_n=h^{-1}\circ\Gamma_n\circ h$, $H_n=h^{-1}\circ K_n\circ h$ and $T_n=h^{-1}S_n$. Then G_n acts discontinuously on $\boldsymbol{\Delta}\cup\boldsymbol{\Delta}^*$. The subregion T_n of $\boldsymbol{\Delta}$ is symmetric about the interval $-1<x<1$, and bounded by the two circular arcs which meet each other at ±1 with the angle $\pi-2\theta_n$. The group H_n coincides with the stabilizer $(G_n)_{T_n}=\{g\in G_n;\ gT_n=T_n\}$ of T_n. Furthermore H_n consists of the transformations

$$z \longrightarrow \frac{(1+\lambda_n^\nu)z+(1-\lambda_n^\nu)}{(1-\lambda_n^\nu)z+(1+\lambda_n^\nu)}, \qquad \nu=0, \pm1, \pm2, \cdots,$$

and (if H_n contains elliptic elements)

$$z \longrightarrow \frac{(1+\lambda_n^\nu)z-(1-\lambda_n^\nu)}{(1-\lambda_n^\nu)z-(1+\lambda_n^\nu)}, \qquad \nu=0, \pm1, \pm2, \cdots,$$

since $h^{-1}\circ\eta\circ h$ belongs to H_n.

As the point at infinity ∞ belongs to Δ^*, in the following we use the spherical metric of $\hat{C}$ when we consider the convergence of functions. To complete the proof it suffices to choose ϕ_n from $T(G_n)$, $n=1, 2, \cdots$, so that ϕ_n converges to $4\alpha(z^2-1)^{-2}$ in Δ^*. Set $\delta=(1-2\alpha)^{1/2}$ with $\mathrm{Re}\,\delta>0$. Then with the assumption that α is in the interior of V, we have that $|\delta-1|<1$. Following Kalme [5] we define a continuous mapping $W_\alpha: \hat{C}\to\hat{C}$ by using the above Möbius transformation h as follows:

$$W_\alpha(z) = \begin{cases} -2i\delta(-i)^\delta/(h(z)\overline{h(z)}^{\delta-1}-(-i)^\delta) & \text{for } z\in\bar{\Delta}, \\ -2i\delta(-i)^\delta/(h(z)^\delta-(-i)^\delta) & \text{for } z\in\Delta^*, \end{cases} \tag{3.2}$$

where we choose an arbitrary, but fixed branch of z^δ in U^*. The mapping W_α is conformal in Δ^*, satisfies that $W_\alpha(z)=z+O(|z|^{-1})$ as $|z|\to\infty$, and in Δ has the Beltrami coefficient $\mu_\alpha(z)=(W_\alpha)_{\bar{z}}/(W_\alpha)_z=(\delta-1)(1-z^2)/(1-\bar{z}^2)$. Thus W_α is a quasiconformal automorphism of $\hat{C}$. Furthermore $\{W_\alpha|_{\Delta^*}, z\}=4\alpha(z^2-1)^{-2}$. We remark that

$$\mu_\alpha(g(z))\overline{g'(z)}/g'(z) = \mu_\alpha(z) \tag{3.3}$$

holds for all $g\in H_n$. Next we construct a sequence of Beltrami coefficients μ_n, $n=1, 2, \cdots$, defined in Δ with the following properties:

(3.4) $\|\mu_n\|_\infty = |\delta-1|$, and μ_n converges to μ_α almost everywhere in Δ, and

(3.5) $\mu_n(g(z))\overline{g'(z)}/g'(z) = \mu_n(z)$ for all $g\in G_n$.

To do this, let $\{g_i\}_{i=0}^\infty$ $(g_0=\mathrm{id})$ be the set of representatives of the left cosets G_n/H_n. Then by using the function μ_α we set

$$\mu_n(z) = \begin{cases} \mu_\alpha(w)g_i'(w)/\overline{g_i'(w)} & \text{for } z=g_i(w),\ w\in T_n, \\ 0 & \text{for } z\in\Delta-\bigcup_{i=0}^\infty g_i(T_n). \end{cases} \tag{3.6}$$

Since $g_iT_n\cap g_jT_n=\emptyset$ for $i\neq j$, μ_n is well defined. By (3.3), (3.6) and the fact that $H_n=(G_n)_{T_n}$, we can see easily that μ_n satisfies both the statements (3.4) and (3.5). The Lebesgue measure of $\Delta-T_n$ diminishes as $n\to\infty$ and eventually becomes 0. Hence μ_n converges to μ_α in measure and then a subsequence

μ_{n_j}, $j=1, 2, \cdots$, converges to μ_α almost everywhere in Δ ([**10**, pp. 91-92]). By replacing the notation $\{\mu_{n_j}\}$ by $\{\mu_n\}$ we obtain the desired sequence.

Let W_n be the quasiconformal automorphism of $\hat{\boldsymbol{C}}$ such that $(W_n)_{\bar{z}}=\mu_n(W_n)_z$ in Δ, $(W_n)_{\bar{z}}=0$ in Δ^* and $W_n(z)=z+O(|z|^{-1})$ as $|z|\to\infty$. Under this normalization at $z=\infty$, for each $R>1$, $|W_n(z)|\leqq 2R$ for $|z|<R$ holds, since W_n is conformal in Δ^* ([**9**, Corollary 1.3]). Then it follows that the family $\{W_n\}$ of $(1+|\delta-1|)/(1-|\delta-1|)$-quasiconformal automorphisms is normal. By abuse of language a uniformly convergent subsequence is denoted again by $\{W_n\}$. By the normalization $W_n(z)=z+O(|z|^{-1})$ as $|z|\to\infty$, the limit function W_∞ is not constant and hence a quasiconformal automorphism of $\hat{\boldsymbol{C}}$ ([**7**, p. 29, Theorem 5.2]). Then we obtain that $W_\infty=W_\alpha$ in $\hat{\boldsymbol{C}}$, because both functions satisfy the same Beltrami equation and the normalization condition at ∞ ([**7**, p. 187, Theorem 5.2]). In particular the conformal mapping $W_n|_{\Delta^*}$ converges uniformly to $W_\alpha|_{\Delta^*}$ in Δ^*, and therefore $\phi_n=\{W_n|_{\Delta^*}, z\}$ converges normally to $4\alpha(z^2-1)^{-2}$ in Δ^*. Finally from (3.5) it follows that ϕ_n belongs to $\boldsymbol{T}(G_n)$. Thus we proved the theorem for α which is in the interior of $\boldsymbol{V}$.

We assume next that α is on the boundary of $\boldsymbol{V}$. We denote by F_α^* the conformal mapping in Δ^* defined by the second expression in (3.2) for $\delta=(1-2\alpha)^{1/2}$. Set $\alpha_k=(1-(1/k))\alpha$ $(k=1, 2, \cdots)$. Then α_k belongs to the interior of $\boldsymbol{V}$, and so we can construct as above a sequence $W_{n,k}$, $n=1, 2, \cdots$, of quasiconformal automorphisms of $\hat{\boldsymbol{C}}$ compatible with G_n which converge uniformly to W_{α_k}. On the other hand, $W_{\alpha_k}|_{\Delta^*}$ converges uniformly to F_α^* in Δ^*. Hence by a suitable choice of sufficiently large $n(k)$, $k=1, 2, \cdots$, the sequence $W_{n(k),k}|_{\Delta^*}$ converges uniformly to F_α^* in Δ^*. Then $\phi_{n(k),k}=\{W_{n(k),k}, z\}$ converges normally to $\{F_\alpha^*, z\}=4\alpha(z^2-1)^{-2}$ in Δ^*. We have already seen that $\phi_{n(k),k}$ belongs to $\boldsymbol{T}(G_{n(k)})$. Thus we complete the proof of Theorem A.

4. A remark on Theorem A.

The question naturally arises whether or not in the statement of Theorem A the sequence $\{\varphi_n\}$ can be chosen so that φ_n converges to φ_α in $\mathrm{cl}\,\boldsymbol{T}(1)$, the closure of $\boldsymbol{T}(1)$ in the Banach space $\boldsymbol{B}(\boldsymbol{U}^*, \{\mathrm{id}\})$. This is true if Γ is either a cyclic group $\langle\gamma\rangle$ generated by a hyperbolic transformation $\gamma(z)=\lambda z$, $\lambda>1$, or an extension of $\langle\gamma\rangle$ of index 2. Indeed in these cases we can see that φ_α, $\alpha\in\boldsymbol{V}$, belongs to the closure of $\boldsymbol{T}(\Gamma)$ in $\boldsymbol{B}(\boldsymbol{U}^*, \Gamma)$ by considering the mapping $W_\alpha\circ h^{-1}$, where W_α and h are as in the previous section. However in general we can give a negative answer to this question.

PROPOSITION 4.1. *In the statement of Theorem A, suppose that Γ is neither a hyperbolic cyclic group $\langle\gamma\rangle$ nor an extension of $\langle\gamma\rangle$ of index 2. Then for each $\alpha\neq 0$ and for any choice of a normally convergent sequence $\{\varphi_n\}$ to φ_α, φ_n does* *not converge to* φ_α *in* $\mathrm{cl}\,\boldsymbol{T}(1)$.

PROOF. From the assumption it follows that Γ and hence Γ_n $(=w_n\circ\Gamma\circ w_n^{-1})$ are not elementary groups. Then the limit set of Γ_n consists of infinitely many points and in particular there are infinitely many hyperbolic fixed points of Γ_n. Since φ_n converges to φ_α normally in $\boldsymbol{U}^*$, there is a number $N>0$ such that the inequality $4(\mathrm{Im}\,z_0)^2|\varphi_n(z_0)|>2|\alpha|>0$ holds for $z_0=-i$ whenever $n>N$. Let η be a hyperbolic element of Γ_n whose attractive fixed point q is neither 0 nor ∞. Since φ_n is a quadratic differential for Γ_n, it follows that

$$\begin{aligned}(4.1)\qquad & 4(\mathrm{Im}\,\eta^\nu(z_0))^2|\varphi_n(\eta^\nu(z_0))-\varphi_\alpha(\eta^\nu(z_0))| \\ & \geqq 4(\mathrm{Im}\,z_0)^2|\varphi_n(z_0)|-4|\alpha|(\mathrm{Im}\,\eta^\nu(z_0))^2/|\eta^\nu(z_0)|^2,\end{aligned}$$

for each integer ν. On the other hand $\eta^\nu(z_0)$ converges to the real number $q(\neq 0)$ as $\nu\to\infty$. Hence $(\mathrm{Im}\,\eta^\nu(z_0))^2/|\eta^\nu(z_0)|^2\to 0$ as $\nu\to+\infty$. Thus by letting $\nu\to+\infty$, we obtain with (4.1) that $\|\varphi_n-\varphi_\alpha\|\geqq 2|\alpha|$ for $n>N$. Hence φ_n does not converge to φ_α in $\mathrm{cl}\,\boldsymbol{T}(1)$. Q.E.D.

5. Proof of Theorem B.

Now we shall give a proof of Theorem B. If Γ contains no hyperbolic elements, then Γ is necessarily elementary, namely the limit set of Γ consists of at most two points. In this case, Γ is of the second kind and then $\boldsymbol{o}(\Gamma)=6$ follows from [**13**]. In particular we have that $\mathcal{O}(\Gamma)=6$. If Γ contains a hyperbolic element, then by Theorem A we can choose a sequence $\{w_n\}_{n=1}^\infty$ in $Q_U(\Gamma)$ and quadratic differentials $\varphi_n\in\boldsymbol{T}(\Gamma_n)$, $\Gamma_n=w_n\circ\Gamma\circ w_n^{-1}$, which converge normally to $\varphi(z)=(-3/2)z^{-2}$, since $-3/2\in\boldsymbol{V}$. Note that the value $\|\varphi\|=6$ is attained at each point on the negative imaginary axis. Hence it follows in particular at $z_0=-i$ that

$$4(\mathrm{Im}\,z_0)^2|\varphi_n(z_0)| \longrightarrow 4(\mathrm{Im}\,z_0)^2|\varphi(z_0)| = 6.$$

On the other hand the Nehari theorem yields that $\|\varphi_n\|\leqq\boldsymbol{o}(\Gamma_n)\leqq 6$. Therefore $\boldsymbol{o}(\Gamma_n)$ converges to 6, and hence the equality $\mathcal{O}(\Gamma)=6$ holds. Thus we complete the proof of Theorem B.

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