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Maximal toral action on aspherical manifolds $\Gamma\backslash G/K$ and G/H

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Introduction.

In this note, we shall consider only topological actions. For a closed aspherical manifold M, it is well known that if a compact connected Lie group G acts on M effectively, then G is a toral group T^s with $s \leqq$ rank of the center $z(\pi_1(M))$ of the fundamental group $\pi_1(M)$ of M (Theorem 5.6 in [**4**]). In [**5**], it was conjectured that if M is a closed aspherical manifold, then

(1) $z(\pi_1(M))$ is finitely generated, say of rank k,

(2) there exists a toral group T^k acting effectively on M.

These have been verified in many cases. For examples, if M is a smooth manifold admitting a Riemannian metric with non-positive sectional curvature or if M is a nilmanifold, then (1) and (2) hold (see [**10**]).

In this note, we shall prove the following

THEOREM A. *The conjectures* (1) *and* (2) *hold for aspherical manifold of type $\Gamma\backslash G/K$, where G is a connected non-compact Lie group, K a maximal compact subgroup of G and Γ a torsion free discrete uniform subgroup of G.*

THEOREM B. *The conjectures* (1) *and* (2) *hold for a compact homogeneous aspherical manifold G/H, where G is a connected non-compact Lie group and H a closed subgroup of G.*

In this note, we shall use the following notations;

1. $\boldsymbol{Z}$, $\boldsymbol{R}$ and $\boldsymbol{C}$ denote the ring of integers, the field of real numbers and the field of complex numbers, respectively.
2. $\tilde{G}$ denotes the universal covering of a Lie group G and $\pi: \tilde{G} \to G$ the covering projection.
3. G^0 denotes the identity component of a Lie group G.
4. $z(G)$ denotes the center of a group G.
5. Lie group is assumed to be connected unless the contrary is stated.

1. Preliminaries.

Let G be a simply connected non-compact Lie group. Then it is well known that G is a semi-direct product of a simply connected semisimple subgroup S and its radical R. Thus every element of G is uniquely written as a product rs $(r\in R,\ s\in S)$ and the product of r_1s_1 and r_2s_2 is given by $(r_1s_1)(r_2s_2)=r_1s_1r_2s_1^{-1}s_1s_2$ and we have the following split exact sequence;

$$1\longrightarrow R\longrightarrow G\underset{i}{\overset{p}{\rightleftarrows}} S\longrightarrow 1.$$

Let Γ be a torsion free discrete uniform subgroup of G and K a maximal compact subgroup of G. It is easy to show that K is semisimple and $R\cap K=1$. Since Γ is torsion free, $\Gamma\cap K=1$. When one considers the manifold $\Gamma\backslash G/K$, it is sufficient to consider the case when S contains no compact normal factors. We list some lemmas which are needed in the sequel.

LEMMA 1 (Corollary 8.28 in [11]). (1) *$\Gamma_R=\Gamma\cap R$ is a discrete uniform subgroup of R.*

(2) *$p(\Gamma)$ is a discrete uniform subgroup of S.*

LEMMA 2 (Corollary 5.18 in [11]). *Let G be a semisimple Lie group without compact normal subgroup and H a closed subgroup with the property* (S) *(e.g. G/H has a finite invariant measure). Then the centralizer of H in G is equal to $z(G)$. In particular, $z(H)$ is contained in $z(G)$.*

LEMMA 3 (Theorems 2.1 and 2.11 in [11]). (1) *Let N be a simply connected nilpotent Lie group and Γ a closed uniform subgroup of N. Then there are no proper connected closed subgroups of N containing Γ.*

(2) *Let N and V be two nilpotent simply connected groups and let H be a uniform subgroup of N. Then any continuous homomorphism $f: H\to V$ can be extended in a unique manner to a continuous homomorphism $\tilde{f}: N\to V$.*

LEMMA 4 (Theorem 1.1 of Chap. VI in [9]). *Let $\underline{g}$ be a non-compact semisimple Lie algebra over $\boldsymbol{R}$ and $\underline{g}=\underline{k}+\underline{p}$ a Cartan decomposition of $\underline{g}$. Suppose (G, K) is any pair associated with $(\underline{g}, \theta)$, where $\theta(T+X)=T-X$ $(X\in\underline{p},\ T\in\underline{k})$ is an involutive automorphism of $\underline{g}$. Then we have*

(1) *K is connected, closed and contains $z(G)$,*

(2) *K is compact if and only if $z(G)$ is finite.*

LEMMA 5 (Theorem 2.3 in Section 2 in Chap. IV in [13]). *Let (G, K) be the pair as in Lemma 4. Then K is its own normalizer and the centralizer $C_G(K)$ of K in G is $z(K)$.*

LEMMA 6. *Let G be a semisimple Lie group and $\pi: \tilde{G}\to G$ the universal cover-*

ing. Then $z(\tilde{G})$ *is equal to* $\pi^{-1}(z(G))$.

PROOF. Let $x\in\pi^{-1}(z(G))$ and consider the continuous map $c_x:\tilde{G}\to\tilde{G}$ defined by $c_x(y)=xyx^{-1}y^{-1}$. Since $\pi(xyx^{-1}y^{-1})=1$, $\operatorname{Im} c_x\subseteqq \operatorname{Ker}\pi$. $\operatorname{Ker}\pi$ being discrete, $c_x(\tilde{G})=1$ and hence $xyx^{-1}y^{-1}=1$, which implies $x\in z(\tilde{G})$. Conversely let $x\in z(\tilde{G})$. Then $\pi(x)\in z(G)$, which implies $x\in\pi^{-1}(z(G))$. Q.E.D.

LEMMA 7. *Let G be a non-compact simple Lie group. Suppose $z(\tilde{G})$ is not finite. Then we have*

(1) *$z(\tilde{G})$ is contained as a lattice in a subgroup L of $\tilde{G}$, which is isomorphic to $\boldsymbol{R}$, and*

(2) *Let $\bar{K}$ be a maximal compact subgroup of $\tilde{G}$. Then L is contained in the centralizer of $\bar{K}$ in $\tilde{G}$.*

PROOF. The following arguments are due to Chap. VI, VIII, X in [**9**]. Since $z(\tilde{G})$ is not finite, G is $PSL(2, \boldsymbol{R})$ (or its finite covering group), $SU(p, q)$, $SO^*(2n)$, $Sp(n, \boldsymbol{R})$, $SO_0(2, q)$, E_6, or E_7. Let K be a subgroup of G such that the pair (G, K) has the property of Lemmas 4 and 5. Then G/K is an irreducible Hermitian space and $z(K)$ is $SO(2)$. It follows from Lemma 6 that $z(G)$ is contained in $L=\pi^{-1}(z(K))$, which is isomorphic to $\boldsymbol{R}$. This proves (1). It follows from arguments in Chap. X in [**9**] (see pp. 451-455 in [**9**]) that $\pi^{-1}(K)$ is isomorphic to $\bar{K}\times\boldsymbol{R}$, where $\bar{K}$ is a maximal compact subgroup of $\tilde{G}$. Since $z(K)=c_G(K)$, we have $\pi(xyx^{-1})=\pi(y)$ for every $y\in\bar{K}$ and $x\in L$. This implies that the image of the continuous map $c:\bar{K}\times L\to\tilde{G}$ defined by $c(y, x)=xyx^{-1}y^{-1}$ is contained in $\operatorname{Ker}\pi$. Since $\operatorname{Ker}\pi$ is discrete and $\bar{K}\times L$ is connected, we have $c(y, x)=1$. This completes the proof of Lemma 7. Q.E.D.

We shall recall some results about solvable Lie groups. Let R be a simply connected solvable Lie group and Γ a discrete uniform subgroup of R. It is well known that there is an exact sequence

$$1 \longrightarrow N \longrightarrow R \longrightarrow \boldsymbol{R}^s \longrightarrow 1,$$

where N is the nilradical of R. It is easy to see that there is a sequence of subgroups of R;

$$N=R_0\subseteqq R_1\subseteqq\cdots\subseteqq R_s=R$$

such that $R_{i+1}=R_i\rtimes\boldsymbol{R}_i$ (semidirect product), where $\boldsymbol{R}_i=\boldsymbol{R}$. In the following, we write the addition of $\boldsymbol{R}$ multiplicatively. Define $\Gamma_i=\Gamma\cap R_i$, $z_{i-1}=z(\Gamma_i)\cap\Gamma_{i-1}$ and $p_i: R_i\to\boldsymbol{R}_i$ the natural projection. Put $\Gamma_N=\Gamma_0$ and $z_N=z_0$. We may write an element of R_i in the form;

$$nx_1x_2\cdots x_i=n\prod^{i}x_j \qquad (n\in N,\ x_j\in\boldsymbol{R}_j).$$

We have the following

LEMMA 8. (1) *Γ_i is a discrete uniform subgroup of R_i.*

(2) *$p_i(\Gamma_i)$ is a discrete uniform subgroup of $\boldsymbol{R}_i$.*

This follows from the standard arguments about Lie group theory (see Chap. 3 in [**3**] and [**11**]).

LEMMA 9. *Let N be a simply connected nilpotent Lie group and Γ_N is a discrete uniform subgroup of N. Suppose $z(\Gamma_N)=\boldsymbol{Z}^n$. Then there exists a subgroup N_0 of N which is isomorphic to $\boldsymbol{R}^n$ and contains $z(\Gamma_N)$ as a lattice.*

This follows from Lemma 3.

LEMMA 10. *Let R be a simply connected solvable Lie group, Γ a discrete uniform subgroup of R, N the nilradical of R and N_0 the subgroup of N which has the property in Lemma 9 for $\Gamma_N=N\cap\Gamma$ and $z(\Gamma)\cap\Gamma_N$. Then we have $rx=xr$ for every $r\in\Gamma$ and $x\in N_0$.*

PROOF. Consider the inner automorphism $c_r: R\rightarrow R$. Since $z(\Gamma)\cap\Gamma_N\subseteqq z(\Gamma)$, we have $z(\Gamma)\cap\Gamma_N\subseteqq N_0\cap c_r(N_0)$. It follows from a result in [**11**] (Lemma 2.4 in [**11**]) that $N_0\cap c_r(N_0)$ is connected and hence $N_0\cap c_r(N_0)=N_0$, which implies $c_r(N_0)=N_0$. Q.E.D.

LEMMA 11. (1) *Let Γ be a group satisfying the exact sequence;*

$$1\longrightarrow\boldsymbol{Z}^t\longrightarrow\Gamma\longrightarrow\boldsymbol{Z}^s\longrightarrow 1.$$

Then there exists a simply connected solvable Lie group R and a closed subgroup D of R such that $\pi_1(R/D)=\Gamma$.

(2) *Let Γ, R and D be as above. Assume $z(\Gamma)$ is not trivial. Then there exist closed subgroups D_1 and D_2 of R which satisfy*

i) *$D_1\lhd D$ and $D_1/D_1^0=z(\Gamma)=\boldsymbol{Z}^k$,*

ii) *D_2/D_1^0 is isomorphic to $\boldsymbol{R}^k$*

and

iii) *$z(\Gamma)$ is contained in D_2/D_1^0 as a lattice.*

PROOF. The following arguments are due to [**1**] (Chap. III, Section 5 in [**1**]).

(1) The arguments in [**1**] (see p. 245) show that there exists a commutative diagram in which the horizontal sequences are exact;

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \boldsymbol{Z}^t & \longrightarrow & \Gamma & \longrightarrow & \boldsymbol{Z}^s & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \boldsymbol{Z}^t\otimes\boldsymbol{C} & \longrightarrow & \Gamma_C & \longrightarrow & \boldsymbol{Z}^s & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \boldsymbol{Z}^t\otimes\boldsymbol{C} & \longrightarrow & R & \longrightarrow & \boldsymbol{Z}^s\otimes\boldsymbol{R} & \longrightarrow & 1.
\end{array}$$

Let D be the subgroup of R generated by the image of Γ and the subgroup I

of $\boldsymbol{Z}^t\times\boldsymbol{C}$ consisting of purely imaginary vectors. Then D is closed in R and $\pi_1(R/D)=D/D^0=D/I=\Gamma$.

(2) Let D_1 be the subgroup of R generated by $z(\Gamma)$ and I. $z(\Gamma)$ satisfies the following exact sequence;

$$1\longrightarrow \boldsymbol{Z}^{t'}\longrightarrow z(\Gamma)\longrightarrow \boldsymbol{Z}^{s'}\longrightarrow 1.$$

It is easy to construct the following commutative diagram;

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \boldsymbol{Z}^{t'} & \longrightarrow & z(\Gamma) & \longrightarrow & \boldsymbol{Z}^{s'} & \longrightarrow & 1\\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \boldsymbol{Z}^{t'}\otimes\boldsymbol{R} & \longrightarrow & \boldsymbol{Z}^{k}\otimes\boldsymbol{R} & \longrightarrow & \boldsymbol{Z}^{s'}\otimes\boldsymbol{R} & \longrightarrow & 1\\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \boldsymbol{Z}^{t'}\otimes\boldsymbol{C} & \longrightarrow & R & \longrightarrow & \boldsymbol{Z}^{s'}\otimes\boldsymbol{R} & \longrightarrow & 1,
\end{array}$$

where $z(\Gamma)=\boldsymbol{Z}^k$. Now let D_2 be the subgroup of R generated by $\boldsymbol{Z}^k\otimes\boldsymbol{R}$ and I. Then $D_2/D_1^0=\boldsymbol{R}^k$ and $z(\Gamma)$ is a lattice of D_2/D_1^0. Q.E.D.

Now we shall consider $M=\Gamma\backslash G/K$, where G is a non-simply connected non-compact Lie group, K a maximal compact subgroup and Γ a torsion free discrete uniform subgroup of G. Let $\pi:\tilde{G}\to G$ be the universal covering of G. Then $\operatorname{Ker}\pi=\pi_1(G)=\pi_1(K)\cong\boldsymbol{Z}^r\times F$, where F is a finite abelian group. Since $\tilde{K}=\pi^{-1}(K)$ is the universal covering of K, $\tilde{K}\cong\boldsymbol{R}^r\times\bar{K}$, where $\bar{K}$ is a simply connected compact semisimple Lie group. Put $\tilde{\Gamma}=\pi^{-1}(\Gamma)$.

We have the following

LEMMA 12. *$\boldsymbol{Z}^r$ and F are central subgroups of G.*

This follows from the fact that $\pi_1(G)$ is a central subgroup of $\tilde{G}$. Let $\tilde{G}=\tilde{R}\circ\tilde{S}$ be the Levi-decomposition of $\tilde{G}$. Define $\tilde{G}^*$, $\tilde{S}^*$, $\tilde{\Gamma}^*$, $\tilde{K}^*$ and $\bar{K}^*$ by $(\ )^*=(\ )/F$. Clearly $\tilde{G}=\tilde{R}\circ\tilde{S}^*$ is the Levi-decomposition of $\tilde{G}^*$. We have the following

LEMMA 13. $\tilde{\Gamma}^*\cap g\tilde{K}^*g^{-1}=\boldsymbol{Z}^r$ *for every* $g\in\tilde{G}^*$.

PROOF. Consider the following commutative diagram in which every horizontal sequence is exact.

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \boldsymbol{Z}^r & \longrightarrow & g\tilde{K}^*g^{-1} & \longrightarrow & \bar{g}K\bar{g}^{-1} & \longrightarrow & 1\\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \boldsymbol{Z}^r & \longrightarrow & \tilde{G}^* & \longrightarrow & G & \longrightarrow & 1\\
 & & \uparrow & & \uparrow & & \uparrow & & \\
1 & \longrightarrow & \boldsymbol{Z}^r & \longrightarrow & \tilde{\Gamma}^* & \longrightarrow & \Gamma & \longrightarrow & 1
\end{array}$$

where $\bar{g}=\pi^*(g)$, $\pi^*:\tilde{G}^*\to G$ the homomorphism induced by π. Since $\tilde{\Gamma}^*$ and Γ are torsion free and $\bar{g}K\bar{g}^{-1}$ is compact, $\tilde{\Gamma}^*\cap g\tilde{K}^*g^{-1}$ is equal to $\boldsymbol{Z}^r$. Q.E.D.

LEMMA 14. (1) *$\Gamma\backslash G/K$ is homeomorphic to $\tilde{\Gamma}^*\backslash\tilde{G}^*/\tilde{K}^*$.*

(2) *The natural map $q: \tilde{G}^*/\bar{K}^*\to\tilde{G}^*/\tilde{K}^*$ is a principal $\boldsymbol{R}^r=\tilde{K}^*/\bar{K}^*$-bundle.*

(3) *The map q induces a map $\tilde{q}: \tilde{\Gamma}^*\backslash\tilde{G}^*/\bar{K}^*\to\tilde{\Gamma}^*\backslash\tilde{G}^*/\tilde{K}^*$ which is a principal $T^r=\boldsymbol{Z}^r\backslash\tilde{K}^*/\bar{K}^*$-bundle.*

PROOF. We omit the proof of (1) and (2). It is clear that $\tilde{q}$ is a fiber bundle with typical fiber $(\tilde{\Gamma}^*\cap K^*)\backslash\tilde{K}^*/\bar{K}^*=\boldsymbol{Z}^r\backslash\tilde{K}^*/\bar{K}^*$. Consider $\boldsymbol{Z}^r\backslash\tilde{K}^*/\bar{K}^*$ as an r-dimensional torus T^r and define a T^r-action on $\tilde{\Gamma}^*\backslash\tilde{G}^*/\bar{K}^*$ by the formula;

$$(\boldsymbol{Z}^r k\bar{K}^*)(\tilde{\Gamma}^*g\bar{K}^*)=\tilde{\Gamma}^*gk^{-1}\bar{K}^*.$$

The well-definedness follows from the fact that $\bar{K}^*\lhd\tilde{K}^*$ and Lemma 12. The action is free. In fact,

$$(\boldsymbol{Z}^r k_1\bar{K}^*)(\tilde{\Gamma}^*g\bar{K}^*)=\tilde{\Gamma}^*g\bar{K}^*\Rightarrow\tilde{\Gamma}^*gk^{-1}\bar{K}^*=\tilde{\Gamma}^*g\bar{K}^*\Rightarrow gk_1^{-1}=xgk'$$
$$(k'\in\bar{K}^*,\ x\in\tilde{\Gamma}^*)$$
$$x=gk_1^{-1}k'^{-1}g^{-1}\in\tilde{\Gamma}^*\cap g\tilde{K}^*g^{-1}.$$

It follows from Lemma 13 that we have $k_1^{-1}=g^{-1}xgk'\in\boldsymbol{Z}^r\bar{K}^*$, which implies that $\boldsymbol{Z}^r k_1\bar{K}^*=1$ in $\boldsymbol{Z}^r\backslash\tilde{K}^*/\bar{K}^*$. It is clear that the orbit space of $\tilde{\Gamma}^*\backslash\tilde{G}^*/\bar{K}^*$ by $\boldsymbol{Z}^r\backslash\tilde{K}^*/\bar{K}^*$ is $\tilde{\Gamma}^*\backslash\tilde{G}^*/\tilde{K}^*$. Q.E.D.

2. The proof of Theorem A when G is simply connected.

In this section, we shall prove Theorem A when G is simply connected. As in Section 1, let $G=R\circ S$ be the Levi-decomposition and $p: G\to S$ the projection. We have the following exact sequence;

$$1\longrightarrow\Gamma_R\longrightarrow\Gamma\longrightarrow p(\Gamma)\longrightarrow 1.$$

It follows from this exact sequence that we have the following exact sequence;

$$1\longrightarrow z(\Gamma)\cap\Gamma_R\longrightarrow z(\Gamma)\longrightarrow p(z(\Gamma))\longrightarrow 1.$$

It is clear that $z(\Gamma)\cap\Gamma_R\subseteqq z(\Gamma_R)$ and $p(z(\Gamma))\subseteqq z(p(\Gamma))$. Since Γ_R is poly-$\boldsymbol{Z}$ group (see [11]), $z(\Gamma)\cap\Gamma_R$ is also a poly-$\boldsymbol{Z}$ group and hence finitely generated. It follows from a result in [11] (Corollary 5.18 in [11]) that $z(p(\Gamma))$ is finitely generated abelian group and hence isomorphic to $\boldsymbol{Z}^k$ for some integer k. We have the following

PROPOSITION 15. (1) *The map $G\to R\times S$; $g=rs\to(r, s)$ is a homeomorphism.*

(2) *The map $f_1: G/K\to R\times(S/K)$; $rsK\to(r, sK)$ is a homeomorphism.*

(3) *The natural map $\Gamma_R\backslash G/K\to\Gamma\backslash G/K$ is a regular covering map with the group $p(\Gamma)$ of covering transformations and hence $\Gamma\backslash G/K\cong p(\Gamma)\backslash(\Gamma_R\backslash G/K)$.*

(4) *The map $g: \Gamma_R\backslash G/K\to(\Gamma_R\backslash R)\times(S/K)$; $\Gamma_R rsK\to(\Gamma_R r, sK)$ is a homeo-*

morphism.

Since these are proved immediately, we shall omit the proof.

Define the action of $p(\Gamma)$ on $(\Gamma_R\backslash R)\times(S/K)$ by

$$(\Gamma_R rs)(\Gamma_R r_1,\ s_1K)=(\Gamma_R rsr_1s^{-1},\ ss_1K).$$

Then the map g is $p(\Gamma)$-equivariant (note the action of $p(\Gamma)\cong\Gamma_R\backslash\Gamma$ on $\Gamma_R\backslash G/K$ is given by $(\Gamma_R rs)(\Gamma_R gK)=\Gamma_R rsgK$). In fact,

$$(\Gamma_R rs)(g(\Gamma_R r_1s_1K))=(\Gamma_R rs)(\Gamma_R r_1,\ s_1K)=(\Gamma_R rsr_1s^{-1},\ ss_1K)=g((\Gamma_R rs)(\Gamma_R r_1s_1K)).$$

It follows that we have the following

PROPOSITION 16. *$\Gamma\backslash G/K$ is homeomorphic to $p(\Gamma)\backslash((\Gamma_R\backslash R)\times(S/K))$.*

Now we shall define a maximal toral action on $N=\Gamma\backslash G/K$. We devide the definition into two steps.

The first step; Let $z(\Gamma)\cap R=\boldsymbol{Z}^n$. We define an action of T^n on $\Gamma_R\backslash R$, which is compatible with the action $p(\Gamma)$.

The second step; Let $p(z(\Gamma))=\boldsymbol{Z}^m$. We define an action of $T^m\times T^n$ on $\Gamma\backslash G/K$.

1. The first step. Let R be a simply connected solvable Lie group and Γ a torsion free discrete uniform subgroup of R. As noted above, we have an exact sequence;

$$1\longrightarrow N\longrightarrow R\longrightarrow \boldsymbol{R}^s\longrightarrow 1,$$

where N is the nilradical of R. First consider the case of $s=1$. We have the following commutative diagram;

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \Gamma_N & \longrightarrow & \Gamma & \longrightarrow & p(\Gamma) & \longrightarrow & 1\\
 & & \uparrow & & \uparrow & & \uparrow & & \\
1 & \longrightarrow & z_N & \longrightarrow & z(\Gamma) & \longrightarrow & p(z(\Gamma)) & \longrightarrow & 1.
\end{array}$$

By the same arguments as in Propositions 15 and 16, we have the following

PROPOSITION 17. (1) *The map $g:\Gamma_N\backslash R\to(\Gamma_N\backslash N)\times\boldsymbol{R}$; $\Gamma_N nx\to(\Gamma_N n,\ x)$ is a homeomorphism.*

(2) *The natural map $\Gamma_N\backslash R\to\Gamma\backslash R$ is a regular covering map with the group $p(\Gamma)$ of covering transformations and hence $\Gamma\backslash R\cong p(\Gamma)\backslash(\Gamma_N\backslash R)$.*

(3) *Define an action of $p(\Gamma)\cong\Gamma_N\backslash\Gamma$ on $(\Gamma_N\backslash N)\times\boldsymbol{R}$ by the formula;*

$$(\Gamma_N nx)(\Gamma_N n_1x_1)=(\Gamma_N nxn_1x^{-1},\ xx_1).$$

Then this is well defined and induces a homeomorphism $h:\Gamma\backslash R\cong p(\Gamma)\backslash((\Gamma_N\backslash N)\times\boldsymbol{R})$.

It follows from Lemma 9 there exists a subgroup N_0 of N such that

(i) $N_0 \subseteqq \boldsymbol{R}^u$ ($u = \mathrm{rank}\, z_N$)

and

(ii) $z_N \subseteqq N_0$ as a lattice.

Now we define an action of $T^u = z_N \backslash N_0$ on $\Gamma \backslash R$.

(1) Define an action of T^u on $(\Gamma_N \backslash N) \times \boldsymbol{R}$ by the formula;

$$(z_N n)(\Gamma_N n_1, x) = (\Gamma_N n n_1, x).$$

This action is easily proved to be well defined and effective.

(2) This action is commutative with the action of $p(\Gamma)$. In fact, we have

$$\begin{aligned}
&(\Gamma_N nx)((z_N n_1)(\Gamma_N n_2, x_2)) = (\Gamma_N nx)(\Gamma_N n_1 n_2, x_2) \\
&= (\Gamma_N nxn_1 n_2 x^{-1}, xx_2) = (\Gamma_N nxn_1 x^{-1} xx_2 x^{-1}, xx_2) \\
&= (\Gamma_N n_1 nxn_2 x^{-1}, xx_2) \qquad \text{(see Lemma 9)} \\
&= (z_N n_1)(\Gamma_N nxn_2 x^{-1}, xx_2) \qquad \text{(see Lemma 10)} \\
&= (z_N n_1)((\Gamma_N nx)(\Gamma_N n_2, x_2)).
\end{aligned}$$

It follows from (1) and (2) that we have defined an action of T^u on $\Gamma \backslash R$.

It is clear that $p(z(\Gamma)) = \boldsymbol{Z}$ or 1. When $p(z(\Gamma)) = \boldsymbol{Z}$, define $A = p(z(\Gamma)) \otimes \boldsymbol{R}$. Then $A/p(z(\Gamma)) = T^1$. We can define an action of $T^u \times T^1$ on $\Gamma \backslash R$ as follows.

(1) Define an action of $T^u \times A$ on $(\Gamma_N \backslash N) \times \boldsymbol{R}$ by the formula;

$$(z_N n, x)(\Gamma_N n_1, x_1) = (\Gamma_N nx_1, x_1 x^{-1}).$$

This action is proved easily to be well defined and effectively.

(2) Define an action of $T^u \times A$ on $\Gamma_N \backslash R$ by the formula;

$$(z_N n, x)(\Gamma_N n_1, x_1) = \Gamma_N nn_1 x_1 x^{-1}.$$

This is well defined. In fact,

$$\begin{aligned}
\Gamma_N n_2 x_2 = \Gamma_N n_1 x_1 &\Rightarrow n_2 x_2 = n' n_1 x_1 \qquad (n' \in \Gamma_N) \\
&\Rightarrow nn_2 x_2 x^{-1} = nn' n_1 x_1 x^{-1} = n' nn_1 x_1 x^{-1} \qquad \text{(by Lemma 9)} \\
&\Rightarrow (z_N n, x)(\Gamma_N n_2, x_2) = \Gamma_N nn_2 x_2 = \Gamma_N nn_1 x^{-1} \\
&\quad = (z_N n, x)(\Gamma_N n_1 x_1).
\end{aligned}$$

(3) The homeomorphism $g: \Gamma_N \backslash R \to (\Gamma_N \backslash N) \times \boldsymbol{R}$ is $(T^u \times A)$-equivariant. In fact,

$$\begin{aligned}
&g((z_N n, x)(\Gamma_N n_1 x_1)) = g(\Gamma_N nn_1 x_1 x^{-1}) = (\Gamma_N nn_1, x_1 x^{-1}) \\
&= (z_N n, x)(\Gamma_N n_1, x_1) = (z_N n, x)(g(\Gamma_N n_1 x_1)).
\end{aligned}$$

It follows from (1) and (3) that the action of $T^u \times A$ on $\Gamma_N \backslash R$ is effective.

(4) The action of $T^u \times A$ on $\Gamma_N \backslash R$ is commutative with the action of $p(\Gamma) \cong \Gamma_N \backslash \Gamma$. To prove this, we need the following lemma;

LEMMA 18. *$xn_1x^{-1}=n_1$ for every $x\in p(\Gamma)$ and $n_1\in N_0$.*

PROOF. This follows from Lemma 3 and the following commutative diagram;

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \Gamma_N & \longrightarrow & \Gamma & \longrightarrow & p(\Gamma) & \longrightarrow & 1 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 1 & \longrightarrow & z_N & \longrightarrow & z(\Gamma) & \longrightarrow & p(z(\Gamma)) & \longrightarrow & 1. \end{array}$$

Q. E. D.

Now we shall prove the assertion (4).

$$\begin{aligned}(\Gamma_N nx)((z_N n_1,\ x_1)(\Gamma_N n_2x_2)) &= (\Gamma_N nx)(\Gamma_N n_1n_2x_2x_1^{-1}) \\ &= \Gamma_N nxn_1n_2x_2x_1^{-1} = \Gamma_N nn_1xn_2x^{-1}xx_2x_1^{-1} \qquad \text{(by Lemma 18)} \\ &= \Gamma_N n_1nxn_2x^{-1}xx_2x_1^{-1} \qquad \text{(by Lemma 10)} \\ &= (z_N n_1,\ x_1)((\Gamma_N nx)(\Gamma_N n_2x_2)).\end{aligned}$$

(5) The group $p(z(\Gamma))$ acts trivially on $p(\Gamma)\backslash(\Gamma_N\backslash R)$. In fact, denote an element of $p(\Gamma)\backslash(\Gamma_N\backslash R)$ by $[\Gamma_N nx]$. Recall $m[\Gamma_N nx]=[\Gamma_N nxm^{-1}]$ $(m\in p(z(\Gamma)))$ $=[\Gamma_N nm^{-1}x]$ ($\boldsymbol{R}$ is abelian). Since $p(nm^{-1}n^{-1}m)=1$, we have $nm^{-1}n^{-1}m=z\in\Gamma_N$ and hence $nm^{-1}=zm^{-1}n$. Thus we have $[\Gamma_N nm^{-1}x]=[\Gamma_N m^{-1}nx]=[(\Gamma_N m^{-1})(\Gamma_N nx)]$ $=[\Gamma_N nx]$.

Next we shall consider the general case. Recall the exact sequence;

$$1\longrightarrow N\longrightarrow R\longrightarrow \boldsymbol{R}^s\longrightarrow 1.$$

As noted in Section 1, we have a sequence of subgroups of R;

$$N=R_0\subset R_1\subset\cdots\subset R_s=R$$

such that $R_i=R_{i-1}\rtimes\boldsymbol{R}_i$ $(\boldsymbol{R}_i=\boldsymbol{R})$.

As in Section 1, we define $\Gamma_i=\Gamma\cap R_i$, $z_{i-1}=z(\Gamma_i)\cap\Gamma_{i-1}$ and $p_i\colon R_i\to\boldsymbol{R}_i$. If $e_i=\operatorname{rank} p_i(z(\Gamma_i))$, then we define $A_i=p_i(z(\Gamma_i))\otimes\boldsymbol{R}$. Clearly $p_i(z(\Gamma_i))\backslash A_i=T^1$. By the same arguments as in the case of $s=1$, we have the following;

PROPOSITION 19. (1) $\Gamma_{i+1}\backslash R_{i+1}\cong p_{i+1}(\Gamma_{i+1})\backslash(\Gamma_i\backslash R_{i+1})$.

(2) $\Gamma_i\backslash R_{i+1}\cong(\Gamma_i\backslash R_i)\times\boldsymbol{R}_{i+1}$.

(3) $\Gamma_{i+1}\backslash R_{i+1}\cong p_{i+1}(\Gamma_{i+1})\backslash((\Gamma_i\backslash R_i)\times\boldsymbol{R}_{i+1})$.

Assume $\Gamma_i\backslash R_i$ admits an action of $T^u\times T^{e_1}\times\cdots\times T^{e_i}$, where $T^u=z_N\backslash N_0$ and $T^{e_j}=p(z(\Gamma_j))\backslash A_j$ $(e_j\neq 0)$, induced by the action of $T^u\times A^{e_1}\times\cdots\times A^{e_i}$ on $(\Gamma_{i-1}\backslash R_{i-1})\times\boldsymbol{R}_i$ given by the formula;

$$(z_N n,\ \prod^i x_j)(\Gamma_{i-1}n_1\prod^i y_j,\ z)=(\Gamma_{i-1}nn_1\prod^{i-1}y_jx_j^{-1},\ zx_i).$$

If we regard $\Gamma_i\backslash R_i$ as $p_i(\Gamma_i)\backslash(\Gamma_{i-1}\backslash R_i)$, the above action is given by the formula;

$$(z_N n,\ \prod^i x_j)(\Gamma_{i-1}n_1\prod^i y_j)=\Gamma_{i-1}nn_1\prod^i y_jx_j^{-1}.$$

Now we define an action of $T^u\times A^{e_1}\times\cdots\times A^{e_{i+1}}$ on $(\Gamma_i\backslash R_i)\times\boldsymbol{R}_{i+1}$ by the formula;

$$(*)\qquad (z_N n, \prod^{i+1} x_j)(\Gamma_i n_1 \prod^{i} y_j, z) = (\Gamma_i n n_1 \prod^{i} y_j x_j^{-1}, z x_{i+1}^{-1}).$$

We should prove that this action is commutative with the action of $p_{i+1}(\Gamma_{i+1}) \cong \Gamma_i \backslash \Gamma_{i+1}$ on $(\Gamma_i \backslash R_i) \times \boldsymbol{R}_{i+1}$ given by the formula;

$$(**)\qquad (\Gamma_i n_1 \prod^{i+1} y_j)(\Gamma_i n_2 \prod^{i} z_j, w) = (\Gamma_i n_1 \prod^{i} y_j y_{i+1} n_2 \prod^{i} z_j y_{i+1}^{-1}, y_{i+1} w).$$

We note the following

PROPOSITION 20.

(1) $(\Gamma_i n_1 \prod^{i} y_j)(\Gamma_i n_2 \prod^{i} z_j) = \Gamma_i n_1 n_2' \prod^{i-1} (y_j (y_{j+1} \cdots y_i) z_j (y_{j+1} \cdots y_i)^{-1}) y_i z_i$,

where $n_i \in N$, $y_j, z_j \in \boldsymbol{R}_j$ *and* $(\prod^{i} y_j) n_2 = n_2' (\prod^{i} y_j)$.

(2) *For every* $j = 1, 2, \cdots, i$ *and* $k \geqq j$, $x_j y_k = y_k x_j$, *where* $x_j \in A_j$, $y_k \in p_k(\Gamma_k)$.

PROOF. (1) follows from direct computations and (2) follows from the fact that the action of $p_k(\Gamma_k)$ on $z(\Gamma_j)$ and hence the action of $p_k(z(\Gamma_k))$ induced by conjugation is trivial.

Now the proof of the commutativity of (*) and (**) is as follows; Put $y_{j+1, i} = y_{j+1} y_{j+2} \cdots y_i$ and $\bar{z}_j = y_{j+1} z_j y_{j+1}^{-1}$.

$$\begin{aligned}
&(z_N n, \prod^{i+1} x_j)((\Gamma_i n_1 \prod^{i+1} y_j)(\Gamma_i n_2 \prod^{i} z_j, w)) \\
&= (z_N n, \prod^{i+1} x_j)(\Gamma_i n_1 \prod^{i} y_j y_{i+1} n_2 \prod^{i} z_j y_{i+1}^{-1}, y_{i+1} w) \\
&= (z_N n, \prod^{i+1} x_j)(\Gamma_i n_1 n_2' \prod^{i} y_j \prod^{i} \bar{z}_j, y_{i+1} w) \\
&= (\Gamma_i n n_1 n_2' \prod^{i-1} y_j y_{j+1, i} \bar{z}_j y_{j+1, i}^{-1} x_j^{-1} y_i \bar{z}_i x_i^{-1}, y_{i+1} w x_{i+1}^{-1}) \\
&= (\Gamma_i n_1 \prod^{i+1} y_j)((z_N n, \prod^{i+1} x_j)(\Gamma_i n_2 \prod^{i} z_j, w)).
\end{aligned}$$

We shall omit the proofs of the well-definedness, effectivity and triviality of the restriction of (**) to $p_{i+1}(z(\Gamma_{i+1}))$. Thus we have defined an action of $T^u \times A^{e_1} \times \cdots \times A^{e_{i+1}}$. By induction, we have completed the first step.

2. The second step. We shall define a maximal toral action on $\Gamma \backslash G/K$, where $G = R \circ S$. Consider the case when S contains no normal factor $\tilde{U}$, where U is one of groups listed in Lemma 7. Then, since $z(p(\Gamma))$ is discrete, rank $z(\Gamma) =$ rank $(\Gamma_R \cap z(\Gamma))$. Put rank $(\Gamma_R \cap z(\Gamma)) = k$. By the arguments at the first step, a k-dimensional toral group $T^k = T^u \times T^{e_1} \times \cdots \times T^{e_s}$ acts on $\Gamma_R \backslash R$ as follows;

$$(*)\qquad (z_N n, \prod^{s} [x_j])(\Gamma_R n_1 \prod^{s} y_j) = \Gamma_R n n_1 \prod^{s} y_j x_j^{-1},$$

where $[x_j]$ denotes an element of $\boldsymbol{Z} \backslash A_j^{e_j}$. Note that if $e_j = 0$ then $x_j = 1$. Define an action of T^k on $(\Gamma_R\backslash R)\times(S/K)$ by

$$(**)\qquad (z_N n, \prod^s[x_j])(\Gamma_R n_1\prod^s y_j, wK)=(\Gamma_R nn_1\prod^s y_j x_j^{-1}, wK).$$

It is easy to show that this action is well defined and effective. The commutativity with the action of $p(\Gamma)$ follows from the same arguments at the first step and the following lemma.

LEMMA 21. (1) *$vx_jv^{-1}=x_j$ for every $x_j\in A_j$ and $v\in p(\Gamma)$.*
(2) *$vn=nv$ for every $n\in N_0$ and $v\in p(\Gamma)$.*

PROOF. This follows from the fact that the action of $p(\Gamma)$ on $z(\Gamma)$ by conjugation is trivial and Lemma 3. Q.E.D.

Thus we have defined an action of T^k on $\Gamma\backslash G/K$.

In general, S is decomposed into a product $S_1\times A$, where A is a product of $\tilde{U}$, where U is one of groups listed in Lemma 7 and S_1 contains no factors of these groups. Then we have $p(z(\Gamma))=\boldsymbol{Z}^a\times F$ (F is a finite abelian group). It follows from results in Section 1 that there exists a subgroup $\boldsymbol{R}^a$ of A which contains $\boldsymbol{Z}^a$ as a lattice.

Let $T^k=T^u\times T^{e_1}\times\cdots\times T^{e_s}$ denote the toral group in the case of $A=1$. Define an action of $T^k\times\boldsymbol{R}^a$ on $\Gamma_R\backslash G/K$ by the formula;

$$(z_N n, \prod^s[x_j], u)(\Gamma_R n_1\prod^s z_j vK)=\Gamma_R nn_1\prod^s z_j x_j^{-1}vu^{-1}K,$$

where $u\in\boldsymbol{R}^a$. In the following, we omit the index s in $\prod^s$. This is well defined; in fact,

$$\begin{aligned}
&\Gamma_R n_1'\prod z_j'v'=\Gamma_R n_1\prod z_j v\\
&\Rightarrow n_1'\prod z_j'v'=rn_1\prod z_j vw \qquad (r\in\Gamma_R,\ w\in K)\\
&\Rightarrow nn_1'\prod z_j'x_j^{-1}v'u^{-1}=nn_1'\prod z_j'v'x_j^{-1}u^{-1}\\
&\qquad\qquad\qquad\qquad\quad =nrn_1\prod z_j vwx_j^{-1}u^{-1}\\
&\qquad\qquad\qquad\qquad\quad =rnn_1\prod z_j x_j^{-1}vwu^{-1} \qquad \text{(by Lemma 7)}\\
&\Rightarrow (z_N n, \prod[x_j])(\Gamma_R n_1\prod z_j v)=(z_N n, \prod[x_j])(\Gamma_R n_1'\prod z_j'v').
\end{aligned}$$

Next define an action of $T^k\times\boldsymbol{R}^a$ on $(\Gamma_R\backslash R)\times(S/K)$ by the formula;

$$(z_N n, \prod[x_j], v)(\Gamma_R n_1\prod z_j, sK)=(\Gamma_R nn_1\prod z_j x_j^{-1}, sv^{-1}K).$$

It is easy to see that this is well defined. The homeomorphism $g:\Gamma_R\backslash G/K\to(\Gamma_R\backslash R)\times(S/K)$ is $(T^k\times\boldsymbol{R}^a)$-equivariant. In fact, we have

$$\begin{aligned}
&g((z_N n, \prod[x_j], v)(\Gamma_R n_1\prod z_j sK))=g(\Gamma_R nn_1\prod z_j x_j^{-1}sv^{-1}K)\\
&=(\Gamma_R nn_1\prod z_j x_j^{-1}, sv^{-1}K)=(z_N n, \prod[x_j], v)(g(\Gamma_R n_1\prod z_j sK)).
\end{aligned}$$

It can also be proved that the action of $T^k\times\boldsymbol{R}^a$ on $(\Gamma_R\backslash R)\times(S/K)$ is effective. In fact, assume $(z_N n, \prod[x_j], v)(\Gamma_R n_1\prod z_j, sK)=(\Gamma_R n_1, \prod z_j, sK)$ for every

$(\Gamma_R n_1, \prod z_j, sK)$. Then we have $n_1\prod z_j = rnn_1\prod z_j x_j^{-1}$ and $s = sv^{-1}w$ $(w\in K,\ r\in\Gamma_R)$ and hence $v=w\in \boldsymbol{R}^a\cap K=1$. If we choose $n_1\prod z_j=1$, then $rn\prod x_j^{-1}=1$ and $n\prod x_j^{-1}\in z_N\times\prod p_i(z(\Gamma_R))$ which implies $(z_N n, \prod[x_j])=1$. This proves that the action of $T^k\times\boldsymbol{R}^a$ on $\Gamma_R\backslash G/K$ is effective. Moreover the action is commutative with the action of $p(\Gamma)$. In fact,

$$
\begin{aligned}
&(z_N n, \textstyle\prod[x_j], v)((\Gamma_R n_1\prod y_j u)(\Gamma_R n_2\prod z_j wK))\\
&=(z_N n, \textstyle\prod[x_j], v)(\Gamma_R n_1\prod y_j u n_2\prod z_j u^{-1}uwK)\\
&=(z_N n, \textstyle\prod[x_j], v)(\Gamma_R n_1 n_2'\prod y_j\prod u z_j u^{-1}uwK)\\
&=(z_N n, \textstyle\prod[x_j], v)(\Gamma_R n_1 n_2'\prod^{s-1} y_j y_{j+1,s}\bar{z}_j y_{j+1,s}^{-1} y_s\bar{z}_s uwK)\\
&\qquad(\text{note } y_{j+1,s}=y_{j+1}\cdots y_s,\ \bar{z}_j=uz_ju^{-1})\\
&=(\Gamma_R nn_1 n_2'\textstyle\prod^{s-1} y_j y_{j+1,s}\bar{z}_j y_{j+1,s}^{-1}x_j^{-1}y_s\bar{z}_s x_s^{-1}uwv^{-1}K)\\
&=(\Gamma_R nn_1 n_2'\textstyle\prod y_j\prod u(z_jx_j^{-1})u^{-1}uwv^{-1}K)\qquad(\text{note that } ux_j=x_ju)\\
&=(\Gamma_R n_1\textstyle\prod y_j u)((z_N n, \prod[x_j], v)(\Gamma_R n_2\prod z_j wK)).
\end{aligned}
$$

It follows that $T^k\times\boldsymbol{R}^a$ acts on $p(\Gamma)\backslash(\Gamma_R\backslash G/K)$. We shall prove that $\boldsymbol{Z}^a$ acts trivially on $p(\Gamma)\backslash(\Gamma_R\backslash G/K)$. Let element of $p(\Gamma)\backslash(\Gamma_R\backslash G/K)$ be written as $[\Gamma_R n\prod z_j wK]$. Recall $m[\Gamma_R n\prod z_j wK]=[\Gamma_R n\prod z_j wm^{-1}]$ for $m\in\boldsymbol{Z}^a$. Since $m\in\boldsymbol{Z}^a \subset z(p(\Gamma))\subset z(S)$, $wm^{-1}=m^{-1}w$ and hence we have $m[\Gamma_R n\prod z_j wK]=[\Gamma_R\prod z_j m^{-1}wK]$. Because $p((n\prod z_j)m^{-1}(n\prod z_j)^{-1}m)=1$, we have $(n\prod z_j)m^{-1}(n\prod z_j)^{-1}m=z\in\Gamma_R$ and hence $[\Gamma_R n\prod z_j m^{-1}wK]=[\Gamma_R m^{-1}(n\prod z_j)wK]=[\Gamma_R n\prod z_j wK]$. This implies that $T^k\times T^a$ acts on $\Gamma\backslash G/K$ effectively. Thus we have proved Theorem A when G is simply connected.

3. The proof of Theorem A when G is not simply connected.

In this section, we shall prove Theorem A when G is not simply connected. We use the same notations as in Section 1. As noted in Section 1, $\tilde{G}^*=\tilde{R}\cdot\tilde{S}^*$, $\tilde{R}\cap\tilde{S}^*=1$ and $\bar{K}^*$ is a maximal compact subgroup of $\tilde{G}^*$. Then the same arguments as in Section 2 show that $z(\tilde{\Gamma}^*)=z(\pi_1(\tilde{\Gamma}^*\backslash\tilde{G}^*/\bar{K}^*))$ is finitely generated, say of rank k' and $\tilde{\Gamma}^*\backslash\tilde{G}^*/\bar{K}^*$ admits an action of $T^{k'}$. Note that $z(\tilde{\Gamma}^*)\cong \boldsymbol{Z}^r\times z(\Gamma)$. In fact, as noted in Section 1, we have an exact sequence;

$$1\longrightarrow \boldsymbol{Z}^r\longrightarrow\tilde{\Gamma}^*\longrightarrow\Gamma\longrightarrow 1,$$

where $\boldsymbol{Z}^r$ is a central subgroup of $\tilde{\Gamma}^*$. It follows that $z(\tilde{\Gamma}^*)\cong\boldsymbol{Z}^r\times z(\Gamma)$. As noted above, $\tilde{\Gamma}^*\backslash\tilde{G}^*/\bar{K}^*$ admits an action of $T^k\times T^r$. It is easy to see that the restriction of the acton of $T^k\times T^r$ to T^r coincides with the principal action of T^r on $\tilde{\Gamma}^*\backslash\tilde{G}^*/\bar{K}^*$. This implies that $\tilde{\Gamma}^*\backslash\tilde{G}^*/\tilde{K}^*$ admits an action of T^k. Thus we have completed the proof of Theorem A.

4. The proof of Theorem B.

Let $M=G/H$ be a compact aspherical manifold. When G is not simply connected, let $\tilde{G}$ be the universal covering of G. Then G/H is homeomorphic to $\tilde{G}/\pi^{-1}(H)$. Thus it is sufficient to consider the case when G is simply connected. Therefore any maximal compact subgroup K of G is simply connected and semisimple. Let N be the subgroup that acts trivially on G/H. Then $N\subset H$ and N is normal in G. Since only a torus among the compact connected Lie group can act effectively on G/H, K is contained in N. Now we have $G/H=(G/N)/(H/N)$, so we can assume that $N=1$, and hence $K=1$. This means G is homeomorphic to $\boldsymbol{R}^n$ for some n. Thus if $\dim H=0$, then M is a manifold of type of $\Gamma\backslash G/K$. Hence Theorem B holds. Next we assume $\dim H>0$. The following facts are known (see [8]).

1. H^0 is solvable.
2. Let $F=N_G(H^0)$, $H_1=F^0H$ and $G_1=H_1^0/(H_1^0\cap H^0)$. Then G/H_1 is homeomorphic to a torus T^n and we have a fiber bundle; $G_1/\Gamma_1\to G/H\to G/H_1$, where $\Gamma_1=(H_1^0\cap H)/(H_1^0\cap H)^0$.
3. G_1 is simply connected.

Since G/H_1 is aspherical and $\dim H_1^0>0$, H_1^0 is also solvable and hence G_1 is solvable. It follows from a result in [**11**] (Proposition 3.10 in [**11**]) that Γ_1 is poly-$\boldsymbol{Z}$ group. It follows from 2 that we have the following exact sequence;

$$1\longrightarrow \Gamma_1\longrightarrow H/H^0\longrightarrow H_1/H_1^0\longrightarrow 1,$$

where $H_1/H_1^0=\boldsymbol{Z}^n$. Γ_1 being a poly-$\boldsymbol{Z}$ group, H/H^0 is also poly-$\boldsymbol{Z}$ group. In other words, M is a closed aspherical manifold with poly-$\boldsymbol{Z}$ fundamental group. If $\dim M\neq 3, 4$, then Theorem B follows from a result in [**10**] (see Chap. 5 in [**10**]).

Now we shall consider the case when $\dim M=3$ or 4.

In his paper ([**16**], [**17**]), V. V. Gorvatsevich has determined all 3 or 4-dimensional homogeneous manifolds. They are given as follows;

1. Torus T^3 or T^4.
2. $\widetilde{SL}(2, \boldsymbol{R})/\Gamma$, Γ: a lattice.
3. $(\widetilde{SL}(2, \boldsymbol{R})/\Gamma)\times S^1$.
4. Solvmanifolds.

Since Theorem B holds for manifolds of type (1), (2) and (3). It is sufficient to consider only manifold $M=R/D$, where R is a simply connected solvable Lie group and D a closed subgroup of R. Let N be the nilradical of R. Then we have a fiber bundle

$$(\#)\quad ND/D\longrightarrow R/D\longrightarrow R/ND$$

where $ND/N=N/N\cap D$ and R/ND is a torus (see [2]). It follows that we have the following exact sequence of fundamental groups;

$$(*)\quad 1 \longrightarrow N\cap D/(N\cap D)^0 \longrightarrow D/D^0 \longrightarrow D/N\cap D \longrightarrow 1, \quad \text{where } D/N\cap D=\boldsymbol{Z}^s.$$

LEMMA 22. (1) *If* $\dim R/D=3$, *then the sequence* $(*)$ *is given by* $1\to\boldsymbol{Z}^t\to D/D^0\to\boldsymbol{Z}^s\to 1$, *where* $t+s=3$.

(2) *If* $\dim R/D=4$ *and* $s>1$, *then* $(*)$ *is given by* $1\to\boldsymbol{Z}^t\to D/D^0\to\boldsymbol{Z}^s\to 1$, *where* $t+s=4$.

This follows immediately from the fact that the fiber $N/N\cap D$ is a circle, or 2-dimensional torus.

First we shall consider the case when $\dim R/D=3$ or 4 and $s\geqq 2$. Put $\Gamma=D/D^0$. It follows from Lemma 11 that there exists a simply connected solvable Lie group S and a closed subgroup C of S such that $\pi_1(S/C)=\Gamma$ and that if $z(\Gamma)\neq 1$, then there exist closed subgroups C_1 and C_2 of S which satisfy

1) $C_1\lhd C$ and $C_1/C^0=z(\Gamma)$,

2) $C_2/C^0=\boldsymbol{R}^k$ ($k=\operatorname{rank} z(\Gamma)$),

and

3) $z(\Gamma)$ is contained in C_2/C^0 as a lattice.

Consider the toral group $T^k=(C_2/C^0)/(C_1/C^0)=C_2/C_1$. Define an action of T^k on S/C by $(xC_1)(yC)=yx^{-1}C$. To show that this is well defined, we need the following

LEMMA 23. *We have* $x^{-1}yxy^{-1}\in C^0$ *for every* $y\in C$ *and* $x\in C_2$.

PROOF. Consider the homomorphism $c_y: S\to S$ defined by $c_y(s)=ysy^{-1}$. This homomorphism leaves C and C^0 invariant, and hence c_y induces an automorphism $\bar{c}_y: C/C^0\to C/C^0$. Since C_1/C^0 is the center of C/C^0, $\bar{c}_y$ is the identity. Let $c_y(C_2)$ be denoted by C_2'. Then c_y induces an automorphism $C_2/C^0\to C_2'/C^0$. Both C_2/C^0 and C_2'/C^0 contain C_1/C^0 as a lattice. It follows from Lemma 3 that C_2/C^0 and C_2'/C^0 are equal, and $\bar{c}_y: C_2/C^0\to C_2/C^0$ is the identity. This implies $\bar{c}_y(xC^0)=yxy^{-1}C^0=xC^0$ and hence $x^{-1}yxy^{-1}\in C^0$. Q.E.D.

COROLLARY. $yx^{-1}=x^{-1}y \pmod{C}$ *for every* $y\in C$ *and* $x\in C_2$.

Now we can show that the action of T^k on S/C defined above is well defined as follows.

$$\begin{aligned} x_1C_1 = x_2C_1 &\Rightarrow x_2 = x_1x\ (x\in C_1) \Rightarrow (x_2C_1)(yC) = yx^{-1}x_1^{-1}C \\ &= yx_1^{-1}x^{-1}zC\ (z\in C^0) = yx_1^{-1}x^{-1}C = (x_1C_1)(yC). \\ y_1C_1 = y_2C_1 &\Rightarrow y_2 = y_1y\ (y\in C) \Rightarrow (xC_1)(y_2C) = y_2x^{-1}C \\ &= y_1yx^{-1}C = y_1x^{-1}C = (xC_1)(y_1C). \end{aligned}$$

This action is effective. In fact, assume $(xC_1)(yC)=yC$ for every y. Then we have $yx^{-1}C=yC$ and hence $x\in C\cap C_2$, which implies $xC_2=C_1$.

Since two solvmanifolds with isomorphic fundamental group are homeomorphic (see [2]), R/D and S/C are homeomorphic and hence R/D admits a maximal toral action.

Lastly we shall consider the case when $\dim R/D=4$, $s=1$ and $N/(N\cap D)$ is not a torus. The natural action of N on $R/D\,(x(rD)=rx^{-1}D)$ has the unique orbit type $N/(N\cap D)$ of dimension 1. It is well known that M is homeomorphic to $\boldsymbol{R}\times_{\boldsymbol{Z}}(N/(N\cap D))$, where $\boldsymbol{Z}$ acts on $\boldsymbol{R}\times(N/(N\cap D))$ as follows;

$$n(v,\ x(N\cap D))=(v-n,\ h^n(x(N\cap D))).$$

where $h: N/(N\cap D)\to N/(N\cap D)$ is an N-equivariant homeomorphism, i.e. $h(x(N\cap D))=xx_0^{-1}(N\cap D)$, $x_0\in N_N(N\cap D)$ (=the normalizer of $N\cap D$ in N). Put $N\cap D=K$. Note that $K^0=N\cap D^0$. Consider the exact sequence of the fundamental groups of (#);

$$1\longrightarrow\pi\xrightarrow{\alpha}\Gamma\xrightarrow{\beta}\boldsymbol{Z}\longrightarrow 1,$$

where $\Gamma=D/D^0$ and $\pi=K/K^0$. Since K is a closed uniform subgroup of N, K^0 is a normal subgroup of N (Corollary to Theorem 2.3 in [11]). Hence it follows from a result in [12] (see the table 1 in [12]) that π is isomorphic to $(\boldsymbol{Z}\times\boldsymbol{Z})\times_\phi\boldsymbol{Z}$, where $\phi: \boldsymbol{Z}\to\mathrm{Aut}(\boldsymbol{Z}\times\boldsymbol{Z})$; $1\to\begin{pmatrix}1 & k\\ 0 & 1\end{pmatrix}$. It can be easily shown that the center $z(\pi)$ is given by $z(\pi)=\left\{\begin{pmatrix}x\\ 0\end{pmatrix}\in\boldsymbol{Z}\times\boldsymbol{Z}\right\}$ (note that we may assume $k\neq 0$).

We shall consider the special case in which $\pi\cap z(\Gamma)=z(\pi)$, in other words, $\beta(z(\pi))=1$. It follows from Lemma 11 that there exist closed subgroups N_1 and N_2 of N such that

(i) $N_1\subset K$ and $N_1/K^0=\boldsymbol{Z}=z(\Gamma)\cap\pi$.

(ii) $K^0\subset N_2$, $N_2/K^0=\boldsymbol{R}$ and N_1/K^0 is a lattice of N_2/K^0.

Consider the action of $T^1=(N_2/K^0)/(N_1/K^0)=N_2/N_1$ on N_2/K defined by $(n_2N_1)(nK)=nn_2^{-1}K$. We show that this action is compatible with homeomorphism h. In fact, we have

$$(n_2N_1)(h(xK))=(n_2N_1)(xx_0^{-1}K)=xx_0^{-1}n_2^{-1}K \quad\text{and}$$
$$h((n_2N_1)(xK))=h(xn_2^{-1}K)=xn_2^{-1}x_0^{-1}K.$$

It follows from the following lemma that we have $x_0n_2x_0^{-1}n_2^{-1}\in K$, which implies that h is equivariant under the action of T^1.

LEMMA 24. *$n_2^{-1}x_0n_2x_0^{-1}\in K$ for every $n_2\in N_2$.*

PROOF. Consider the homomorphism $c_{x_0}: K/K^0\to K/K^0$ defined by $c_{x_0}(kK^0)=x_0kx_0^{-1}K^0$. Clearly c_{x_0} induces the identity on N_1/K^0. Since $c_{x_0}(N_2/K^0)$ and N_2/K^0 contain N_1/K^0 as a lattice, it follows from Lemma 3 that c_{x_0} is the identity on N_2/K^0, in other words, $x_0 n_2^{-1} K^0 = n_2 K^0$ for every $n_2 \in N_2$. Q.E.D.

Next we shall consider the general case; i.e. $\beta(z(\pi)) = \boldsymbol{Z}$. Let $\beta(z(\Gamma)) = n_0 \boldsymbol{Z} \subset \boldsymbol{Z}$. Define an action of $\boldsymbol{R}$ on $\boldsymbol{R} \times_{\boldsymbol{Z}} (N/K)$ by the formula;

$$t[x, nK] = [x+t, nK],$$

where $[x, nK]$ denotes the orbit of (x, nK). It is easy to see that this action is well defined. It is also proved that $\beta(z(\Gamma)) = n_0 \boldsymbol{Z}$ is the ineffective kernel of this action. In fact, we have

$$n_0[x, nK] = [x+n_0, nK] = [x, nx_0^{n_0}K].$$

The following lemma shows that $[x, nx_0^{n_0}K] = [x, nK]$, which implies that the group $\boldsymbol{R}/n_0\boldsymbol{Z}$ acts on $\boldsymbol{R} \times_{\boldsymbol{Z}} (N/K)$.

LEMMA 25. $[x, nx_0^{n_0}K] = [x, nK]$.

PROOF. Consider the following commutative diagram;

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \pi & \longrightarrow & \Gamma & \longrightarrow & \boldsymbol{Z} & \longrightarrow & 1 \\
 & & \| & & & & & & \\
 & & K/K^0 & & \uparrow & & \uparrow & & \\
 & & \uparrow & & & & & & \\
1 & \longrightarrow & z(\Gamma)\cap\pi & \longrightarrow & z(\Gamma) & \longrightarrow & \beta(z(\Gamma)) & \longrightarrow & 1. \\
 & & & & & & \| & & \\
 & & & & & & n_0\boldsymbol{Z} & &
\end{array}$$

Because the lower exact sequence is central, $n_0\boldsymbol{Z}$ acts on $z(\Gamma)\cap\pi$ trivially, i.e. $n_0(n_1K^0) = n_1x_0^{n_0}K^0 = n_1K^0$. In particular, we have $x_0^{-n_0}K^0 = K^0$ and hence $x_0^{n_0}K^0 = K^0$. Q.E.D.

It is not difficult to show that the action of $\boldsymbol{R}/n_0\boldsymbol{Z}$ is commutative with the action of T^1 and $(\boldsymbol{R}/n_0\boldsymbol{Z}) \times T^1$ acts on $\boldsymbol{R} \times_{\boldsymbol{Z}} (N/K)$. Thus $M = \boldsymbol{R} \times_{\boldsymbol{Z}} (N/K)$ admits a maximal torus action. This completes the proof of Theorem B.

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