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On the rationality of complex homology 2-cells: I

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Introduction.

In [V] Van de Ven raised the following question: Let V be an irreducible smooth algebraic surface/$\boldsymbol{C}$, with $H_i(V, \boldsymbol{Z})=(0)$ for $i>0$. Is V rational? Following Van de Ven, we call such a surface V, a complex homology 2-cell.

It can be easily shown that if V is a complex homology 2-cell then

(i) V is an affine surface,

(ii) for any smooth projective completion X of V, both the irregularity $q(X)$ and the geometric genus $p_g(X)$ vanish, the divisor $D=X-V$ is simply-connected and the irreducible components of D generate freely the divisor class group, Pic X,

(iii) a smooth projective completion X of V can be chosen such that $D=X-V$ has at worst ordinary double point singularities.

For the proof of these, see [F], §1 and §2. In [F] it is also proved that if $\bar{k}(V)\leqq 1$, then (a homology 2-cell) V is rational. An affirmative answer to Van de Ven's question would follow from:

(*) "*Let X be an irreducible, smooth projective surface/$\boldsymbol{C}$ with the geometric genus $p_g(X)=0$. Let D be a reduced (not necessarily connected) curve on X with at worst ordinary double point singularities. Suppose each connected component of* Supp$D$ *is simply-connected and the irreducible components of D generate the divisor class group* Pic$X$. *Then X is rational.*"

In this paper we settle the case of a general type surface in (*) (see Theorem below). The elliptic case will be considered in a subsequent paper.

THEOREM. *Let X and D be as in (*). Then X is not a surface of general type.*

We have the following consequences of the theorem.

COROLLARY 1. *The Chow group of a complex homology 2-cell V is trivial.*

In view of the properties of a complex homology 2-cell discussed above we have to only verify that any 0-cycle on V is rationally equivalent to zero. But this follows immediately from the main theorem in [**B-K-L**] which asserts that any 0-cycle of degree 0 on X is trivial.

Also if X is rational or elliptic with $p_g(X)=0$ then $SA_0(X)=0$, by [**B-K-L**]. In [**M-S**], it is shown that if $SA_0(X)=0$ then any algebraic vector bundle on X splits as a direct sum of a trivial subbundle and a line bundle. Hence we have the next corollary.

COROLLARY 2. *All algebraic vector bundles on a complex homology 2-cell are trivial.*

This is a generalization of C. S. Seshadri's result on the triviality of algebraic vector bundles on $\boldsymbol{C}^2$. The first non-trivial example of a smooth, contractible surface/$\boldsymbol{C}$ was given by C. P. Ramanujam in [**R,** § 3]. In [**G-M**], R. V. Gurjar and M. Miyanishi have constructed infinitely many non-isomorphic complex homology 2-cells and smooth contractible surfaces.

Let X be as in (*). Then some connected component of D supports an effective divisor C with $C^2>0$. Since every connected component of D is simply-connected by assumption, by a generalization of the Lefschetz hyperplane section theorem proved in [**N**], we see that $\pi_1(X)$ is trivial. Thus, the geometric genus and the irregularity of X both vanish. These observation will be tacitly used in the following sections.

We will briefly indicate some of the ideas used in the proof of the theorem.

Our proofs depend heavily on Y. Miyaoka's inequality ([**M**], Theorem 1.1). We assume that the surface X is not rational. Then $K+D$ has a Zariski-decomposition, so that Miyaoka's inequality applies. In § 2, by studying the blowing-down process $\pi: X \to X''$, where X'' is the minimal model for the function field of X, we deduce an auxiliary inequality relating various integral invariants of D and X. This auxiliary inequality (2.8) plays a central role in our proof. A careful interpretation of terms involved in (2.8) yield a bound on the number of components of the exceptional set $\mathcal{E}$ of π that lie outside D. The geometric Lemma 4.1, about the effectivity of $\pm K$ is of general interest. It plays another important role in our proof. To begin with, this lemma gives better estimates of terms in (2.8). At this stage, by repeated application of Miyaoka's inequality we can reduce the problem to the case when $\mathcal{E} \subseteq D$. However, beyond this, the proof becomes heavily computational and mainly uses the unimodularity of D and lemmas in § 4. So we must prepare ourselves with some technical lemmas in § 5 and § 6, converting these properties of D into something easily checkable.

After that, it follows that the number of components $b_2(D)$ of D is bounded above. In the general type case, we have $b_2(D) \leqq 12$. In § 7 we complete the proof of Theorem, the bulk of which goes into handling the situation $\mathcal{E} \subseteq D$.

§ 1. Miyaoka's inequality.

1.1. NOTATIONS, CONVENTIONS etc. Let Y be a smooth projective surface and C a reduced curve on Y. By a "component" of C we always mean an irreducible component of C. By a (-1)-curve we mean an exceptional curve of the 1st kind and by contraction we mean (a succession of) blowing-down of (-1)-curves. We say C is NC (normal crossings) if all the components of C are smooth and C has at worst ordinary double point singularities. C is MNC (minimal with normal crossings) if it is NC and after blowing-down any (-1)-curve in C, the image of C is not NC. Let K_Y denote the canonical divisor of Y. Suppose C is NC and K_Y+C has a Zariski decomposition $K_Y+C=P+N$ with P and N denoting positive and negative parts respectively. For the definition and properties of the Zariski decomposition, see ([**F**], § 6). By Miyaoka's inequality, we have,

$$0 \leqq \chi_{\text{top}}(Y)-\chi_{\text{top}}(C)-\frac{1}{3}(K_Y+C)^2+\frac{1}{12}N^2,$$

where χ_{top} denotes the topological Euler characteristic. (See Theorem 1.1 of [**M**].)

1.2. With Y and C as above, assume further that $p_g(Y)=0=q(Y)$ and all components of C are rational. Let $b_i=b_i(C)$ denote the i^{th} betti number of C (i. e. b_0=number of connected components, b_1=number of lin. ind. 1-cycles, b_2= number of components). Then we have

$$\chi_{\text{top}}(Y) = \beta_2(Y)+2\,; \quad \chi_{\text{top}}(C) = b_0-b_1+b_2$$

$$K_Y^2 = 10-\beta_2(Y)\,; \quad K_Y\cdot C+C^2 = 2(b_1-b_0)$$

where $\beta_i(Y)$ denotes the i^{th} betti number of Y. The last equality follows from repeated application of the adjunction formula. Thus the inequality yields:

$$0 \leqq 4\beta_2+b_1-b_0-4-3b_2-K_Y\cdot C+\frac{1}{4}N^2. \tag{1.3}$$

First we have the following:

1.4. LEMMA. *Suppose C has a component C_0 such that $C_0\cdot(C-C_0)\leqq 1$. Then $N^2<0$.*

PROOF. Clearly $(K_Y+C)\cdot C_0<0$. Hence by elementary properties of the Zariski decomposition, C is a component of N. This implies $N^2<0$.

1.5. Let now Y be a smooth projective surface with $p_g(Y)=q(Y)=0$ and C a reduced curve on Y with all it's components smooth and rational (but C need not be NC). Introduce the notation:

$$M(Y, C) = 4\beta_2+b_1-b_0-4-3b_2-K_Y\cdot C$$

where β_2, b_i are as above. Then if C is also NC and $K_Y+C=P+N$ is the Zariski decomposition, the inequality (1.3) reads as follows:

$$0 \leqq M(Y, C)+\frac{1}{4}N^2. \tag{1.6}$$

The following lemma will be useful in the applications of the above inequality.

1.7. LEMMA. *Suppose C is* MNC *and $\phi: Y \to Y'$ is a non-trivial composite of contractions of (-1)-curves such that $C'=\phi(C)$ has all it's components smooth and $C=\phi^{-1}(C')$. Then $M(Y, C) \leqq M(Y', C')-1$.*

PROOF. Write $\phi=\phi_r \circ \cdots \circ \phi_1$, $r \geqq 1$, where each ϕ_i contracts a (-1)-curve E_i, $Y_0=Y$, $C_0=C$, $Y_i=\phi_i(Y_{i-1})$ and $C_i=\phi_i(C_{i-1})$. Since each component of C_i is smooth, E_i intersects each component of C_{i-1}, transversally in at most one point. C being MNC, it follows that $E_1 \cdot (C_0-E_1) \geqq 3$ and $E_i \cdot (C_{i-1}-E_i) \geqq 2$ for $i \geqq 2$. To see the last inequality it suffices to note that if $E_i \cdot (C_{i-1}-E_i) \leqq 1$ then, from the analysis of blowing-ups we see easily that C will not be MNC.

We have $-K_Y \cdot C \leqq -K_{Y_1} \cdot C_1 - 2$ and $-K_{Y_{i-1}} \cdot C_{i-1} \leqq -K_{Y_i} \cdot C_i - 1$ for $i \geqq 2$. On the other hand, $b_0(C_i)=b_0(C_{i-1})$, $b_1(C_i)=b_1(C_{i-1})$, $b_2(C_i)=b_2(C_{i-1})-1$, $\beta_2(Y_i)=\beta_2(Y_{i-1})-1$ for $i \geqq 1$. Hence $M(Y, C) \leqq M(Y_1, C_1)-1 \leqq M(Y_2, C_2)-1 \leqq \cdots \leqq M(Y_r, C_r)-1=M(Y', C')-1$. This completes the proof of the lemma.

1.8. REMARK. If C and C' are as in (1.7) and C has a component C_0 satisfying $C_0 \cdot (C-C_0) \leqq 1$, then from (1.4) and (1.7) above we get $0 < M(Y', C')-1$.

§ 2. An auxiliary inequality.

2.1. Now let X and D be as in (*). We shall assume that X is not rational for the rest of the paper and arrive at a contradiction. By assumption D is a tree of rational curves, $\beta_1(X)=0$, $p_g(X)=0$. We can contract (-1)-curves E on X with $E \cdot D=1$, without changing the hypothesis on X and D. Thus we may assume that for each (-1)-curve E on X, $E \cdot D \geqq 2$. In particular, it follows that D is MNC. For some n (in fact for $n=2$) $|nK| \neq \emptyset$, $(K=K_X)$ and so $K+D$ has Zariski decomposition. Now we can apply (1.6) above with $Y=X$ and $C=D$. Our purpose in this section is to obtain a rough upper bound for $\beta_2=\beta_2(X)$. The following lemma is an easy consequence of non-rationality of X.

2.2. LEMMA. (i) *Any two (-1)-curves on X are disjoint.*
(ii) *$C^2<0$ for any component C of D.*

2.3. Let now X'' be the minimal model for the function field of X, $\pi: X \to X''$ be the composite of contractions of (-1)-curves, $D''=\pi(D)$. Let $\mathcal{E}$ be the exceptional set for π. Write $\pi=\varphi_m \circ \cdots \circ \varphi_1$, where each φ_j is a contraction of a (-1)-curve E_j.

Let $\phi_0=\mathrm{id}_X$ and $\phi_j=\varphi_j\circ\cdots\circ\varphi_1$ for $j\geqq 1$. The integer $\beta(E_j)=(\phi_{j-1}(D)-E_j)\cdot E_j$ is the branching number of E_j, w.r.t. $\phi_{j-1}(D)$. In view of the lemma above, we can rearrange φ_j's in such a way that if $\pi_1=\varphi_{n_1}\circ\cdots\circ\varphi_1=\phi_{n_1}$, then

a) $D'=\pi_1(D)$ has all the components (still) smooth,

b) for each $j>n_1$, $E_j\cdot C\geqq 2$ for at least one component C of $\phi_{j-1}(D)$.

Let $X'=\pi_1(X)$, $\pi_2=\varphi_m\circ\cdots\circ\varphi_{n_1+1}: X'\to X''$ and $\mathcal{E}_i$ be the exceptional set for π_i, $i=1, 2$. Write $m=n_1+n_2$. Clearly $b_2(\mathcal{E}_i)=n_i$, $b_2(\mathcal{E})=n_1+n_2$. From now on for any irreducible curve C_i'' on X'', we shall denote it's proper transform on X (resp. on X') by C_i (resp. C_i') etc. We denote the components of $\mathcal{E}_1$ by $\{L_i\}_{1\leq i\leq n_1}$ and those of $\mathcal{E}_2$ by $\{E_i'\}_{1\leq i\leq n_2}$. The proper transform of E_i' on X will be denoted by E_i. For any component C_i of $\mathcal{E}$ we define $\beta(C_i)=\beta(\phi_{j-1}(C_i))$ where j is such that $\phi_{j-1}(C_i)$ is a (-1)-curve (and by virtue of 2.2 (i), this is well defined). We shall now introduce some subsets of $\mathcal{E}$ and some more notations:

$$R_2=\bigcup\{L_i\mid\beta(L_i)=2\},\quad R_3=\bigcup\{L_i\mid\beta(L_i)=3\},\quad R_4=\bigcup\{L_i\mid\beta(L_i)\geqq 4\},$$

$$S=\mathcal{E}_2\cap D',\quad r_i=b_2(R_i),\quad s_2=b_2(S),\quad e_1=n_1-b_2(\mathcal{E}_1\cap D),$$

$$\sigma=n_2-\sum_{E'\in S}(E'^2+2).$$

Let now $D=\{D_r\}$, $D'=\{D_s'\}$ and $D''=\{D_t''\}$. As stated above D_t and D_t' will denote the proper transforms of D_t'' on X and X' respectively and D_s will denote the proper transform of D_s' in X. Write K, K' and K'' for the canonical divisors of X, X', X'' resp. For $1\leqq j\leqq n_1$, let now φ_j contract $\phi_{j-1}(L_j)$ where $L_j\subset\mathcal{E}_1\cap R_i$, for some $i=2$, 3 or 4. (Note, $\mathcal{E}_1=R_2\cup R_3\cup R_4$.) Then clearly,

$$\sum_r((\phi_{j-1}(D_r))^2+2)\leqq\begin{cases}\sum_r((\phi_j(D_r))^2+2)-i & \text{if } L_j\not\subset D\\ \sum_r((\phi_j(D_r))^2+2)-i+1 & \text{if } L_j\subset D.\end{cases}\tag{2.4}$$

Here we take $(\phi_{j-1}(D_r))^2=0$ (resp. $(\phi_j(D_r))^2=0$) if $\phi_{j-1}(D_r)$ (resp. $\phi_j(D_r)$) is a point. By the adjunction formula we have $-K\cdot D=\sum_r(D_r^2+2)=\sum_r((\phi_0(D_r))^2+2)$ and $-K'\cdot D'=\sum_s(D_s'^2+2)=\sum_s((\pi_1(D_s))^2+2)=\sum_r((\phi_{n_1}(D_r))^2+2)$. Now by repeated application of 2.4, for $j=1,\cdots,n_1$ and the fact that $n_1=r_2+r_3+r_4$, we obtain

$$-K\cdot D\leqq -K'\cdot D'-2r_2-3r_3-4r_4+b_2(\mathcal{E}_1\cap D)$$

and hence,

$$-K\cdot D\leqq -K'\cdot D'-r_3-2r_4-n_1-e_1.\tag{2.5}$$

Let $\{p_{t,i}\}$ be all the singular points of D_t'', including the infinitely near ones, and let the multiplicities at these be $m_{t,i}$. Note that $m_{t,i}\geqq 2$. By the genus formula, $(D_t''^2+2)+D_t''\cdot K''=\sum_i m_{t,i}(m_{t,i}-1)$. Since D_t' is the smooth model for D_t'', we have

$$D_t'^2+2\leqq D_t''^2+2-\sum_i m_{t,i}^2=-\sum_i m_{t,i}-D_t''\cdot K''.$$

Now let

$$\tau=\sum_{t,i} m_{t,i}-2n_2, \qquad \lambda=\sum_t D_t''\cdot K''.$$

Then, $\sum_t(D_t'^2+2)\leqq-\tau-2n_2-\lambda$. Since $-K'\cdot D'=\sum_s(D_s'^2+2)=\sum_t(D_t'^2+2)+\sum_{E'\in S}(E'^2+2)$, from (2.5) we get,

$$-K\cdot D\leqq-\lambda-\tau-\sigma-n_2-n_1-r_3-2r_4-e_1. \tag{2.6}$$

Now applying (1.3) with $Y=X$, $C=D$, we obtain,

$$0\leqq 4\beta_2-b_0-3b_2-4-\lambda-\tau-\sigma-n_2-n_1-r_3-2r_4-e_1+\frac{1}{4}N^2. \tag{2.7}$$

Now note that $\beta_2''=\beta_2(X'')=\beta_2(X)-n_1-n_2$. All the quantities in (2.7), except possibly $(1/4)N^2$, are integers. So we can rewrite the above as

$$3(b_2-\beta_2)+b_0+\lambda+\sigma+\tau+e_1+r_3+2r_4\leqq\beta_2''-5. \tag{2.8}$$

Since X'' is a minimal surface with $q(X'')=0=p_g(X'')$, we should have $\beta_2''\leqq 10$. Further, if X'' is of general type, then $\beta_2''\leqq 9$, and hence we get

$$3(b_2-\beta_2)+b_0+\lambda+\sigma+\tau+e_1+r_3+2r_4\leqq 4. \tag{2.9}$$

These are the auxiliary inequalities mentioned in the introduction. Finally,

2.10. LEMMA. (i) *Each term on the left hand side of* (2.8) *is non-negative.*
(ii) *If there is an equality in* (2.8) *then*

$$K\cdot D\geqq 4\beta_2-b_0-3b_2-5.$$

PROOF. (i) $b_2-\beta_2$ is non-negative because the cohomology classes of the components of D generate $H^2(X, \mathbf{Z})$. Each $m_{t,i}\geqq 2$ and there are at least n_2 singular points, and hence we see that τ is non-negative. X'' is minimal and birationally non-ruled, so λ is non-negative. Other terms are non-negative by definition.

(ii) Equality in (2.8) means we have

$$\begin{aligned}0&=4\beta_2-b_0-3b_2-5-\lambda-\tau-\sigma-n_2-n_1-r_3-2r_4-e_1\\&\geqq 4\beta_2-b_0-3b_2-5-K\cdot D\end{aligned}$$

and hence the result.

§ 3. Interpretation of terms in (2.8).

3.1. LEMMA. (i) *To each (-1)-curve E_i' in S there exist $D_i'\subset D'$ such that $E_i'\cdot D_i'\geqq 2$ and some point $x_i\in E_i'\cap D_i'$ such that either $b_2(\pi_1^{-1}(x_i))\geqq 2$ and $\pi_1^{-1}(x_i)$ contains a curve $L_i\in R_3\cup R_4$ or $\pi_1^{-1}(x_i)=\{L_i\}$ for some (-1)-curve L_i which is not in D. In particular if p=number of (-1)-curves in S, then $p\leqq n_1$.*

(ii) $\tau=0 \Rightarrow$ *all* $m_{t,i}$*'s are equal to* 2.

(iii) $\sigma=0 \Rightarrow \mathcal{E}_2=S$ *and* $\mathcal{E}_2$ *is a disjoint union of* (-1)*-curves, and hence* $n_2=p\leqq n_1$. *Similarly* $\sigma=1 \Rightarrow$ *either* $\mathcal{E}_2-D'=\{E_1'\}$ *and* S *is a disjoint union of* (-1)*-curves, or,* $\mathcal{E}_2=S$ *consists of disjoint union of some* (-1)*-curves and* $E_1'\cup E_2'$, *with* $E_1'^2=-2$, $E_2'^2=-1$ *and* $E_1'\cdot E_2'=1$.

PROOF. (i) Let E_i' be any (-1)-curve in S. By definition of $\mathcal{E}_2$, there exists $D_i'\subset D'$ with $D_i'\cdot E_i'\geqq 2$. Since D is NC and $E_i'\subset D'$, it follows that for all $x\in D_i'\cap E_i'$ such that $(D_i'\cdot E_i')_x\geqq 2$ we have the said property for $\pi_1^{-1}(x_i)$. If $(D_i'\cdot E_i')_x=1$ for all $x\in D_i'\cap E_i'$ then $D_i'\cup E_i'$ is not simply connected. Since D is simply connected, it follows that for some $x_i\in D_i'\cap E_i'$, we have $(D_i'\cdot E_i')_x\geqq 2$ and so we are done.

(ii) By definition, $2n_2+\tau=\sum m_{t,i}\geqq 2n_2$. Since $m_{t,i}$'s are at least n_2 in number, the claim follows.

(iii) $\sigma=n_2-\sum_{E'\in S}(E'^2+2)$. Since $b_2(S)\leqq n_2$ and $E'^2\leqq -1$ for each $E'\in S$, $\sigma=0$ implies that $E'^2=-1$ for all $E'\in S$ and $b_2(S)=n_2$. Hence $\mathcal{E}_2=S$ and $\mathcal{E}_2$ is a disjoint union of (-1)-curves. If $\sigma=1$ then either $b_2(S)=n_2-1$ or $b_2(S)=n_2$. In the former case, we further have $E'^2=-1$ for all $E'\in S$ as before. In the latter case $\mathcal{E}_2=S$ and all except one curve, say E_1' in S are (-1)-curves and $E_1'^2=-2$. Clearly all the (-1)-curves are disjoint and $E_1'\cdot E_2'=1$ for one of these (-1)-curves E_2'. And then it is clear that E_1' is disjoint from other (-1)-curves.

3.2. LEMMA. *Suppose* $\lambda\geqq 2$. *Then* $b_2=\beta_2$ *and hence* D *is unimodular; also* $r_4=0$.

PROOF. By (2.8) we have

$$3(b_2-\beta_2)+b_0+\lambda+\sigma+\tau+e_1+r_3+2r_4 \leqq 5\,.$$

Since $b_0\geqq 1$ and $\lambda\geqq 2$ by assumption, $b_2=\beta_2$ follows. Now suppose $r_4\geqq 1$. Then $r_4=1$, $e_1=0$, $r_3=0$, $b_0=1$, $\sigma=0$, $\beta_2''=10$. Let $R_4=\{L_1\}$. Then since $e_1=0$, $L_1\subset D$. Since $r_3=0$ and D is MNC, it follows that L_1 is the only (-1)-curve on X. L_1 meets at least four other components say D_1, D_2, D_3, D_4 of D. If $\varphi_1\colon X\to\varphi_1(X)$ contracts L_1, then $\varphi_1(D_i)$, $i=1, 2, 3, 4$, meet other $\varphi_1(D_j)$ transversally. Since $r_3=0$ and $R_4=\{L_1\}$ $\varphi_1(D_i)$ are not (-1)-curves. It follows that $\varphi_1(X)=X'$. Now since $\sigma=0$, by 3.1 it follows that $n_2=0$ and hence $X'=X''$, $b_2=\beta_2=11$. Now consider $C=D-L_1$. Then $b_0(C)\geqq 4$, $b_1(C)=0$, $b_2(C)=10$. Since we have an equality in (2.8), now we have $K\cdot D\geqq 5$ (by 2.10(ii)). Hence $K\cdot C=K\cdot D+1\geqq 6$. Thus $M(X, C)\leqq 44-4-4-30-6=0$, contradicting (1.8). Hence $r_4=0$ as claimed.

§4. Some lemmas about $\pm K$ being effective.

In this section we prove a somewhat general result which will be useful in dealing with many cases.

4.1. LEMMA. *Let Y be a smooth projective surface with $q(Y)=0$ and $C=\bigcup_{i\geq 0} C_i$ a reduced, connected curve on Y. Suppose that there is a canonical divisor $K=\sum_{i\geq 0}\alpha_i C_i$ supported on C and the connected components of $\bigcup_{i\geq 1} C_i$ can be contracted to rational singular points on a normal projective surface W by a morphism $\phi: Y\to W$. Assume further that either* (a) *ϕ is a minimal resolution of singularities of W or* (b) *C is* MNC. *Then K or $-K$ is effective, i.e., $\alpha_0\geqq 0$ implies $\alpha_i\geqq 0$, for all i, and $\alpha_0\leqq 0$ implies $\alpha_i\leqq 0$.*

PROOF. (a) ϕ is birational and W is normal. Hence by the projection formula, for any line bundle $\mathcal{L}$ on W, $\phi_*\phi^*\mathcal{L}\approx\mathcal{L}$. Hence $\phi^*: \mathrm{Pic}\, W\to \mathrm{Pic}\, Y$ is injective. Since $q(Y)=0$, $\mathrm{Pic}\, Y$ is finitely generated and so $\mathrm{Pic}\, W$ is also finitely generated. It follows that $H^1(W, \mathcal{O}_W)=(0)$. Since W has only rational singularities, $\chi(Y, \mathcal{O}_Y)=\chi(W, \mathcal{O}_W)$ (see [A]).

This implies that $\phi^*: H^2(W, \mathcal{O}_W)\xrightarrow{\sim} H^2(Y, \mathcal{O}_Y)$. By duality, $\dim H^0(Y, K_Y)=\dim H^0(W, K_W)$. Given a regular 2-form ω on Y, ω can be thought of as a rational 2-form on W, which is clearly regular on W. We thus get an injective homomorphism $H^0(Y, K_Y)\to H^0(W, K_W)$ which is then surjective also.

We consider the rational 2-form ω with divisor K on Y. Suppose $\alpha_0\geqq 0$. Then the divisor of ω on W is $\alpha_0\phi(C_0)\geqq 0$. By the arguments above, ω is also regular on Y, i.e., $\alpha_i\geqq 0$ for all i.

Now suppose $\alpha_0\leqq 0$. Let $\Lambda=\sum_{i\geq 1}\alpha_i C_i$ and assume first that $\bigcup_{i\geq 1} C_i$ is connected. The intersection form on $\bigcup_{i\geq 1} C_i$ is negative definite and by assumption, $C_i^2\leqq -2$ for $i\geqq 1$. Each $C_i\approx \boldsymbol{P}^1$ and hence $K\cdot C_i\geqq 0$ for $i\geqq 1$.

Now $\Lambda\cdot C_i=K\cdot C_i-\alpha_0 C_0\cdot C_i\geqq 0$ for $i\geqq 1$. Hence $\Lambda\cdot C_i\geqq 0$ for all $i\geqq 1$. It is easy to see from this that $\alpha_i\leqq 0$ for $i\geqq 1$ as required (see 5.3).

In general, if $\bigcup_{i\geq 1} C_i$ is not connected, we argue with each connected component and deduce the same conclusion.

(b) This follows easily from (a). We factor ϕ as $\phi_1: Y\to Y_1$ and $\phi_2: Y_1\to W$ where ϕ_2 is the minimal resolution of singularities of W. The divisor of ω on Y_1 will have the required property. Now analysing the effect of a blowing-up, on the canonical divisor and using the fact that C is MNC, we get the required result.

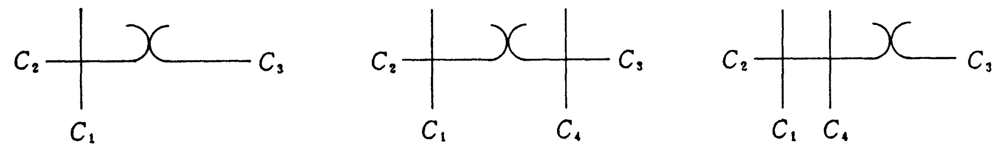
REMARK. The above lemma is valid even without the assumption $q(Y)=0$. We do not, however, use this more general result in the paper.

4.2. LEMMA. *Assume that X'' is of general type. Then there are at least two components D''_{t_1}, D''_{t_2} with $K''\cdot D''_{t_i}>0$. In particular, $\lambda\geqq 2$ and hence $b_2=\beta_2$, $r_4=0$, $\sigma+\tau+r_3+e_1\leqq 1$.*

PROOF. For large n, $|nK''|$ defines a birational morphism $\phi: X''\to W$ onto a normal projective surface, which contracts all (-2)-curves on X'', to finitely many rational double points and X'' is the minimal resolution of singularities of W. Since components of D'' generate $\mathrm{Pic}X''$, there is a canonical divisor K'' supported on D''. Note that for any curve C on X'', $K''\cdot C\geqq 0$ and $K''\cdot C=0$ iff C is a (-2)-curve. Thus if there is at most one curve D_i'' with $K''\cdot D_i''>0$, then by Lemma 4.1 above, either K'' or $-K''$ is effective. Then either $p_g(X'')\neq 0$ or $|nK''|=\emptyset$ for all $n>0$. This contradiction proves the first part of the assertion and also that $\lambda\geqq 2$. Rest of the conclusion of 4.2 follow by 3.2.

The following technical extensions of 4.1 will be needed in §7, and also later, in the elliptic case.

4.3. LEMMA. *Let Y be a smooth, projective, minimal surface with $q(Y)=0$. Let C be one of the following configurations of smooth rational curves on Y:*



(1) (2) (3)

with $\{-C_2^2, -C_3^2\}=\{2, 3\}$ or $\{3, 3\}$ and $-C_4^2=2$ or 3, $C_2\cdot C_3=2p$ for some point p, $C_1\cdot C_2=1$, $C_1\cdot C_3=0$ ($C_1\cdot C_4=0$ and either $C_2\cdot C_4=0$ and $C_3\cdot C_4=1$ or $C_2\cdot C_4=1$ and $C_3\cdot C_4=0$ as the case may be). Suppose $0\neq K_Y\sim\sum t_iC_i+\Lambda$ ($t_i\in\boldsymbol{Z}$) for some divisor Λ such that $\mathrm{Supp}(\Lambda)\cap C_i=\emptyset$ for $i\geqq 2$, $C_1\not\subset\mathrm{Supp}\Lambda$, and C_1 intersects each connected component of $\mathrm{Supp}\Lambda$. Further suppose that $\mathrm{Supp}\Lambda$ can be contracted to a finite number of rational singularities on a normal surface. Then K_Y or $-K_Y$ is effective.

PROOF. We shall show that $t_1>0$ implies all t_i ($i=2, 3, 4$) are nonnegative and $t_1\leqq 0$ implies all t_i ($i=2, 3, 4$) are nonpositive. Then as in the proof of 4.1, it would follow that K_Y or $-K_Y$ is effective.

So let $a_i=-C_i^2$. By computing $K_Y\cdot C_i$ and using adjunction formula we obtain the following relations:

Figure (1).

$$\left.\begin{aligned} a_3-2&=-a_3t_3+2t_2\\ a_2-2&=2t_3-a_2t_2+t_1\end{aligned}\right\}\Rightarrow\left\{\begin{aligned} t_3&=(2t_2-a_3+2)/a_3\\ t_1&=\frac{(a_2a_3-4)}{a_3}(t_2+1)\end{aligned}\right.$$

and so for the three different values of $(a_2, a_3)=(2, 3)$, $(3, 2)$ and $(3, 3)$ we obtain $t_1=2/3\cdot(t_2+1)$, t_2+1, and $5/3\cdot(t_2+1)$ respectively. Note that $t_i\in\boldsymbol{Z}$ and hence our assertion follows, in this case.

Figure (2).

$$\left.\begin{array}{l} a_4-2=-a_4t_4+t_3 \\ a_3-2=t_4-a_3t_3+2t_2 \\ a_2-2=2t_3-a_2t_2+t_1 \end{array}\right\} \Rightarrow \left\{\begin{array}{l} t_3=a_4-2+a_4t_4 \\ t_2=\dfrac{a_3a_4-a_3-2}{2}+\dfrac{(a_3a_4-1)}{2}t_4 \\ t_1=\dfrac{a_2a_3a_4-a_2a_3-4a_4+4}{2}+\dfrac{a_2a_3a_4-a_2-4a_4}{2}t_4 \end{array}\right.$$

and for the six different values of $(a_2, a_3, a_4)=(2, 3, 2)$, $(2, 3, 3)$, $(3, 2, 2)$, $(3, 2, 3)$, $(3, 3, 2)$ and $(3, 3, 3)$ we have $t_1=1+t_4$, $2+2t_4$, $(2+t_4)/2$, $(4+3t_4)/2$, $(5+7t_4)/2$ and $5+6t_4$. Thus, again note that $t_i \in \boldsymbol{Z}$ to obtain the assertion above, in this case too.

Figure (3).

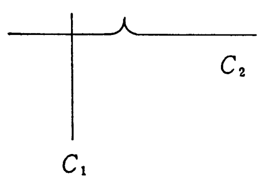
$$\left.\begin{array}{l} a_4-2=-a_4t_4+t_2 \\ a_3-2=-a_3t_3+2t_2 \\ a_2-2=t_4+2t_3-a_2t_2+t_1 \end{array}\right\} \Rightarrow \left\{\begin{array}{l} t_4=(t_2-a_4+2)/a_4 \\ t_3=(2t_2-a_3+2)/a_3 \\ t_1=\dfrac{(a_2a_3a_4-4a_4-a_3)}{a_3a_4}t_2+\dfrac{a_2a_3a_4+a_3a_4-4a_4-2a_3}{a_3a_4} \end{array}\right.$$

and again for the above six values of (a_2, a_3, a_4) we obtain

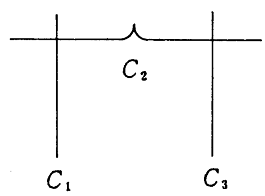
$$t_1=(t_2+4)/6,\ (t_2+3)/3,\ (t_2+2)/2,\ (2t_2+4)/3,\ (7t_2+10)/6 \text{ and } (4t_2+6)/3,$$

and the conclusion of the lemma follows.

4.4. LEMMA. *With the same hypothesis as in* 4.3 *except that now let* C *be one of the following two curves:*



(1)



(2)

where C_2 *is a rational curve with one ordinary cusp,* $C_2^2=-1$, C_1 *and* C_3 *are smooth rational curves,* $C_3^2=-2$ *or* -3. *Then* K_Y *or* $-K_Y$ *is effective.*

PROOF. Argue exactly as above with the following equations now:

Figure (1). $\quad t_1=t_2+1.$

Figure (2).

$$\left.\begin{array}{l} a_3-2=-a_3t_3+t_2 \\ 1=t_3-t_2+t_1 \end{array}\right\} \Rightarrow \left\{\begin{array}{l} t_3=(t_2-a_3+2)/a_3 \\ t_1=((a_3-1)t_2+(2a_3-2))/a_3 \end{array}\right.$$

and hence $t_1=t_2/2$ or $t_1=(2t_2+4)/3$.

§ 5. Small trees that are "rational".

5.1. Let C be NC on a smooth complex surface. Associated to C, is its dual weighted graph $\Gamma(C)$. In what follows, we will identify C with $\Gamma(C)$. For instance, we say C (or $\Gamma(C)$) is negative definite, by which we mean the intersection form of C is negative definite. For an abstract weighted graph Γ, with vertices $\{u\}$ and weights $\Omega_u \in \boldsymbol{Z}$, $u \in \Gamma$, we may visualize Γ as the dual graph of a curve C, which is NC on a smooth complex surface. Then geometric operations like blowing-up and blowing-down on Γ make sense. For instance a graph Γ is minimal if every vertex $u \in \Gamma$ with $\Omega_u = -1$ is joined to at least three other vertices.

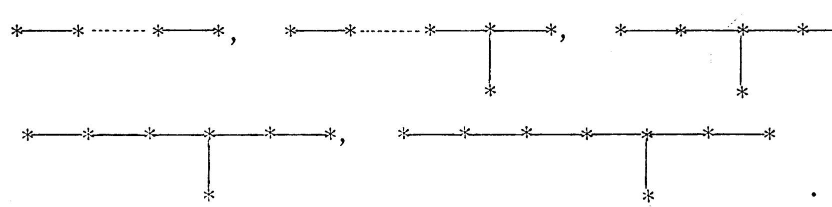
5.2. We shall now recall some results from [**A**]. Let C be a connected curve on a smooth surface, such that the intersection form on the components of C is negative definite. An effective divisor Z supported on C is called the fundamental cycle of C, if Z is the smallest nonzero divisor with the property: $Z \cdot C_i \leqq 0$ for all components C_i of C. By Theorem 3 of [**A**], C contracts to a rational singular point on a normal surface if and only if the arithmetic genus $p_a(Z)=0$. If this happens, we say C is "rational". (Thus by our convention an abstract weighted graph Γ is rational if $\Gamma = \Gamma(C)$ for some rational C.) We have:

5.3. Lemma. *Let C be a connected curve with negative definite intersection form.* (i) *If Z' is a nonzero divisor supported on C having the property: $Z' \cdot C_i \leqq 0$ for each component C_i of C, then Z' is effective.* (ii) *If C is "rational", then any connected curve C' contained in C is also "rational".*

Proof. (i) This is easy (see the proof of Proposition 2 in [**A**]).

(ii) This is a direct consequence of Proposition 1 in [**A**], which says that C is "rational" if and only if every positive divisor Z' supported on C has $p_a(Z') \leqq 0$.

5.4. Lemma. *Let Γ be a connected tree of smooth rational curves (Γ is* NC *by assumption) with all weights $\leqq -2$ and having one of the following configurations:*



Then Γ is rational.

Proof. Since all weights are $\leqq -2$, Γ is negative definite. Let Γ' be the tree obtained by replacing all the weights by -2 in Γ. Then Γ' corresponds to the minimal resolution of singularity of a rational double point. Now clearly, Γ can be thought of as a subtree of a suitable blown-up $\tilde{\Gamma}'$ of Γ'. Hence by 1.1 (ii) above, Γ is "rational".

§ 6. Some lemmas about tips and branches.

6.1. LEMMA. *Let T be a weighted tree with unimodular intersection form. For any vertex $v_0 \in T$, let $T - \{v_0\} = \amalg_{j=1}^{k} T_j$. Then the discriminants $d(T_j)$ (i.e., the modulus of the determinant of the intersection matrix of T_j) are pairwise coprime.*

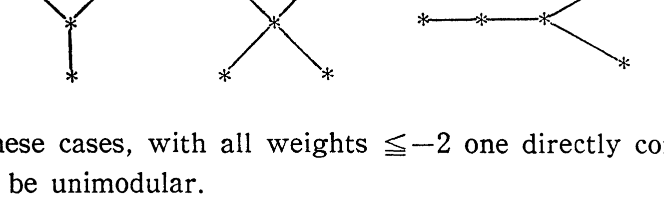
REMARK. T_j are called branches of T at v_0. Note that one of the $d(T_j)$'s may be equal to zero. But then the assertion of the lemma implies that all other $d(T_j)$ are equal to 1.

PROOF. For any weighted tree Γ, let $\boldsymbol{Z}(\Gamma)$ be the free abelian group on the vertex set of Γ and let $R(\Gamma)$ be the set of rows of the intersection matrix of Γ. Then $R(\Gamma)$ is identified with a subset of $\boldsymbol{Z}(\Gamma)$ in an obvious way. Also, $d(\Gamma) \neq 0$ if and only if the quotient group $\boldsymbol{Z}(\Gamma)/(R(\Gamma))$ is a finite group, and then the order of this group is equal to $d(\Gamma)$.

In particular, since $d(T)=1$, $R(T)$ generates the whole group $\boldsymbol{Z}(T)$. Let $\varphi : \boldsymbol{Z}(T) \to \boldsymbol{Z}(T)/(v_0) \cong \bigoplus \boldsymbol{Z}(T_j)$ be the canonical morphism. Then we have $\varphi(R(T)) = \amalg_j R(T_j) \amalg \{\varphi(R_0)\}$ where R_0 is the row corresponding to the vertex v_0. It follows that $\bigoplus_j \boldsymbol{Z}(T_j)/(R(T_j))$ is a cyclic group generated by the image of $(\varphi(R_0))$. Hence $\boldsymbol{Z}(T_j)/(R(T_j))$ are all cyclic groups of order pairwise coprime. The conclusion of the lemma follows.

6.2. LEMMA. *Let T be a unimodular tree with all weights $\leqq -2$. Then $\#(T) \geqq 6$.*

PROOF. Clearly T cannot be linear. Thus, $\#(T) \leqq 5$ forces T to have one of the following configurations:



In each of these cases, with all weights $\leqq -2$ one directly computes and sees that T cannot be unimodular.

By a tip C_0 of C we mean a component of C such that $C_0 \cdot (C - C_0) = 1$. In the language of trees a tip is a free vertex. We have:

6.3. LEMMA. *Let C be* MNC *such that its dual graph T is a unimodular tree, with all weights $\Omega_v \leqq -1$. Then*

(i) *for any two tips v_1, v_2 joined to a single vertex v_0 in T, $(\Omega_{v_1}, \Omega_{v_2})=1$,*

(ii) *if $v_1, \cdots, v_k$ are tips joined to a single vertex v_0 in T, then $\sum_{i=1}^{k} \Omega_{v_i} \leqq -\theta_k$, where $\theta_k = 2+3+5+\cdots+p_k$ is the sum of first k prime numbers.*

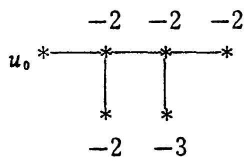
PROOF. (i) Since C is MNC, it follows that for any tip v_i in T $\Omega_{v_i} \leqq -2$. Now apply (2.1), and note $d(\{v_i\}) = |\Omega_{v_i}|$.

(ii) follows from (i).

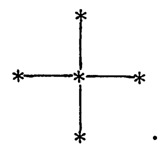
6.4. LEMMA. *Let Γ be a unimodular tree, $v_0 \in \Gamma$ be a vertex, Γ_1 be a branch of Γ at v_0, $u_0 \in \Gamma_1$ be the vertex joined to v_0 in Γ.*

(a) *Suppose Γ_1 is not "rational", $\#(\Gamma_1)=5$ and $\Omega_u \leqq -2$ for all $u \in \Gamma_1$. Then $\exists$ three vertices u_1, u_2, $u_3 \in \Gamma_1$ which are tips of Γ and such that $\sum_{i=1}^{3} \Omega_{u_i} \leqq -10$.*

(b) *Suppose Γ_1 is not "rational", $\#(\Gamma_1)=6$, $\Omega_{u_0} \leqq -2$ and the weight set $\Omega(\Gamma_1 - \{u_0\})$ is either $\{-2, -2, -2, -2, -2\}$ or $\{-2, -2, -2, -2, -3\}$. Then Γ_1 is the following tree:*

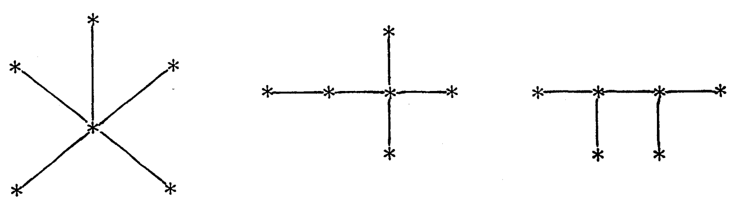


PROOF. (a) By 5.4, Γ_1 has the configuration



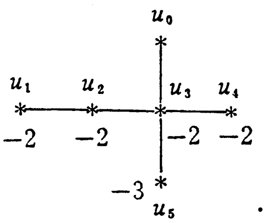
Thus no matter where the vertex u_0 is located at least three of the vertices say u_1, u_2, u_3 of Γ_1 are going to be tips of Γ. Now the conclusion of (a) follows from 6.3.

(b) By 5.4, Γ_1 should have one of the following configurations:



(1) (2) (3)

In (1), no matter where u_0 is located, we get at least four vertices of Γ_1 as tips of Γ. The weights at these vertices are pairwise coprime by 6.1, which is impossible. In (2), by the same argument, Γ_1 is forced to be



But then both of the branches $\overset{u_1}{\underset{-2}{*}}\!-\!\!-\!\!\overset{u_2}{\underset{-2}{*}}$ and $\overset{u_5}{\underset{-3}{*}}$ at u_3, have discriminant=3, contradicting 6.1. In (3), again by the same argument, it follows that u_0 has to be one of tips of Γ_1 and then the weights have to be as claimed in the lemma.

§ 7. Proof of Theorem completed.

7.1. We have to show that X is not of general type. So, assuming on the contrary, we have $\beta_2(X'')\leqq 9$. Since Pic X is generated by components of D, there is a canonical divisor supported on D. This fact will be used implicitly whenever we want to apply results of § 4. By 4.2 and 3.2 we have $\lambda\geqq 2$, $b_2=\beta_2$, $r_4=0$ and $\sigma+\tau+r_3+e_1\leqq 1$.

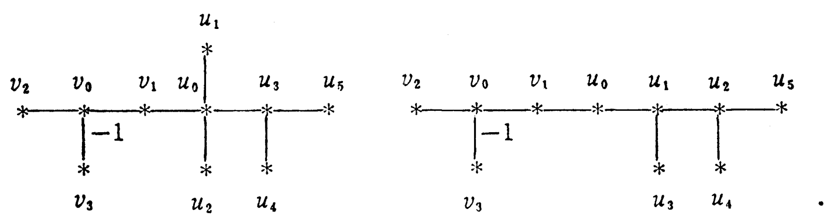
So we first consider the case $e_1=1$. We then have equality in 2.8 and hence by 2.10, we have $K\cdot D\geqq\beta_2-6$. Take $C=D\cup\{L_1\}$ where $\{L_1\}=\mathcal{E}_1-D$. Then since $r_3=0$ we have $L_1\in R_2$ and hence $b_1(C)\leqq 1$. $M(X, C)\leqq 4\beta_2+1-1-4-3\beta_2-3-(\beta_2-7)=0$. Since D is unimodular, it is not linear. Hence C has tips and so by 1.7, $0\leqq M(X, C)-1=-1$ which is absurd.

Thus from now on we will assume that $e_1=0$, so that $\sigma+\tau+r_3\leqq 1$.

7.2. Consider the case when $r_3=0$. Since D is MNC, $r_3=r_4=e_1=0$, it follows that D is free from (-1)-curves and $n_1=0$. By 3.1 ((i) and (iii)), it follows that $n_2\leqq 1$ and hence $\beta_2\leqq 10$. Since $\beta_2=b_2$, each connected component of D is unimodular, and hence by 6.2, has at least 6 components. It follows that D is connected. Let T be the dual tree of D. Then $\exists$ a vertex $v_0\in T$ such that each connected component of $T-\{v_0\}$ has at most 5 vertices. By 4.1 at least one of these components say Γ_1 is not "rational". By 5.4, $\#(\Gamma_1)\geqq 5$ and hence $\#(\Gamma_1)=5$. If $u_0\in\Gamma_1$ is joined to v_0 in T then by 6.4(a), we have $u_1, u_2, u_3\in\Gamma_1$ with $\sum_{i=1}^{3}\Omega_{u_i}\leqq -10$. Now let $T'=T-\Gamma_1$. Then since $\Gamma_1-\{u_0\}$ has all its branches "rational" and $T-\{u_0\}=T'\amalg(\Gamma_1-\{u_0\})$. It follows from 4.1 that T' is not "rational". Since $\#(T')\leqq 5$, we obtain $\#(T')=5$ and $\exists v_1, v_2, v_3\in T'$ such that $\sum_{i=1}^{3}\Omega_{v_i}\leqq -10$, as before. But then one easily checks that $K\cdot D\geqq 8$ which contradicts (1.3).

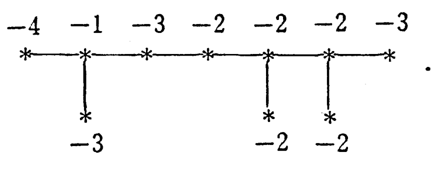
7.3. Thus from now on, we shall assume $r_3=1$, so that $b_0=1$, $\sigma+\tau=0$, $\lambda=2$, $\beta_2''=9$. (As before since we have equality in 2.8, we have $K\cdot D\geqq\beta_2-6$ and hence $K\cdot D=\beta_2-6$). Since $e_1=0$, $\mathcal{E}_1\subset D$ and since $\sigma=0$, $\mathcal{E}_2=S\subset D'$ (by 3.1). Hence $\mathcal{E}\subset D$ since $r_3=1$, $(r_4=0)$ it follows that there is a unique (-1)-curve L_0 on X and $\{L_0\}=R_3$. Let L_1, L_2 and L_3 be the components of D that meet L_0. We shall consider three subcases according as $\#(\mathcal{E})=1$, 2 or $\geqq 3$.

7.4. First consider the subcase when $\#(\mathcal{E})=1$, i.e., $\mathcal{E}=\{L_0\}$. It follows that $L_i^2\leqq -3$ for $i=1, 2, 3$. Let T be the dual graph of D and $v_i\in T$ be the vertices corresponding to L_i in T. Then $T-\{v_0\}$ has precisely three branches T_1, T_2, T_3, say. One of these, say, T_1 is not "rational" and is free from (-1)-curves. So $\#(T_1)\geqq 5$. Now if $T'=T_2\cup T_3\cup\{v_0\}$, it is easily seen that T' is rational. Hence as before $T_1-\{v_1\}$ should have a branch Γ_1 which is not "rational". If $\#(\Gamma_1)=5$, using 6.4(a) we easily conclude that $\lambda\geqq 4$, which is absurd. So $\#(\Gamma_1)=6$. But now, notice that not all $L_i^2=-3$, $i=1, 2, 3$, for then after blowing down L_0 we obtain three (-2)-curves passing through the same point on a minimal surface of general type, which is absurd. Thus at least one $L_i^2\leqq -4$. So in Γ_1 we can have at most one (-3)-curve (since $\lambda=2$); all other weights being -2. Thus by 6.4(b), we have the following two possible configurations for T:



(1) (2)

In (1), we should have $(\Omega_{u_4}, \Omega_{u_5})=1$ and $(\Omega_{u_1}, \Omega_{u_2})=1$ which is impossible since all weights in Γ_1, except perhaps one, are -2. By the same reason, from configuration (2) we obtain that T should look like:

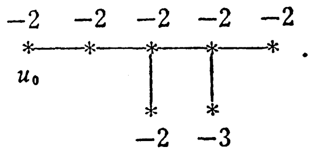


But then $d(T)\neq\pm 1$, contradicting the unimodularity of T.

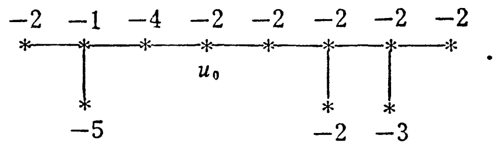
7.5. Now consider the case when $\#(\mathcal{E})=2$. This happens if and only if one of L_i^2's, say, $L_3^2=-2$ and $L_1^2\leqq -4$, $L_2^2\leqq -4$. After blowing-down L_0 and L_3, the images $\{L_1'^2, L_2'^2\}$ of $\{L_1, L_2\}$ are smooth rational curves tangentially meeting in a single point on $X'=X''$ (a minimal surface of general type). Hence both $L_i'^2$ cannot be equal to -2. Hence $K'\cdot L_i'>0$ for some $i=1, 2$. By 4.2, there are at least two components of D' such that $D'_s \cdot K' > 0$. Since $\lambda = 2$ it follows that there are precisely two components $\{D'_1, D'_2\}$ such that $D'_i \cdot K' > 0$, (and hence $D'_i \cdot K' = 1$). Also one of the L'_i belongs to $\{D'_1, D'_2\}$.

Since $r_3 = 1$ and $r_4 = 0$, L_3 is a tip of D. Taking T, T_i and Γ_1 as in 7.4 we infer as before, that $7 \geqq \#(\Gamma_1) \geqq 6$. Let $u_0 \in \Gamma_1$ be the vertex joined to v_1 $(v_1 \leftrightarrow L_1)$. We claim that $\Gamma_1 - \{u_0\}$ has a branch Γ_2 which is not "rational", assume on the contrary. On X', we have a situation as in 4.3(1) or 4.3(2) according as $\#(\Gamma_1) = 7$ or 6 (with the correspondence $u_0 \leftrightarrow C_1$, $L'_1 \leftrightarrow C_2$, $L'_2 \leftrightarrow C_3$). Hence $\pm K$ is effective, which is absurd. Hence the claim.

Now the possibility of $\#(\Gamma_2) \leqq 5$ is ruled out exactly as before, by 6.4(a). Hence $\#(\Gamma_2) = 6$. Now by repeated application of 6.4(b) and 6.3(i) we obtain Γ_1 as



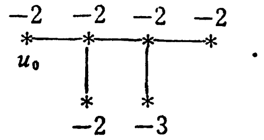
Since $(\Omega_{v_2}, \Omega_{v_3}) = 1$ by 6.3, it follows that $\Omega_{v_2} = -5$ and $\Omega_{v_1} = -4$. Hence T looks like



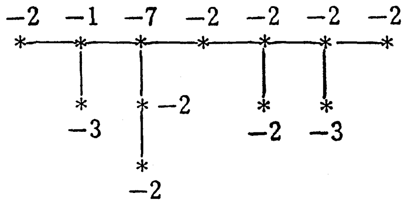
But then $d(T) \neq 1$.

7.6. Finally consider the case when $\#(\mathcal{E}) \geqq 3$. This happens if and only if $L_3^2 = -2$, and one of the L_1^2 or L_2^2 is -3, say, $L_2^2 = -3$. After blowing down L_0 and L_3 it follows that we obtain X'. The image L'_2 of L_2 is now a (-1)-curve meeting L_1 tangentialy in a single point. Hence $L'_2 \in \mathcal{E}_2$. By 3.1, $\mathcal{E}_2 = S$ is a disjoint union of (-1)-curves and hence it follows that $\mathcal{E}_2 = \{L'_2\}$. Thus after blowing down L'_2 we obtain the minimal surface X''; $\#(\mathcal{E}) = 3$; $\beta_2 = 12$; $K \cdot D = 6$. The image L''_1 of L_1 on X'' is a cuspidal curve $L''_1 \cdot K'' > 0$. By 4.2, it follows that there is one more component D''_1 of D'' such that $D''_1 \cdot K'' > 0$. Since $\lambda = 2$, we have $D''_1 \cdot K'' = 1 = L''_1 \cdot K''$ and $D''_i \cdot K'' = 0$ for all other components of D''. Hence $L''^2_1 = -1$, $(L_1^2 = -7)$, $D''^2_1 = -3$, and $D''^2_i = -2$ for all other components of D''.

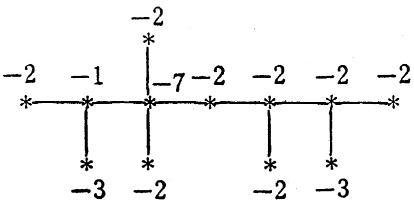
Clearly L_3 is a tip of $D\,(r_3 = 1,\ r_4 = 0)$. Since $K \cdot D = 6$, it follows that L_2 is also a tip of D. Thus if T, T_i and Γ_1 are as in 7.4, arguing similarly, we conclude that $\#(\Gamma_1) \geqq 6$. Suppose $\#(\Gamma_1) = 6$. Then again by 6.4(b) one sees that Γ_1 is like:



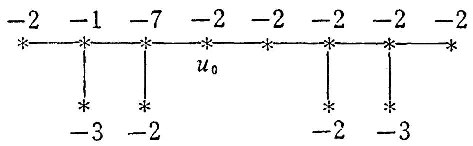
It follows that T looks like:



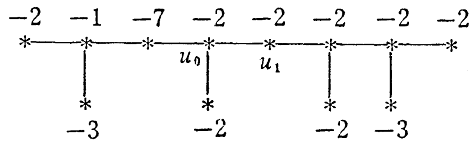
or



neither of which is unimodular. So we may assume that $\#(\Gamma_1) \geqq 7$. Again if $u_0 \in \Gamma_1$ is the vertex joined to L_1 then $\Gamma_1 - \{u_0\}$ should have a branch Γ_2 which is not rational (by 4.4). Hence as before $\#(\Gamma_2)=6$ and T itself looks like:

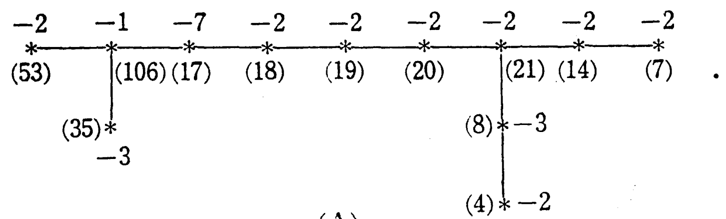


which is not unimodular. Thus $\#(\Gamma_1)=8$. As before $\Gamma_1 - \{u_0\}$ has a branch Γ_2 which is not rational and then $\#(\Gamma_2) \geqq 6$. If $\#(\Gamma_2)=6$, then T looks like:

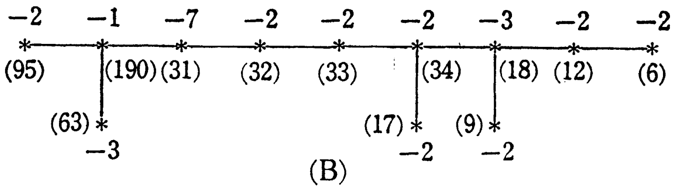


which is not unimodular. Hence $\#(\Gamma_2)=7$.

We now claim that T has one of the following two configurations and compute K. In the diagrams below the numbers in the bracket indicate the coefficient of the corresponding curve in a linear equivalence expression for K, which shows that in each case K is effective. This contradicts the assumption $p_g=0$ and so completes the proof of the theorem:

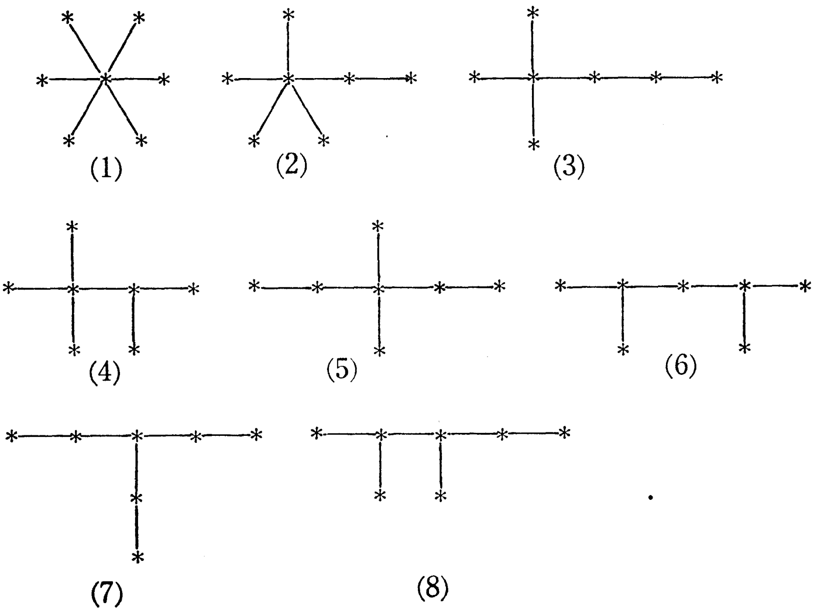


(A)

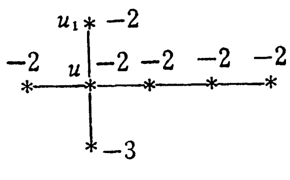


(B)

Let Γ_1' be the tree obtained from Γ_1 by changing the weight at u_0 to $\Omega_{u_0}+1$. Then, from the corresponding properties of T, it is easily seen that Γ_1' is unimodular and has one positive eigenvalue. It then follows that $\Omega'_{u_0}=-1$. (For otherwise, $\Omega'_{u_0}=-2$ in which case $\Omega_{u_0}=-3$ and hence $\Omega_w=-2$ for all $w\in\Gamma_2$. Hence Γ_1' will be of even type. By a well known result, the signature of the intersection form of Γ_1' is divisible by 8. But $\#(\Gamma_1')=8$ and Γ_1' has precisely one positive eigenvalue.) Γ_2 is identified with $\Gamma_1'-\{u_0\}$, and the weight set of Γ_2 is $\{-2, -2, -2, -2, -2, -2, -3\}$. Let u_1 be the vertex of Γ_2 joined to $u_0\in\Gamma_1'$. Since Γ_2 is not rational, it has one of the following configurations:

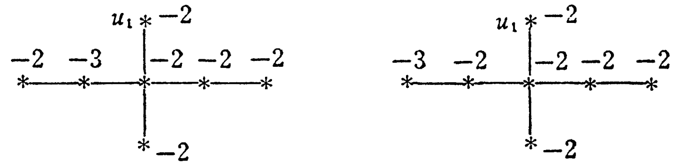


We apply 6.1 repeatedly. Thus the possibilities (1), (2) and (4) are quickly ruled out. In (3), first we see that Γ_2 should be like:



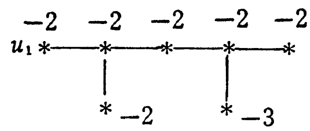
and then the two branches $\overset{-2}{*}$ and $\overset{-2}{*}$—$\overset{-2}{*}$—$\overset{-2}{*}$ at u have both even discriminants.

In (5), first we are forced to take one of the tips as u_1 and then Γ_2 has only the following two possibilities:



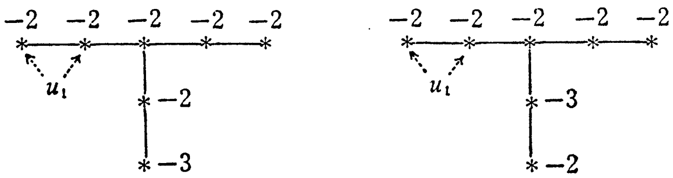
with $d(\Gamma_1')=17$ or 23 respectively.

In (6) again one of the tip should be u_1 and so Γ_1 is forced to be

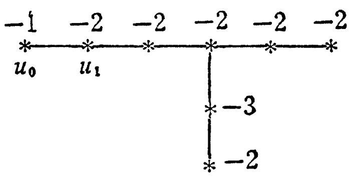


with $d(\Gamma_1')=6$.

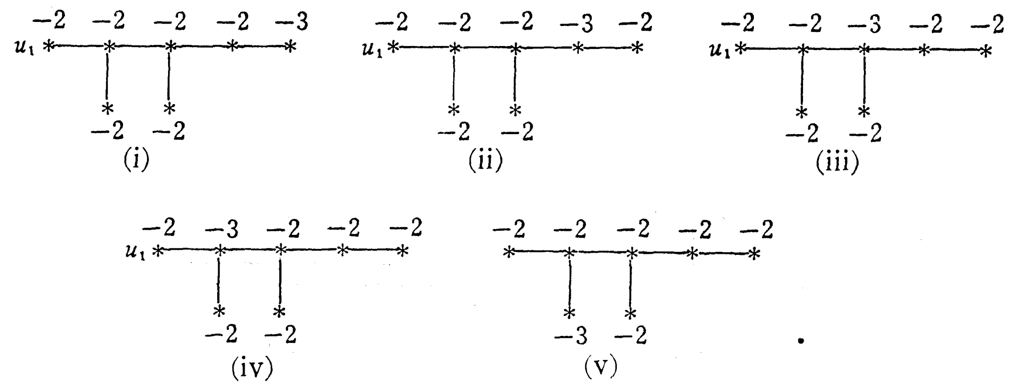
In (7), (upto symmetry) there are only two possible places for the weight -3, yielding the two configurations:



In each case u_1 is forced to be one of the vertices indicated by the arrow, yielding in all, 4 possibilities for Γ_1'. Amongst these, we see only one of them is unimodular, viz.:



This in turn yields the configuration (A) for T. Finally consider the case (8). There are only five possibilities for the location of the (−3)-curve. The location of u_1 is also decided by the same criteria (viz. 6.1) except in the last one:



In (i), (ii) and (iv) one checks that $d(\Gamma_1')=11$, 9, and 6 respectively. In (v) one checks that no matter which vertex is u_1, Γ_1' is not unimodular. In (iii) we do obtain a unimodular configuration for Γ_1' which yields (B) for T as required. This completes the proof of the theorem.

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