Polarized manifolds of degree three and Δ-genus two

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Introduction.

This paper is a continuation of [F4], and we use the same terminology there. As we saw in [F4], in order to complete the classification of polarized manifolds (M, L) of Δ-genus two, we should study the following cases:

1) $L^n = 1$.
2) $L^n \geq 2$, $\dim \mathrm{Bs}|L| = 1$ and the fibration given by $|L|$ is not hyperelliptic (cf. [F4; §3]).
3) $L^n = 2$ or 3 and $\dim \mathrm{Bs}|L| \leq 0$.

The cases 1) and 2) remain to be "mystery" at present. Partial results about the case 3) are found in [F0]. Here we study the case $L^n = 3$. So $h^0(M, L) = n + 1$.

Suppose that $\dim \mathrm{Bs}|L| \geq 1$. Then $n \leq 3$ by [F4; (1.17)]. Moreover, if $n = 3$, then (M, L) is a Segre product of $(P^1, \mathcal{O}(1))$ and a polarized surface (S, A) with $A^2 = \Delta(S, A) = 1$. Thus, in this case, the problem is reduced to the study of such polarized surfaces.

If $\dim \mathrm{Bs}|L| \leq 0$, we have two cases:

a) $|L|$ has no base point.
b) $|L|$ has (finitely many) base points.

In case a), M is a triple covering of P^n. Hence the theory in [L], [M] and [F7] applies. In particular, when $n \geq 4$, M is a triple section of an ample line bundle over P^n. However, when $n \leq 3$, there are many triple covers not of triple section type. The classification of them is still unknown.

The main purpose of this paper is the study of the case b). We will show that $|L|$ has only one base point p, and the blow-up of M at p becomes a double covering of P^n. Double coverings are much easier to study than triple coverings (cf. [F1]). However, the covering is not a finite morphism here, and the branch locus may have singularities. By a careful analysis of these troubles we obtain a classification theory and precise structure theorems of such polarized manifolds. See (1.3), (1.7), (2.17), (3.15), (4.15) for details.

§1. Generalities.

(1.1) Throughout this paper let (M, L) be a polarized manifold with $\dim M$ $=n\geq 2$, $L^n=3$ and $h^0(M, L)=n+1$. So $\varDelta(M, L)=2$. We further assume $\dim \mathrm{Bs}|L|\leq 0$ and $g(M, L)\geq 2$, since otherwise the structure of (M, L) is well understood by [**F4**].

(1.2) If $\mathrm{Bs}|L|=\varnothing$, $|L|$ gives a finite morphism $\rho: M\to \boldsymbol{P}^n$ of degree three such that $\rho^*\mathcal{O}(1)=L$. So $\mathcal{A}=\rho_*\mathcal{O}_M$ is a locally free sheaf of rank three on $P=\boldsymbol{P}^n$. Actually $\mathcal{A}$ is a sheaf of $\mathcal{O}_P$-algebra and we have the trace map $\tau: \mathcal{A}\to \mathcal{O}_P$ and the natural injection $\iota: \mathcal{O}_P\to \mathcal{A}$. Clearly $\tau\cdot\iota$ is 3-times of the identity. Hence $\mathcal{A}\simeq\mathcal{O}_P\oplus\mathcal{E}$ for $\mathcal{E}=\mathrm{Ker}(\tau)\simeq\mathrm{Coker}(\iota)$. This is called the *Tschirnhausen module* in [**M**]. The type of M is determined by this vector bundle $\mathcal{E}$.

If $n>3$, then $\mathcal{E}\simeq\mathcal{O}(-k)\oplus\mathcal{O}(-2k)$ for some $k>0$ and M is a triple section of the total space of the line bundle $\mathcal{O}(k)$ over P (cf. [**L**] or [**F7**]). For $n\leq 3$, we do not have a classification of such vector bundle $\mathcal{E}$.

(1.3) From now on, throughout this paper, we assume $\mathrm{Bs}|L|\neq\varnothing$. Then we have the following

THEOREM. 1) *Any general member of* $|L|$ *is non-singular.*
2) $\mathrm{Bs}|L|$ *consists of a single point* p.
3) *Let* $\pi: M_1\to M$ *be the blow-up at* p *and let* E *be the exceptional divisor over* p. *Then* $\mathrm{Bs}|\pi^*L-E|=\varnothing$.

PROOF. 1) is a special case of [**F3**; (2.5)]. In order to prove 2) and 3) we use the induction on n. When $n=2$, let C be a general member of $|L|$. Let F be the fixed part of $|L|_C$ and write $L_C=F+X$. If $\deg(F)>1$, then $\deg(X)\leq 3-\deg(F)\leq 1$ and $h^0(C, X)=\dim|L|_C+1=h^0(M, L)-1=2$. This implies $C\simeq \boldsymbol{P}^1$, contradicting $g(C)=g(M, L)\geq 2$. Hence $\deg(F)=1$. This implies 2) and 3).

When $n>2$, let D be a general member of $|L|$, let p be the unique base point of $|L_D|$ and let $\pi: M_1\to M$ be the blow-up at p. The proper transform D_1 of D in M_1 is a member of $\varLambda=|\pi^*L-E|$ and is isomorphic to the blow-up of D at p. The restriction mapping $H^0(M, L)\to H^0(D, L_D)$ is surjective since $\varDelta(D, L)\leq 1$ would imply $g(D, L)\leq 1$. Applying the induction hypothesis to (D, L), we infer that $\mathrm{Bs}|\varLambda|_{D_1}=\varnothing$. So $\mathrm{Bs}\varLambda=\varnothing$, which implies 2) too. Q.E.D.

(1.4) The linear system $\varLambda$ gives a morphism $\rho: M_1\to P\simeq\boldsymbol{P}^n$ of degree two. We will denote the pull-backs of $\mathcal{O}_P(1)$ by H. For example, $H=\pi^*L-E$ on M_1. We have $H_E=\mathcal{O}(1)$ on $E\simeq\boldsymbol{P}^{n-1}$ and hence $S=\rho(E)$ is a hyperplane in P. Write $\rho^*S=\varepsilon E+E^*+D$, where E^* is the sum of components mapped onto S and D is the sum of components mapped onto sets of codimension greater than one.

(1.5) LEMMA. *Let x be a point on P, $X=\rho^{-1}(x)$ and suppose that $\dim X>0$. Then X is an irreducible curve with $EX=1$ and $x\in S$. Moreover $X\subset E^*$ or $X\subset D$.*

PROOF. For $t\gg0$, $\pi^*tL-E=tH+(t-1)E$ is ample on M_1, hence so is the restriction E_X to X, since $H_X=0$. This implies $X\cap E\neq\emptyset$ and $x\in S$. Since $E\simeq S$, $X\cap E$ is a single point. So $\dim X=1$ by the ampleness of E_X. Let Λ_x be the linear subsystem of $|\mathcal{O}_P(1)|$ consisting of hyperplanes passing x and let $D_1,\cdots,D_{n-1}$ be general members of $\rho^*\Lambda_x$. Then $Z=D_1\cap\cdots\cap D_{n-1}$ contains X and $\dim Z=1$. If X is not irreducible, then $EZ\geqq2$ since each component of X meets E. But $EZ=H^{n-1}E=1$. Therefore X is irreducible and $EX=1$. Since $0=HX=(\varepsilon E+E^*+D)X$, we have $E^*X<0$ or $DX<0$. So $X\subset E^*$ or $X\subset D$.

(1.6) COROLLARY. *For any component Y of D, $\rho(Y)=\rho(E\cap Y)\simeq E\cap Y$ is a divisor on S.*

(1.7) PROPOSITION. *One of the following conditions is valid:*
1) *Both E^* and D are irreducible and reduced, $\varepsilon=1$ and $[E^*]_E=D_E=\mathcal{O}_E(1)$.*
2) *E^* is irreducible and reduced, $E\cap E^*=\emptyset$, $\varepsilon=1$ and $D_E=\mathcal{O}_E(2)$.*
3) *$E^*=0$, $\varepsilon=2$ and $D_E=\mathcal{O}_E(3)$.*

PROOF. Suppose that $E^*=0$. Then $2=H^n\{M_1\}=H^{n-1}\cdot\rho^*S=\varepsilon H^{n-1}E=\varepsilon$. Moreover $\mathcal{O}_E(1)=[\rho^*S]_E=[2E+D]_E=-2H+D_E$. So $D_E=\mathcal{O}_E(3)$ and we are in case 3).

Suppose that $E^*\neq0$. Then $H^{n-1}E^*>0$ and $2=H^{n-1}\cdot\rho^*S=\varepsilon+H^{n-1}E^*$. So $\varepsilon=1=H^{n-1}E^*$. Hence E^* is irreducible and reduced. If $E^*\cap E=\emptyset$, then $\mathcal{O}_E(1)=[\rho^*S]_E$ implies $D_E=\mathcal{O}_E(2)$ and we are in case 2).

If $E^*\cap E\neq\emptyset$, we have $E_E^*=\mathcal{O}_E(s)$ for some $s>0$. By (1.6), $Y_E\neq0$ for any component Y of D. $\mathcal{O}_E(1)=[\rho^*S]_E$ implies $D_E=\mathcal{O}_E(2-s)$. So $s=1$, D is prime and we are in case 1), unless $D=0$. We will derive a contradiction assuming $D=0$. By virtue of (1.3), we reduce the problem to the case $n=2$ by induction on n. In this case $D=0$ implies that ρ is a finite morphism. Therefore its branch locus B is a smooth divisor on P. We have $\rho^*S=E+E^*$ and $EE^*=2$. Hence $SB=4$ and the intersection multiplicity at any point of $S\cap B$ is even. In particular B is of degree four and the canonical bundle K_1 of M_1 is $\rho^*K^P+2H=-H$. So $KL=K_1\cdot\pi^*L=K_1(E+H)=-H(E+H)=-3$, while $2g(M,L)-2=(K+L)L=KL+3$. This contradicts $g(M,L)\geqq2$.

Thus we complete the proof of (1.7). The above three cases will be studied further in the following sections.

§2. Type $(-)$.

Throughout this section we assume that the condition (1.7; 3) is satisfied. We employ the same notation as in §1. Thus $\rho^*S=2E+D$ and $D_E=\mathcal{O}_E(3)$.

(2.1) For the moment, until (2.9), we study the case $n=2$. If D is irreducible and $D=\mu X$ for some prime divisor X, we have $EX=1$ by (1.5). So $\mu=3$ since $DE=3$. Then $0=HX=(2E+3X)X=2+3X^2$, which is absurd. Thus this case is ruled out. Hence $D=X_1+X_2+X_3$ or X_1+2X_2, where X_i's are curves as in (1.5) and $x_i=\rho(X_i)$ are different points on S.

(2.2) For the moment, until (2.6), we assume $D=X_1+X_2+X_3$. Then $0=HX_i=(2E+D)X_i=2+X_i^2$ and $(H-X_i)^2=0$ for any i. We claim $\mathrm{Bs}|H-X_i|=\varnothing$. To see this, let S_1 and S_2 be general hyperplanes in P passing x_i and let $\rho^*S_j=C_j+\mu_jX_i$. Then $0\leq C_1C_2=(H-\mu_1X_i)(H-\mu_2X_i)=2-2\mu_1\mu_2$. So $\mu_1=\mu_2=1$ and $C_1C_2=0$. Hence $C_j\in|H-X_i|$ and $C_1\cap C_2=\varnothing$. This proves the claim. Thus the scheme-theoretical fiber ρ^*x_i is the Cartier divisor X_i.

(2.3) Let P' be the blowing-up of P at the three points x_i and let Z_i be the exceptional curve over x_i. By the observation in (2.2) we infer that ρ factors through P'. Moreover, $f^*Z_i=X_i$ for the morphism $f:M_1\to P'$. Clearly $S'=f(E)$ is the proper transform of S on P' and $(S')^2=-2$. By (1.5), X_i's are the only curves contracted to a point by ρ. Since $f(X_i)=Z_i$, f contracts no curve and is a finite morphism of degree two. So the branch locus B of f is a smooth divisor on P' and $B\in|2F|$ for some $F\in\mathrm{Pic}(P')$. Since $\rho^*S'=\rho^*S-\sum X_i=2E$, S' is a component of B. Therefore $B=S'+B'$ for some smooth divisor B' with $S'\cap B'=\varnothing$.

Set $F=\beta H-\gamma_1Z_1-\gamma_2Z_2-\gamma_3Z_3$ for some integers β,γ_i. Then $[B']=2F-(H-Z_1-Z_2-Z_3)=(2\beta-1)H-\sum_i(2\gamma_i-1)Z_i$ and $0=S'B'=(2\beta-1)-\sum_i(2\gamma_i-1)$. So $\beta=\gamma_1+\gamma_2+\gamma_3-1$. Note also that $0\leq B'Z_i=2\gamma_i-1$ and hence $\gamma_i\geq1$.

(2.4) Let K_1 be the canonical bundle of M_1. Then $K_1=f^*(K'+F)$ for the canonical bundle K' of P'. Since $K'=-3H+Z_1+Z_2+Z_3$, we have $K_1=(\beta-3)H-\sum_i(\gamma_i-1)X_i$. Hence $K_1H=2\beta-6$, $K_1E=(\beta-3)-\sum_i(\gamma_i-1)=-1$, $KL=K_1\pi^*L=K_1(H+E)=2\beta-7$ and $2g(M,L)-2=(K+L)L=2\beta-4$. So $g(M,L)=\beta-1$ and $\beta=\gamma_1+\gamma_2+\gamma_3-1\geq3$.

(2.5) Conversely, for any three points x_i on a line S in $P\simeq P^2$ and for any positive integers $\gamma_1,\gamma_2,\gamma_3$ with $\gamma_1+\gamma_2+\gamma_3\geq4$, there are polarized surfaces (M,L) of the above type. Indeed, for $\beta=\gamma_1+\gamma_2+\gamma_3-1$, we have $\mathrm{Bs}|(2\beta-1)H-\sum_i(2\gamma_i-1)Z_i|=\varnothing$ on P'. So any general member B' of this linear system is smooth. Let M_1 be the double covering of P' branched along $B=B'+S'$. Since S' is a (-2)-curve, its inverse image E on M_1 is a (-1)-curve. Contracting E to a point we get M. Moreover $H+E$ is the pull-back of a line bundle L on M. It remains to show the ampleness of L. Since B meets Z_i at $Z_i\cap S'$ transversally, $X_i=f^{-1}(Z_i)$ is irreducible for each i. Therefore $(H+E)Y>0$ for any curve Y in M_1 except E. So Nakai's criterion applies since $L^2=(H+E)^2=3$.

(2.6) Using the above description of the structure of (M, L), we can compute various invariants of it.

Since $f_*\mathcal{O}_{M_1}=\mathcal{O}\oplus\mathcal{O}[-F]$ on P', we have $h^q(M, \mathcal{O}_M)=h^q(P', -F)$ for $q>0$. Note that $F-S'/2=\sum_i(\gamma_i-1/2)(H-Z_i)$ is nef and big on P'. Hence $h^q(P', -F)=0$ for $q<2$ by Kawamata-Viehweg's vanishing theorem. So $h^1(M, \mathcal{O}_M)=0$. It is easy to compute $p_g(M)=h^2(P', -F)=h^0(P', K'+F)$ and $\chi(M, \mathcal{O}_M)$.

In order to calculate $c_2(M)$, note that $[B']=\sum_i(2\gamma_i-1)(H-Z_i)$ is nef and big on P'. So B' is connected and $e(B')=-(K'+B')B'=-8(\gamma_2\gamma_3+\gamma_3\gamma_1+\gamma_1\gamma_2)+12(\gamma_1+\gamma_2+\gamma_3)-12$. Hence $c_2(M)=c_2(M_1)-1=2c_2(P')-e(B')-e(S')-1=8(\gamma_2\gamma_3+\gamma_3\gamma_1+\gamma_1\gamma_2)-12(\gamma_1+\gamma_2+\gamma_3)+21$. It is easy to compute $c_1(M)^2=K_1^2+1$.

These invariants are related by Noether's formula, so we can omit one of the above computations.

The Kodaira dimension $\kappa(M)$ depends on γ_i's. By symmetry we may assume $\gamma_1\leq\gamma_2\leq\gamma_3$. Since $K_1=-H+\sum_i(\gamma_i-1)(H-X_i)=H+2E+\sum_i(\gamma_i-2)(H-X_i)$, $\kappa(M)=2$ if $\gamma_1\geq2$. If $\gamma_1=\gamma_2=1$, then $K_1(H-X_3)=-2$ and $\kappa(M)<0$ since $\mathrm{Bs}|H-X_3|=\emptyset$. So we assume $1=\gamma_1<\gamma_2$. Then $K_1=(\gamma_2-2)(H-X_2)+(\gamma_3-2)(H-X_3)+2E+X_1$. Hence $\kappa(M)=2$ if $\gamma_2>2$. When $\gamma_2=2$, we easily see that $(\gamma_3-2)(H-X_3)$ is the semipositive part of the Zariski decomposition of K_1. Therefore $\kappa(M)=1$ if $\gamma_3>2$ while $\kappa(M)=0$ if $\gamma_3=2$. In the last case M is a blowing-up of a $K3$-surface at a point. Thus we have:

$\kappa(M)<0$ if $(\gamma_1, \gamma_2, \gamma_3)=(1, 1, \gamma_3)$,

$\kappa(M)=0$ if $(\gamma_1, \gamma_2, \gamma_3)=(1, 2, 2)$,

$\kappa(M)=1$ if $(\gamma_1, \gamma_2, \gamma_3)=(1, 2, \gamma_3)$ with $\gamma_3>2$,

$\kappa(M)=2$ otherwise.

Finally we show that M is topologically simply connected. To see this, let B_0 be a general member of $|3H-Z_1-Z_2-Z_3|$ and let $\tau: T\to P'$ be the finite double covering with branch locus B_0+S'. Similarly as in (2.5), $E_0=\tau^{-1}(S')$ is a (-1)-curve on T and we get a smooth surface T^- by contracting E_0 to a point. Moreover $E_0+\tau^*H$ is the pull-back of an ample line bundle L^- on T^- such that $(L^-)^2=3$. We easily see that the canonical bundle of T^- is $-L^-$ and T^- is a cubic surface. Since B_0 is general, it intersects B' normally, hence τ^*B' is a smooth divisor on T. This is mapped isomorphically onto an ample divisor B^- on T^- since $\tau^*B'\cap E_0=\emptyset$. Now, let V be the normalization of the fiber product of T and M_1 over P'. Then V is a double covering of T with branch locus τ^*B', hence is birational to the double covering V^- of T^- with branch locus B^-. We have $\pi_1(V^-)\simeq\pi_1(T^-)\simeq\{1\}$ by [F6]. So V is simply connected. On the other hand, $V\to M_1$ is a ramified double covering. So $\pi_1(V)\to\pi_1(M_1)$ is surjective and hence M is simply connected.

(2.7) Now we study the case $D=X_1+2X_2$ until (2.9). We have $X_1^2=-2$ and

$\mathrm{Bs}|H-X_1|=\emptyset$ similarly as in (2.2), while $X_2{}^2=-1$ since $0=X_2\cdot\rho^*S=(2E+D)X_2$ $=2+2X_2{}^2$. Let S_1 and S_2 be general hyperplanes in P passing x_2 and let $\rho^*S_j=$ $S_j{}'+\mu_jX_2$. Since $E\cdot\rho^*S_j=1$, we have $\mu_j=1$ and $S_j{}'\cap E=\emptyset$. Then $0=X_2\cdot\rho^*S_j$ $=X_2S_j{}'-1$, so $X_2\cap S_j{}'=y_j$ is a single point. Moreover $y_1=y_2$ since $S_1{}'S_2{}'=$ $(H-X_2)^2=1$ and $S_1{}'\cap S_2{}'\subset\rho^{-1}(S_1\cap S_2)=X_2$. This point $y=y_1=y_2$ is the unique base point of $|H-X_2|$. Note that $y\notin E$.

(2.8) Let M_2 be the blowing-up of M_1 at y and let Y be the exceptional curve over y. Let $X_2{}''$ be the proper transform of X_2, while the proper transform of E is denoted by E by abuse of notation since $y\notin E$. The intersection of X_2 and $S_j{}'$ at y is transverse since $X_2S_j{}'=1$. In particular X_2 is smooth at y. So $X_2{}''=X_2-Y$ on M_2 and $S_j{}''=S_j{}'-Y$ for the proper transform $S_j{}''$ of $S_j{}'$ on M_2. Then $S_1{}''S_2{}''=(S_1{}'-Y)(S_2{}'-Y)=0$ and hence $S_1{}''\cap S_2{}''=\emptyset$. Since $[S_j]_{M_2}$ $=S_j{}''+X_2{}''+2Y$, the scheme-theoretical fiber of $M_2\to P$ over x_2 is the Cartier divisor $X_2{}''+2Y$.

Let P' be the blowing-up of P at x_1, x_2 and let Z_i be the (-1)-curve over x_i. Similarly as in (2.3), we have a morphism $\rho_2\colon M_2\to P'$ such that $\rho_2{}^*Z_1=X_1$, $\rho_2{}^*Z_2=X_2{}''+2Y$ and $\rho_2(E)$ is the proper transform S' of S on P'. Since $(X_2{}'')^2$ $=-2$ and $X_2{}''Y=1$, we have $Z_2X_2{}''=0$ and $Z_2Y=-1$. Hence $x_3=\rho_2(X_2{}'')$ is a point while $\rho_2(Y)=Z_2$. Clearly $x_3\in Z_2$ and $x_3\in S'$ since $X_2{}''\cap E\neq\emptyset$. So x_3 is the point $Z_2\cap S'$. Since $\rho_2{}^*Z_2=X_2{}''+2Y$, $\rho_2{}^*S'=(2E+D)-\rho_2{}^*Z_1-\rho_2{}^*Z_2=$ $2E+X_2{}''$ and $E\cap Y=\emptyset$, the scheme-theoretical fiber of ρ_2 over x_3 is the Cartier divisor $X_2{}''$.

Let P'' be the blowing-up of P' at x_3 and let Z_3 be the (-1)-curve over x_3. Then we have a morphism $f\colon M_2\to P''$ such that $f^*Z_3=X_2{}''$. We easily see that f is a finite morphism. The branch locus B of f is a smooth member of $|2F|$ for some line bundle F on P''.

Let S'' and $Z_2{}''$ be the proper transforms of S' and Z_2 on P'' respectively. Then $f^*S''=2E$ and $f^*Z_2{}''=2Y$. Hence S'' and $Z_2{}''$ are components of B. So $B=S''+Z_2{}''+B''$ for some smooth divisor B'' such that $B''\cap S''=B''\cap Z_2{}''=\emptyset$.

Set $F=\beta H-\gamma_1Z_1-\gamma_2Z_2-\gamma_3Z_3$ for some integers β, γ_1, γ_2 and γ_3. Then $[B'']$ $=(2\beta-1)H-(2\gamma_1-1)Z_1-2\gamma_2Z_2-(2\gamma_3-2)Z_3$, $0=B''Z_2{}''=2\gamma_2-2\gamma_3+2$ and $0=B''S''=$ $(2\beta-1)-2(\gamma_1+\gamma_2+\gamma_3)+3$. So $\gamma_3=\gamma_2+1$ and $\beta=\gamma_1+2\gamma_2$. Moreover $B''Z_1\geq 0$ and $B''Z_3\geq 0$ imply $\gamma_1\geq 1$ and $\gamma_2\geq 0$.

Let K_2 and K'' be the canonical bundles of M_2 and P'' respectively. Then $K_2=f^*(K''+F)=(\beta-3)H-(\gamma_1-1)X_1-(\gamma_2-1)(X_2{}''+2Y)-(\gamma_3-1)X_2{}''$. So $K_2H=$ $2\beta-6$ and $K_2E=K_1E=-1$. Hence $KL=K_2(H+E)=2\beta-7$ and $g(M,L)=\beta-1$. Therefore $\beta\geq 3$.

Conversely, similarly as in (2.5), for any two points x_1, x_2 on a line S in P and for any integers γ_1, γ_2 with $\gamma_1\geq 1$, $\gamma_2\geq 0$ and $\beta=\gamma_1+2\gamma_2\geq 3$, we can construct examples of polarized surfaces of the above type. Details are left to the

reader.

(2.9) We can compute numerical invariants of (M, L) similarly as in (2.6).

When $\gamma_2 > 0$, we have essentially the same formulae as in (2.6). For example $c_2(M) = 2c_2(P'') - e(S'') - e(Z_2'') - e(B'') - 2 = 16\gamma_1\gamma_2 + 8\gamma_2^2 - 4\gamma_1 - 16\gamma_2 + 8$ and M is simply connected. To see this, let B_0 be a general member of $|H - Z_1|$ on P'' and let $\tau : T \to P''$ be the finite double covering with branch locus $B_0 + S'' + Z_2''$. Then $\tau^{-1}(S'')$ and $\tau^{-1}(Z_2'')$ are (-1)-curves and we get a P^1-bundle T^- over P^1 by contracting them to smooth points. In fact T^- is Σ_1 since $\tau^{-1}(Z_1)$ is a (-2)-curve meeting $\tau^{-1}(S'')$. Moreover, τ^*B'' is smooth and is mapped isomorphically onto an ample divisor B^- on T^-. The double covering V^- of T^- branched along B^- is simply connected by [F6], and is birational to the normalization V of the fiber product of T and M_2 over P''. Hence V and M are simply connected since $V \to M_2$ is a ramified double covering.

However, when $\gamma_2 = 0$, the situation is very different. In fact $B'' \in |(2\beta - 1)(H - Z_1)|$ and B'' consists of proper transforms of $(2\beta - 1)$ lines in P passing x_1. The linear system $|H - X_1|$ gives a P^1-fibration of M_2 over a hyperelliptic curve C of genus $\beta - 1$, having the unique singular fiber $E + X_2'' + Y$. Hence (M, L) is a scroll over C and $h^1(M, \mathcal{O}_M) = \beta - 1 = g(M, L)$ and $c_2(M) = 8 - 4\beta$. It is easy to see that M is obtained from $C \times P^1$ by a so-called elementary transformation.

As for the Kodaira dimension, we have $\kappa(M) < 0$ if $\gamma_2 = 0$. We have $K_2 = (\gamma_1 - 1)(H - X_1) + (2\gamma_2 - 3)(H - X_2) + 2E + X_1 + Y$, $\mathrm{Bs}|H - X_1| = \varnothing$ and $H - X_2$ is nef and big. So $\kappa(M) = 2$ if $\gamma_2 > 1$. If $\gamma_2 = 1$, then $K_2 = (\gamma_1 - 1)(H - X_1) - X_2''$ and $2K_2 = (2\gamma_1 - 3)(H - X_1) + 2E + 2Y$. So $\kappa(M) < 0$ if $\gamma_1 = 1$ and $\kappa(M) = 1$ if $\gamma_1 > 1$.

(2.10) Now we study the case $n \geq 3$.

LEMMA. $D \cap E$ *is an irreducible hypersurface of degree three in* $E \simeq P^{n-1}$.

PROOF. Let W_1 be a general member of $|H|$. Then $W = \pi(W_1)$ is a smooth member of $|L|$ and W_1 is the blowing-up of W at the unique base point of $|L_W|$. Thus, cutting several times successively, we reduce the problem to the case $n = 3$.

If we cut once more, then we are in the situation (2.1). So we infer that any general line in $E \simeq P^2$ meets D at two or three points. In particular $\mathrm{Supp}(D \cap E)$ is not a line.

Let $D = \sum \mu_\lambda D_\lambda$ be the prime decomposition of D and let $C_\lambda = \rho(D_\lambda)$. Then, by (1.5), $D_\lambda \cap E \simeq C_\lambda$ and $D_\lambda = \rho^{-1}(C_\lambda)$. Since $D_E = \mathcal{O}_E(3)$ and $D \cap E$ is not a line, our assertion is valid unless $\mu_\lambda = 1$ and C_λ is a line in $S \simeq P^2$ for some λ. Assuming this, we will derive a contradiction.

Let T be a general plane in $P \simeq P^3$ such that $T \cap S = C_\lambda$. Since $\rho^{-1}(C_\lambda) = D_\lambda$,

we have $\rho^*T = \nu D_\lambda + T'$ in $\mathrm{Div}(M_1)$ for some effective divisor T' containing no component of D. Restricting this relation to E, we infer $\nu = 1$ and $T' \cap E = \varnothing$. Take another component D_α of D. Then $C_\alpha \cap T \neq \varnothing$. So $D_\alpha \cap T' \neq \varnothing$ since $\rho(T') = T$ and $D_\alpha = \rho^{-1}(C_\alpha)$. Clearly $\rho(D_\alpha \cap T') \subset C_\alpha \cap T$ and $C_\alpha \cap T$ is a finite set. On the other hand $\dim(D_\alpha \cap T') \geq 1$. Hence $D_\alpha \cap T'$ contains a curve X of the type (1.5). So $T' \cap E \supset X \cap E \neq \varnothing$, contradicting $T' \cap E = \varnothing$.

(2.11) $C = \rho(D)$ is a hypercubic in $S \simeq P^{n-1}$ and $D \cap E \simeq C$. Let T be a general hypercubic in P such that $T \cap S = C$. Then $\rho^*T = \nu D + T'$ for some effective divisor T' on M_1. Restricting to E we infer $\nu = 1$ and $T' \cap E = \varnothing$. Since $\rho^*S = 2E + D$, we have $\rho^*C = \rho^*T \cap \rho^*S = D$ in the scheme theoretical sense. Hence we get a morphism $f : M_1 \to P'$ onto the blowing-up P' of P along C such that $f^*Z = D$, where Z is the exceptional divisor. Since $DX = -2EX = -2$ for any fiber X of ρ of the type (1.5), $f(X)$ is not a point and hence f is a finite morphism.

(2.12) Let y be a point on the singular locus Σ of P' and let $p : P' \to P$ be the projection. Clearly $x = p(y)$ is a singular point of C. Take an affine coordinate $(z_1, \cdots, z_n)$ of a neighborhood U of x in P such that x is the origin $(0, \cdots, 0)$ and S is the divisor $z_1 = 0$. Then C is defined by $z_1 = \phi(z_2, \cdots, z_n) = 0$ for some polynomial ϕ of degree three. So $p^{-1}(U) \simeq \{(z_1, \cdots, z_n, (\xi_0 : \xi_1)) \in U \times P^1 | z_1 : \phi(z) = \xi_0 : \xi_1\}$. By the Jacobian criterion we see that y must be the point $(0, \cdots, 0, (1 : 0))$. By this observation we infer $\Sigma \simeq p(\Sigma)$ and $p(\Sigma)$ is the singular locus of C. In particular $\dim \Sigma \leq n - 3$. So P' is normal by Serre's criterion, since it has only hypersurface singularities.

(2.13) Now we apply the theory in [**F1**; §2]. There is a holomorphic involution ι of M_1 such that $M_1/\iota \simeq P'$. Any hypersurface singularity of such quotient P' must be of pure dimension $n - 2$. Hence P' is non-singular by (2.12). This in turn implies that C is smooth. Furthermore the branch locus B of f is a smooth member of $|2F|$ for some $F \in \mathrm{Pic}(P')$.

(2.14) $S' = f(E)$ is the proper transform of S in P' and is a component of B. Set $B = B' + S'$ and $F = \beta H - \gamma Z$. Then $[B'] = (2\beta - 1)H - (2\gamma - 1)Z$. Since $B' \cap S' = \varnothing$, we obtain $2\beta - 1 = 3(2\gamma - 1)$ and $\beta = 3\gamma - 1$. Unlike the case $n = 2$, B' is always connected since $3H - Z$ is nef and big on P'.

(2.15) Conversely, for any smooth hypercubic C in a hyperplane S of $P \simeq P^n$ and for any smooth member B' of $|(2\gamma - 1)(3H - Z)|$ on P', we can construct a polarized manifold (M, L) of the above type. Details are left to the reader.

(2.16) We can calculate various invariants of (M, L) similarly as in the case $n = 2$. For example, if K_1 and K' are the canonical bundles of M_1 and

P', then $K_1 = f^*(K'+F) = (3\gamma-n-2)H-(\gamma-1)D$. So $(K+(n-1)L)L^{n-1} = (K+(n-1)L)(E+H)^{n-1}=(K_1+(n-1)H)H^{n-1}=6\gamma-6$. Hence $g(M, L)=3\gamma-2$, so $\gamma\geq2$. Note also that we get a polarized surface of the type (2.2) if we take general members of $|L|$ $(n-2)$-times successively. Therefore M is simply connected by Lefschetz theorem.

(2.17) SUMMARY OF RESULTS. Let things be as in §1 and suppose further that $\rho^*S=2E+D$ for some effective divisor D such that $\dim(\rho(D))<n-1$.

If $n\geq3$, then $C=\rho(D)$ is a smooth hypercubic in S and ρ gives a finite double covering $f: M_1\to P'$ onto the blowing-up P' of P along C. Let Z be the exceptional divisor over C and let S' be the proper transform of S on P'. Then $S'=f(E)$ and the branch locus B of f is of the form $S'+B'$, where B' is a smooth connected member of $|(2\gamma-1)(3H-Z)|$ such that $B'\cap S'=\emptyset$. γ is an integer such that $\gamma\geq2$ and $g(M, L)=3\gamma-2$. The numerical invariants of M are explicitly computable.

If $n=2$, then $C=\rho(D)$ is two or three points on the line S in P. If C is three points x_1, x_2, x_3, then M_1 is a finite double covering of the blowing-up P' of P at these points. Let Z_i be the (-1)-curve over x_i and let S' be the proper transform of S on P'. Then $f(E)=S'$ and the branch locus B of f is of the form $S'+B'$, where B' is a smooth connected member of $|\sum_i(2\gamma_i-1)(H-Z_i)|$. Here γ_i's are positive integers such that $\gamma_1+\gamma_2+\gamma_3>3$ and $g(M, L)=\gamma_1+\gamma_2+\gamma_3-2$.

The case in which C is two points can be viewed usually as a degeneration of the above case, where two of the three points x_i are infinitely near (for details, see (2.7) and (2.8)). But there is another exceptional possibility in which the branch locus of ρ consists of several lines passing a point (see the case $\gamma_2=0$ in (2.9)).

M is simply connected except the final case, in which (M, L) is a scroll over a hyperelliptic curve.

§3. Type (∞).

Throughout this section we assume the condition (1.7; 2). Thus $\rho^*S= E+E^*+D$ and $E\cap E^*=\emptyset$.

(3.1) Since $E\cap E^*=\emptyset$, any fiber X of the type (1.5) is not contained in E^*. Therefore $E^*\simeq S$ by Zariski's Main Theorem.

(3.2) For the moment, until (3.8), we study the case $n=2$. Since $DE=2$, we have $D=X_1+X_2$ or $2X_1$, where X_i's are curves of the type (1.5).

(3.3) From now on, until (3.7), we assume $D=X_1+X_2$ and $x_i=\rho(X_i)$ are different points. Let S_α be a general line passing x_i and set $\rho^*S_\alpha=\nu X_i+C_\alpha$.

Then $\nu=1$ and $X_iE^*=1$ since $1=E^*(\nu X_i+C_\alpha)$. Therefore $0=HX_i=(E+E^*+X_1+X_2)X_i=2+X_i^2$. Hence, similarly as in (2.2), $(H-X_i)^2=0$ and $\mathrm{Bs}|H-X_i|=\varnothing$ for each i.

(3.4) Let P' be the blowing-up of P at x_1 and x_2 and let Z_i be the (-1)-curve over x_i. Then, similarly as in (2.3), we have a finite double covering $f: M\to P'$ such that $f^*Z_i=X_i$. The branch locus B of f is a smooth member of $|2F|$ for some $F\in\mathrm{Pic}(P')$. For the proper transform S' of S, we have $f(E)=S'$, $(S')^2=-1$ and $f^*S'=E+E^*$. Note that $B\cap S'=\varnothing$ since $E\cap E^*=\varnothing$. So E^* is mapped onto a (-1)-curve on M, which is sometimes denoted by E^* by abuse of notation.

(3.5) Set $F=\beta H-\gamma_1 Z_1-\gamma_2 Z_2$ for some integers $\beta, \gamma_1, \gamma_2$. Then $\beta=\gamma_1+\gamma_2$ since $0=BS'=2F(H-Z_1-Z_2)$. We have $2\gamma_i=BZ_i>0$ since $X_i=f^*Z_i$ is irreducible. So F is nef and big, and hence B is connected.

Let K_1 and K' be the canonical bundles of M_1 and P' respectively. Then $K_1=f^*(K'+F)=(\beta-3)H-(\gamma_1-1)X_1-(\gamma_2-1)X_2$. Therefore $(K+L)L=(K+L)H=(K_1+H)H=2\beta-4$. So $g(M, L)=\beta-1=\gamma_1+\gamma_2-1$. Thus γ_i's are positive integers with $\gamma_1+\gamma_2>2$.

S' is a (-1)-curve on P' and we get $\boldsymbol{P}^1\times\boldsymbol{P}^1$ by contracting S' to a point. Then B is mapped onto a divisor of bidegree $(2\gamma_1, 2\gamma_2)$, which is denoted by B^-. Moreover, M is the blowing-up of the double covering M^- of $\boldsymbol{P}^1\times\boldsymbol{P}^1$ branched along B^-, and the exceptional divisor is identified with E^*.

(3.6) Conversely, for any integers γ_1, γ_2 as above and any smooth member B of $|2F|$ with $F=\gamma_1(H-Z_1)+\gamma_2(H-Z_2)$, we can construct a polarized surface (M, L) of the above type, provided that there exists a point z_i on each Z_i such that the intersection multiplicity of B and Z_i at z_i is odd. This last condition is necessary in order that $f^{-1}(Z_i)$ is irreducible. Any general member of $|2F|$ satisfies actually this condition since $\mathrm{Bs}|2F|=\varnothing$.

Alternately, we get M by blowing-up at a point on a double covering M^- of $\boldsymbol{P}^1\times\boldsymbol{P}^1$ branched along a smooth divisor of bidegree $(2\gamma_1, 2\gamma_2)$. In this case $L=A-E^*$, where A is the pull-back of $\mathcal{O}(1, 1)$ on $\boldsymbol{P}^1\times\boldsymbol{P}^1$ and E^* is the exceptional curve of the blowing-up $M\to M^-$.

(3.7) Similarly as in (2.6), we can compute various numerical invariants of (M, L). For example $c_2(M)=8\gamma_1\gamma_2-4\gamma_1-4\gamma_2+9$. Moreover M is simply connected since it is birational to a double covering of $\boldsymbol{P}^1\times\boldsymbol{P}^1$. As for the Kodaira dimension, we have:

$\kappa(M)<0$ if $\gamma_1=1$ or $\gamma_2=1$,

$\kappa(M)=0$ if $\gamma_1=\gamma_2=2$,

$\kappa(M)=1$ if $2=\gamma_1<\gamma_2$ or $2=\gamma_2<\gamma_1$,

$\kappa(M) = 2$ if $\gamma_1 > 2$ and $\gamma_2 > 2$.

(3.8) We now study the case $D=2X$ for a curve X of the type (1.5). Similarly as in (3.3), we have $XE^* = 1$ and $X^2 = -1$. So, by the method in (2.7), we infer that $\mathrm{Bs}\,|H-X|$ is a single point y on X off E and E^*. Thus we are in a situation as in (2.8). To be precise, let M_2 be the blowing-up of M_1 at y, let Y be the (-1)-curve over y and let X'' be the proper transform of X on M_2, while the pull-backs of E and E^* are denoted by E and E^* by abuse of notation. Let P' be the blowing-up of P at x, let Z_1 be the (-1)-curve over x and let S' be the proper transform of S. Let x_2 be the point $Z_1 \cap S'$ and let P'' be the blowing-up of P' at x_2. Let Z_2 be the (-1)-curve over x_2 and let S'' and Z_1'' be the proper transforms of S' and Z_1 on P'' respectively. Then ρ gives a finite double covering $f: M_2 \to P''$ such that $f^*Z_2 = X''$, $f^*Z_1'' = 2Y$ and $f^*S'' = E+E^*$. The branch locus B of f is a member of $|2F|$ for some $F \in \mathrm{Pic}(P'')$ and $B = Z_1'' + B''$ for some smooth divisor B'' with $B'' \cap Z_1'' = B'' \cap S'' = \varnothing$.

Set $F = \beta H - \gamma_1 Z_1 - \gamma_2 Z_2 = \beta H - \gamma_1 Z_1'' - (\gamma_1 + \gamma_2) Z_2$. Since $B''Z_1'' = B''S'' = 0$, we have $\gamma_2 = \gamma_1 + 1$ and $\beta = 2\gamma_1 + 1$. Let K_2 and K'' be the canonical bundles of M_2 and P'' respectively. Then $K_2 = f^*(K'' + F) = (\beta-3)H - (\gamma_1-1)(X''+2Y) - (\gamma_2-1)X''$, $(K+L)L = (K_2+H-Y)H = 2(\beta-2)$ and hence $g(M, L) = 2\gamma_1$. So γ_1 is a positive integer. We easily see that B'' is connected.

If we contract the (-1)-curve S'' on P'', we get Σ_2 this time. The (-2)-curve on it is Z_1'' and we get singular conic C by contraction of it. B'' is mapped isomorphically onto a divisor B^- on C. On the other hand, we get a surface M^- by contracting the (-1)-curve on M which is identified with E^*. There exists a natural morphism $f^-: M^- \to C$, which is a finite morphism of degree two branched along B^-. Thus, for some appropriate polarization, M^- is of the type (*II) in [F1]. So $\kappa(M) = 2$ and M is simply connected by [F1]. Other numerical invariants are easily calculated.

Conversely, for any $\gamma_1 > 0$ and any smooth member B'' of $|(2\gamma_1+1)(2H-Z_1''-2Z_2)|$, we can construct polarized surface of the above type. Details are left to the reader.

(3.9) Now we study the case $n \geq 3$. We first prove the following

LEMMA. $C = \rho(D)$ *is of degree two in* $S \simeq P^{n-1}$.

PROOF. Similarly as in (2.10), we reduce the problem to the case $n=3$. Since $D \cap E \simeq C$ and $D_E = \mathcal{O}_E(2)$, it suffices to derive a contradiction assuming that C is a line in $S \simeq P^2$.

Set $X = \rho^{-1}(C) = D_{\mathrm{red}}$ and let S_1, S_2 be general planes in P containing C. Then $\rho^*S_j = \nu_j X + T_j$ for some effective divisor T_j. Restricting to E we infer

that $\nu_j=1$, $X_E=\mathcal{O}_E(1)$ and $T_j\cap E=\varnothing$. Similarly we have $T_j\cap E^*=\varnothing$. Thus $l=T_1\cap T_2$ does not meet E and $l\subset\rho^{-1}(S_1\cap S_2)=X$. So the restriction $\rho_l:l\rightarrow C$ is a finite morphism. Moreover this is birational since we are in a situation (3.8) if we cut by a general member of $|H|$. Hence ρ_l is an isomorphism.

Let M_2 be the blowing-up of M_1 along l, let E_l be the exceptional divisor over l and let X'', T_j'' be the proper transforms of X, T_j respectively. Let $\rho_2:M_2\rightarrow P$ be the induced morphism. Then $\rho_2{}^*S=E+E^*+2X''+2E_l$ and $\rho_2{}^*S_j=X''+2E_l+T_j''$. The scheme-theoretical intersection $T_1\cap T_2$ is locally Macaulay and is smooth at general points. Hence it is l in the strong sense, which implies $T_1''\cap T_2''=\varnothing$. Thus $\rho_2{}^*C=X''+2E_l$ in the scheme-theoretical sense. So we get a morphism $\rho':M_2\rightarrow P'$ onto the blowing-up P' of P along C such that $(\rho')^*Z_1=X''+2E_l$ for the exceptional divisor Z_1 over C. Moreover $\rho'(E)$ is the proper transform S' of S on P' and $(\rho')^*S'=E+E^*+X''$. For $C_2=Z_1\cap S'$, we have $(\rho')^{-1}(C_2)=X''$ in the scheme-theoretical sense. Hence we have a morphism $f:M_2\rightarrow P''$ onto the blowing-up P'' of P' along C_2 such that $f^*Z_2=X''$, where Z_2 is the exceptional divisor over C_2. In view of (1.5), we infer that f is a finite double covering.

Since $f^*Z_1''=2E_l$ for the proper transform Z_1'' of Z_1 on P_2, Z_1'' is a component of the branch locus B of f, which is a smooth member of $|2F|$ for some $F\in\mathrm{Pic}(P'')$. We have also $f^*S''=E+E^*$ for the proper transform S'' of S' on P''. So $B=Z_1''+B''$ and $B''\cap Z_1''=B''\cap S''=\varnothing$.

Set $B''=bH-b_1Z_1-b_2Z_2=bH-b_1Z_1''-(b_1+b_2)Z_2$ in $\mathrm{Pic}(P'')$. Note that $Z_1''\simeq Z_1\simeq C\times\boldsymbol{P}_\sigma^1$ and $[-Z_1]_{Z_1}=H_\sigma-H$, where H_σ is the pull-back of $\mathcal{O}(1)$ of $\boldsymbol{P}_\sigma^1$. So, in $\mathrm{Pic}(Z_1'')$, we have $Z_2=H_\sigma$ and $Z_1''=H-2H_\sigma$. Hence $B''\cap Z_1''=\varnothing$ implies $b=b_1=b_2$. On the other hand, $S''\simeq S\simeq\boldsymbol{P}^2$ and $Z_1''\cap S''=\varnothing$, $[Z_2]_{S''}=\mathcal{O}(1)$. So $B''\cap S''=\varnothing$ implies $b=b_1+b_2$. Combining them we get $b=b_1=b_2=0$, which is clearly absurd. Thus we complete the proof of the lemma.

(3.10) Lemma. *C is irreducible.*

Indeed, if C is a union of two hyperplanes, then we can derive a contradiction as in (2.10).

(3.11) Similarly as in (2.11), we get a finite morphism $f:M_1\rightarrow P'$ onto the blowing-up P' of P along C such that $f^*Z=D$ for the exceptional divisor Z over C. By the method in (2.12) and (2.13), we infer that C is smooth and the branch locus B of f is a smooth member of $|2F|$ for some $F\in\mathrm{Pic}(P')$. For the proper transform S' of S, we have $f^*S'=E+E^*$. Hence $S'\cap B=\varnothing$.

(3.12) By (3.11) we may set $F=\gamma(2H-Z)$ for some integer γ. Let K_1 and K' be canonical bundles of M_1 and P' respectively. Then $K_1=f^*(K'+F)=(2\gamma-n-1)H-(\gamma-1)D$, $(K+(n-1)L)L^{n-1}=(K+(n-1)L)H^{n-1}=(K_1+(n-1)H)H^{n-1}$

$=4\gamma-4$ and $g(M, L)=2\gamma-1$. Hence $\gamma\geq2$.

The normal bundle of $S'\simeq P^{n-1}$ in P' is $\mathcal{O}(-1)$. Hence it is contracted to a smooth point on another manifold Q. In fact, it is easy to see and well known that Q is a hyperquadric and $\mathcal{O}_Q(1)=2H-Z$ in $\mathrm{Pic}(P')$. The image of B in Q is a member of $|\mathcal{O}_Q(2\gamma)|$. Let $M^-\to Q$ be the double covering branched along this divisor. Then M is the blowing-up of M^- at a point, and the exceptional divisor is identified with E^* on M_1. Since M^- is of the type (II) in [**F1**], we can easily calculate various numerical invariants of (M, L). For example M is simply connected.

(3.13) Since every fiber of $D\to C$ is irreducible, every fiber of $Z\to C$ meets B at some point with odd multiplicity. This implies $n\leq\gamma+1$.

To see this, it suffices to derive a contradiction assuming $n=\gamma+2$. We use the method in [**F2**; $(17.6)\sim(17.11)$]. The normal bundle $\mathfrak{N}$ of C in P is $H_\alpha\oplus2H_\alpha$, where H_α is the restriction of $H=\mathcal{O}_P(1)$ to C. Hence $Z\simeq P_C(\mathfrak{N}^\vee)\simeq P_C(\mathcal{E})$ for $\mathcal{E}=H_\alpha\oplus\mathcal{O}_C$. Let $H_\zeta=H(\mathcal{E})$ be the tautological line bundle on $P_C(\mathcal{E})$. Then the normal bundle of Z is $2H_\alpha-H_\zeta$. Therefore the restriction B_Z is a member of $2\gamma H_\zeta$. The corresponding section in $H^0(Z, 2\gamma H_\zeta)\simeq H^0(C, S^{2\gamma}\mathcal{E})$ does not vanish at any point on C. So this gives a trivial subbundle of $S^{2\gamma}\mathcal{E}$, and a section b of $V=P(S^{2\gamma}\mathcal{E}^\vee)\to C$.

Let $\mu: G=P(S^\gamma\mathcal{E}^\vee)\to V$ be the map defined by square. If $b(x)\in\mathrm{Im}(\mu)$ for some $x\in C$, then the fiber of $Z\to C$ over x meets B with even multiplicity at every point. This should be ruled out. So, in order to derive a contradiction, we will show $I>0$ for the intersection number I of $b(C)$ and $\mu(G)$ in V. Note that the dimensions are right and I is well-defined.

Let H_σ and H_τ be the tautological line bundles on V and G respectively. Similarly as in [**F2**; (17.9)], the class of $b(C)$ in the Chow ring of V is $\prod_{j=1}^{2\gamma}(H_\sigma+jH_\alpha)$. Since $\mu^*H_\sigma=2H_\tau$, we have $I=\prod_{j=1}^{2\gamma}(2H_\tau+jH_\alpha)$ in $\mathrm{Chow}(G)$. Similarly as in [**F2**; (17.11)], we obtain $I=2^{\gamma+1}\prod_{t=1}^{\gamma}(2t-1)>0$ using elementary formulas. Thus we conclude $n\leq\gamma+1$.

(3.14) Conversely, for any integer γ with $\gamma\geq n-1$, we can construct examples of polarized manifolds of the above type by taking a general member B of $|2\gamma(2H-Z)|$. Details are left to the reader.

(3.15) SUMMARY OF RESULTS. Let things be as in (1.7; 2). Then M is the blowing-up at a point of a finite double covering M^- of a (possibly singular) hyperquadric Q. The pull-back of $\mathcal{O}_Q(1)$ to M is $L+E^*$, where E^* is the exceptional divisor.

If $n\geq3$, then Q is smooth and the branch locus B of $M^-\to Q$ is a smooth member of $|\mathcal{O}_Q(2\gamma)|$ for some integer γ such that $g(M, L)=2\gamma-1$ and $\gamma\geq n-1$.

If $n=2$ and $Q\simeq P^1\times P^1$, then B is a smooth divisor of bidegree (γ_1, γ_2) and

$g(M, L)=\gamma_1+\gamma_2-1$.

If Q is a singular quadric surface, then B is a member of $|\mathcal{O}_Q(2\gamma+1)|$ and $g(M, L)=2\gamma$ for some positive integer γ. The point over the singular point of Q is an isolated fixed point of the sheet-changing involution of M^-.

§4. Type (+).

Throughout this section we assume the condition (1.7; 1). Since $D_E=\mathcal{O}(1)$, $C=\rho(D)$ is a hyperplane in S.

(4.1) When $n=2$, we have $EE^*=DE=DE^*=1$ and hence $D^2=-2$. So $\mathrm{Bs}|H-D|=\varnothing$ and we have a morphism $f: M_1\to P'$ onto the blowing-up P' of P at C such that $f^*Z=D$ for the exceptional curve Z over C. Then M_1 is a finite double covering of P' and the branch locus B of f is a smooth member of $|2F|$ for some line bundle F on P'. We have $f^*S'=E+E^*$ for the proper transform S' of S. So B meets S' at the point $f(E\cap E^*)$ with multiplicity 2. Hence, if we set $F=\beta H-\gamma Z$, we have $1=FS'=\beta-\gamma$. Let K_1 and K' be the canonical bundles of M_1 and P' respectively. Then $K_1=f^*(K'+F)=(\gamma-2)H-(\gamma-1)D$, $(K+L)L=(K_1+H)H=2\gamma-2$ and $g(M, L)=\gamma\geqq2$. So F is ample and B is connected.

Conversely, for any $\gamma\geqq2$, we can construct examples of polarized surfaces of the above type in the following way. Take a smooth member B of $|2F|$ as above. $P'\simeq\Sigma_1$ is a P^1-bundle over P^1_ξ and the map $B\ (\subset P')\to P^1_\xi$ is of degree two. Take a point on the branch locus of this map and let S' be the fiber of $P'\to P^1_\xi$ over this point. The pull-back of S' on the finite double covering of P' branched along B is a sum of two (-1)-curves. Contracting one of them to a smooth point, we get a polarized surface (M, L) of the desired type.

As for numerical invariants, we first note that M is a rational surface. Indeed, $|H-D|$ gives the natural mapping $M_1\to P^1_\xi$, and any general fiber of it is a double covering of P^1 branched at two points. So this is a P^1-fibration. Its fiber is singular exactly where $B\to P^1_\xi$ is ramified. Easy computation gives $g(B)=2\gamma$ and $c_2(M)=4\gamma+5$.

(4.2) From now on, we assume $n\geqq3$. Let S_α be a general hyperplane in P such that $S\cap S_\alpha=C$ and let $\rho^*S_\alpha=\nu D+T$ for some effective divisor T. Restricting to E we infer $\nu=1$ and $T\cap E=\varnothing$. So T contains no fiber of the type (1.5). Hence $\rho_T: T\to S_\alpha$ is a finite morphism. On the other hand, $E^*\to S$ is a birational morphism and $D\cap E^*$ is mapped onto C. So, restricting $\rho^*S_\alpha=D+T$ to E^*, we infer that $\dim\rho(T\cap E^*)<n-2$. Hence $\dim(T\cap E^*)<n-2$. This implies $T\cap E^*=\varnothing$ since T and E^* are Cartier divisors. Therefore $\rho^{-1}(C)=\rho^{-1}(S\cap S_\alpha)=\rho^*S\cap\rho^*S_\alpha=D$ in the ideal-theoretical sense. Thus we obtain a

morphism $\rho_1 \colon M_1 \to P'$ onto the blowing-up P' of P along C such that $\rho_1{}^*Z_1=D$, where Z_1 is the exceptional divisor over C. However, unlike the cases in the preceding sections, ρ_1 is not always finite. If there is a fiber X of the type (1.5) such that $X \subset E^*$ and $X \cap D = \emptyset$, then $\rho_1(X)$ is a point. In fact, any fiber of ρ_1 of positive dimension must be of this type.

(4.3) For the moment, until (4.13), we study the case $n=3$.

The scheme theoretical intersection $l = E \cap E^*$ is a line in $E \simeq \boldsymbol{P}^2$. So the intersection is transverse and E^* is non-singular along l. Hence the singular locus Σ of E^* does not contain a positive dimensional fiber of ρ. On the other hand $\rho(\Sigma)$ is finite since $E^* \to S$ is birational. Therefore Σ is at most a finite set. So E^* is normal by Serre's criterion. Note also that l is a Cartier divisor on E^* with $l^2 = -1$ since $[E]_E = \mathcal{O}_E(-1)$ and $El = -1$.

(4.4) CLAIM. *Let $E^\#$ be the minimal resolution of E^*. Then one of the following conditions is satisfied:*
a) *$E^\# = E^*$ and E^* is the blowing-up of S at two points on $\rho(l)$.*
b) *Σ consists of a single ordinary double point. The map $E^\# \to S$ factors through the blowing-up S_1 of S at the point $x_1 = \rho(\Sigma) \in \rho(l)$. Moreover $E^\#$ is the blowing-up of S_1 at the point $x_2 = Y \cap l_1$, where Y is the (-1)-curve over x_1 and l_1 is the proper transform of $l_0 = \rho(l)$. The proper transform Y_1 of Y on $E^\#$ is the (-2)-curve lying over Σ and the (-1)-curve Y_2 over x_2 is mapped onto the unique positive dimensional fiber of $E^* \to S$.*

PROOF. Suppose that $E^* \to S$ has two positive dimensional fibers X_1 and X_2. Let $\tilde{X}_i$ be the proper transforms of them on $E^\#$. Clearly $x_i = \rho(X_i)$ are points on $l_0 = \rho(l)$ and $E^\# \to S$ factors through the blowing-up S_2 of S at these points. The proper transform l_2 of l_0 is a (-1)-curve on S_2. On the other hand, l lifts to a (-1)-curve $\tilde{l}$ on $E^\#$ since $l \cap \Sigma = \emptyset$. So we infer that $E^\# \to S_2$ is étale over l_2 by $l_2{}^2 = \tilde{l}^2$. This implies that $\tilde{X}_i$ is the proper transform of the (-1)-curve on S_2 over x_i. By the minimality of the resolution, the exceptional curve of the final blowing-up of $E^\# \to S$ over x_i is not contracted to a point by $E^\# \to E^*$. So it must be one of $\tilde{X}_i$'s. Thus we conclude $S_2 = E^\# = E^*$, which proves a).

Suppose that there is only one positive dimensional fiber X of $E^* \to S$. Let $\tilde{X}$ be the proper transform of X on $E^\#$. $x_1 = \rho(X)$ is a point on the line $l_0 = \rho(l)$ and $\rho^\# \colon E^\# \to S$ factors through the blowing-up S_1 of S at x_1. Then $l_1{}^2 = 0$ for the proper transform l_1 of l_0 on S_1. Since $E^\# \to S_1$ is étale except over the (-1)-curve Y_1, this map factors through the blowing-up S_2 of S_1 at $x_2 = Y_1 \cap l_1$. Then the proper transform l_2 of l_1 is a (-1)-curve on S_2. Hence $E^\# \to S_2$ is étale over l_2. This implies that $\tilde{X}$ is the proper transform of the (-1)-curve Y_2 on S_2 over x_2. Similarly as in case a), we infer $E^\# = S_2$ by the minimality of the resolution. The proper transform $\tilde{Y}_1$ of Y_1 on S_2 is the (-2)-curve con-

tracted to Σ on E^*. Thus the condition b) is satisfied.

(4.5) For the moment, until (4.8), we consider the case (4.4; a). The positive dimensional fibers of $\rho_1: M_1 \to P'$ are the two exceptional curves X_1 and X_2 of $E^* \to S$. For $x_i = \rho_1(X_i)$, we have $\rho_1^{-1}(x_i) = X_i$ in the set-theoretical sense. In fact this is true in the stronger scheme-theoretical sense.

We will verify this assertion at the point $y_i = X_i \cap l$, since it is much easier to check at other points. Take a local parameter (u, v, w) of M_1 at y_i such that E^*, E and X_i are defined at y_i by $w=0$, $v=0$ and $u=w=0$ respectively. Let S_α (resp. S_β) be general plane in P containing l_0 (resp. x_i). Then $D_1 = \rho_1^* S$, $D_2 = \rho_1^* S_\alpha$ and $D_3 = \rho_1^* S_\beta$ are defined by $vw=0$, $uv + w(\phi_1 + v\phi_2) = 0$ and $u + w\phi_3 = 0$ at y_i respectively, where ϕ_j's are holomorphic functions in a neighborhood of y_i such that $\phi_1(y_i) \neq 0$, since $D_1|_E = l$, $D_2|_{E^*} = l + X_i$ and $D_3|_{E^*} = X_i$. So elementary computation yields $\rho_1^* x_i = D_1 \cap D_2 \cap D_3 = X_i$ in the scheme-theoretical sense.

(4.6) Let P'' be the blowing-up of P' at the points x_1, x_2 and let Y_i be the exceptional divisor over x_i. Let M_2 be the blowing-up of M_1 along the two curves X_1, X_2 and let V_i be the exceptional divisors over X_i. By (4.5) we have a morphism $f: M_2 \to P''$ such that $f^* Y_i = V_i$.

Since $0 = X_i \cdot \rho^* S = X_i(E + E^* + D)$, we have $E^* X_i = -1$. Moreover X_i is a (-1)-curve on E^*. So the normal bundle of X_i in M_1 is $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$. This implies $V_i \simeq \boldsymbol{P}^1 \times \boldsymbol{P}^1$ and the normal bundle is of bidegree $(-1, -1)$. So the restriction of f to V_i is a finite double covering of $Y_i \simeq \boldsymbol{P}^2$ with branch locus being a smooth conic. This implies that f itself is a finite double covering. Its branch locus B is a smooth member of $|2F|$ for some line bundle F on P'' whose restriction to Y_i is $\mathcal{O}(1)$.

(4.7) Set $F = \beta H - \gamma Z_1 - Y_1 - Y_2$, where Z_1 denotes the total transform of $Z_1 \in \mathrm{Pic}(P')$ by abuse of notation. Recall that $Y_i \cap Z_1 = \emptyset$. The proper transform S'' of S is a member of $|H - Z_1 - Y_1 - Y_2|$ and $f^* S'' = E_2 + E_2^*$ where E_2 and E_2^* are the proper transforms of E and E^* on M_2 respectively. Clearly $E_2^* \simeq E^*$, $l_2 = E_2^* \cap E_2 \simeq l$ and $E_2 \to E$ is the blowing-up at the two points $y_i = X_i \cap E$. So E_2 and E_2^* meet along l_2 transversally. From this we infer that the restriction of B to S'' is $2l''$ where $l'' = f(l_2)$. Hence $\beta - \gamma = 1$.

Let K_2 and K'' be the canonical bundles of M_2 and P'' respectively. Then $K_2 = f^*(K'' + F) = (\gamma - 3)H - (\gamma - 1)D + V_1 + V_2$, $(K + 2L)L^2 = (K_1 + 2H)H^2 = (K_2 + 2H)H^2 = 2\gamma - 2$ and so $\gamma = g(M, L) \geq 2$. If we cut things by a general hyperplane in P, we will be in a situation as in (4.1). In particular B is connected.

REMARK. Let $M_1 \to W \to P'$ be the Stein factorization of ρ_1 and let B' be the image of B by the map $P'' \to P'$. The above observations imply that W is

the double covering of P' branched along B', which has ordinary double points x_1, x_2. Hence W itself has ordinary double points at them. The map $M_2 \to W$ is the blow-up at these points and V_i's are the exceptional divisors. The desingularization $M_2 \to M_1 \to W$ is of the type which is well-known in three-dimensional geometry.

(4.8) Conversely, for any $\gamma \geq 2$, we can find a smooth member B of $|2H - 2\gamma(H - Z_1) - 2Y_1 - 2Y_2|$ on P'' such that $B \cap Y_i$ are smooth conics and the restriction of B to S'' is $2l''$ for the proper transform l'' of the line passing x_1 and x_2. Taking the double covering M_2 of P'' branched along B and blowing-down M_2 suitably, we get a polarized 3-fold of the desired type. Details will be left to the reader, but we just make a remark here.

$V_i = f^* Y_i$ is isomorphic to $P^1 \times P^1$ and its normal bundle in M_2 is of bidegree $(-1, -1)$. So it can be blown-down to P^1 along two directions. On the other hand $f^* S''$ is the sum of two components both of which are isomorphic to S''. By a contraction $V_i \to P^1$, a (-1)-curve on one of these components is contracted to a point, but the other component remains the same. The contraction $V_i \to P^1$ along the other direction has the opposite effect. Thus, if we blow down V_1 and V_2 correctly, then we are in a desired situation. But if we blow down in the wrong way, both components are blown down to Σ_1.

(4.9) Now we consider the case (4.4 ; b). Let X be the fiber of $E^* \to S$ over x_1. Then X is the unique fiber of $\rho_1 : M_1 \to P'$ of positive dimension. Note that $X \simeq P^1$ and $-E^* X = EX = 1$.

Let M_2 be the blowing-up of M_1 along X and let V be the exceptional divisor over X. The proper transform of E^* will be denoted by $E^\#$ since it is isomorphic to the minimal resolution of E^*. Moreover the restriction of $E^\#$ to V as a divisor is $X_2 + Y$, where X_2 (resp. Y) is a section (resp. the fiber over the double point of E^*) of the projection $V \to X$. Note that X_2 and Y are identified with the curves Y_2 and Y_1' on $E^\#$ in (4.4 ; b). In particular, $X_2^2 = -1$, $X_2 Y = 1$ and $Y^2 = -2$ on $E^\#$. Since the restriction of V to $E^\#$ is $X_2 + Y$, we have $V X_2 = 0$ and $V Y = -1$. On the other hand, the total transform of E^* on M_2 is $E^\# + V$ and so $(E^\# + V) X_2 = E^* X = -1$, $(E^\# + V) Y = 0$. Hence $E^\# X_2 = -1$ and $E^\# Y = 1$. So $X_2^2 = -2$ on V since $X_2 Y = 1$. This implies that $V \simeq \Sigma_2 = P(\mathcal{O}(2) \oplus \mathcal{O})$ and that X_2 is the unique (-2)-section of it. Since $V X_2 = 0$ and $V Y = -1$, the conormal bundle of V is the tautological bundle $\mathcal{O}(1) = X_2 + 2Y$. This implies that the normal bundle of $X \simeq P^1$ in M_1 is $\mathcal{O}(-2) \oplus \mathcal{O}$.

(4.10) If $x = \rho_1(X)$, the fiber of $M_2 \to P'$ over x is V in the set-theoretical sense. Furthermore, similarly as in (4.5), we infer that this is true in the scheme-theoretical sense too. The key point here is to observe $E^\# \cap \rho_1^{-1}(x) = E^\# \cap V$ in the scheme-theoretical sense.

Thus we have a morphism $\rho_2: M_2 \to P''$ onto the blowing-up P'' of P' at x such that $\rho_2^*(Z_2) = V$, where Z_2 is the exceptional divisor over x.

(4.11) The restriction of ρ_2 to $E^\#$ is isomorphic to the mapping $E^\# \to S_1$ in (4.4; b), which is the contraction of $X_2 = Y_2$ to a smooth point x_2. The restriction of ρ_2 to $V \simeq \Sigma_2 = \{(\xi_0 : \xi_1), (\zeta_0 : \zeta_1 : \zeta_2) \in P_\xi^1 \times P_\zeta^2 \mid \zeta_1 : \zeta_2 = \xi_0^2 : \xi_1^2\}$ is isomorphic to the mapping $\Sigma_2 \to P_\zeta^2$ of degree two induced by the second projection. So X_2 is the unique fiber of ρ_2 of positive dimension. We claim $\rho_2^{-1}(x_2) = X_2$ in the scheme-theoretical sense.

Indeed, this is obvious set-theoretically. The proper transform E' of E on M_2 is isomorphic to the blow-up of E at the point $E \cap X$. So $E' \simeq \rho_2(E') = \rho_2(E^\#) \simeq S_1$. Hence $\rho_2^{-1}(x_2) \cap E'$ is a simple point in the scheme-theoretical sense. So the claim is true at this point. Recall that $\rho_2^*(\rho_2(E^\#)) = E' + E^\#$. Examining the mapping $E^\# \to \rho_2(E^\#)$, we infer $\rho_2^{-1}(x_2) \cap E^\# = X_2$ in the strong sense. Combining these observations we prove the claim.

(4.12) Now we get a morphism $f: M_3 \to P'''$ with $f^* Z_3 = E_3$, where M_3 and P''' are the blowing-ups of M_2 and P'' along/at X_2 and x_2, and E_3 and Z_3 are the exceptional divisors over X_2 and x_2 respectively. Since $E^\# X_2 = -1$ and $X_2^2 = -1$ on $E^\#$, the normal bundle of X_2 in M_2 is $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$. This implies $E_3 \simeq P^1 \times P^1$ and the conormal bundle of it is of bidegree $(1, 1)$. So the restriction $E_3 \to Z_3$ of f is a finite double covering. Hence f itself is a finite double covering.

The branch locus B of f is a smooth member of $|2F|$ for some line bundle F on P'''. The restriction of B to $Z_3 \simeq P^2$ is a smooth conic. The proper transform Z_2' of Z_2 is the blowing-up Σ_1 of P^2 at a point, and the restriction of B to Z_2' is the sum of two fibers of $\Sigma_1 \to P^1$. Hence we may set $F = \beta H - \gamma Z_1 - Z_2 - Z_3$, where Z_i denote the total transforms and hence $Z_2 = Z_2' + Z_3$.

Let S_3 be the proper transform of S on P'''. Then $f^* S_3 = \tilde{E} + \tilde{E}^*$ and $\tilde{E} \simeq \tilde{E}^* \simeq S_3$, where $\tilde{E}$ and $\tilde{E}^*$ are the proper transforms of E and E^* on M_3 respectively. So the restriction of the divisor B to S_3 is $2l_2$, where l_2 is the proper transform of the line $\rho_1(E \cap E^*)$ on S. This implies $\beta = \gamma + 1$. Similarly as before, we have $\gamma = g(M, L) \geq 2$. This implies that B is connected.

Conversely, for any integer $\gamma \geq 2$, we can construct examples of polarized manifolds of the above type. Details are left to the reader.

(4.13) REMARK. The situation (4.4; b) can be viewed as a degeneration of the case (4.4; a), where the two points x_1 and x_2 are infinitely near. Similar degeneration of double coverings can be constructed locally as follows.

Let $S = S_0$ be a plane in $P \simeq P^3$, let $l = l_0$ be a line in S and let x_1 be a point on l. We assume that there is a divisor $B = B_0$ on P which is defined by $u^k w + v^2 + vw = 0$ in a neighborhood of x_1, where k is a positive integer and

(u, v, w) is a local parameter system of P at x_1 such that S and l are defined by $\{w=0\}$ and $\{v=w=0\}$ at x_1 respectively. So B has an isolated singularity at x_1 and is tangent to S along l.

Let P_1 be the blowing-up of P at x_1 and let Z_1 be the exceptional divisor over x_1. Let B_1, S_1 and l_1 be the proper transforms of B, S and l respectively. Then B_1 has an isolated singularity at $x_2=l_1\cap Z_1$ if $k>1$. Moreover, if (u_1, v_1, w_1) is the local parameter system at x_2 such that $u=u_1$, $v=u_1v_1$ and $w=u_1w_1$, then B_1, S_1 and l_1 are defined by $u_1^{k-1}w_1+v_1^2+v_1w_1=0$, $w_1=0$ and $v_1=w_1=0$ respectively.

Next we blow up at x_2 and repeat this process similarly. Precisely speaking, the center of the i-th blowing-up $P_i\to P_{i-1}$ is the point $x_i=l_{i-1}\cap S_{i-1}$, where l_{i-1} and S_{i-1} are the proper transforms of l and S on P_{i-1} respectively. If we take the local parameter system (u_i, v_i, w_i) at x_{i+1} such that $u_{i-1}=u_i$, $v_{i-1}=u_iv_i$ and $w_{i-1}=u_iw_i$, then the proper transform B_i of B_0 is defined by $u_i^{k-i}w_i+v_i^2+v_iw_i=0$, provided $i\leq k$. In particular B_k is non-singular on P_k, while x_k is an ordinary double point of B_{k-1}.

Set $P'=P_k$ and let Z_i' be the proper transform on P' of the exceptional divisor Z_i over x_i. Then, for each $i<k$, Z_i' is isomorphic to Σ_1 and the restriction of $B'=B_k$ to Z_i' consists of two different fibers of $\Sigma_1\to P^1$. The restriction of B' to $Z_k\simeq P^2$ is a smooth conic.

Let $f: M_k\to P'$ be the finite double covering branched along B' and set $Y_i=f^*Z_i'$. Then $Y_k\simeq P^1\times P^1$ and its normal bundle is of bidegree $(-1, -1)$. The divisor $Y_{k-1}\cap Y_k$ is of bidegree $(1, 1)$. For each $i<k$, Y_i is isomorphic to Σ_2, the P^1-bundle $P(\mathcal{O}(2)\oplus\mathcal{O})$ over P_σ^1. Its normal bundle is $2H_\sigma-2H_\tau$, where H_τ is the tautological line bundle and H_σ is the pull-back of the $\mathcal{O}(1)$ of P_σ^1. Note that the (-2)-section is the unique member of $|H_\tau-2H_\sigma|$ and is identified with $Y_{i+1}\cap Y_i$ while $Y_{i-1}\cap Y_i$ is another section of $Y_i\simeq\Sigma_2\to P_\sigma^1$ and belongs to $|H_\tau|$.

Since the restriction of the divisor B' to $S'=S_k$ is $2l'$, we have $f^*S'=\tilde{E}+\tilde{E}^*$ for some divisors $\tilde{E}$, $\tilde{E}^*$ isomorphic to S'. $\tilde{E}\cap Y_k$ is a fiber of one of the two rulings of $Y_k\simeq P^1\times P^1$ and $\tilde{E}^*\cap Y_k$ is a fiber of the other ruling. They are mapped isomorphically to the exceptional curve of the final blowing-up $S'=S_k\to S_{k-1}$. For $i<k$, $\tilde{E}\cap Y_i$ is a fiber of $Y_i\simeq\Sigma_2\to P_\sigma^1$ and $\tilde{E}^*\cap Y_i$ is another fiber. They are (-2)-curves on $\tilde{E}$ and $\tilde{E}^*$.

Since the normal bundle of Y_k is of bidegree $(-1, -1)$, we can contract Y_k to P^1 in two directions. We choose the blow-down $\pi_k: M_k\to M_{k-1}$ such that $\pi_k(\tilde{E}^*)\simeq\tilde{E}^*$ and $\tilde{E}\to\pi_k(\tilde{E})$ is the contraction of the (-1)-curve $Y_k\cap\tilde{E}$ on $\tilde{E}$. The image $\pi_k(Y_{k-1})$ is isomorphic to $Y_{k-1}\simeq\Sigma_2$ and its normal bundle in M_{k-1} is $-H_\tau$. Hence it can be contracted to P^1 and let $\pi_{k-1}: M_{k-1}\to M_{k-2}$ be the blowing-down. Contracting the images of Y_k, Y_{k-1}, $\cdots$, Y_1 successively in this

way, we get a birational mapping $\pi: M_k \to M_{k-1} \to \cdots \to M_0$. Then $E = \pi(\tilde{E})$ is isomorphic to $S \simeq \boldsymbol{P}^2$ and $E^* = \pi(\tilde{E}^*)$ is obtained by contracting the $(k-1)$ (-2)-curves $Y_i \cap \tilde{E}^*$ $(i=1, \cdots, k-1)$ to a rational double point of type A_{k-1}. f yields a morphism $\rho: M_0 \to P$ of degree two and $X = \rho^{-1}(x_1)$ is the unique fiber of ρ of positive dimension.

In case (4.4; a) (resp. (4.4; b)), we are in a situation as above where $k=1$ (resp. $k=2$).

(4.14) Now we consider the case $n \geq 4$. We claim that this cannot happen.

To show this, it suffices to rule out the case $n=4$ since any general member of $|L|$ is a polarized manifold of the same type of dimension $n-1$. So we consider the case $n=4$.

Set $W = \{x \in P' \mid \dim \rho_1^{-1}(x) > 0\}$. Then $W \subset S'$ for the proper transform S' of S on P'. Since any fiber of $D \to C$ is mapped onto a curve by ρ_1, any fiber over W does not meet D. Hence the image of W in $S \simeq S'$ does not meet C. This implies that W is a finite set since C is a hyperplane in $S \simeq \boldsymbol{P}^3$. Therefore $W \cap T = \emptyset$ for any general member T of $|H|$ on P'.

On the other hand, if $N_1 = \rho_1^*(T)$ and if N is its image in M, then (N, L_N) is a polarized 3-fold of the type (4.3). So, by the observation (4.4), the mapping $N_1 \to T$ has one or two positive dimensional fiber(s). This contradicts $W \cap T = \emptyset$.

Thus we prove the claim.

(4.15) SUMMARY OF RESULTS. Let things be as in (1.7; 1). Then $n \leq 3$ and $C = \rho(D)$ is a hyperplane in $S \simeq \boldsymbol{P}^{n-1}$. Let P' be the blowing-up of P along C. When $n=2$, M_1 is a finite double covering of P' branched along a smooth connected member B of $|2H + 2\gamma(H-Z)|$, where Z is the exceptional curve over C and $\gamma = g(M, L) \geq 2$. B is tangent to the proper transform S' of S on P' at any point on $B \cap S'$. M is a rational surface.

When $n=3$, there is a morphism $\rho_1: M_1 \to P'$ of degree two. $W = \{x \in P' \mid \dim \rho_1^{-1}(x) > 0\}$ consists of one or two point(s) and $W \cap Z = \emptyset$. If W has two points, the local structure of ρ_1 at each point of W looks like that in the case $k=1$ in (4.13). If W is one point, ρ_1 has a local structure as in the case $k=2$ in (4.13). In any case the branch locus B' of ρ_1 is a member of $|2H + 2\gamma(H-Z)|$ with $\gamma = g(M, L) \geq 2$, is non-singular off W, and is tangent to S' along the proper transform of the line $\rho(E \cap E^*)$.

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