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On the rationality of complex homology 2-cells: II

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§ 8. Introduction.

The purpose of this paper is to prove:

8.1. THEOREM. *Let X be an irreducible, smooth projective surface/$\boldsymbol{C}$, with the geometric genus $p_g(X)=0$. Let D be a reduced (not necessarily connected) curve on X with at worst ordinary double point singularities. Suppose each connected component of* Supp D *is simply connected and the irreducible components of D generate the divisor class group* Pic(X). *Then X is rational.*

In [**G-S**] we showed that such a surface X cannot be a surface of general type. Thus 8.1 follows from:

8.2. THEOREM. *Let X and D be as in* 8.1. *Assume further that X is an elliptic surface. Then X is rational.*

We will refer to the paper [**G-S**] by Part I. As already indicated in Part I, one easily deduces from 8.1, that any complex homology 2-cell is rational, thus answering affirmatively a question of Van de Ven. (For other consequences of 8.1 see Part I.) The reader is assumed to be familiar with Part I of this paper, the notations and conventions of which we continue to use here also. We shall now briefly outline the proof of 8.2 here.

8.3. Recall that (in Part I), we begin with the assumption that X is not rational (or, equivalently, that $|nK|\neq\emptyset$ for some $n>0$) so that $K+D$ has Zariski decomposition: $K+D=P+N$. Without loss of generality we also assume that there are no (-1)-curves E on X such that (i) $E\cdot D=1$ or (ii) $E\cdot D=2$ and E meets two different connected components of D. We then apply Miyaoka's inequality to (X, D). Then, by studying the blowing-down process $\pi: X\to X''$, where X'' is the minimal model for the function field of X, we obtain the auxiliary inequality

$$(2.8') \qquad 3(b_2-\beta_2)+b_0+\lambda+\sigma+\tau+e_1+r_3+2r_4 \leqq \beta_2''-5$$

where each term on the left hand side is nonnegative.

8.4. We now assume that X is an elliptic surface, $\varphi: X\to \boldsymbol{P}^1$, and arrive at a sequence of contradictions thereby proving 8.2. Now β_2'', on the right hand side of (2.8) is the second betti number of X'', which is a minimal elliptic surface, with $p_g(X'')=0=q(X'')$. Hence $\beta_2''=10$. (Note that if X is of general type then $\beta_2''\leqq 9$. So there is an increase in the right hand side of (2.8), as compared with the general type case and this contributes to the enormous increase in our task here.) Except for 4.2, all the results in § 1-§ 6 (of Part I) hold here also. Our first aim is to prove the analogue of 4.2 here also, viz., to show that $\lambda\geqq 2$. This is achieved in § 9 by showing that D has at least two horizontal components with respect to φ. Once we show that $\lambda\geqq 2$, by 3.2, we obtain that $b_2=\beta_2$, D is unimodular, $r_4=0$, and $\sigma+\tau+e_1+r_3\leqq 2$.

In § 12, we show that $e_1=0$ and in § 13 we will show that $r_3\geqq 1$. In § 14, we show that $(r_3, \sigma)\neq(1, 1)$. In § 15, we show that $r_3\geqq 2$. Since, any way $r_3\leqq 2$, in § 16 and § 17 we show that even the assumption $r_3=2$ leads to a contradiction.

In each of these cases we first get hold of the geometry of $D\cup\mathcal{E}$ (where $\mathcal{E}$ is the exceptional set for $\pi: X\to X''$). In § 12-§ 14, often, we apply Miyaoka's inequality to the curve $D\cup\mathcal{E}$, to arrive at a contradiction. Note that at the end of § 14, we have $\sigma=0$, $e_1=0$, $r_3\geqq 1$, from which it follows that $\mathcal{E}\subset D$. Thus in § 15-§ 17 we apply Miyaoka's inequality to a suitable subset of D. This is rather fruitful only if we have equality in (2.8). Further, we need to estimate the N^2-term in the Zariski decomposition. For this purpose we use results of [**F**] which we have collected in § 10 in a manner suitable for our purpose. The unimodularity of D and the fact that $\pm K$ is not effective are heavily used. In § 11, we have collected various technical lemmas, similar to those in § 5 and § 6, which enable us to use these two properties.

§ 9. Number of horizontal components.

9.0. Let X and D be as in 8.2. We assume X is not rational, so that $|nK|\neq\emptyset$ for some $n>0$. Since X is simply connected (see [**G-S**] p. 3), we have an elliptic fibration $\varphi: X\to\boldsymbol{P}^1$. Let $\pi: X\to X''$ be the composite of contraction of (-1)-curves where X'' is the minimal model for the function field of X. It follows easily that φ defines the minimal elliptic fibration $\varphi'': X''\to\boldsymbol{P}^1$, $\varphi=\varphi''\circ\pi$. An irreducible curve C on X is called vertical if C is contained in a fibre of φ; otherwise it is called horizontal. Our main aim here is to show that D has at least two horizontal components, equivalently, $D''=\varphi''(D)$ has at least two horizontal components with respect to φ''. On the way, we also determine multiplicities of the singular fibres of φ''.

Note that components of D generate Pic X and hence D should have at least one horizontal component. We shall first prove:

9.1. LEMMA. *There are exactly two multiple singular fibres of* φ'' *(and hence, of* φ*), say* m_1P_1, m_2P_2 *(respectively,* m_1F_1, m_2F_2*) with* $\{m_1, m_2\}=\{2, 3\}$ *or* $\{2, 5\}$. *Further, if* $K''\cdot H''\leqq 2$ *for some horizontal component* H'', *then* $\{m_1, m_2\}=\{2, 3\}$.

PROOF. Let $\{m_iP_i\}_{1\leqq i\leqq r}$ be the multiple fibres of φ''. By the simply connectivity of X'' it follows that $r\leqq 2$ and m_1 and m_2 are coprime (see Proposition 2 of [**K**]). Since $p_g=0=q$, we have the canonical bundle formula

$$K''=\varphi''^{-1}(\mathcal{O}_{P^1}(-1))\otimes[P_1]^{(m_1-1)}\otimes\cdots\otimes[P_r]^{(m_r-1)}.$$

Thus if $r\leqq 1$, it follows that $|nK''|=\emptyset$ for all n and hence $|nK|=\emptyset$ for all n, contradicting our basic assumption. Hence $r=2$.

Now if P denotes a general fibre of φ'', we have the linear equivalence

$$P\sim m_1P_1\sim m_2P_2$$

and hence

$$K''\sim(m_2-1)P_2-P_1\sim(m_1-1)P_1-P_2.$$

Thus K'' is numerically equivalent to the $\boldsymbol{Q}$-divisor mP where $m=(m_1m_2-m_1-m_2)/m_1m_2$. Now let H'' be a horizontal component of D''. Then $1\leqq K''\cdot H''=m(P\cdot H'')$. Since m_1 and m_2 are coprime it follows that $m_1m_2-m_1-m_2$ divides $K''\cdot H''$. On the other hand we have, by (2.8), $\lambda\leqq 4$ (since $b_0\geqq 1$) and hence $K''\cdot H''\leqq 4$. Hence $\{m_1, m_2\}=\{2, 3\}$ or $\{2, 5\}$, and if $K''\cdot H''\leqq 2$ for some H'', then $\{m_1, m_2\}=\{2, 3\}$, as claimed.

9.2. LEMMA. *Suppose* $\beta_2(X)=b_2(D)$. *Then* $H_1(X-D, \boldsymbol{Z})=(0)$ *and consequently no fibre of* φ *is completely contained in* D.

PROOF. Since components of D generate $\mathrm{Pic}\,X$, $\beta_2(X)=b_2(D)$ implies that components of D form a free basis for $\mathrm{Pic}\,X$ and hence $H^2(X, \boldsymbol{Z})\to H^2(D; \boldsymbol{Z})$ is an isomorphism. From the cohomology exact sequence of (X, D), we get $H^3(X, D; \boldsymbol{Z})=(0)$. Hence, by duality, $H_1(X-D; \boldsymbol{Z})=(0)$.

Now suppose F_0 is a fibre of φ such that $F_0\subset D$. Then, since D is simply connected, F_0 is simply connected, and so, in particular, $F_0\neq m_iF_i$, where m_1F_1 and m_2F_2 are the multiple fibres of φ. Let $p_i=\varphi(F_i)$ and let $f: \Delta\to\boldsymbol{P}^1$ be a cyclic covering, ramified on p_0 and p_1 with ramification index m_1 ($\geqq 2$). After normalization of $X\times_{P^1}\Delta$, we obtain a ramified cyclic covering $\tilde{f}: \tilde{X}\to X$ with ramification locus contained in F_0 and hence an m_1-fold cyclic, unramified covering of $X-D$. This contradicts the fact that $H_1(X-D, \boldsymbol{Z})=(0)$. Hence no fibre of φ is completely contained in D. This completes the proof of the lemma.

9.3. LEMMA. D'' *(and hence* D*) has at least two horizontal components. In particular,* $\lambda\geqq 2$, $\{m_1, m_2\}=\{2, 3\}$ *and* $6|(P\cdot H'')$ *for any horizontal component* H''

and any general fibre P of φ''. Further, if there are only two horizontal components, then exactly one component from each fibre of φ is not contained in D.

PROOF. Assume that D'' has only one horizontal component, H''. We first claim that φ'' has no fibres of type $m\mathrm{I}_0$, with $m>1$. Suppose m_1P_1 is such a fibre. Then $P_1\sim\sum_i\alpha_iD_i''+\alpha H''$, where $D_i''\neq H''$ are other components of D'' (which are vertical). Intersecting P_1 with a general fibre, we see that $\alpha=0$. Since each component of D'' is rational, P_1 is not contained in D''. On the other hand, linear equivalence relations between fibre components of φ'' are all generated by relations of the type $P\sim P'$ for any two scheme-theoretic fibres of φ''. Hence in any such relation P_1 should occur in multiples of m_1, which contradicts the relation, $P_1\sim\sum_i\alpha_iD_i''$, above.

Now let $S_1, \cdots, S_k$ (respectively, $T_1, \cdots, T_l$) denote the simply connected (respectively, not simply-connected) singular fibres of φ. By the above observation, it follows that $\chi_{\mathrm{top}}(T_j)=b_2(T_j)$ and hence we have

$$\beta_2(X)+2=\chi_{\mathrm{top}}(X)=\sum_{i=1}^{k}(b_2(S_i)+1)+\sum_{j=1}^{l}b_2(T_j)\,. \tag{9.4}$$

Since, by simply-connectivity of D, none of the T_i is completely contained in D, we get

$$b_2(D)\leqq\sum_{i=1}^{k}b_2(S_i)+\sum_{j=1}^{l}(b_2(T_j)-1)+1\,. \tag{9.5}$$

Together with (9.4), this yields $k+l\leqq 3+\beta_2(X)-b_2(D)$. If $k=0$, then clearly no fibre of φ is contained in D. However, by 9.1, it follows that $l\geqq 2$. Since $\beta_2(X)\leqq b_2(D)$, it follows that $k\leqq 1$, and if $k=1$ then $\beta_2(X)=b_2(D)$. Thus if $k\neq 0$, we can appeal to 9.2 to conclude that no fibre of φ is contained in D. But then we have

$$b_2(D)\leqq\sum_{i=1}^{k}(b_2(S_i)-1)+\sum_{j=1}^{l}(b_2(T_j)-1)+1<\operatorname{rank}(\operatorname{Pic}X)$$

which is absurd.

Hence D'' has at least two horizontal components, and hence $\lambda=\sum_t D_t''\cdot K''\geqq 2$. Now by (2.8) we have $\lambda\leqq 4$, and hence $K''\cdot H_i''\leqq 2$ for some horizontal component. Hence by 9.1, $\{m_1, m_2\}=\{2, 3\}$, $K''\cdot H_i''=(1/6)\cdot(P\cdot H_i'')$ and hence $6\,|\,P\cdot H_i''$. By (2.8) it now follows that $b_2=\beta_2$ and hence by 9.2, no fibre of φ is completely contained in D.

Finally let there be exactly two horizontal components. Then we have

$$b_2(D)\leqq\sum_{i=1}^{k}(b_2(S_i)-1)+\sum_{j=1}^{l}(b_2(T_j)-1)+2=\operatorname{rank}\operatorname{Pic}(X)\leqq b_2(D)\,.$$

Hence equality holds everywhere. Thus exactly one component from each fibre of φ is missing from D. This completes the proof of the lemma.

9.6. REMARK. Note that, unlike D, D'' may contain a fibre of φ''. The following lemma gives some idea of such a situation and will be very useful later. We use Kodaira's list of singular fibres of an elliptic fibration. For any fibre P and a component C of D let $\mu(C)$ denote the multiplicity of C in P.

9.7. LEMMA. *Suppose P is a fibre of φ'' contained in D'' and is not of type II^*. Assume that*

(i) *there is at most one point $x \in P$ which is worse than an ordinary double point singularity of D'' and*

(ii) *if x exists, then P is of type $m\mathrm{I}_1$, II, III or IV with $x \in P$ being the singularity of P and at most one horizontal component of D'' passes through x.*

Then $b_1(D'') \geqq 2$.

PROOF. First assume x exists. By 9.3, there is a horizontal component H'' of D'' not passing through x. If P is of type $m\mathrm{I}_1$, then $m=2$ or 3, $6|(P\cdot H'')$ and hence $P \cap H''$ consists of at least two points. Since $b_1(m\mathrm{I}_1)=1$, we have $b_1(D'') \geqq b_1(P \cup H'') \geqq 2$. If P is of type II, III or IV, then it follows that $P \cap H''$ consists of at least six points and hence $b_1(D'') \geqq b_1(P \cup H'') \geqq 5$.

Next assume that x does not exist. (Clearly P is not of type $m\mathrm{I}_0$). Hence P is of type $m\mathrm{I}_b$, ($m=2$ or 3, $b \geqq 1$), I_b^*, III^* or IV^*. Hence for every component C of P, $\mu(C) \leqq 4$, $P \cap H_i''$ consists of points which are all ordinary double points of D''. Hence it follows that $b_1(P \cup H_i'') \geqq 1$ for each horizontal component of D''. Hence $b_1(D'') \geqq b_1(P \cup H_1'' \cup H_2'') \geqq 2$. This completes the proof of the lemma.

9.8. LEMMA. *Suppose $\beta_2(X)=b_2(D)$ and there are precisely two horizontal components of D. Let F_0 be a fibre of φ different from the multiple fibres m_1F_1 and m_2F_2, and let C_0 be the component of F_0 not contained in D. Then the multiplicity $\mu(C_0)$ of C_0 in F_0 is not equal to m_1 or m_2.*

PROOF. If not, say $\mu(C_0)=m_1$. Then as in 9.2, we can construct an m_1-fold unramified cyclic cover of V which is absurd.

9.9. LEMMA. *Let Y be a normal, projective surface/$\boldsymbol{C}$ having exactly one singular point y. Suppose $\tilde{Y} \xrightarrow{\pi} Y$ is a minimal resolution of singularity. Assume further the following conditions:*

i) *$\mathcal{O}_{Y,y}$ is a rational singularity*

ii) *$\mathcal{O}_{Y,y}$ is a unique factorization domain and*

iii) *$p_g(\tilde{Y}) = 0 = q(\tilde{Y})$.*

Then locally analytically, the singularity at y is E_8-rational double point, i.e., given by $\{z_1^2+z_2^3+z_3^5=0\}$.

PROOF. There exists an affine neighbourhood U of y such that $\Gamma(U, \mathcal{O}_Y)$ is a U.F.D. Since $p_g(\tilde{Y})=0$, any topological 2-cycle is algebraic. Let $\pi^{-1}(y)$

$=\bigcup_{i=1}^{r} L_i$ where L_i are the irreducible components. We can then find divisors $\Delta_1, \cdots, \Delta_s$ supported on $\tilde{Y}-\pi^{-1}(U)$ such that $L_1, \cdots, L_r, \Delta_1, \cdots, \Delta_s$ form a free basis of $H^2(\tilde{Y}, \boldsymbol{R})$. Therefore the intersection matrix of these divisors can be assumed to be unimodular. Since the sets $L_1 \cup \cdots \cup L_r$ and $\Delta_1 \cup \cdots \cup \Delta_s$ are disjoint, it follows that the intersection matrix $(L_i \cdot L_j)$ is unimodular. Since the singularity of y is rational, this implies that analytically the singularity is the E_8-rational double point, by results of Brieskorn [**B**].

§ 10. Estimating N^2.

10.0. Here we briefly recall and modify certain results and terminologies from § 3 and § 6 of [**F**] which will help us to estimate the term N^2. Let Y be a smooth projective surface and C be a reduced curve on Y. For our purpose we shall assume that all components of C are smooth rational curves. Hence, we shall drop the word 'rational' from Fujita's terminology. We shall also assume that C is minimal with normal crossings (MNC). For any curve Γ on Y, let $Q(\Gamma)$ denote the subspace of Pic $Y \otimes \boldsymbol{Q}$ generated by components of Γ.

10.1. For any component C_0 of C the branching number is given by $\beta(C_0) = C_0 \cdot (C - C_0)$. C_0 is called a tip of C if $\beta(C_0)=1$. A sequence Γ of components $\{C_1, \cdots, C_r\}$, $r \geqq 1$, is called a twig of D if $\beta(C_1)=1$, $\beta(C_j)=2$ and $C_{j-1} \cdot C_j = 1$ for $2 \leqq j \leqq r$. We denote Γ by $[a_1, \cdots, a_r]$ where $a_i = -C_i^2$. Γ is a maximal twig if there is a (unique) component C_0 of C such that $C_r \cdot C_0 = 1$ and $\beta(C_0) \geqq 3$. Then C_0 is called the branching component of Γ. For any twig Γ, $\bar{\Gamma}$ denotes the curve $C_2 \cup \cdots \cup C_r$ ($\bar{\Gamma} = \emptyset$ if $r=1$), and $e(\Gamma) = d(\bar{\Gamma})/d(\Gamma)$ where $d(-)$ denotes the discriminant, ($d(\phi)=1$, by convention). A sequence Γ is called a club if $\beta(C_1)=\beta(C_r)=1$, $\beta(C_j)=2$, $2 \leqq j \leqq r-1$ and $C_j \cdot C_{j+1}=1$, $1 \leqq j \leqq r-1$.

10.2. A connected component Δ of C is called an abnormal (rational) club if

(i) Δ has a unique component C_0 with $\beta(C_0)=3$,

(ii) Δ has three maximal twigs $\Gamma_1, \Gamma_2, \Gamma_3$ with C_0 as their branching component,

(iii) Δ is negative definite, and

(iv) $d(\Gamma_1)^{-1} + d(\Gamma_2)^{-1} + d(\Gamma_3)^{-1} > 1$.

By 6.19 of [**F**] it follows that an abnormal club Δ can be contracted to a rational singularity, i. e. Δ is 'rational' according to our Definition 5.2.

10.3. Assume now that $K+C$ is pseudo effective, so that, by 6.13, of [**F**], all (rational) clubs and maximal twigs of C are negative definite (contractible). If Γ is a maximal twig of C, then Bk(Γ) (bark of Γ) is the element $N_1 \in \boldsymbol{Q}(\Gamma)$ such that $N_1 \cdot C_1 = -1$, $N_1 \cdot C_j = 0$ for $j \geqq 2$ (equivalently, $N_1 \cdot C_i = (K+C) \cdot C_i$ for $1 \leqq i \leqq r$). If Γ is a club of C then Bk(Γ) is an element $N_2 \in \boldsymbol{Q}(\Gamma)$ such that

$N_2 \cdot C_1 = N_2 \cdot C_r = -1$ and $N_2 \cdot C_j = 0$, $2 \leqq j \leqq r-1$ (equivalently, $N_2 \cdot C_i = (K+C) \cdot C_i\ \forall C_i$ in Γ). For an isolated club $\Gamma = \{C_1\}$, we have $\mathrm{Bk}(\Gamma) = 2(-C_1^2)^{-1} C_1$. For any connected component Λ of C we define $\mathrm{Bk}(\Lambda) = \sum_i \mathrm{Bk}\,\Gamma_i$ where Γ_i are all maximal twigs and clubs of Λ. We define $\mathrm{Bk}(C) = \sum \mathrm{Bk}(\Lambda_j)$ where Λ_j are all connected components of C.

For an abnormal club Λ of C we define $\mathrm{Bk}^*(\Lambda)$ as the element $N \in \boldsymbol{Q}(\Lambda)$ such that $N \cdot L = (K+C) \cdot L = (K+\Lambda) \cdot L$ for every component L of Λ. If Λ is a connected component which is not an abnormal club of C then we take $\mathrm{Bk}^*(\Lambda) = \mathrm{Bk}(\Lambda)$. Finally we define $\mathrm{Bk}^*(C) = \sum \mathrm{Bk}^*(\Lambda_j)$ where Λ_j are all the connected components of C.

We introduce the notation $\mathrm{bk}(\Lambda)$ (respectively $\mathrm{bk}^*(\Lambda)$) for the rational number $(\mathrm{Bk}(\Lambda)) \cdot (\mathrm{Bk}(\Lambda))$ (respectively, $(\mathrm{Bk}^*(\Lambda)) \cdot (\mathrm{Bk}^*(\Lambda))$). Then clearly $\mathrm{bk}(C) = \sum_i \mathrm{bk}(\Gamma_i)$ where Γ_i are all clubs and maximal twigs of C. We have

10.4. LEMMA. *Let C be as in* 10.0 *and* 10.3, *and $\Gamma = [a_1, \cdots, a_r]$, $r \geqq 1$ be a twig of C.*

(i) *If Γ is a maximal twig of C, then $\mathrm{bk}(\Gamma) = -e(\Gamma)$. In particular, $\mathrm{bk}(\Gamma) \leqq -1/a_1$.*

(ii) *If Γ is a club of C then $\mathrm{bk}(\Gamma) \leqq -(1/a_1 + 1/a_r)$.*

(iii) *If Γ is a club of C with $r=1$, 2 or 3 then $\mathrm{bk}(\Gamma) = -4/a_1$, $-(a_1+a_2+2)/(a_1a_2-1)$ or $-a_2(a_1+a_3)/(a_1a_2a_3 - a_1 - a_3)$, respectively.*

(iv) *If $\{L_1, \cdots, L_k\}$ is a set of tips of C, then $\mathrm{bk}(C) \leqq \sum_{i=1}^k 1/(L_i^2)$.*

(v) *For any abnormal club Λ of C we have $\mathrm{bk}^*\Lambda < \mathrm{bk}(\Lambda)$. Hence $\mathrm{bk}^*(C) < \mathrm{bk}(C)$.*

PROOF. (i) That $\mathrm{bk}(\Gamma) = -e(\Gamma)$ is proved in 6.16 of [F]. Since $d(\Gamma) = a_1 d(\bar{\Gamma}) - d(\bar{\bar{\Gamma}})$ it follows that

$$\mathrm{bk}(\Gamma) = -e(\Gamma) = -d(\bar{\Gamma})/(a_1 d(\bar{\Gamma}) - d(\bar{\bar{\Gamma}})) \leqq -d(\bar{\Gamma})/a_1 d(\bar{\Gamma}) = -1/a_1 .$$

(ii) If $r_1 = 1$, then $\mathrm{bk}(\Gamma) = -4/a_1$ and so we are done. In general, let $N_1 = \sum \lambda_i C_i$, $N_2 = \sum \mu_i C_i \in \boldsymbol{Q}(\Gamma)$ be defined as in 10.3. Then $N_2 = \mathrm{Bk}(\Gamma)$ whereas N_1 has the numerical property of $\mathrm{Bk}(\Gamma)$ if Γ were a maximal twig with a_1 as its tip. Hence by 6.16 of [F], $\lambda_1 = -N_1^2 \geqq 1/a_1$ by (i) above. On the other hand we have $(N_2 - N_1) \cdot C_i \leqq 0$ for every component C_i of Γ and hence $N_2 \geqq N_1$ (by 5.3). In particular $\mu_1 \geqq \lambda_1 \geqq 1/a_1$. By symmetry, we get $\mu_r \geqq 1/a_r$. Now $\mathrm{bk}(\Gamma) = N_2 \cdot N_2 = -(\mu_1 + \mu_r) \leqq -(1/a_1 + 1/a_r)$.

(iii) Compute directly.

(iv) Follows easily from (i) and (ii) above.

(v) Let Λ be an abnormal club of C as in 10.2, $N = \mathrm{Bk}^*\Lambda$, and let $N_i = \mathrm{Bk}(\Gamma_i)$, $i=1, 2, 3$. Let $\theta = N - (N_1 + N_2 + N_3)$. Then we have $\theta \cdot L = 0$ for every component of $\Gamma_1 \cup \Gamma_2 \cup \Gamma_3$, and $\theta \cdot C_0 = (K+C) \cdot C_0 - (d(\Gamma_1)^{-1} + d(\Gamma_2)^{-1} + d(\Gamma_3)^{-1}) < 0$, by 6.16 of [**F**]. Hence $\theta>0$ (by 5.3). Now $N^2=N_1^2+N_2^2+N_3^2+\theta^2$ and $\theta^2=\theta\cdot(\lambda C_0)$ where $\lambda\geqq 0$ is the coefficient of C_0 in N. Since $\theta\cdot C_0<0$, it follows that $\theta^2<0$. Hence $\mathrm{bk}^*(\Lambda)=N^2<N_1^2+N_2^2+N_3^2=\mathrm{bk}(\Lambda)$. Hence $\mathrm{bk}^*(C)<\mathrm{bk}(C)$. This completes the proof of the lemma.

Finally the following lemma will enable us to use the above estimates in our situation.

10.5. LEMMA. *Suppose C is a union of trees of rational curves on a smooth projective surface Y, C is* MNC *and all (-1)-curves on Y are contained in C. For any set $E_1, \cdots, E_r$ of (-1)-curves, $r\geqq 0$, let $\underset{\sim}{C}$ be the reduced curve $\underset{\sim}{C}=C-(E_1+\cdots+E_r)$. Assume that $|nK_Y|\neq\emptyset$ for some $n>0$, and $K_Y+\underset{\sim}{C}=\underset{\sim}{P}+\underset{\sim}{N}$ is the Zariski decomposition. Then $\underset{\sim}{N}=\mathrm{Bk}^*(\underset{\sim}{C})$. In particular $\underset{\sim}{N}^2\leqq\mathrm{bk}(\underset{\sim}{C})$.*

PROOF. By 6.20 of [**F**], if $\underset{\sim}{N}\neq\mathrm{Bk}^*(\underset{\sim}{C})$ then there exists an exceptional curve E on Y, not contained in $\underset{\sim}{C}$, such that one of the following holds:

(i) $\underset{\sim}{C}\cdot E=0$.

(ii) $\underset{\sim}{C}\cdot E=1$ and E meets a component of $\mathrm{Bk}^*(\underset{\sim}{C})$.

(iii) $\underset{\sim}{C}\cdot E>1$ and E meets *precisely* two components of D, one of which is a tip of a (rational) club of D.

Note that the word "precisely" in (iii) is not there in [**F**], though from the proof of 6.20, this is obvious. Now, since C is MNC, every exceptional curve E_i not in $\underset{\sim}{C}$ meets at least three components of $\underset{\sim}{C}$. Hence $\underset{\sim}{N}=\mathrm{Bk}^*(\underset{\sim}{C})$. That $\underset{\sim}{N}^2\leqq\mathrm{bk}(\underset{\sim}{C})$ follows from 10.4 (v) above. This completes the proof of the lemma.

§11. Some technical lemmas.

11.0. Here we collect a number of technical results similar to those in §4, §5 and §6. The first one below, is analogous to 4.3 and will be used in conjunction with 4.1.

11.1. LEMMA. *Let Y be a smooth projective surface, C be a curve on Y, with $C=\sum_{i=0}^k C_i$, where either* (a) *C_i are smooth rational curves pairwise intersecting transversally as shown in* (a) *Figure* 1, *or* (b) *C_i are smooth rational curves, $C_1\cdot C_2=2$, $C_0\cdot C_1=C_0\cdot C_2=1$ or* (c) *C_0 is a smooth rational curve; C_1 is a rational curve with an ordinary cusp; $C_0\cdot C_1=2$. Suppose, in each of these cases, we have a canonical divisor*

$$K_Y=\sum_{i=0}^k t_iC_i+\Lambda\,,\qquad t_i\in\boldsymbol{Z}$$

for some divisor Λ with the property $\mathrm{supp}(\Lambda)\cap C_i=\emptyset$, $i\geqq 1$, and $C_0\not\subset\mathrm{supp}(\Lambda)$. Set $a_i=-C_i^2$. Then:

(a) *In Figure* 1(a), *let (a_1, a_2, a_3) be any of the six triples* (2, 2, 3), (3, 2, 3), (4, 2, 3), (2, 2, 4), (3, 2, 4) *or* (2, 3, 4). *Then* $t_0>0$ (*resp.* $t_0\leqq 0$) *implies* $t_1>0$, $t_2>0$, $t_3>0$ (*resp.* $t_1<0$, $t_2<0$, $t_3<0$).

(b) *In Figure* 1(b) *if* $(a_1, a_2)=(2, 3)$, *then* $t_0>0$ (*respectively*, $t_0\leqq 0$) *implies* $t_1>0$ *and* $t_2>0$ (*respectively* $t_1\leqq 0$ *and* $t_2\leqq 0$).

(c) *In Figure* 1(c) *if* $C_1^2=-1$, *then* $t_0>0$ (*respectively* $t_0\leqq 0$), *implies* $t_1>0$ (*respectively* $t_1<0$).

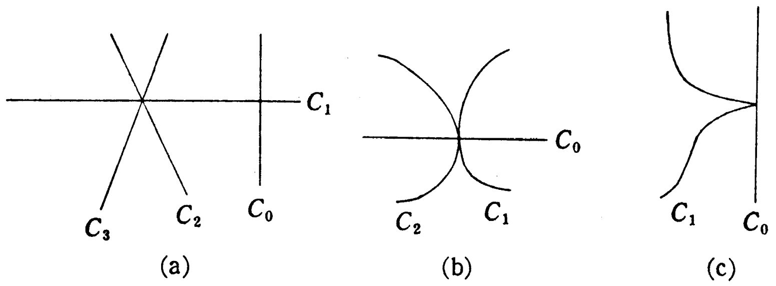


Figure 1.

PROOF. (a) Compute $K_Y\cdot C_i$ and use adjunction formula to obtain the equations:

$$\begin{cases} a_1-2 = t_0+t_2+t_3-a_1t_1 \\ a_2-2 = t_1+t_3-a_2t_2 \\ a_3-2 = t_1+t_2-a_3t_3 . \end{cases}$$

These, in turn, yield

$$(a_2a_3-1)t_2 = (a_3+1)t_1-(a_2a_3-a_3-2),$$
$$(a_2a_3-1)t_3 = (a_2+1)t_1-(a_2a_3-a_2-2),$$
$$(a_2a_3-1)t_0 = (a_1a_2a_3-a_1-a_2-a_3-2)t_1+(a_1a_2a_3-a_1-a_2-a_3-2).$$

Using the fact that t_i are all integers, we easily check the assertion of (a) for any of the triples in the statement of (a).

(b) In Figure 1(b), note that $C_1\cdot C_2=2$ and so we obtain

$$\begin{cases} 0 = -2t_1+2t_2+t_0 \\ 1 = -3t_2+2t_1+t_0 \end{cases}$$

and hence $t_1=(5t_0-2)/2$; $t_2=2t_0-1$, and the conclusion follows easily.

(c) Here again, the adjunction formula yields $C_1\cdot K_Y=1=2t_0-t_1$ so that $2t_0=1+t_1$ and again the conclusion follows easily.

The next lemma helps us to compute the discriminant of certain weighted trees. The idea used here occurs in [**R**].

11.2. LEMMA. (a) *Suppose* Γ *is a weighted tree*, $u_0\in\Gamma$ *is a vertex such that* $\Gamma-\{u_0\}$ *is negative* (*or positive*) *definite. Then* Γ *can be diagonalized in such a way that if* p/q *is the diagonal entry at* u_0 *with* p, q *coprime, then*

$d(\Gamma)(:=|\det(\Gamma)|)=|p|$.

(b) *Suppose Λ is a tree, $v_0, u_0 \in \Lambda$ are any two adjacent vertices, Γ is the branch of Λ at v_0 containing u_0, $\Gamma'=\Lambda-\Gamma$ is the branch at u_0 containing v_0. Suppose both Γ and Γ' can be diagonalized as in* (a) *above with diagonal entries at u_0 and v_0 being p/q and p'/q' respectively. Suppose that $p'\neq 0$ (or $p\neq 0$). Then $d(\Lambda)=|pp'-qq'|$.*

PROOF. (a) We induct on the number of vertices of Γ. Let $\Gamma-\{u_0\}=\Gamma_1 \amalg \cdots \amalg \Gamma_k$, $k\geqq 1$, with $u_i \in \Gamma_i$, being joined to u_0 in Γ. Then each Γ_i can be diagonalized in such a way that if p_i/q_i is the entry at u_i, then $d(\Gamma_i)=|p_i|$. By the definiteness, $p_i\neq 0$. Hence we can further diagonalize the whole of Γ by taking the diagonal entry at u_0 as $\Omega_{u_0}-\sum_{i=1}^{k}(q_i/p_i)=p/q$, say, where $q=p_1\cdots p_k$. Thus $d(\Gamma)=|\det(\Gamma)|=|(p/q)d(\Gamma_1)\cdots d(\Gamma_k)|=|p|$ as claimed.

(b) Since $p'\neq 0$, using the given diagonalization of Γ and Γ' we can further diagonalize Λ, by changing the entry at u_0 to $p/q-q'/p'=(pp'-qq')/p'q$. If $\Gamma_1, \cdots, \Gamma_k$ are the branches of Γ at u_0, then we have $|p|=d(\Gamma)=|(p/q)d(\Gamma_1)\cdots d(\Gamma_k)|$ and hence $|q|=d(\Gamma_1)\cdots d(\Gamma_k)$. Hence $d(\Lambda)=d(\Gamma_1)\cdots d(\Gamma_k)\cdot(|pp'-qq'|/p'q)\cdot d(\Gamma')=|pp'-qq'|$ as required.

11.3. LEMMA. *Let Γ be a unimodular tree with all weights$\leqq -1$, $v\in\Gamma$ be a vertex, Γ_1 be a branch of Γ at v, with $w\in\Gamma_1$ joined to v in Γ. Suppose Γ_1 satisfies the following conditions:*

(i) $\#(\Gamma_1)=6$, $\Omega_w \leqq -2$;

(ii) *The weight-set $\Omega(\Gamma_1-\{w\})$ of $\Gamma_1-\{w\}$, is one of the following four sets:* $\{-2, -2, -2, -2, -2\}$, $\{-3, -2, -2, -2, -2\}$, $\{-3, -3, -2, -2, -2\}$ *or* $\{-4, -2, -2, -2, -2\}$.

(iii) *Γ_1 is not rational* (*see* 5.2).

Then we have:

(a) *Γ_1 is one of the trees listed in Figure* 2.

(b) *Γ_1 is negative definite and we can diagonalize Γ_1 as in* 11.2(a) *so that if p/q is the entry at w then $|p|=|\det \Gamma_1|=d(\Gamma_1)$.*

(c) *For $\Omega_w=-2$, p/q takes the values,* $-16/23$, $-4/17$, $-4/9$, $-27/26$, $-32/23$, $-13/17$, $-13/42$ *and for $\Omega_w=-3$, p/q takes the values* $-39/23$, $-21/17$, $-13/9$, $-53/26$, $-55/23$, $-40/17$, $-55/42$ *respectively. And,*

(d) *If $\{\Lambda_i\}$ are the maximal twigs of Γ_1, not containing w, then* $\sum_i \mathrm{bk}(\Lambda_i)\leqq -7/6$.

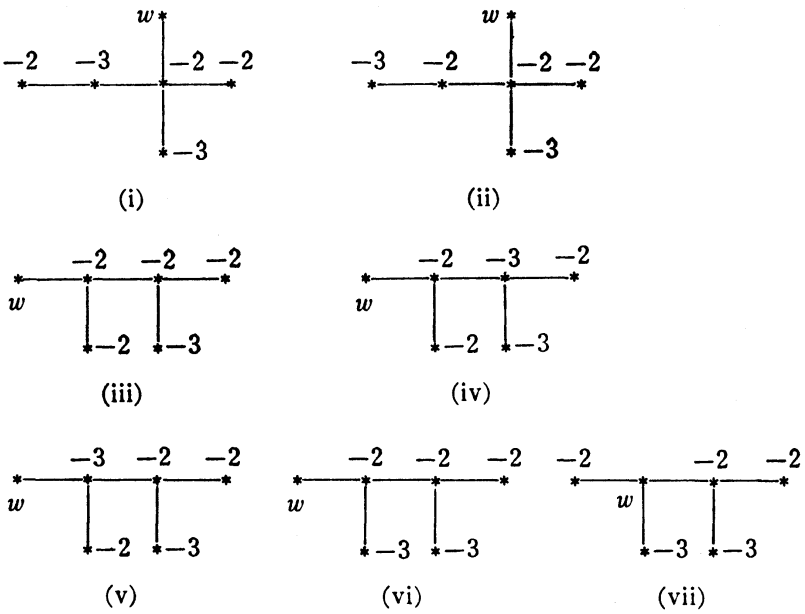


Figure 2.

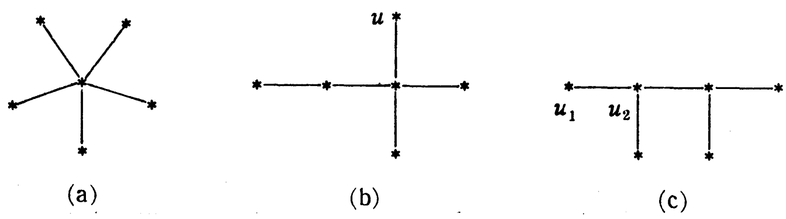


Figure 3.

PROOF. Since Γ_1 is nonrational, by 5.4, it follows that Γ_1 should have one of the configurations shown in Figure 3. By 6.3 (i) configuration (a) is ruled out easily. For the same reason, in (b) it follows that the tip u must be same as w, i.e. $u=w$. And then if $v_1 \in \Gamma_1$ is the branch point, the twigs of Γ at v_1 (which are the twigs of Γ_1 at v_1 not containing w) should have coprime discriminants. Thus it is easily seen that Γ_1 should be either (i) or (ii) of Figure 2.

Finally, in (c) of Figure 3, upto symmetry there are two choices for the location of w, viz., $w=u_1$ or $w=u_2$. The first choice gives precisely the four possibilities (iii)-(vi) for Γ_1 and the second choice precisely the last one in Figure 2. This proves statement (a). Statements (b), (c) and (d) are directly checked.

11.4. LEMMA. *Let Γ be a unimodular tree $v \in \Gamma$ be a vertex at which Γ_1 is a branch of Γ with $w \in \Gamma_1$ being joined to v in Γ. Suppose Γ_1 satisfies the following conditions:*

(i) $\#(\Gamma_1)=7$.

(ii) *$\Omega_w \leqq -2$ and $\Omega(\Gamma_1-\{w\})=\{-2,-2,-2,-2,-2,-2\}$ or $\{-3,-2,-2,-2,-2,-2\}$ and*

(iii) *Γ_1 is not rational.*

Then we have;

(a) *Γ_1 is one of the trees listed in Figure* 4.A *and Figure* 4.B.

(b) *$\Gamma_1-\{w\}$ is negative definite and we can diagonalize Γ_1, as in* 11.2(a), *so that if p/q is the entry at w, then $d(\Gamma_1)=|p|$.*

(c) *For $\Omega_w=-2$, p/q takes the values* $-3/7$, $-12/13$, $-3/22$, $+4/7$, $-4/13$, $-20/21$, $-16/13$, $+1/7$, $+1/30$, $+1/42$, $+1/18$, $+1/4$, $+1/10$, $+4/3$, $16/7$, $4/13$ *and for $\Omega_w=-3$, p/q takes the values* $-10/7$, $-25/13$, $-25/22$, $-3/7$, $-17/13$, $-41/21$, $-29/13$, $-6/7$, $-29/30$, $-41/42$, $-17/18$, $-3/4$, $-9/10$, $+1/3$, $+9/7$, $-9/13$ *respectively.*

(d) *If $\{A_i\}$ are maximal twigs of Γ_1 not containing w, then* $\sum_i \mathrm{bk}(A_i) \leqq -16/15$.

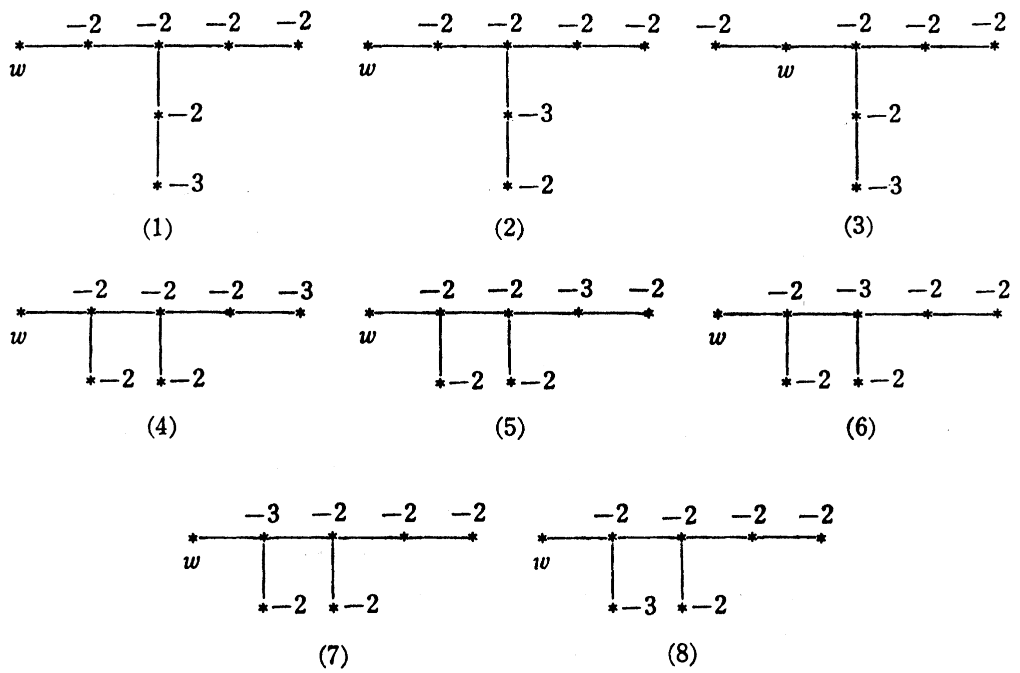
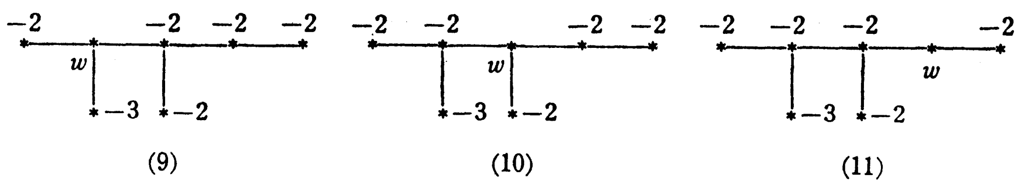


Figure 4.A.



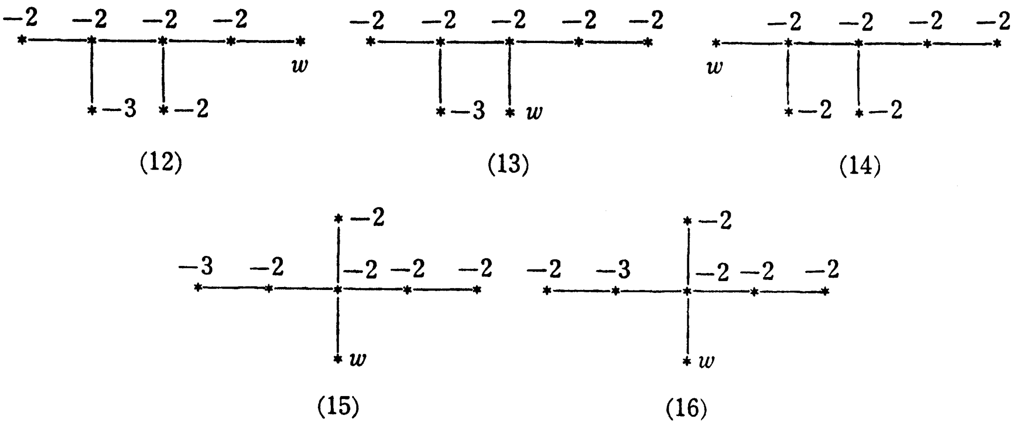


Figure 4.B.

PROOF. We argue exactly as in 11.3. By 5.4, first of all we see that the non-rationality of Γ_1 implies that Γ_1 has one of the configurations (a)-(h) in Figure 5. We then employ 6.1 repeatedly, taking these configurations, one by one, first to determine the possible locations for w, and then with each such location of w, the possible weights at other vertices.

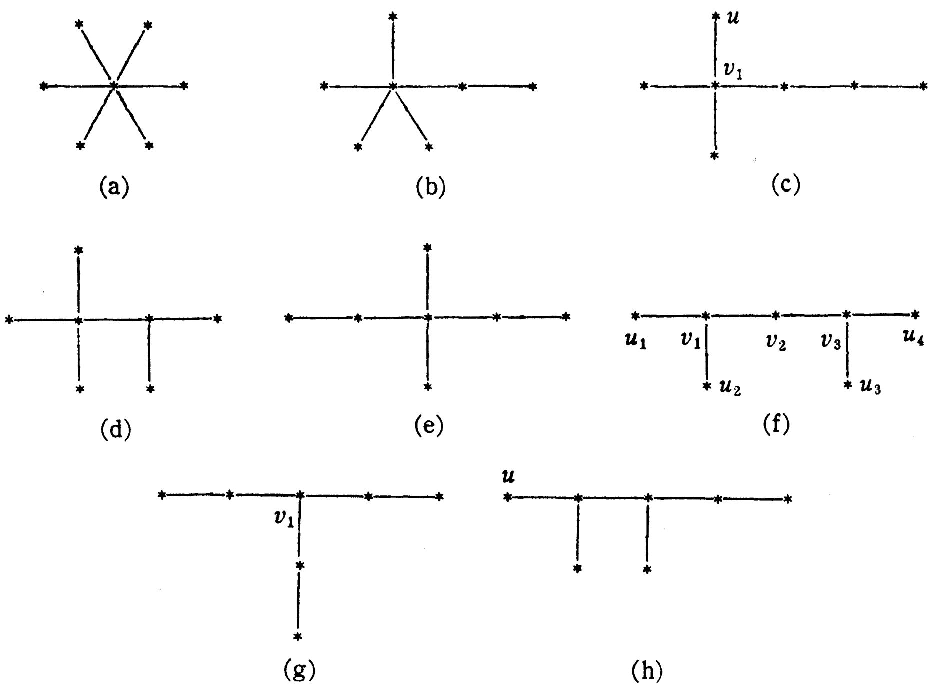


Figure 5.

The configurations (a), (b) and (d) are quickly ruled out. Configuration (c) is ruled out as follows: Upto symmetry, we may assume that $w=u$. Then the two isolated twigs of Γ_1 at v_1 should have weights -2 and -3 respectively.

This forces the third twig at v_1 to be [2, 2, 2] with discriminant even. This contradicts 6.1. Configuration (e) gives the possibilities (15) and (16) of Figure 4. In (f), we first see that w should be one of the four tips, which are symmetrical and hence we may assume $w=u_1$. Then it follows that $\{\Omega_{u_3}, \Omega_{u_4}\}=\{-2, -3\}$, and all other weights are -2. But now at v_1, Γ has two branches, viz. $\{u_2\}$ and $\{v_2, v_3, u_3, u_4\}$ with even discriminants, contradicting 6.1. Thus configuration (f) is also ruled out.

Consider now, the configuration (g). There are, upto symmetry, three different possible locations for w. The possibility $w=v_1$ is ruled out easily. The other two possibilities precisely yield the first three configurations in Figure 4.

Finally, consider the configuration (h). If $\Omega(\Gamma_1-\{w\})=\{-2, -2, -2, -2, -2, -2\}$, then upto symmetry there is a unique choice for w, viz., $w=u$. This then yields configuration (14) of Figure 4. Now let $\Omega(\Gamma-\{w\})=\{-3, -2, -2, -2, -2, -2\}$. Then upto symmetry, there are six different locations for w. If $w=u$, then the (-3)-curve can be located in six different places. Five of these are listed in (4)-(8), and the sixth one is ruled out easily. For the remaining five locations of w, it is easily seen that the location of (-3)-curve is uniquely determined, giving the trees (9)-(13). This proves statement (a).

Statements (b), (c) and (d) are directly checked. This completes the proof of Lemma 11.4.

§ 12. The case $e_1\geqq 1$.

12.0. Returning to the proof of 8.2, note that we have $\lambda\geqq 2$ (by 9.3), and hence $r_4=0$, D is unimodular, $b_2=\beta_2$ (by 3.2), and $r_3+e_1+\sigma+\tau\leqq 2$, by (2.8). By (1.3) we now obtain $K\cdot D\leqq\beta_2-6=b_2-6$. In this section we shall dispose off the case when $e_1\geqq 1$. However, we shall first prove:

12.1. LEMMA. *Suppose D is unimodular and has no (-1)-curves. Then $b_2=b_2(D)\geqq 12$.*

PROOF. Assume on the contrary that, $b_2\leqq 11$. Since D is unimodular, each connected component of D is also unimodular. Hence, by 6.2, it follows that D is connected. Let T be the dual tree of D; $v_0\in T$ be a vertex such that all branches of T at v_0 have less than six vertices. By 4.1, at least one of them, say T_1, is not rational. Hence by 5.4, $\#(T_1)=5$, and by 6.4 (a), there exist $u_1, u_2, u_3\in T_1$ which are tips of T such that $\sum_{i=1}^{3}\Omega_{u_i}\leqq -10$. It follows that $\sum_{u\in T_1}\Omega_u\leqq -14$. Let $u_4\in T_1$ be the vertex joined to v_0 in T. Again, by 4.1, $T-u_4$ should have a branch Γ which is non-rational. Since $\#(T_1-\{u_4\})=4$, it follows that Γ is disjoint from $T_1-\{u_4\}$. Clearly $\#(\Gamma)\leqq 6$. Since $\sum_{u\in T_1}\Omega_u\leqq -14$, and $K\cdot D\leqq 5$, it follows that all components in Γ are (-2)-curves except perhaps one which may be a (-3)-curve. Hence by 6.4, it follows that $\#(\Gamma)=6$, and T should be the tree shown in Figure 6. But then one easily checks that T is not unimodular. This contradiction completes the proof of the lemma.

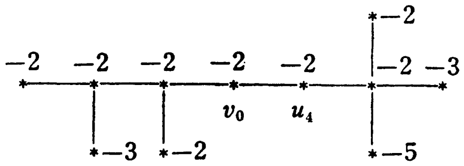


Figure 6. $d=149$.

12.2. We now claim that $e_1 \leqq 1$. Assume the contrary that $e_1=2$. Then we have $r_3=0$ and an equality in (2.8). Hence, by 2.10, $K\cdot D=\beta_2-6$. Let L_1, L_2 be the two components of $\mathcal{E}_1-D$. Then $L_i \subseteqq R_2$ since $r_3=0=r_4$. We claim that $L_1 \cap L_2=\emptyset$. For, if $L_1 \cap L_2 \neq \emptyset$, then, we may assume $L_1^2=-1$, say, so that $L_1\cdot L_2=1$, $L_2^2 \leqq -2$. Since components of D generate $\operatorname{Pic} X$, $L_2\cdot D \geqq 1$. Clearly $L_1\cdot D=2$. Hence it follows that $\beta(L_2) \geqq 3$ contradicting the observation that $L_2 \subseteqq R_2$. So $L_1 \cap L_2=\emptyset$. Now it follows easily that for the curve $C= D\cup L_1 \cup L_2$, we have $b_1(C) \leqq 2$, $K\cdot C \geqq K\cdot D-2=\beta_2-8$, and $b_2(C)=b_2(D)+2=\beta_2+2$. Hence

$$M(X, C) \leqq 4\beta_2+2-1-4-3(\beta_2+2)-(\beta_2-8) = -1$$

contradicting (1.6). Hence $e_1 \leqq 1$ as claimed.

For the rest of the section we shall assume that $e_1=1$ and hence $b_0+\lambda+\sigma+\tau+r_3 \leqq 4$. Let L_0 be the unique component of $\mathcal{E}_1-D$.

12.3. Suppose first that $L_0 \subseteqq R_3$. Then $r_3 \geqq 1$ and hence $r_3=1$, $b_0=1$, $\lambda=2$, $\sigma+\tau=0$ and we have equality in (2.8). Hence by 2.10, $K\cdot D=\beta_2-6$. Let D_1, D_2, D_3 be the three distinct components of D such that $L_0\cdot D_i=1$. Suppose first that all the three points $L_0 \cap D_i$ are distinct. Then we claim that $D_i^2 \leqq -3$, for $i=1, 2, 3$. If not, say, $D_1^2=-2$. Then, it follows that $D_1 \subseteqq \mathcal{E}_1$ and hence $D_1 \subseteqq R_2$. This means that D_1 is an isolated component of D and hence $1=b_0 \geqq 2$ which is absurd. It follows that after blowing down L_0, we obtain the surface X'. Indeed we claim that $X'=X''$. For, first of all $\sigma=0$ implies, by 3.1, that $\mathcal{E}_2 \subseteqq D'$ has at most one component. Clearly this should be one of D_1', D_2' or D_3'. But we have just seen that $D_i^2 \leqq -3$ and hence $D_i'^2 \leqq -2$. Hence $X'=X''$. In particular, it follows that $\beta_2=11$ and D has no (-1)-curves. By the lemma above, this contradicts the unimodularity of D.

Hence the three points $L_0 \cap D_i$, $i=1, 2, 3$, cannot be all distinct. Since D is *NC* all the three points cannot coincide. Hence it must be the case that $L_0 \cap D_1 = L_0 \cap D_2 \neq L_0 \cap D_3$, say. But then for $C=D\cup L_0$, we have $b_1(C)=1$, $b_2(C)=\beta_2+1$, $K\cdot C=K\cdot D-1=\beta_2-7$ and hence

$$M(X, C) = 4\beta_2+1-1-4-3(\beta_2+1)-(\beta_2-7) = 0.$$

We now note that C has tips (since D is not linear) and hence by 1.4, and 1.6, we have a contradiction.

Thus, we have shown that $L_0 \subseteqq R_2$.

12.4. Now let D_1 and D_2 be the components of D such that $D_i \cdot L_0=1$. We shall first show that $L_0 \cap D_1 \neq L_0 \cap D_2$. For, if $L_0 \cap D_1 = L_0 \cap D_2$, then for $C = D \cup L_0$, $b_1(C)=0$, $b_2(C)=\beta_2+1$, and $K \cdot C = K \cdot D - 1 \geqq \beta_2 - 8$ by 2.6 and hence

$$M(X, C) \leqq 4\beta_2-1-4-3(\beta_2+1)-(\beta_2-8) = 0$$

which again leads to a contradiction as above. Hence $L_0 \cap D_1 \neq L_0 \cap D_2$. Here again, if $K \cdot D \geqq \beta_2-6$, then, as above we will have $M(X, C) \leqq 0$ which leads to a contradiction. So we assume that $K \cdot D \leqq \beta_2 - 7$. If $b_0=1$, from (2.10) we see that there is no equality in (2.8) and hence $\sigma+\tau+r_3=0$. If $b_0>1$, we get $b_0+\lambda+\sigma+\tau+r_3 \leqq 4$. Thus $b_0=2$, $\lambda=2$, $r_3=0$, $\sigma=0$ and $\tau=0$. Now by 3.1, it follows that $n_2 \leqq 1$, and $\mathcal{E}_2 \subseteqq D'$.

12.5. Suppose $n_2=1$. Let $\mathcal{E}_2=\{E_1'\}$, $E_1'^2=-1$. Since $\tau=0$, there should exist $\{D_i'\}_{1 \leq i \leq s}$, in D', such that $E_1' \cdot D_1'=2$, $E_1' \cdot D_i'=1$, $2 \leqq i \leqq s$. Clearly, after blowing down E_1', we obtain the minimal surface X''. So one can easily see that $K' \cdot D' = s+\lambda = s+2$ and hence

$$M(X', D') \leqq 44+1-b_0-4-36-(s+2) = 3-s-b_0.$$

Further, since D is not linear, it follows that D' has tips. Hence $M(X', D')>0$, by 1.6, and 1.4. Hence $s=1$, $b_0=1$. Note that E_1' should meet D_1' in two distinct points (otherwise $r_3>0$).

Thus on X'', all components of D'' are smooth except D_1'' which has a node at $\pi_2(E_1')=x$, and all the singularities of D'' are ordinary double point singularities. By 3.1 (i), it follows that $L_0 \subseteqq \pi^{-1}(x)$. So if F is the singular fibre of φ, containing L_0 it follows that $F-L_0 \subseteqq D$, by 9.3. Hence, the fibre P of φ'', through x is contained in D''. Since no other component of D'' passes through x, we conclude that $P_{\mathrm{red}}=D_1''$. But now we can appeal to 9.7 to conclude that $b_1(D'') \geqq 2$ which is absurd, since $b_1(D'')=b_1(D')=1$. Thus we have shown that $n_2=0$, i.e., $X''=X'$.

12.6. It follows that D' is NC, $b_1(D')=b_1(D \cup L_0)=1$, $b_2(D')=11$, $\lambda=2$. Let now $\pi_1(L_0)=x \in X'$. Then by 9.3, it follows that the fibre P through x of $\varphi''(=\varphi')$ is contained in $D''=D'$. Since $b_1(D'')=1$, by 9.7, we conclude that P is of type II*. Further more, if H_1'' and H_2'' are the two horizontal components, then since $\lambda=2$, we have $H_i'' \cdot K''=1$, $i=1, 2$. Hence $H_i'' \cdot P=6$. Again, using the fact that $D'=D''$ is NC and $b_1(D'')=1$, we infer that one of the horizontal

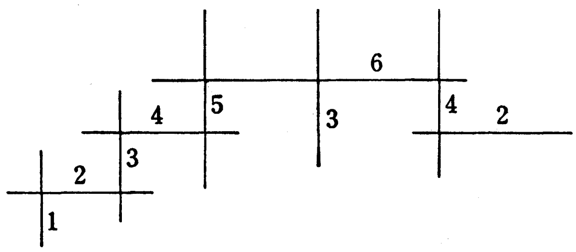


Figure 7. The fibre $P=\mathrm{II}^*$.

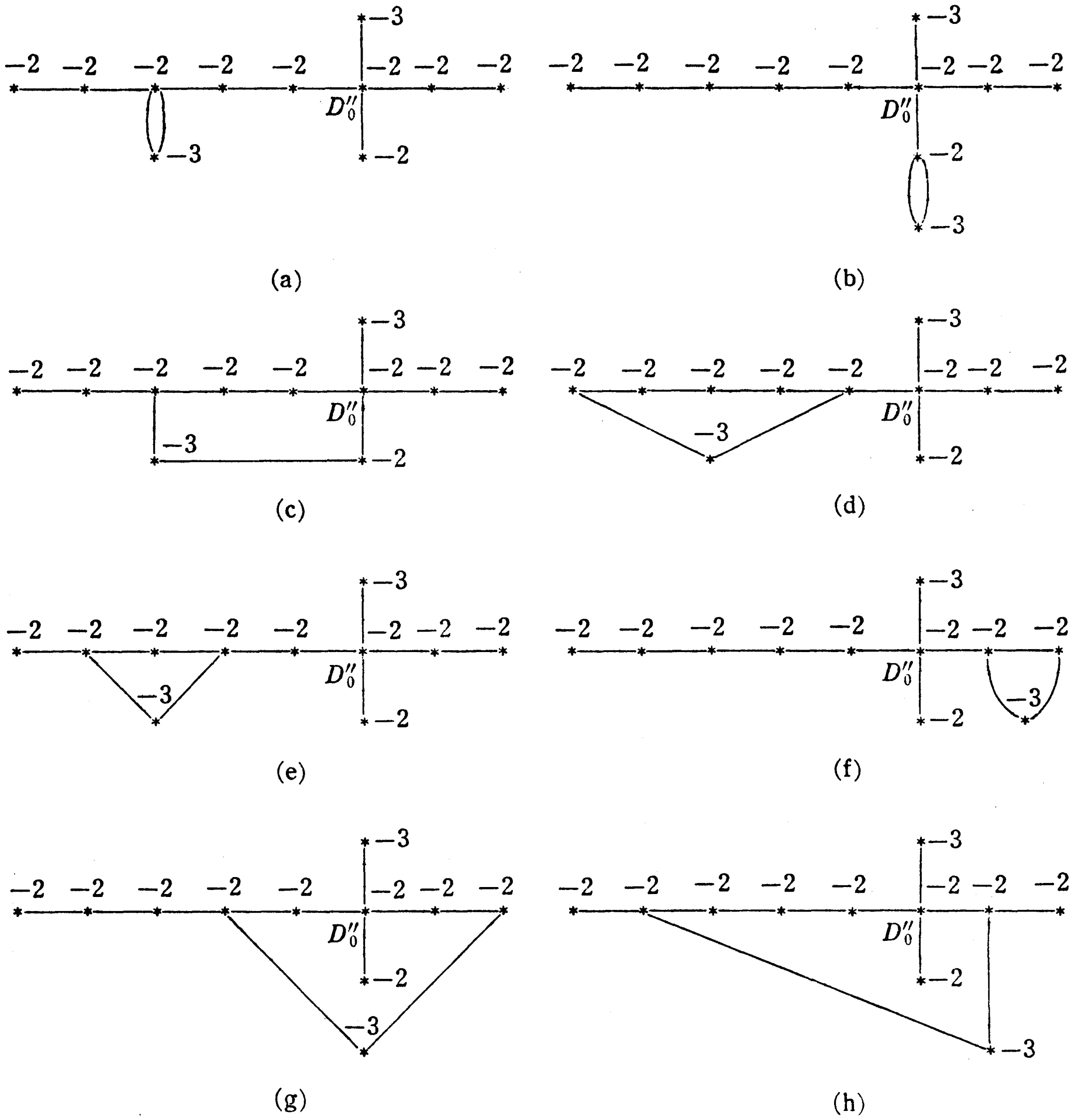


Figure 8.

components say H_1'' meets D_0'' transversely, where D_0'' is the component of P with $\mu(D_0'')=6$. The other component H_2'' will meet two (not necessarily distinct) components D_i'' and D_j'' of P transversely, so that $\mu(D_i'')+\mu(D_j'')=6$. This yields eight possible configuration for the dual graph of D', as shown in Figure 8. Consider the configuration (a). It follows that if D_0 is the proper transform of D_0'' on X, then D has two branches [3] and [2, 2] at D_0 both having discriminant 3. This contradicts 6.1. Thus configuration (a) is ruled out. Exactly for the same reason, (b), (c), (d) and (e) are also ruled out. In (f), D will have two branches [3] and [2, 2, 2, 2, 2] at D_0 with discriminants 3 and 6 respectively. This again contradicts 6.1. In (g) and (h), we consider the curve $C=D''-D_0''$. Then $b_0(C)=3$, $b_1(C)=0$, $b_2(C)=10$, $K\cdot C=2$, and so

$$M(X'', C)=40-3-4-30-2=1\,.$$

If T_1, T_2 and T_3 denote the components of C, then by 10.4, $\mathrm{bk}(C)=\sum \mathrm{bk}(T_i)<-2-4/3-(1/2+1/2+1/2)<-4$. Hence by 1.6 and 10.5, we have,

$$0\leqq M(X'', C)+\frac{1}{4}N^2\leqq 1+\mathrm{bk}(C)<1-1=0$$

which is absurd.

This proves that $e_1=0$.

§ 13. The case $r_3=0$.

13.0. We shall now dispose of the case $r_3=0$. Since D is MNC, $r_3=r_4=e_1=0$ implies that $n_1=0$, i.e., $X=X'$, and D is free from (-1)-curves. By 12.1, it follows that $\beta_2=b_2\geqq 12$ and hence $n_2\geqq 2$. Hence, by 3.1, it follows that $\sigma\geqq 2$ and hence $\sigma=2$, and $n_2=2$. Let $\mathcal{E}_2=\{E_1', E_2'\}$.

13.1. Suppose $E_1'\cap E_2'=\emptyset$. Then, it follows that $E_1'^2=E_2'^2=-1$. Since $\tau=0$ (and $n=2$), $m_{t,i}=2$ for all (t, i). Hence both E_i' are outside D. Also it follows that there exist $\{D_{i,j}'\}_{1\leqq j\leqq s_i}\subseteqq D'$ such that $E_i'\cdot D_{i,1}'=2$, $E_i'\cdot D_{i,j}'=1$, $2\leqq j\leqq s_i$, $s_i\geqq 1$, for $i=1, 2$. Note that for a fixed i, $D_{i,j}'$ are distinct. Now, there is an equality in (2.8) and hence $K\cdot D=\beta_2-6=6$. On the other hand using the fact that $\lambda=2$, one also computes easily that $K\cdot D=s_1+s_2+4$. Hence $s_1=s_2=1$. Now for the curve $C=D'\cup E_1'\cup E_2'$, we have $b_0(C)=1$, $b_1(C)\leqq 2$, $b_2(C)=b_2(D')+2=14$ and $K\cdot C=4$. Hence,

$$M(X', C')\leqq 48+2-1-4-42-4=-1$$

contradicting (1.6).

13.2. Hence $E_1'\cap E_2'\neq\emptyset$ and we may assume that $E_1'^2=-1$, $E_1'\cdot E_2'=1$, $E_2'^2=-2$. Again by 3.1, $E_i'\not\subset D'$, and there exist components $\{D_{i,j}'\}$ of D' such that $E_1'\cdot D_{1,1}'=2$, $E_1'\cdot D_{1,j}'=1$, $2\leqq j\leqq s_1$, $(s_1\geqq 1)$, and $E_2'\cdot D_{2,j}'=1$, for $1\leqq j\leqq s_2$, $s_2\geqq 0$.

Now for the curve $C=D'\cup E_1'\cup E_2'$, it follows that, $b_0(C)=1$, $b_1(C)\leqq s_1+s_2$, $b_2(C)=14$, and $K\cdot C=2s_1+s_2+3$. Hence,

$$M(X',\ C)\leqq 48+s_1+s_2-1-4-42-(2s_1+s_2+3)<0$$

contradicting (1.6).

Thus, we have shown that $r_3\geqq 1$.

§ 14. The case $r_3=1=\sigma$.

14.0. So far we have proved that $r_4=e_1=0$ and $r_3\geqq 1$. Hence $\sigma\leqq 1$. In this section we shall show that $\sigma=0$. Assuming that $\sigma=1$, we have $r_3=1$, $b_0=1$, $\lambda=2$, $\tau=0$, $b_2=\beta_2$, $K\cdot D=\beta_2-6$, etc. from (2.8). Let L_0 be the unique (-1)-curve in R_3. Note that since $e_1=0$, $\{L_0\}=R_3\subset D$. Let L_1, L_2, L_3 be the three components of D such that $L_0\cdot L_i=1$.

14.1. Suppose $L_i^2\leqq -3$, $i=1, 2, 3$. Then, it follows that after blowing-down L_0, we obtain X'. Now $\sigma=1$ implies, by 3.1, that $\mathcal{E}_2=\{E'\}$ is not contained in D, and there exist components $\{D_i'\}$ of D' such that $E'\cdot D_1'=2$, and $E_1'\cdot D_j'=1$, $2\leqq j\leqq s$, $s\geqq 1$. It follows that $\beta_2(X')=11=b_2(D')$. Since $\lambda=2$, one easily computes that $K'\cdot D'=s+3$, $K\cdot D=s+5=\beta_2-6=6$. Hence, $s=1$. Take $C'=D'\cup E'$. Then $K'\cdot C'=3$, $b_1(C')\leqq 1$, and hence

$$M(X',\ C')\leqq 44+1-1-4-36-3=1\,.$$

Now consider the (nontrivial) contraction $\varphi_1: X\to X'$. After blowing up at $E'\cap D_1'$ if necessary, we obtain a contraction $\alpha: \tilde{X}\to X$ such that $\tilde{C}=\alpha^{-1}(C)$ is NC. Then $M(\tilde{X},\ \tilde{C})\leqq M(X,\ C)$. And as in the proof of 1.7 we see that $M(X,\ C)\leqq M(X',\ C')-1\leqq 0$. Hence $M(\tilde{X},\ \tilde{C})\leqq 0$. On the other hand since C has tips so does $\tilde{C}$. Hence by (1.4) and (1.6) $M(\tilde{X},\ \tilde{C})>0$, which is absurd.

Thus, it follows that $L_i^2=-2$, for some $i=1, 2, 3$, say $L_3^2=-2$. Since $r_3=1$, it follows that $L_3\subseteqq R_2$ and hence L_3 should be a tip of D. Clearly $L_1^2\leqq -3$, and $L_2^2\leqq -3$.

14.2. Now, suppose that $L_1^2\leqq -4$, $L_2^2\leqq -4$. Then, it follows that after blowing down L_0 and then L_3, we obtain X'. Now arguing exactly as in 14.1, we obtain $\{E_1'\}=\mathcal{E}_2$, $E_1'\not\subset D'$ etc., and for $C'=D'\cup E_1'$, $M(X',\ C')\leqq 1$ which leads to a contradiction, as above. Hence we may assume that $L_2^2=-3$, so that the image L_2' of L_2 on X', is a (-1)-curve. Note that $L_1'\cdot L_2'=2$, $L_1'\cap L_2'=\{x\}$, say. By 3.1, it follows that $\mathcal{E}_2$ consists of one more component, besides L_2', say, $\mathcal{E}_2=\{E_1',\ L_2'\}$ and we must have one of the following two cases:

(a) E_1' and L_2' are disjoint, $E_1'^2=-1$, and $E_1'\not\subset D'$, or

(b) $E_1'\cdot L_2'=1$, $E_1'^2=-2$, and $E_1'\subset D'$.

14.3. Consider the case (a) above. It follows that there exist components $\{D'_{i,j}\}_{1\leq j\leq s_1}$ of D' with $E'_1\cdot D'_{1,1}=2$, $E'_1\cdot D'_{1,j}=1$, $2\leqq j\leqq s_1$, $s_1\geqq 1$. Let $\{D'_{2,j}\}_{1\leq j\leq s_2}$, be the components of D', such that $L'_2\cdot D'_{2,j}=1$. (Clearly $D'_{2,j}\neq L'_1$.) Then one easily computes that $K'\cdot D'=2+(s_1-1)+2+s_2+2-1=s_1+s_2+4$. Again take $C'=D'\cup E'_1$. Then $K'\cdot C'=K'\cdot D'-1=s_1+s_2+3$, $b_1(C')\leqq s_1$ and hence $M(X', C')$ $=48+s_1-1-4-39-(s_1+s_2+3)=1-s_2\leqq 1$. This again leads to a contradiction as in 14.1, using the fact that C' has tips.

14.4. Consider the case (b). Let now, $\{D'_{1,j}\}$ and $\{D'_{2,j}\}$ be components of D' such that $D'_{2,j}\cdot L'_2=1$, $D'_{2,j}\neq E'_1$, $1\leqq j\leqq s_2$, $s_2\geqq 0$. $D'_{1,j}\cdot E'_1=1$, $D'_{1,j}\neq L'_2$, $1\leqq j\leqq$ $\leqq s_1$, $s_1\geqq 0$. Again, one easily, computes that $K\cdot D=s_1+2s_2+8=\beta_2-6=8$, and hence $s_1=0=s_2$. In particular, it follows that E_1, the proper transform of E'_1 on X is a (-2)-curve and is a tip of D, and L_2 intersects only E_1 and L_0. Thus the curve $C=D-\{L_0\}$ has three connected components T_1, T_2, T_3 say, where $T_3=\{L_3\}=[2]$, and $T_2=\{L_2\cup E_1\}=[2, 3]$. On the other hand we have:

$$M(X, C)=56-3-4-39-9=1\,.$$

Hence, by 10.5, and 1.6, $\mathrm{bk}(C)\geqq -4$. On the other hand $\mathrm{bk}(C)=\sum\mathrm{bk}(T_i)=$ $\mathrm{bk}(T_1)-7/5-2$. We shall presently show that $\mathrm{bk}(T_1)<-3/5$ which leads to a contradiction.

As before, by 4.1, it follows that T_1 is non-rational, and hence has at least three tips. One easily sees that L''_1 is horizontal and so $L''_1\cdot K''>0$. Since there is another horizontal component in D'', say, L''_4, it follows that $L''_1\cdot K''=L''_4\cdot K''$ $=1$ $(\lambda=2)$. It follows that $L_1^2=-9$ and $L_4^2=-3$, and all other components of T_1 are (-2)-curves. Hence by 10.4, $\mathrm{bk}(T_1)\leqq -(1/9+1/3+1/2)<-3/5$ as required.

This completes the proof of the claim $\sigma=0$.

14.5. REMARK. At this stage, it is not hard to see that there are only finitely many possibilities for the dual graph T of D.

§ 15. The case $r_3=1$.

15.0. To sum-up, so far, we have proved that $r_4=e_1=\sigma=0$ and $r_3\geqq 1$. Of course, $3\geqq\lambda\geqq 2$, $b_2=\beta_2$, $b_0\geqq 1$, $b_0+r_3+\tau+\lambda\leqq 5$. In this section we shall dispose off the case $r_3=1$. So, assume now that $r_3=1$, so that $b_0+\tau+\lambda\leqq 4$. Since $\sigma=0$, by 3.1, there is a unique (-1)-curve L_0 on X, $\{L_0\}=R_3\subset D$ and there are precisely three components of D which meet L_0, say, $L_i\cdot L_0=1$, $i=1, 2, 3$.

Again, by 3.1, it follows that either $\mathcal{E}_2=\emptyset$ or $\mathcal{E}_2=\{E'_1\}\subset D'$. In 15.1-15.4 we shall show that $\mathcal{E}_1$ has at least two components. Equivalently, we will show that one of the L_i, $i=1, 2, 3$, is a (-2)-curve. Then in 15.5-15.9 we will show that $\mathcal{E}_2\neq\emptyset$ and in the rest of the section we investigate the case when $\mathcal{E}_2\neq\emptyset$.

For a tree Γ of smooth rational curves let us introduce the notation:

$$\lambda(\Gamma) := -\sum_{C\in\Gamma}(C^2+2) = \sum_{C\in\Gamma}(C\cdot K).$$

15.1. So, assume that $L_i^2\leqq -3$ for $i=1, 2, 3$. It follows that after blowing-down L_0, we obtain the surface $X'=X''$, (using 3.1). Since $b_1(D')=b_1(D)=0$, it follows from 9.7 (or otherwise) that not all L_i are (-3)-curves, $i=1, 2, 3$. (For, if so, $L_1'\cup L_2'\cup L_3'$ will be a full fibre of φ'' contained in D''.) Thus $L_i^2\leqq -4$ for some $i=1, 2, 3$.

15.2. Let now T_1, T_2, T_3 be the branches of T at L_0, with $L_i\in T_i$, $i=1, 2, 3$. As argued in 12.1, it follows from 4.1 that one of the T_i say T_1, is non-rational. Since $\lambda\leqq 3$, it follows, from 5.4 and 6.4 that $\#(T_1)\geqq 6$. Since $\#(T)=b_2(D)=11$, one easily checks that $T_2\cup\{L_0\}\cup T_3$ is rational. Hence, again by 4.1, it follows that $T_1-\{L_1\}$ should have a branch Γ_1 which is non-rational, and then again $\#(\Gamma_1)\geqq 6$. Since $L_i^2\leqq -4$ for some i, it follows that $\lambda(\Gamma_1)\leqq 2$.

15.3. Suppose $\#(\Gamma_1)=6$. By Lemma 11.3, it follows that Γ_1 is one of the trees in Figure 2 where w is the vertex joined to L_1, in T_1. Using 6.1, we list the possibilities for $\Gamma_1'=T-\Gamma_1$ in Figure 9. We see that these trees can be diagonalized in such a way that if p'/q' is the entry at L_1 then $d(\Gamma_1')=|p'|$, and p'/q' has values $-3/8$, $-11/8$, $-11/13$, $-6/13$, $-1/7$, $-1/4$, $-4/7$, $-8/7$, $-5/4$, $-1/10$, $-4/15$, $-11/10$ and $-5/14$. With p/q given by Lemma 11.3, using 11.2, we see that $d(T)=|pp'-qq'|\neq 1$, for any of these values.

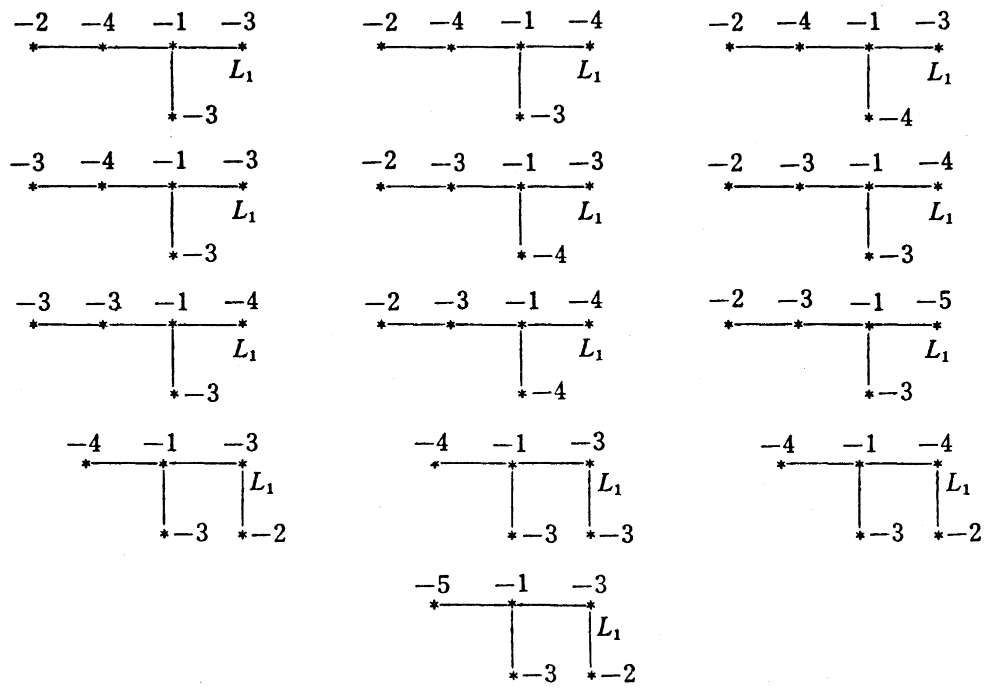


Figure 9.

15.4. Thus we have shown that $\#(\Gamma_1)\geqq 7$. Indeed $\#(\Gamma_1)=7$, $\#(T_1)=8$, so that $\#(T_2)=\#(T_3)=1$. We can now apply 11.1(a) with $Y=X'$, $C_i=L_i'$, $i\geqq 1$ and C_0 as the image of the curve dual to $w\in\Gamma_1$. Since $\lambda\leqq 3$, one easily sees that the triple $(a_1, a_2, a_3)=(-L_1'^2, -L_2'^2, -L_3'^2)$ is one of the six triples mentioned in 11.1(a). Since K' is supported on D', this leads to a contradiction by 11.4.

Thus as claimed in 15.0, one of the L_i is a (-2)-curve, say, $L_3^2=-2$. Then clearly $L_1^2\leqq -3$; $L_2^2\leqq -3$. We shall now show (in 15.5-15.9) either L_1 or L_2 is a (-3)-curve.

15.5. Suppose that $L_1^2\leqq -4$, $L_2^2\leqq -4$. Clearly, $R_3=\{L_0\}$, and so $T_3=\{L_3\}=[2]$. After blowing-down L_0 and L_3 we obtain the surface $X'=X''$. So $b_2=12$. As before, using 9.7, we see that not both L_1 and L_2 are (-4)-curves. By 4.1, either T_1 or T_2, say T_1 is non-rational, and again by 5.4 and 6.4, $\#(T_1)\geqq 6$, so that $\#(T_2\cup\{L_0, L_3\})\leqq 6$. So one easily sees that $T_2\cup\{L_0, L_3\}$ is rational and hence, again by 4.1, $T_1-\{L_1\}$ should have a branch Γ_1 which is non-rational and as before $\#(\Gamma_1)\geqq 6$. Let $u_0\in\Gamma_1$ be the vertex joined to L_1 in T. Since $L_1''^2+L_2''^2\leqq -5$, it follows that $\lambda(\Gamma_1)\leqq\lambda-1$.

15.6. Suppose further that $\lambda=3$. Then we have $K\cdot D=12-6=6$. Hence, for $C=D-\{L_0\}$, we have $K\cdot C=7$, $b_0(C)=3$, $b_1(C)=0$, $b_2(C)=11$. Hence $M(X, C)=48-3-4-33-7=1$. By 1.6 and 10.5, it follows that $\mathrm{bk}(C)\geqq -4$. On the other hand $\mathrm{bk}(C)=\mathrm{bk}(T_1)+\mathrm{bk}(T_2)-2$. Below, we shall show in 15.7, that $\mathrm{bk}(T_1)+\mathrm{bk}(T_2)<-2$ which is absurd, thus proving that $\lambda=2$.

15.7. Clearly, $\{-L_1''^2, -L_2''^2\}=\{2, 3\}, \{3, 3\}, \{2, 4\}$. ($\lambda(\Gamma_1)>0$, for otherwise Γ_1 will be contained in a fibre of φ'', hence rational. Thus $\{-L_1''^2, -L_2''^2\}\neq\{3, 4\}$.) Accordingly, $\{-L_1^2, -L_2^2\}=\{4, 5\}, \{5, 5\}, \{4, 6\}$. Thus it follows that the weight set $\Omega(T_1\cup T_2)$ is one of the following:

$$\{-5, -4, -3, -3, -2, -2, -2, -2, -2, -2\},$$
$$\{-5, -4, -4, -2, -2, -2, -2, -2, -2, -2\},$$
$$\{-5, -5, -3, -2, -2, -2, -2, -2, -2, -2\} \quad \text{or}$$
$$\{-4, -6, -3, -2, -2, -2, -2, -2, -2, -2\}.$$

Suppose $T_1\cup T_2$ has at least six tips. Then by 10.4 it follows that $\mathrm{bk}(T_1\cup T_2)\leqq -\sum_{i=1}^{6}(1/a_i)$, for some $\{a_1, \cdots, a_6\}\subset\Omega(T_1\cup T_2)$, which is easily seen to be less than -2. So we may assume that $T_1\cup T_2$ has fewer than six tips.

Suppose $\#(\Gamma_1)=6$. Then by 11.3, it follows that T_1 has at least 4 tips. Hence $t_2=\#(T_2)=1$. This means $T_2=[5]$, $\mathrm{bk}(T_2)=-4/5$. And it is easily checked that $\mathrm{bk}(\Gamma_1)<-6/5$ as desired. So we may assume that $\#(\Gamma_1)\geqq 7$.

Suppose $\lambda(\Gamma_1)\leqq 1$. If $t_2=2$ then by 11.4, it follows that $T_1\cup T_2$ has at least six tips, except in the case (1) or (2) of 11.4. In these two cases we directly check that $d(T)\neq\pm 1$, for various possibilities of T. If $t_2=1$, then $T_2=[5]$ and one easily verifies that $\mathrm{bk}(T_1)<-6/5$. Hence $\lambda(\Gamma_1)=2$.

Now let $t_2\neq 1$. Since $\lambda(\Gamma_1)=2$ it follows that $\{-L_1''^2, -L_2''^2\}=\{2, 3\}$ and hence we can apply 4.3(2) with $L_2''=C_3$, $L_1''=C_2$, to conclude that $\Gamma_1-\{u_0\}$ should have branch Γ_2 which is not rational, where $u_0\in\Gamma_1$ is the vertex joined L_1 in T_1; $\#(\Gamma_2)\geqq 6$, and hence $\#(\Gamma_2)=6$. Again, by Lemma 11.3, it follows that $(T_1\cup T_2)$ has at least six tips. Hence we must have $t_2=1$.

It follows that $T_2=\{L_2\}=[5]$, so that $\mathrm{bk}(T_2)=-4/5$. Thus we have to show that $\mathrm{bk}(T_1)<-6/5$. Looking at $\Omega(T_1)$, this is obvious if T_1 has at least four tips. Thus, since T_1 is non-rational, we have to consider the case when T_1 has precisely three tips. Let Λ_1, Λ_2, Λ_3 be its maximal twigs. Since $T_1-\{L_1\}$ is also nonrational, we may assume that $L_1\in\Lambda_1$. Further, if Λ_2 and Λ_3 consist of only (-2)-curves then by 6.1, it follows that $\#(\Lambda_2)+\#(\Lambda_3)\geqq 3$ and hence $\mathrm{bk}(\Lambda_2)+\mathrm{bk}(\Lambda_3)\leqq -(1/2+2/3)$ so that $\mathrm{bk}(T_1)\leqq -(1/2+2/3+1/6)<-6/5$. So we may assume that Λ_2 has a (-3) or (-4)-curve. In particular, it follows that $L_1^2\neq -6$, i.e. $L_1''^2=-2$ or -3.

Thus we apply 4.3(1) with $Y=X''$, $C_3=L_2''$ and $C_2=L_1''$. Along with 4.1, this implies that $\Gamma_2=T_1-\{L_1, u_0\}$ is nonrational where u_0 is the vertex joined to L_1, in T_1. Note that $\#(\Gamma_2)=7$ and hence by 5.4 it follows that T_1 itself has the configuration as shown in Figure 10. Further if $L_1^2=-5$, then $\lambda(\Gamma_2)\leqq 1$ and so, by 11.4 Γ_2 is (1) or (2) of Figure 4. Hence $\mathrm{bk}(\Lambda_2)+\mathrm{bk}(\Lambda_3)\leqq -16/15$ and hence $\mathrm{bk}(T_1)\leqq -16/15-1/5<-6/5$. Finally, if $L_1^2=-4$, then it follows that $\Lambda_2=[4, 2]$, $[3, 3]$ or $[3, 2]$, and hence $\Lambda_3=[2, 2]$ always (using 6.1). But then $\mathrm{bk}(T_1)\leqq -(1/4+2/7+2/3)<-6/5$ as desired, in 15.6.

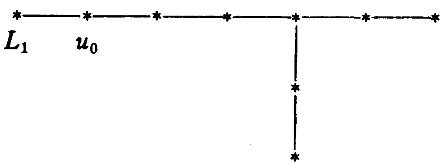


Figure 10.

15.8. Thus we have $\lambda=2$, and so $\lambda(\Gamma_1)\leqq 1$. Suppose, now that $\#(\Gamma_1)=6$. Then by 11.3, Γ_1 should be as in (iii) of Figure 2, with $u_0=w$, $\Omega_w=-2$. $d(\Gamma_1)=4$, and $\lambda(\Gamma_1)=1$. Hence we have $\{L_1''^2, L_2''^2\}=\{-2, -3\}$, i.e. $\{L_1^2, L_2^2\}=\{-4, -5\}$, $L_0^2=-1$ and all other components of $\Gamma_1'=T-\Gamma_1$ are (-2)-curves. Also, since $d(\Gamma_1)=4$, by 6.1, all other branches of T at L_1 should have odd discriminants. Using 6.1, at other vertices of T also, it follows that the only possibility for T is the one shown in Figure 11. But one easily checks that even this tree is not unimodular. Hence $\#(\Gamma_1)\geqq 7$.

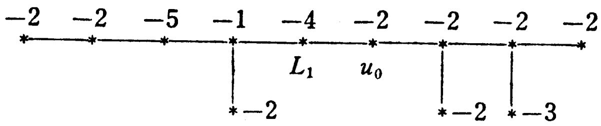


Figure 11.

15.9. We now apply 4.3, with $Y=X''$, C_1 as the component dual to $u_0 \in \Gamma_1$, $C_2=L_1''$ and $C_3=L_2''$. It follows that $\Gamma_1-\{u_0\}$ has a branch Γ_2 which is non-rational, $\#(\Gamma_2)\geqq 6$. Let $u_1 \in \Gamma_2$ be the vertex joined to u_0. If $\#(\Gamma_2)=6$, then Γ_2 is as in (iii) of Figure 2 with $u_1=w$, $\Omega_w=-2$, $d(\Gamma_2)=4$. Arguing as in 15.8 it follows that $\Gamma_2'=T-\Gamma_2$ is one of the configurations shown in Figure 12. We can diagonalize each of these tree, with the diagonal entry p'/q' at u_0 taking the values 1/1, +1/2 and +4/1 respectively, where as the diagonal entry p/q at $u_1=w$, for Γ_2 is $-4/9$. Hence $d(T)=|pp'-qq'|\neq 1$, which proves that $\#(\Gamma_2)\geqq 7$.

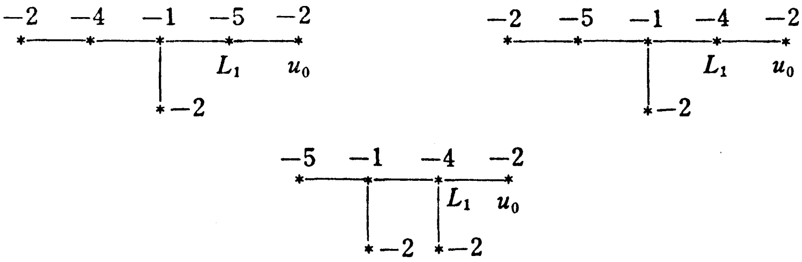


Figure 12.

Thus $\#(\Gamma_2)=7$; $\#(T_1)=9$, so that L_2, L_3 are tips of T and hence by 6.1, L_2^2 is odd and so $L_2^2=-5$. Lemma 11.4 gives various possibilities for Γ_2 with $w=u_1$. Since Γ_2 is non-rational, $\lambda(\Gamma_2)>0$ and hence $\lambda(\Gamma_2)=1$. Then Γ_2' should be the tree in Figure 13. We can diagonalize this with diagonal entry p'/q', at u_0 given by $p'/q'=-1/2$. Using the value of p/q from 11.4, we see that $d(T)=|pp'-qq'|\neq 1$, for any of the values of p/q. This contradiction, then proves that either $L_1^2=-3$ or $L_2^2=-3$, as claimed in 15.4.

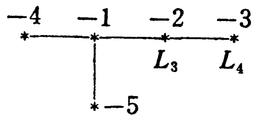


Figure 13.

15.10. Thus in the remaining paragraphs we shall assume that $L_2^2=-3$. It follows that after blowing down L_0, L_3 and L_2 successively, we obtain X''. The image L_1'' of L_1 is a rational curve with a cusp; $L_1''^2+L_1''\cdot K''=0$. We claim that either $\lambda=3$ and L_2 is a tip of D or $\lambda=2$ and L_2 is linear in D (i.e., L_2

meets at most two components of D). For if L_2 meets s components of D, then one easily computes that $K\cdot D=\lambda+s+3$. On the other hand $K\cdot D\leqq\beta_2-6=7$ and $K\cdot D=\beta_2-6$ if $\lambda=3$ (by 2.8 and 2.10). Hence $s\leqq 4-\lambda$ which proves the claim.

In particular, through the cusp of L_1'' there passes at most one other component of D''. Hence by 9.7, we conclude that L_1'' is horizontal; $L_1''\cdot K''=1$ or 2. Accordingly, $L_1^2=-7$ or -8. In 15.11, below we shall dispose off, the case $\lambda=3$.

15.11. Assume now that $\lambda=3$. As seen above, L_2 is a tip of D. We now apply (1.3) to $C=D-\{L_0\}$. Letting T_1, T_2, T_3 denote the three branches of C, as before, this yields; $-4\leqq \mathrm{bk}(C)\leqq \mathrm{bk}(T_1)-4/3-2$, and hence $\mathrm{bk}(T_1)\geqq -2/3$.

On the other hand, it follows that L_1'' is horizontal. Since $\lambda=3$ and there is at least one more horizontal component of D'', which is of course a smooth rational curve, it follows that the weight set $\Omega(T_1)$ is one of the following:

$$\{-8, -3, -2, -2, -2, -2, -2, -2, -2, -2\},$$
$$\{-7, -4, -2, -2, -2, -2, -2, -2, -2, -2\} \quad \text{or}$$
$$\{-7, -3, -3, -2, -2, -2, -2, -2, -2, -2\}.$$

Moreover T_1 is non-rational and hence it has at least three tips. Hence, as seen before, $\mathrm{bk}(T_1)<-2/3$. This contradiction shows that $\lambda=2$.

15.12. So, from now on, we have $\lambda=2$. In 15.12-15.14, we shall show that L_2 is a tip of D. Assuming the contrary, as seen in 15.10, it follows that L_2 will meet another component, say L_4, $L_2\cdot L_4=1$, and $L_4^2=-3$ or -4; $L_1^2=-7$ (because there is one more horizontal curve) $K\cdot D=\lambda+s+3=7$. Hence for $C=D-\{L_0\}$, by (1.3), we obtain, $\mathrm{bk}(C)\geqq -4$. We shall show that $\mathrm{bk}(T_1)+\mathrm{bk}(T_2)<-2$ thereby arriving at a contradiction and proving that L_2 is a tip.

Note that the weight set

$$\Omega(T_1\cup T_2)=\{-7, -4, -3, -2, -2, -2, -2, -2, -2, -2, -2\} \quad \text{or}$$
$$\{-7, -3, -3, -3, -2, -2, -2, -2, -2, -2, -2\}.$$

Hence, if $T_1\cup T_2$ has more than five tips then clearly $\mathrm{bk}(T_1)+\mathrm{bk}(T_2)<-2$ (by 10.4). So we shall assume that $T_1\cup T_2$ has at most five tips and hence at least one of them is linear. On the other hand, at least one of them is non-rational and hence $T_1\cup T_2$ has at least four tips.

15.13. Assume first that T_2 is linear. Then T_2 has two tips and so T_1 has three tips, $T_1-\{L_1\}$ has a non-rational branch Γ_1 and hence Γ_1 also has three tips. By 11.3, it follows that $\#(\Gamma_1)\geqq 7$, and if $\#(\Gamma_1)=7$ then by 11.4, it is either (1) or (2) of Figure 4, with $w\in\Gamma_1$ being the vertex joined to L_1 and

$\lambda(\Gamma_1)=1$. It follows that $T_2=[3, 3, 2]$ and hence $\mathrm{bk}(T_2)=-15/13$, by 10.4. By 11.4, we have $\mathrm{bk}(T_1)\leqq -16/15$ and hence $\mathrm{bk}(T_1)+\mathrm{bk}(T_2)<-2$ as required. Thus $\#(T_1)=8$. Again T_1 should have a (-3)-curve and so $T_2=[3, 3]$, $d(T_2)=8$. This contradicts 6.1.

15.14. It follows that T_1 is linear. Note that $\Gamma_1=T_2-L_2$ is connected. It should be non-rational and as usual we must have $\#(\Gamma_1)\geqq 6$. Suppose $\#(\Gamma_1)=6$, so that $\#(T_2)=7$. By 11.3 and 10.4, $\mathrm{bk}(T_2)\leqq -7/6-1/3=-3/2$. On the other hand, $\#(T_1)=4$, and $T_1-\{L_1\}$ consists of (-2)-curves only i.e. $T_1=[7, 2, 2, 2]$. Hence $\mathrm{bk}(T_1)\leqq -(1/7+1/2)=-9/14$. Thus $\mathrm{bk}(T_1)+\mathrm{bk}(T_2)<-2$. Suppose now that $\#(\Gamma_1)=7$. By 11.4 it follows that $\mathrm{bk}(T_2)\leqq -(16/15+1/3)=-7/5$. Also $\#(T_1)=3$, $d(T_1)$ is odd and so $T_1=[7, 3, 2]$, or $[7, 2, 2]$. Hence $\mathrm{bk}(T_1)\leqq -9/14$ so that $\mathrm{bk}(T_1)+\mathrm{bk}(T_2)<-2$.

Now suppose $\#(\Gamma_1)=8$. Then it follows that $T_1=[7, 2]$ (since if $T_1=[7, 3]$ then $d(T_1)$ is even). Hence $\mathrm{bk}(T_1)=-11/13$. Note that L_4 is not a tip of T_2. Hence one of the tips of T_2 is a (-2)-curve. Hence $\mathrm{bk}(T_2)\leqq -(1/3+1/3+1/2)$ $=-7/6$ and again we have $\mathrm{bk}(T_1)+\mathrm{bk}(T_2)<-2$.

So finally let $\#(\Gamma_1)=9$, so that $T_1=[7]$; $\mathrm{bk}(T_1)=-4/7$. We have to show that $\mathrm{bk}(T_2)<-10/7$. We can now apply 11.1 (c), with $Y=X''$, $C_0=L_4''$, and $C_1=L_1''$. Combined with 4.1, this implies $\Gamma_1-\{L_4\}$ is non-rational. Also if T_2 has four tips then clearly $\mathrm{bk}(T_2)\leqq -(1/3+1/3+1/2+1/2)=-5/3<-10/7$. So T_2 has only 3 tips. Consequently $\Gamma_1-\{L_4\}$ has three tips. By 5.4, $\Gamma_1-\{L_4\}$ has one of the two configurations in Figure 14. Consequently, T_2 itself has one of the configurations shown in Figure 15. Note that $L_4^2=-3$, for otherwise $L_4^2=-4$ and $\Gamma_1-\{L_4\}$ will consist of only (-2)-curves. The rest of the weights are determined by using 6.1.

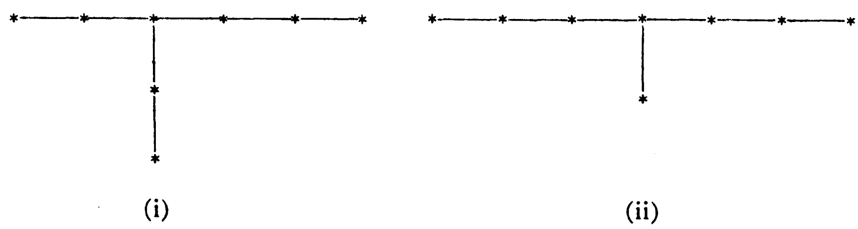


Figure 14.

Let Λ_1, Λ_2 and Λ_3 be the three maximal twigs of T_2, with $L_2\in\Lambda_3$. Then $\mathrm{bk}(T_2)=\sum_{i=1}^{3}\mathrm{bk}(\Lambda_i)$. Now consider Figure 15 (i), in which $\Lambda_3=[3, 3, 2, 2, 2]$, $\Lambda_2=[2, 2]$ and $\Lambda_1=[2, 3]$ or $[3, 2]$. Hence $\mathrm{bk}(T_2)\leqq -(2/5+2/3+9/23)<-10/7$ as required. In Figure 15 (ii), if Λ_1 and Λ_2 consist of only (-2)-curves then $\mathrm{bk}(\Lambda_1)+\mathrm{bk}(\Lambda_2)\leqq -(3/4+2/3)$ where as $\mathrm{bk}(\Lambda_3)\leqq -1/3$. Hence $\mathrm{bk}(T_2)<-10/7$ as required. So we may assume that, say Λ_1 has a (-3)-curve. Then $\Lambda_3=$ $[3, 3, 2, 2]$, $\mathrm{bk}(\Lambda_3)=-7/18$. One easily checks that $\mathrm{bk}(\Lambda_1)+\mathrm{bk}(\Lambda_2)\leqq -(2/5+3/4)$

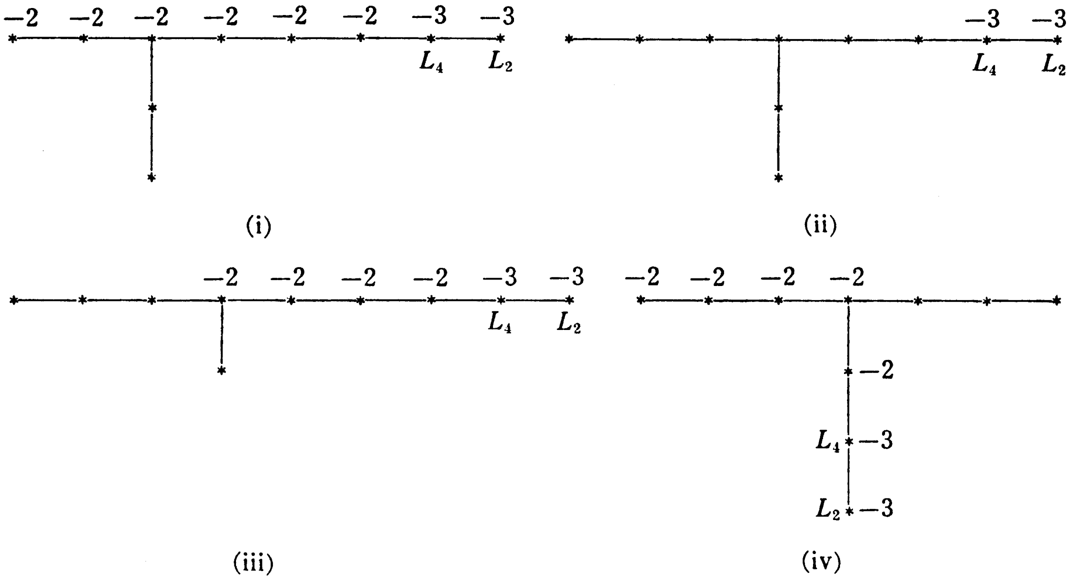


Figure 15.

or $\leqq -(3/7+2/3)$ and hence again $\mathrm{bk}(T_2)<-10/7$.

In Figure 15 (iv), $\mathrm{bk}(\Lambda_3)=-5/13$, $\mathrm{bk}(\Lambda_2)=-3/4$ and $\mathrm{bk}(\Lambda_1)\leqq -3/7$ and hence $\mathrm{bk}(T_2)<-10/7$. Finally in Figure 15 (iii), one checks that $\mathrm{bk}(T_2)<-10/7$ unless $\Lambda_1=[3,2,2]$, $\Lambda_2=[2]$. In this last case one directly computes $d(T)$ and sees that T is not unimodular.

Thus the claim in 15.12 that L_2 is a tip has been proved.

15.15. As before, we have a branch Γ_1 of $T_1-\{L_1\}$ which is non-rational and $\#(\Gamma_1)\geqq 6$; $\lambda(\Gamma_1)\leqq 1$. If $\#(\Gamma_1)=6$, by 11.3, Γ_1 is the tree (iii) in Figure 2, with $d(\Gamma_1)=4$; $\lambda(\Gamma_1)=1$. Another branch of T at L_1 is $L_0\cup L_3\cup L_2$, and the rest consists of three (-2)-curves, all contained in T_1. Thus it is easily seen that there will be one more branch of T at L_1 with even discriminant which contradicts 6.1.

Hence $\#(\Gamma_1)\geqq 7$. Suppose $\#(\Gamma_1)=7$. Then by 11.4, the possibilities for Γ_1 are given in Figure 4, with $w\in\Gamma_1$ being the vertex joined to L_1. If $\lambda(\Gamma_1)\leqq 1$, it follows that $\Gamma_1'=T-\Gamma_1$ is as shown in Figure 16, (using 6.1) which can be diagonalized with diagonal entry at $L_1=-1/3=(p'/q')$. With the values of p/q given by 11.4, we see that $|pp'-qq'|\neq 1$ for any of these possibilities. If $\lambda(\Gamma_1)$

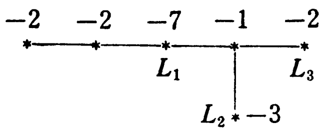


Figure 16.

$=0$, then Γ_1 is the unique configuration (14) of Figure 4 and it is easily verified that $d(T)\neq 1$.

15.16. Hence $\#(\Gamma_1)\geqq 8$. We can now apply Lemma 4.4, to conclude that $\Gamma_1-\{u_0\}$ should have a branch Γ_2 which is non-rational where u_0 is the vertex of Γ_1 joined to L_1. Let $u_1\in\Gamma_2$ be the vertex joined to u_0. As before, $\#(\Gamma_2)\geqq 6$, and if $\#(\Gamma_2)=6$, then Γ_2 is the configuration (iii) of Figure 2 with $u_1=w$, $\Omega_w=-2$, and $p/q=-4/9$. Also $\Gamma_2'=T-\Gamma_2$ is one of the configurations shown in Figure 17. Neither of these possibilities yields a unimodular T and hence $\#(T_2)\geqq 7$.

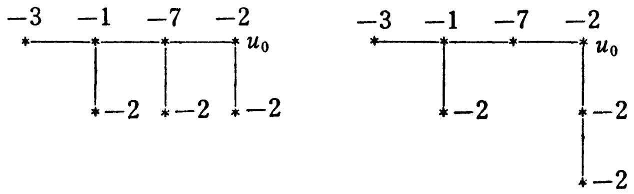


Figure 17.

15.17. Suppose $\#(\Gamma_2)=7$. Then 11.4 gives the possibilities for Γ_2 with $w=u_1$ etc. as before whereas one easily sees that $\Gamma_2'=T-\Gamma_2$ is one of the configurations shown in Figure 18 (again using $\lambda(\Gamma_2)=1$), with the diagonal entry p'/q' at u_0 taking the values $p'/q'=0/1$, $-1/2$ respectively. One easily checks that none of these yield $|pp'-qq'|=1$ and so $\#(\Gamma_2)\geqq 8$.

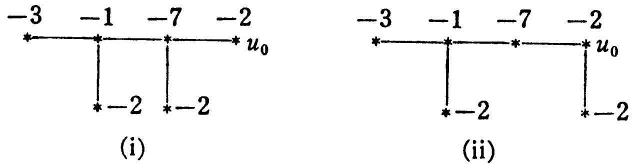


Figure 18.

15.18. Hence $\#(\Gamma_2)=8$. Note that L_1 is a horizontal component and so we shall also denote it by H_1. Since Γ_2 is non-rational, it follows that Γ_2 contains the other horizontal component H_2, $\lambda(\Gamma_2)=1$. In particular, $\Omega_{u_0}=-2$.

We first claim that all connected components of $\Gamma_2-\{u_1\}$ are rational. This is obvious, if each of them has less than 6 vertices. Now suppose Γ_3 is a branch of $\Gamma_2-\{u_1\}$ with $\#(\Gamma_3)\geqq 6$. If $\#(\Gamma_3)=6$, then Γ_3 is not rational would imply, by 11.3, that Γ_3 is as in (iii) of Figure 2 with $\Omega_w=-2$ and hence T itself will be as shown in Figure 19 which is clearly not unimodular.

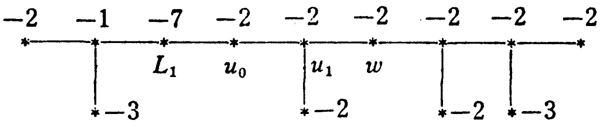


Figure 19.

When $\#(\Gamma_3)=7$, the non-rationality of Γ_3 implies, by 11.4, that Γ_3 is one of the trees shown in Figure 4. Again not all curves in Γ_3 are (-2)-curves and so $\lambda(\Gamma_3)=1$. Hence it follows that $\Gamma_3'=T-\Gamma_3$ is as shown in Figure 20 with p'/q' at u_1 equal to -1. With p/q at w for Γ_3 being given by 11.4 (c),

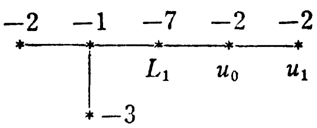


Figure 20.

we see that $|pp'-qq'|=1$ if and only if ($|p+q|=1$ and hence) $p/q=-12/13$, $-20/21$ or $-3/4$. In the last mentioned case, it turns out that $\lambda(\Gamma_3)=2$ and so we are left with only the first two cases, when T itself has configuration as shown in Figure 21 (a) and (b). In each case, a direct computation shows that K is effective. In Figure 21, the numbers in bracket give the respective coefficients for the canonical divisor. Thus, we have shown that all branches of $\Gamma_2-\{u_1\}$ are rational.

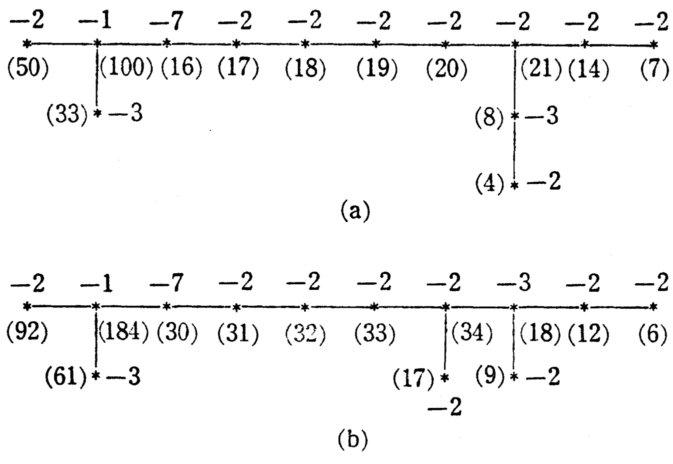


Figure 21.

15.19. Consider the linear equivalence

$$K'' \sim \mu_1 H_1'' + \mu_2 H_2'' + \sum \lambda_i C_i''$$

on X'', where C_0'' and H_1'' are the components corresponding to u_0 and L_1. All the C_i'' are (-2)-curves and vertical. H_1'' is a rational curve with a cusp, $(H_1'')^2=-1$, so that $K''\cdot H_1''=1$. H_2'' is a (-3)-curve and $K''\cdot H_2''=+1$. Hence it follows that $\mu_1=-\mu_2$, and hence $\mu_2\neq 0$. Intersecting with H_1'' and C_0'' we obtain

$$K\cdot H_1''=1=-\mu_1+\lambda_0\,,$$
$$K\cdot C_0''=0=\mu_1-2\lambda_0+\lambda_1$$

and hence

$$\left.\begin{aligned}\lambda_0&=\mu_1+1\\ \lambda_1&=\mu_1+2\end{aligned}\right\}.$$

As in the proof of 4.1, since all branches of $\Gamma_2-\{u_1\}$ are rational, it follows that if $\lambda_1\leqq 0$ then $\mu_2\leqq 0$ and $\lambda_i\leqq 0$ for all $i\geqq 1$ and $\lambda_1\geqq 0$ then $\mu_2\geqq 0$ and $\lambda_i\geqq 0$ for all $i\geqq 1$.

Thus, if $\lambda_1\leqq 0$ it follows that $-K$ is effective. Suppose $\lambda_1>0$, $\lambda_0\geqq 0$ and $\mu_1\geqq -1$. Hence either K is effective or $\mu_1=-1$. Thus we have obtained a linear equivalence

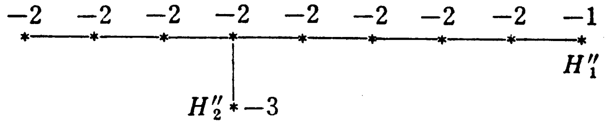
$$K''\sim H_2''-H_1''+Z$$

where Z is supported on the vertical components of D''. Now blow-down all the vertical components of D'' to finitely many rational double-points on a normal projective surface Y, and let $\pi: X''\to Y$ be the contraction. Since D'' is a tree, it follows that $\pi(H_1'')\cap\pi(H_2'')=\{y_0\}$, a singleton set and y_0 is a singularity of Y. We claim y_0 is the only singularity of Y. For if $y\neq y_0$ is any other singularity of Y, then $y\in\pi(H_2'')$ and $\pi(H_1'')$ does not pass through y. On the other hand $K_Y\sim\pi(H_2'')-\pi(H_1'')$ and K_Y is locally principal on Y. Since every divisor on Y is linearly equivalent to an integral combination of $\pi(H_1'')$ and $\pi(H_2'')$, it follows that $\mathcal{O}_{Y,y}$ is a unique factorization domain. By Lemma 9.9, y is an E_8-singularity, which is not possible since π contracts, in all, eight curves only.

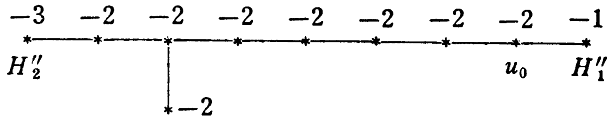
Thus we have shown that y_0 is the only singularity of Y, i.e., all the eight vertical components of D'' form a connected curve, and both H_1'' and H_2'' are tips of D''.

15.20. Let F_0 be the fibre of φ'' containing $\pi^{-1}(y_0)$. From Kodaira's list of possible singular fibres of φ'', we see that F_0 is of one of the three types: mI_9, I_4^* or II*. In the first two cases, it follows that $\pi^{-1}(y_0)$ has configuration A_8 and D_8 respectively. If F_0 is of type II*, then by 9.8, it follows that the component of F_0, not contained in D'' should occur with multiplicity 1 and hence $\pi^{-1}(y_0)$ is actually the configuration E_8.

We can now list all possibilities for the configuration of D''. Using the unimodularity of D'', only two possibilities, as shown in Figure 22, may occur.



(a)



(b)

Figure 22.

In both the cases one easily computes and checks that K is effective; this completely disposes off the case $r_3=1$.

§ 16. The case $r_3=2$ and the two components meet.

16.0. So far, we have proved that $r_4=e_1=\sigma=0$, $r_3=2$ and hence $b_0=1$, $\lambda=2$, $\tau=0$ and D is unimodular. In this section we shall show that the two components of R_3 do not meet. So, we assume that the two components meet, and denote them by L_0 and L_3. We can further assume that $L_0^2=-1$. If follows that $L_0\cdot L_3=1$; $L_3^2=-2$. Let L_1, L_2 be the other two components of D such that $L_0\cdot L_1=L_0\cdot L_2=1$. It follows that there is precisely one component $L_4(\neq L_0)$ such that $L_3\cdot L_4=1$, $L_1^2\leqq -3$; $L_2^2\leqq -3$. Also since there is equality in 2.8, we have $K\cdot D=\beta_2-6$. Setting $C=D-\{L_0\}$, it follows that $M(X, C)=1$ and hence $\mathrm{bk}(C)\geqq -4$. In the sequel, we shall show that $\mathrm{bk}(C)<-4$ thus arriving at a contradiction and thereby showing that the two components of R_3 do not meet. In this, often we use the unimodularity of T.

16.1. Blow-down L_0 and L_3 to obtain a surface X_1. We shall assume in 16.1-16.3 that X_1 is minimal, (i.e., $X_1=X'=X''$) and show that $\mathrm{bk}(C)<-4$. By 9.7, it follows that $(L_1''^2, L_2''^2)\neq(-2, -2)$. Since $\lambda=2$, and there are two horizontal components, $\{L_1''^2, L_2''^2\}=\{-3, -3\}$ or $\{-2, -3\}$, so that $\{L_1^2, L_2^2\}=\{-5, -5\}$ or $\{-4, -5\}$. Also $L_4^2=-3$ or -4. Setting $T-\{L_0\}=T_1\amalg T_2\amalg T_3$, with $L_i\in T_i$, $i=1, 2, 3$, it follows, by 4.1, as usual, that at least one of the T_i is non-rational.

16.2. Suppose T_3 is non-rational. Then, as seen before, it follows that $\#(T_3)\geqq 6$ and so $\#(T_1\cup T_2\cup\{L_0\})\leqq 6$. Hence $T_1\cup T_2\cup\{L_0\}$ is easily seen to be rational, using 5.4 and 6.1. Hence it follows that $\Gamma_1=T_3-\{L_3\}$ is itself non-rational; $\#(\Gamma_1)\geqq 6$; $\lambda(\Gamma_1-\{L_4\})\leqq 1$.

If $\#(\Gamma_1)=6$, then by 11.3, Γ_1 is (iii) of Figure 2 with $w=L_4$, $\Omega_w=-3$. It follows that $\{L_1^2, L_2^2\}=\{-4, -5\}$. Keeping in mind 6.1 we obtain only two possibilities for T as shown in Figure 23. But neither of these is unimodular. Hence $\#(\Gamma_1)\geqq 7$.

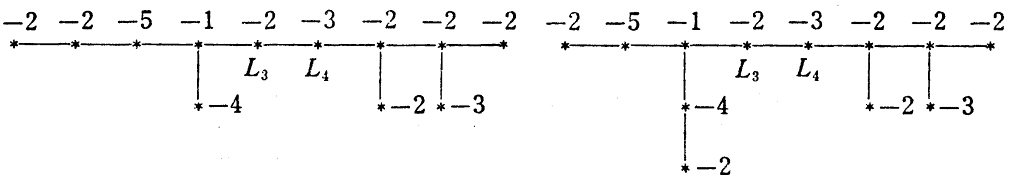


Figure 23.

If $\#(\Gamma_1)=7$, then 11.4 gives all possibilities for Γ_1 with $w=L_4$. It follows that except perhaps for (14) of Figure 4 (i.e. when $\lambda(\Gamma_1-\{L_4\})=0$), $\Omega_w=-3$ and $\{L_1^2, L_2^2\}=\{-4, -5\}$, and in (14), we may even have $\Omega_w=-4$ and $\{L_1^2, L_2^2\}=\{-4, -5\}$ or $\Omega_w=-3$ and $\{L_1^2, L_2^2\}=\{-5, -5\}$. Thus, we list up the configurations for $\Gamma_1'=T-\Gamma_1$ in Figure 24. We can diagonalize each of these with diagonal entry p'/q' at L_3 taking values $-2/9$, $-1/18$, $-7/26$, respectively. With p/q at L_4 given by 11.4 we see that $|pp'-qq'|\neq 1$ for any of these values and hence by 11.2 $d(T)\neq 1$. Hence $\#(\Gamma_1)\geqq 8$.

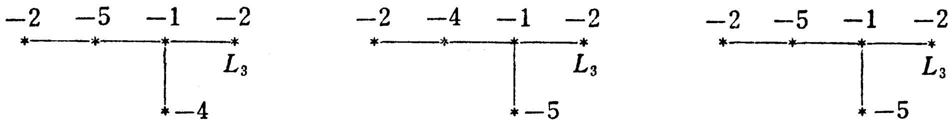


Figure 24.

In fact $\#(\Gamma_1)=8$, $\#(T_3)=9$, so that both L_1 and L_2 are tips of T. In particular it follows that $\{L_1^2, L_2^2\}=\{-4, -5\}$, by 6.1. We can now apply 11.1 (b) with $Y=X_1$, $C_1=L_1''$, $C_2=L_2''$, and $C_0=L_4''$. Along with 4.1, this yields that $\Gamma_1-\{L_4\}$, has a branch Γ_2 which is non-rational.

Now, if $\#(\Gamma_2)=6$, then Γ_2 is as in (iii) of Figure 2 with $d(\Gamma_2)=4$ and there is a tip of T which is joined to L_4 and which is a (-2)-curve, contradicting 6.1. Hence $\#(\Gamma_2)=7$. Since $\lambda(\Gamma_2)=1$, Γ_2 is determined by 11.4, whereas $\Gamma_2'=T-\Gamma_2$ is the configuration in Figure 25 with the diagonal entry p'/q' at L_4 being $+5/2$. With the value of p/q at $w\in\Gamma_2$ given by 11.4, we see that $|pp'-qq'|=1$ if and only if T is the configuration shown in Figure 26. But then one easily sees that

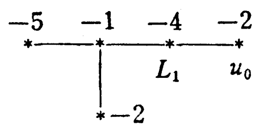


Figure 25.

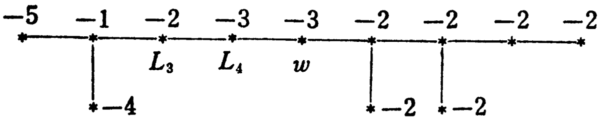


Figure 26.

$$\mathrm{bk}(C) = \mathrm{bk}(T_1)+\mathrm{bk}(T_2)+\mathrm{bk}(T_3) = -(1+4/5+8/13+1/2+1/2+2/3) < -4$$

as claimed.

16.3. We may now assume that T_1 is non-rational; (the case T_2 being non-rational is similar). As before, one first concludes that $T_1-\{L_1\}$ has a branch Γ_1 which is non-rational, $\#(\Gamma_1)\geqq 6$; $\lambda(\Gamma_1)\leqq 1$. Further, if $\#(\Gamma_1)=6$, then Γ_1 is as in (iii), Figure 2. Clearly $\#(T_1)\leqq 8$. If $\#(T_1)=8$, then it follows that there is a (-2)-curve in T, which is a tip of T and is joined to L_1, contradicting 6.1, since $d(\Gamma_1)$ is even. If $\#(T_1)=7$ one easily determines that $\Gamma_1'=T-\Gamma_1$ is one of the trees shown in Figure 27 with p'/q' at L_1 taking values 15/4, 13/8 and 12/7 respectively. With p/q at $w\in\Gamma_1$ equal to $-4/9$, we see that $|pp'-qq'|\neq 1$ and hence $\#(\Gamma_1)\geqq 7$.

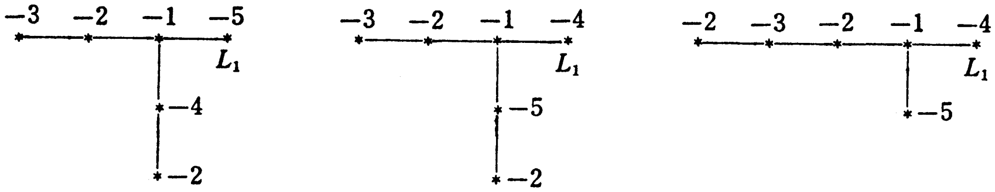


Figure 27.

Indeed $\#(\Gamma_1)=7$, $\lambda(\Gamma_1)\geqq 1$ and it is given by 11.4, where as Γ_1' is now given by Figure 28 with p'/q' at L_1 taking the value 5/3. Again, with p/q being given by 11.4, we see that $|pp'-qq'|=1$ if and only if $p'/q'=5/3$, and $p/q=4/7$. This yields the unique possibility for T as shown in Figure 29. But then again one easily computes that $\mathrm{bk}(C)<-4$, as claimed in 16.1.

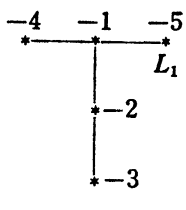


Figure 28.

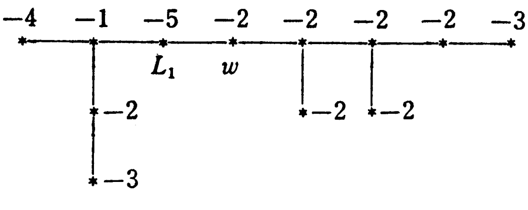


Figure 29.

16.4. Thus we can now assume that X_1 is not minimal. It follows that one of the images of L_1, L_2 and L_4 is a (-1)-curve on X_1. Indeed, we claim that the image of L_4 is exceptional. For, suppose that the image of L_1 is exceptional. Then, (since $\sigma=0$), clearly $X_1=X'$, and $\{L_1'\}=\mathcal{E}_2$, so that after blowing down L_1', we obtain the minimal surface X''. L_2'' is a cuspidal curve on X''. One easily computes that $K\cdot D\geqq 8$ whereas we know that $K\cdot D=\beta_2-6=7$, which is absurd. The case when the image of L_2 is exceptional on X_1 is symmetrical. Hence the image of L_4 is exceptional on X_1.

16.5. Blow-down L_0, L_3 and L_4 successively, to obtain a surface X_2. Then clearly $X_2=X'$. Indeed we claim $X_2=X'=X''$. For, to begin with since L_4 is in R_2, L_4 is a tip of D. L_1' and L_2' meet in a single point x, such that $L_1'\cdot L_2'=3$, and through x, no other component of D' passes. Hence, if at all X' is not minimal, then either L_1' or L_2' is exceptional on X_1'. But then we see that $\tau>0$ which contradicts our earlier observation that $\tau=0$. Hence $X_2=X'=X''$. In particular, $\beta_2=13$. Clearly at least, one of $\{L_1'', L_2''\}$ is horizontal and hence $\{L_1''^2, L_1''^2\}=\{-2, -3\}$ or $\{-3, -3\}$, so that $\{L_1^2, L_2^2\}=\{-5, -6\}$ or $\{-6, -6\}$, $L_3^2=-2$, $L_4^2=-2$; $T_3=[2, 2]$, $d(T_3)=3$.

As before, it follows that T_1 or T_2 is non-rational; say, T_1 is non-rational; $\#(T_1)\geqq 6$. And then $T_1-\{L_1\}$ itself has a branch Γ_1 which is non-rational; $\#(\Gamma_1)\geqq 6$, $\lambda(\Gamma_1)=1$.

16.6. If $\#(\Gamma_1)=6$, then by 11.3, Γ_1 is (iii) of Figure 2, with $\lambda(\Gamma_1)=1$ and hence $\{L_1^2, L_2^2\}=\{-5, -6\}$. It follows that $\mathrm{bk}(T_1)\leqq-(1/6+4/3)=-3/2$, and $\mathrm{bk}(T_2)\leqq-(1/6+1/2)=-2/3$. Since $\mathrm{bk}(T_3)=-2$, we have shown that $\mathrm{bk}(C)<-4$ as required.

If $\#(\Gamma_1)=7$, then by 11.4, we have $\mathrm{bk}(T_1)\leqq-(1/6+16/15)$. Now $\#(T_2)=1$, or 2, so that $T_2=[5]$, $[5, 2]$, or $[6, 2]$. ($T_2\neq[6]$, by 6.1), and hence $\mathrm{bk}(T_2)\leqq-4/5$ and hence again we obtain $\mathrm{bk}(C)<-4$.

16.7. Finally suppose $\#(\Gamma_1)=8$. Then $\#(T_2)=1$ and hence $T_2=[5]$, $L_1^2=-6$, $\Omega(T_1)=\{-6, -3, -2, -2, -2, -2, -2, -2, -2\}$. We should now show that $\mathrm{bk}(T_1)<-6/5$. This is clear if T_1 has more than three tips. Hence we may assume that T_1 has precisely three tips. (Clearly L_1 is one of them.) Let Λ_1, Λ_2, Λ_3 be the three maximal twigs of T_1 with $L_1\in\Lambda_1$. Since Γ_1 is non-rational it follows that T_1 should have one of the configurations shown in Figure 30.

In (i) we may assume that $\Lambda_2=[2, 2]$. Then $\Lambda_3=[2, 3]$ or $[3, 2]$ and $\Lambda_1=[6, 2, 2, 2]$. Hence

$$\mathrm{bk}(T_1)=\mathrm{bk}(\Lambda_1)+\mathrm{bk}(\Lambda_2)+\mathrm{bk}(\Lambda_3)\leqq-(4/21+2/3+2/5)=-44/35<-6/5.$$

In (ii), if Λ_2 (or Λ_3) is $[2, 2, 2]$ then clearly $\mathrm{bk}(T_1)\leqq-(1/6+1/3+3/4)<-6/5$. So, we may assume that Λ_2 has a (-3)-curve and hence $\Lambda_1=[6, 2, 2]$, $\Lambda_3=[2, 2]$, and $\Lambda_2=[2, 2, 3]$, $[2, 3, 2]$ or $[3, 2, 2]$. So, in any case $\mathrm{bk}(T_1)\leqq-(3/16+2/3+3/7)<-6/5$. In (iii) we may assume that $\Lambda_2=[2, 2, 2]$ and so we are done. In (iv), if $\Lambda_2=[2, 2, 2]$ we are done. So we may assume that $\Lambda_3=$

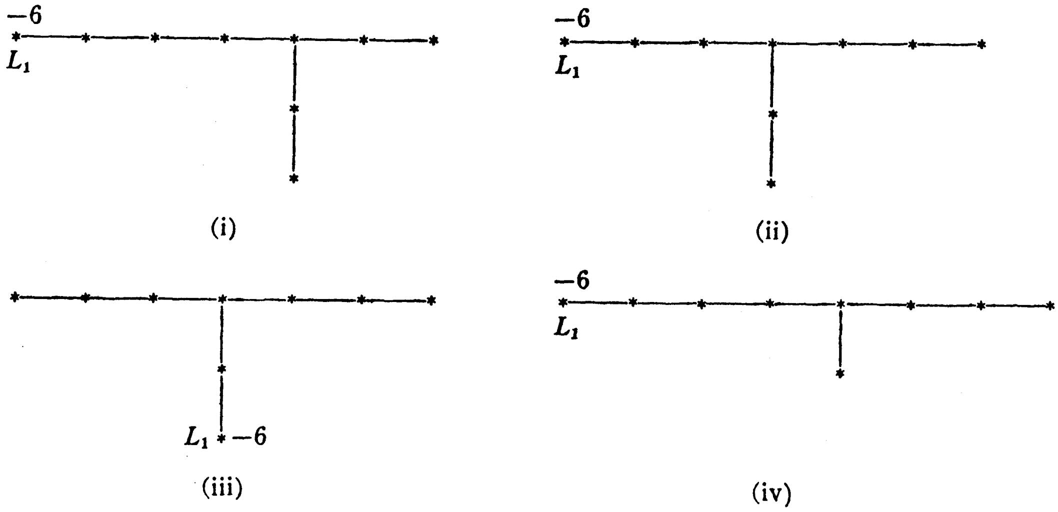


Figure 30.

[2], $\Lambda_1=[6, 2, 2, 2]$ and $\Lambda_2=[2, 2, 3]$, $[2, 3, 2]$ or $[3, 2, 2]$. In the first two cases $\mathrm{bk}(\Lambda_2)\leqq -5/8$ and so $\mathrm{bk}(T_1)\leqq -(4/21+5/8+1/2)<-6/5$. In the third case, we do not get $\mathrm{bk}(T_1)<-6/5$ as needed. However, the tree T is now as shown in Figure 31. One easily checks that this is not unimodular and so this case does not exist. Thus we have shown that $\mathrm{bk}(C)<-4$, in all cases, and hence the two components of R_3 are disjoint, as claimed in 16.0.

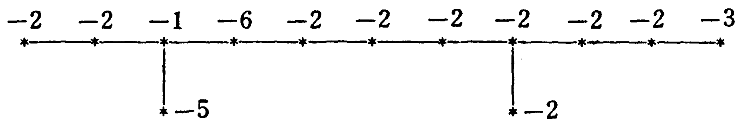


Figure 31. $d=16$.

§ 17. Completion of the proof of the Theorem 8.2.

17.0. So far, we have proved that $r_4=e_1=\sigma=\tau=0$, $b_0=1$, $\lambda=2$, D is unimodular, $K\cdot D=\beta_2-6$, $r_3=2$ and the two components of R_3 are disjoint. We shall denote these two components by $L_{1,0}$ and $L_{2,0}$. Now for the curve $C=D-L_{0,1}-L_{0,2}$ we have $b_0(C)=5$, $b_1(C)=0$, $b_2(C)=\beta_2-2$, and $K\cdot C=\beta_2-4$. Hence by 1.3 and 10.5, $\mathrm{bk}(C)\geqq -4$. As before, here also, we shall estimate $\mathrm{bk}(C)$ directly, and show that $\mathrm{bk}(C)<-4$, which will complete the proof of the theorem.

Let now, $\{L_{i,j}\}_{1\leqq j\leqq 3}$ be the other components of D meeting $L_{i,0}$, i.e., $L_{i,0}\cdot L_{i,j}=1$ $(i=1, 2)$, and $L_1=\bigcup_{j=0}^{3}L_{1,j}$ and $L_2=\bigcup_{j=0}^{3}L_{2,j}$. Since D is simply connected, it follows that $L_1\cap L_2$ is either empty or a single point or an irreducible curve. In the last case we shall choose the labeling so that $L_1\cap L_2=L_{1,1}=L_{2,1}$.

Thus $L_{1,1} \neq L_{2,1}$ means $L_{1,j} \neq L_{2,k}$ for any j and k. Let $C = T_1 \amalg T_2 \amalg T_3 \amalg T_4 \amalg T_5$, $\#(T_s) = t_s$. Then $\mathrm{bk}(C) = \sum_{s=1}^{5} \mathrm{bk}(T_s)$.

17.1. Let X_1 be the surface obtained by contracting $L_{1,0}$ and $L_{2,0}$. Assume, first that X_1 is minimal, $X_1 = X' = X''$. Then, $\beta_2 = 12$, $L_{i,j}^2 \leqq -3$, $1 \leqq j \leqq 3$. We shall make two subcases, viz., (i) $L_{1,1} = L_{2,1}$, (ii) $L_{1,1} \neq L_{2,1}$.

Consider the case $L_{1,1} = L_{2,1}$. Then, it follows that $L_{1,1}^2 = -4$ or -5. Then using 9.7, it follows that the dual graph of the curve $L = L_1 \cup L_2$ is one of the three trees shown in Figure 32. Note also that the five components of $L - L_{1,0} - L_{2,0}$ belong to distinct T_s. In (a) and (b) all components of $D - L$ are (-2)-curves whereas in (c), $D - L$ has one (-3)-curve and all other (-2)-curves.

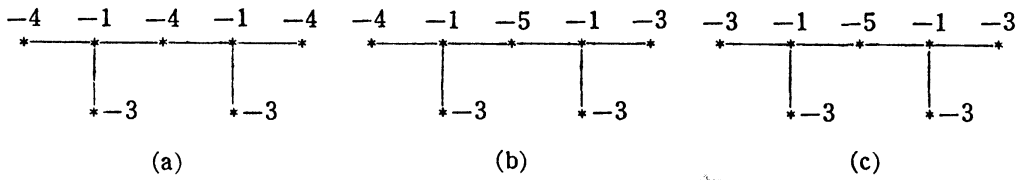


Figure 32.

Thus if $t_s \leqq 2$, then $\mathrm{bk}(T_s) \leqq -1$ unless $T_s = [5]$ or $[5, 3]$. Hence, if $t_s \leqq 2$ for more than three $s \in \{1, 2, 3, 4, 5\}$, say $t_s \leqq 2$ for $s \leqq 4$, then clearly $\sum_{s=1}^{4} \mathrm{bk}(T_s) \leqq -(1+1+1+5/7)$. On the other hand $\mathrm{bk}(T_5) < -1/2$ and hence $\sum_{s=1}^{5} \mathrm{bk}(T_s) < -4$.

So we may assume $t_1 \geqq 3$, and $t_2 \geqq 3$. Since $\sum t_s = 10$, it follows that $t_1 + t_2 \leqq 7$ and $t_3 + t_4 + t_5 \leqq 4$. Here again $\mathrm{bk}(T_1) + \mathrm{bk}(T_2) < -1$ and so if $T_s \neq [5]$ or $[5, 3]$ for any $s = 3, 4, 5$, then we are done. So assume that $T_5 = [5]$ or $[5, 3]$. Then it follows that $\Omega(T_1 \cup T_2)$ does not contain -5 and so it follows that $\mathrm{bk}(T_1) + \mathrm{bk}(T_2) < -(1/4 + 1/2 + 1/3 + 1/2)$, (by 10.4, (iii)) and hence $\sum_{s=1}^{5} \mathrm{bk}(T_s) < -(1/4 + 1/2 + 1/3 + 1/2 + 1 + 1 + 5/7) < -4$ as required.

17.2. Now consider the case $L_{1,1} \neq L_{2,1}$. Now using 9.7, we conclude that $\Omega(L_1) = \Omega(L_2) = \{-1, -3, -3, -4\}$ and $\Omega(C) = \{-4, -4, -3, -3, -3, -3, -2, -2, -2, -2\}$. Note that there is a unique branch T_s of C say, T_1 which meets both $L_{1,0}$, $L_{2,0}$, and $t_1 \geqq 2$. Now if $t_s \leqq 2$, then $\mathrm{bk}(T_s) \leqq -1$ and hence as in 17.1, we may assume, that $t_s \geqq 3$ for at least two $s \in \{1, 2, 3, 4, 5\}$. So let $t_2 \geqq 3$. Further if t_3, t_4 or t_5, say, $t_3 \geqq 3$, then $t_4 = t_5 = 1$, $t_1 = 2$, $t_2 = t_3 = 3$. Clearly $T_1 = [4, 4]$, $[4, 3]$ or $[3, 3]$ and hence $\mathrm{bk}(T_1) \leqq -10/15$. Also $\mathrm{bk}(T_2) \leqq -(1/4 + 1/2)$, $\mathrm{bk}(T_3) \leqq -(1/4 + 1/2)$ where as $\mathrm{bk}(T_4) \leqq -1$, $\mathrm{bk}(T_5) \leqq -1$, and hence $\sum \mathrm{bk}(T_s) < -4$. Finally, consider the case where $t_s \leqq 2$ for $s = 3, 4, 5$. Then $\mathrm{bk}(T_s) \leqq -1$ and in any case $\mathrm{bk}(T_1) < -1/2$, $\mathrm{bk}(T_2) < -1/2$ so that $\sum \mathrm{bk}(T_i) < -4$.

Thus we have shown that $\mathrm{bk}(C) < -4$ if X_1 is minimal.

17.3. Now suppose X_1 is not minimal so that $L_{i,j}^2 = -2$ at least for one

(i, j). Since $r_4=0$, if $L_{1,1}=L_{2,1}$ then $L_{1,1}$ is never in $\mathcal{E}_1$ and so we may assume that $L_{1,2}^2=-2$. Contract the image of $L_{1,2}$ on X_1, to obtain a surface X_2. We shall now assume that X_2 is minimal, i.e. $X_2=X'=X''$, and show that $\mathrm{bk}(C)<-4$.

It follows that $\beta_2=13$, and $L_{1,2}$ is a tip of D as $r_3=2$. As before we now make two subcases (i) $L_{1,1}=L_{2,1}$ and (ii) $L_{1,1}\neq L_{2,1}$.

17.4. Suppose $L_{1,1}=L_{2,1}$. It follows that L has one of the configurations shown in Figure 33, with $D-L$ consisting of (-2)-curves in (a), (b) and (c), and having one (-3)-curve and other (-2)-curves in (d).

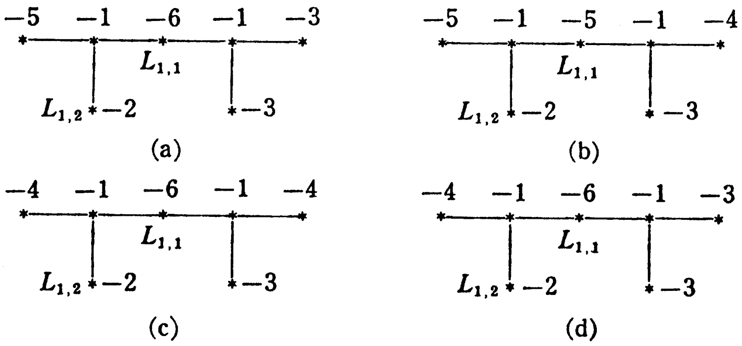


Figure 33.

We set $T_5=\{L_{1,2}\}=[2]$, so that $\mathrm{bk}(T_2)=-2$. Thus we have to show that $\sum_{s=1}^4 \mathrm{bk}(T_s)<-2$. If $t_s=1$, then clearly $\mathrm{bk}(T_s)\leqq -4/6$. If $T_s\geqq 2$, then by 10.4, it easily follows that $\mathrm{bk}(T_s)\leqq -(1/6+1/3)=-1/2$ and, for at least one of the s, $\mathrm{bk}(T_s)<-1/2$. Hence $\sum_{s=1}^4 \mathrm{bk}(T_s)<-2$ as required.

17.5. Now consider the case $L_{1,1}\neq L_{2,1}$. As before, it follows that $\Omega(L_1)=\{-5, -4, -2, -1\}$ and $\Omega(L_2)=\{-4, -3, -3, -1\}$. So that $\Omega(C)$ is $\{-5, -4, -4, -3, -3, -2, -2, -2, -2, -2, -2\}$. Taking $T_5=\{L_{1,2}\}$, here also we have to show that $\sum_{s=1}^4 \mathrm{bk}(T_s)<-2$. As in 17.2, there is a unique T_s, say T_1, which meets both $L_{1,0}$ and $L_{2,0}$. Then for T_2, T_3, T_4, it easily follows that $\mathrm{bk}(T_s)\leqq -(1/5+1/2)=-7/10$. Hence $\sum_{s=1}^4 \mathrm{bk}(T_s)<-2$ as required.

17.6. We may now assume that X_2 is not minimal. This means that the image of $L_{1,1}$, $L_{1,3}$ or one of $L_{2,j}$ on X_2 is a (-1)-curve. However, using the fact that $K\cdot D=\beta_2-6$, we can see that if $L_{1,1}=L_{2,1}$ then its image on X_2 cannot be a (-1)-curve (for, the contraction of the image of $L_{1,1}$ gives X'' by 3.1, $K\cdot D=8$. This is a contradiction). So we may assume that $L_{1,3}$ is a (-1)-curve or $L_{2,2}$ is a (-1)-curve. In the latter case, i.e. if $L_{2,2}$ is a (-1)-curve then it follows (by $r_4=0$) that $L_{2,2}$ is a tip of D, $L_{2,2}^2=-2$, and hence clearly $\mathrm{bk}(C)<\mathrm{bk}(\{L_{1,2}\})+\mathrm{bk}(\{L_{2,2}\})=-4$. So we may assume that $L_{2,j}^2\leqq -3$.

Hence, the image of $L_{1,3}$ on X_2 is a (-1)-curve. Again using the fact $K\cdot D=\beta_2-6$, it follows that $L_{1,3}$ is also a tip of D as above, $L_{1,3}^2=-3$. Thus taking $T_4=\{L_{1,3}\}$, we have to show that $\sum_{s=1}^{3}\mathrm{bk}(T_s)<-2/3$.

Contract the image of $L_{1,3}$ on X_2 to obtain a surface X_3. It is easily seen that X_3 is minimal by 3.1, i.e. $X_3=X''$. Suppose $L_{1,1}\in T_1$. Then it follows easily that for any component D_i of T_2 or T_3, we have $D_i^2\geqq -4$ and hence $\mathrm{bk}(T_2)\leqq -1/2$, $\mathrm{bk}(T_3)\leqq -1/2$ proving thereby that $\sum_{s=1}^{3}\mathrm{bk}(T_s)<-2/3$ as required.

This completes the proof of 8.2.

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