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Spaces which contain a copy of the rationals

Dedicated to Professor Yukihiro Kodama on his 60-th birthday

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1. Introduction.

The main purpose of this paper is to study what kind of space contains a (closed) copy of $\boldsymbol{Q}$, where $\boldsymbol{Q}$ is the space of rationals with the usual topology. We show that every non-scattered Lašnev space contains a copy of $\boldsymbol{Q}$ and every non-scattered sequential space with character less than $\boldsymbol{b}$ contains a copy of $\boldsymbol{Q}$, where $\boldsymbol{b}$ is the minimum cardinal of an unbounded subfamily of ${}^{\omega}\omega$ (see [2]). In addition, let X_n $(n<\omega)$ be arbitrary regular topological spaces. If $\boldsymbol{Q}$ is embedded in $\prod_{n<\omega}X_n$ as a closed subset, then there exists an $n<\omega$ such that X_n contains a copy of $\boldsymbol{Q}$, where ω is the first infinite ordinal number. Moreover if we assume Martin's axiom (MA), the statement holds for any infinite cardinal number κ less than $\boldsymbol{c}$ $(=2^{\omega})$ instead of ω. The following theorems are of similar form to the last theorem.

(1) *If $\beta\omega$ is embedded in $\prod_{\alpha<\kappa}X_\alpha$ $(\kappa<\mathrm{cf}(\boldsymbol{c}))$, then there exists an $\alpha<\kappa$ such that X_α contains a copy of $\beta\omega$, where $\beta\omega$ is the Stone-Čech compactification of ω with the discrete topology.*

This theorem was proved by Malyhin [6] for the case $\kappa=\omega$ and by van Douwen-Przymusinski [3] for the other case.

(2) (Nogura-Tanaka [8]) *If S (S_2) is embedded in $\prod_{\alpha<\kappa}X_\alpha$ $(\kappa<b)$, then there exist $\alpha_1, \alpha_2, \cdots, \alpha_n$ such that $\prod_{i=1}^{n}X_{\alpha_i}$ contains a copy of S (S or S_2, respectively), where S is a sequential fan and S_2 is Arens' space* (see [1] or [8]).

We note that the closedness of embedding in our last theorem can not be dropped, because the product of infinitely many non-degenerate topological spaces contains a copy of $\boldsymbol{Q}$.

By a mapping we mean a continuous, surjective function and by a space a regular T_1 topological space.

2. Nowhere dense mappings.

A mapping $f: Z \to X$ is said to be *nowhere dense* if $\operatorname{int} f^{-1}(x) = \emptyset$ for all $x \in X$. It is easy to see that if f is nowhere dense then Z and X have no isolated point.

The following lemma is used throughout this paper.

LEMMA 2-1. *Let $f: \boldsymbol{Q} \to X$ be a nowhere dense mapping. Then X contains a copy of $\boldsymbol{Q}$.*

PROOF. Let $Seq = \{\langle n_1, n_2, \cdots, n_k\rangle : n_i < \omega, 1 \leqq i \leqq k, k < \omega\}$. For $s = \langle n_1, n_2, \cdots, n_k\rangle \in Seq$ we say that the length of s is k, and for $n < \omega$ $s * \langle n\rangle$ denotes the sequence $\langle n_1, n_2, \cdots, n_k, n\rangle$.

By induction on the length of elements of Seq, for each $s \in Seq$ we shall define a point $q_s \in \boldsymbol{Q}$, an open neighborhood U_s of q_s in $\boldsymbol{Q}$ and an open neighborhood V_s of $f(q_s)$ in X as follows. Choose $q_{\langle\,\rangle} = 0$, $U_{\langle\,\rangle} = \boldsymbol{Q}$, $V_{\langle\,\rangle} = X$, where $\langle\,\rangle$ is the empty sequence. Suppose that we have gotten q_s, U_s and V_s for $s \in Seq$ with the length less than or equal to k satisfying $f(U_s) \subset V_s$. Choose a sequence $\{q_{s*\langle n\rangle} : n < \omega\}$ satisfying:

(1) $\{q_{s*\langle n\rangle} : n < \omega\}$ converges to q_s,

(2) $\{q_{s*\langle n\rangle} : n < \omega\} \subset U_s$,

(3) $f(q_{s*\langle n\rangle}) \neq f(q_s)$ for $n < \omega$ and $f(q_{s*\langle n\rangle}) \neq f(q_{s*\langle m\rangle})$ for $n \neq m$.

We show the existence of such a seqence. Let $\{U_n : n < \omega\}$ be an open neighborhood base of q_s in $\boldsymbol{Q}$ such that $U_{n+1} \subset U_n \subset U_s$ and $U_n \setminus \overline{U_{n+1}} \neq \emptyset$ for all $n < \omega$. Since f is nowhere dense, we can take $q_{s*\langle 0\rangle} \in (U_0 \setminus \overline{U}_1) \setminus f^{-1}\{f(q_s)\}$ and $q_{s*\langle n\rangle} \in (U_n \setminus \overline{U_{n+1}}) \setminus f^{-1}\{f(q_s)\} \cup f^{-1}\{f(q_{s*\langle k\rangle}) : k < n\}$ inductively. It is easy to see that the sequence $\{q_{s*\langle n\rangle} : n < \omega\}$ has the properties (1)-(3). Take an open neighborhood $V_{s*\langle n\rangle}$ of $f(q_{s*\langle n\rangle})$ so that

(4) $V_{s*\langle n\rangle} \cap V_{s*\langle m\rangle} = \emptyset$ for $n \neq m$ and $f(q_s) \notin \overline{V_{s*\langle n\rangle}}$ for $n < \omega$.

(Note that $\{f(q_{s*\langle n\rangle}) : n < \omega\}$ is a discrete subspace of X.) Next, take open sets $U_{s*\langle n\rangle} \subset U_s$ satisfying:

(5) $q_{s*\langle n\rangle} \in U_{s*\langle n\rangle}$;

(6) $f(U_{s*\langle n\rangle}) \subset V_{s*\langle n\rangle}$;

(7) The diameter of $U_{s*\langle n\rangle}$ is less than $1/n$.

Now we have gotten q_s, U_s and V_s for all $s \in Seq$.

Let $Y = \{q_s : s \in Seq\}$. Since Y is a countable metrizable space without an isolated point, Y is homeomorphic to $\boldsymbol{Q}$. We show that the restriction of f, i.e. $f|Y : Y \to f(Y)$, is a homeomorphism. Clearly $f|Y$ is one-to-one and continuous. We show that $f|Y$ is open. Let U be an open neighborhood of q_s.

Then, there exists an $n<\omega$ such that $\bigcup\{U_{s*\langle m\rangle}: m\geqq n\}\subset U$ by (1) and (7). By (4), $f(q_s)\notin\overline{\bigcup\{V_{s*\langle k\rangle}: k\leqq n-1\}}$. By (3) and (4), we get

$$\begin{aligned} f(q_s) &\in f(Y)\cap(V_s\setminus\bigcup\{V_{s*\langle k\rangle}: k\leqq n-1\}) \\ &\subset f(Y)\cap(V_s\setminus\bigcup\{f(U_{s*\langle k\rangle}): k\leqq n-1\})\subset f(Y)\cap f(U\cap U_s). \end{aligned}$$

Hence $f|Y$ is an open mapping. The proof is complete.

Any first countable space without an isolated point has a countable dense in itself subset. Therefore we get,

PROPOSITION 2-2 (Folklore). *Let X be a non-scattered first countable space. Then X contains a copy of $\boldsymbol{Q}$.*

A closed image of a metrizable space is said to be *Lašnev.* It is known that there exists a Lašnev space which has no first countable point [**5**]. Nevertheless we have the following theorem.

THEOREM 2-3. *Let X be a Lašnev space. If X is not scattered, then X contains a copy of $\boldsymbol{Q}$.*

PROOF. Let $f: M\to X$ be a closed map from a metric space M. Without loss of generality we may assume that X has no isolated point. By Theorem 4 of [**5**], we may also assume that f is irreducible. (A map f is said to be *irreducible* if any non-empty open set of M contains the full pre-image of some point $x\in X$.) Choose $m(x)\in f^{-1}(x)$ for all $x\in X$ and put $Y=\{m(x): x\in X\}$. We show that Y has no isolated point. Assume the contrary and let $m(x_0)$ be an isolated point in Y. There exists an open neighborhood U of $m(x_0)$ in M such that $U\cap Y=\{m(x_0)\}$. Since f is irreducible and $f|Y$ is one-to-one, $f^{-1}(x_0)\subset U$ and $f^{-1}(x)\cap U=\emptyset$ for every $x\neq x_0$. Thus $X\setminus f(X\setminus U)=\{x_0\}$ is an open set. This is impossible since X has no isolated point. Thus Y has no isolated point. By Proposition 2-2 Y contains a subspace Z homeomorphic to $\boldsymbol{Q}$. Now it is easy to show that $f|Z: Z\to f(Z)$ is a nowhere dense mapping. Therefore, $f(Z)$ contains a copy of $\boldsymbol{Q}$ by Lemma 2-1.

A space X is *sequential,* if the following hold: A subset A of X is closed if and only if the limit point of any convergent sequence in A also belongs to A. The character $\chi(x)$ of x $(\in X)$ is the least cardinal of a neighborhood base of x.

THEOREM 2-4. *Let X be a sequential space without isolated points. If the character $\chi(x)<\boldsymbol{b}$ for each $x\in X$, then X contains a copy of $\boldsymbol{Q}$.*

PROOF. By the sequentiality of X, we can get $x_s\in X$ and an open neighborhood U_s of x_s for each $s\in Seq$ so that the following hold:

(1) $\{x_{s*\langle k\rangle}: k<\omega\}$ converges to x_s and $\{x_{s*\langle k\rangle}: k<\omega\}\subset U_s$;

(2) $U_{s*\langle k\rangle}\subset U_s$ for any $k<\omega$ and $U_{s*\langle j\rangle}\cap U_{s*\langle k\rangle}=\emptyset$ for $j\neq k$.

From now on we work in the subspace $\{x_s: s\in S\}$ $(=X^*)$ and hence we let $\{U_\alpha: \alpha<\kappa\}$ be a base of X^*, where $\kappa<\boldsymbol{b}$. For each $\alpha<\kappa$, define $f_\alpha: Seq\to\omega$ by:

$$f_\alpha(s)=\min\{k<\omega: x_s\in U_\alpha \text{ implies } x_{s*\langle j\rangle}\in U_\alpha \text{ for every } j\geqq k\}.$$

Since $\kappa<\boldsymbol{b}$, there exists $h: Seq\to\omega$ such that $f_\alpha(s)\leqq h(s)$ for almost all $s\in Seq$ for each α. Now, let

$$T=\{s\in Seq: h(s|i)\leqq s_i \text{ for every } i\leqq lh(s)\} \quad\text{and}\quad Y=\{x_s: s\in T\},$$

where $lh(s)$ is the length of s $(=(s_1, \cdots, s_{lh(s)}))$ and $s|i=\langle s_1, \cdots, s_{i-1}\rangle$. By definition, $\langle\ \rangle$ belongs to T and $s*\langle k\rangle\in T$ for almost all k for each $s\in T$. Therefore, Y is nonempty and without isolated points. We want to show that Y is first countable, but the next claim assures it.

CLAIM. *For any $s\in T$, $\alpha<\kappa$, $x_s\in U_\alpha$, there exists $k<\omega$ such that $s*\langle j\rangle*t\in T$ implies $x_{s*\langle j\rangle*t}\in U_\alpha$ for any $j\geqq k$ and $t\in Seq$.*

By the definition of h, there exists a finite subset H of Seq such that $f_\alpha(t)\leqq h(t)$ for every $t\in Seq\setminus H$. Pick k so that $f_\alpha(s)\leqq k$ and for any $j\geqq k$ any extension of $s*\langle j\rangle$ does not belong to H. We show this k is the desired one in the claim by induction on the length of $t\in Seq$. Let $j\geqq k$. In case $t=\langle\ \rangle$, the conclusion clearly holds. In case $t=\bar{t}*\langle m\rangle$ and $s*\langle j\rangle*\bar{t}*\langle m\rangle\in T$, $s*\langle j\rangle*\bar{t}\in T$ and hence $x_{s*\langle j\rangle*\bar{t}}\in U_\alpha$ by induction hypothesis. Then, $x_{s*\langle j\rangle*\bar{t}*\langle i\rangle}\in U_\alpha$ holds for any $i\geqq f_\alpha(s*\langle j\rangle*\bar{t})$. By the property of k, $s*\langle j\rangle*\bar{t}$ does not belong to H and hence $f_\alpha(s*\langle j\rangle*\bar{t})\leqq h(s*\langle j\rangle*\bar{t})\leqq m$, which implies the conclusion.

REMARK 2-4. Every Lašnev space is Fréchet. (A space X is said to be *Fréchet* if $x\in\bar{A}$ for $A\subset X$, then there exists a sequence in A converging to the point x.) Therefore it is natural to ask whether every non-scattered Fréchet space contains a copy of $\boldsymbol{Q}$. In the appendix we shall show the existence of a non-scattered Fréchet space containing no copy of $\boldsymbol{Q}$ in a strong sense under the continuum hypothesis. We do not know whether the set theoretic hypothesis is necessary or not. On the other hand, if one drops the condition about the cardinality of characters in Theorem 2.4, one can easily get a sequential space which has no isolated point but does not contain a copy of $\boldsymbol{Q}$. In fact the space S_ω defined in [1] has such a property. We leave its easy proof to the reader.

3. Embedding to product spaces.

THEOREM 3-1. *Let X and Y be spaces. If $X\times Y$ contains a copy of $\boldsymbol{Q}$, then either X or Y contains a copy of $\boldsymbol{Q}$.*

PROOF. Assume $\boldsymbol{Q}\subset X\times Y$. Let p_X and p_Y be the projections from $X\times Y$ to X and Y respectively. Let $p_X|\boldsymbol{Q}: \boldsymbol{Q}\to X$ be the restriction of p_X to $\boldsymbol{Q}$. Without loss of generality we may assume $p_X|\boldsymbol{Q}$ is surjective. If $\mathrm{int}_{\boldsymbol{Q}} p_X|\boldsymbol{Q}^{-1}(x)\neq\emptyset$ for some $x\in X$, then $p_X|\boldsymbol{Q}^{-1}(x)$ contains a copy of $\boldsymbol{Q}$, therefore Y contains a copy of $\boldsymbol{Q}$. We have nothing to do in this case. Assume $\mathrm{int}_{\boldsymbol{Q}} p_X|\boldsymbol{Q}^{-1}(x)=\emptyset$ for every $x\in X$. Then $p_X|\boldsymbol{Q}$ is a nowhere dense mapping. Therefore, X contains a copy of $\boldsymbol{Q}$ by Lemma 2-1.

A space X satisfies the *countable chain condition*, if there exists no uncountable pairwise disjoint family of non-empty open subsets.

LEMMA 3-2 (MA). *Let Z be a space of cardinality less than c which has no isolated points and satisfies the countable chain condition, and X_α $(\alpha<\kappa)$ be arbitrary spaces, where $\kappa<\boldsymbol{c}$. If Z is embedded in $\prod_{\alpha<\kappa}X_\alpha$ as a closed subset, then there exist a non-empty open set U of Z and an index $\alpha\in\kappa$ such that the mapping $p_\alpha|U: U\to p_\alpha(U)$ is nowhere dense, where $p_\alpha: \prod_{\alpha<\kappa}X_\alpha\to X_\alpha$ is the projection. Moreover, if Z is countable and $\kappa=\omega$, we do not need* MA.

PROOF. Assume the contrary, i.e., for any $\alpha<\kappa$ and non-empty open subset U of Z there exists $x\in p_\alpha(U)$ such that $\mathrm{int}_Z p_\alpha|U^{-1}(x)\neq\emptyset$. Then $\bigcup\{\mathrm{int}_Z p_\alpha|Z^{-1}(p_\alpha(z)): z\in Z\}$ is a dense open subset of Z for each $\alpha<\kappa$. Let $\mathfrak{F}$ be the set of all finite subsets of κ and for each $F\in\mathfrak{F}$ let $\pi_F: Z\to\prod_{\alpha\in F}X_\alpha$ be the restriction of the projection to Z. Then,

$$\mathrm{int}_Z\pi_F^{-1}(x)=\bigcap_{i=1}^{n}\mathrm{int}_Z p_{\alpha_i}^{-1}(x_{\alpha_i})$$

for each $x=(x_{\alpha_1}, x_{\alpha_3}, \cdots, x_{\alpha_n})\in\pi_F(Z)$. Consequently,

(1) $\bigcup\{\mathrm{int}_Z\pi_F^{-1}(\pi_F(z)): z\in Z\}$ is also dense open.

Let $\boldsymbol{P}=\{p\in\prod_{\alpha\in F}X_\alpha: \mathrm{int}_Z\pi_F^{-1}(p)\neq\emptyset,\ F\in\mathfrak{F}\}$ and $p\leqq q$ if and only if p is an extension of q for $p, q\in\boldsymbol{P}$. Note that p is incompatible with q if and only if

$$\mathrm{int}_Z\pi_{\mathrm{dom}\,p}{}^{-1}(p)\cap\mathrm{int}_Z\pi_{\mathrm{dom}\,q}{}^{-1}(q)=\emptyset\,,$$

where $\mathrm{dom}\,p$ denotes the domain of p. Then, $\boldsymbol{P}$ satisfies the countable chain condition by the countable chain condition of Z. Set $D_z=\{p: \pi_{\mathrm{dom}\,p}(z)\neq p\}$ for $z\in Z$ and $D_\alpha=\{p: \alpha\in\mathrm{dom}\,p\}$ for $\alpha\in\kappa$. We first show that D_z is dense in $\boldsymbol{P}$ for all $z\in Z$. Let $p\in\boldsymbol{P}\setminus D_z$. Then $\pi_{\mathrm{dom}\,p}(z)=p$. Since Z has no isolated point and $\mathrm{int}_Z\pi_{\mathrm{dom}\,p}{}^{-1}(p)$ is a non-empty open subset of Z, $\mathrm{int}_Z\pi_{\mathrm{dom}\,p}{}^{-1}(p)\setminus\{z\}$ is non-empty. Hence there exist $F\in\mathcal{F}$ and an open set U in $\prod_{\alpha\in F}X_\alpha$ such that

(2) $\operatorname{dom}p\subset F$,

(3) $\emptyset\neq\pi_F{}^{-1}(U)\subset\operatorname{int}_Z\pi_{\operatorname{dom}p}{}^{-1}(p)$,

(4) $z\notin\pi_F{}^{-1}(U)$.

Using (1) we get $z'\in Z$ such that $\pi_F{}^{-1}(U)\cap\pi_F{}^{-1}(\pi_F(z'))\neq\emptyset$. Put $q=\pi_F(z')$. Then $q\in U$ and hence $q\leqq p$ by (2) and (3). Since $\pi_F(z)\neq q$ by (3) and (4), $q\in D_z$. These show that D_z is dense in $\boldsymbol{P}$. Next we show that D_α is dense in $\boldsymbol{P}$ for each $\alpha<\kappa$. Let $p\in\boldsymbol{P}$ and $F=\{\alpha\}\cup\operatorname{dom}p$. By (1), there exists $z\in\operatorname{int}_Z\pi_F{}^{-1}(\pi_F(z))\cap\operatorname{int}_Z\pi_{\operatorname{dom}p}{}^{-1}(p)$. Let $q=\pi_F(z)$. Then $q\in\boldsymbol{P}$, $\operatorname{dom}q=F$ and $\operatorname{int}_Z\pi_F{}^{-1}(q)\subset\operatorname{int}_Z\pi_{\operatorname{dom}p}{}^{-1}(p)$. We get $q\leqq p$ and $q\in D_\alpha$, therefore D_α is dense in $\boldsymbol{P}$.

Put $\varDelta=\{D_z: z\in Z\}\cup\{D_\alpha:\alpha<\kappa\}$. Then $\varDelta$ is a family of dense subsets of $\boldsymbol{P}$ with $\operatorname{card}\varDelta\leqq\kappa<\boldsymbol{c}$. Let G be a generic filter for $\varDelta$. (In the case $\kappa=\omega$, we do not need MA because $\varDelta$ is countable.) Then $\bigcup G\in\prod_{\alpha<\kappa}X_\alpha$ because, for each $\alpha\in\kappa$, there exists $p\in G$ satisfying $\alpha\in\operatorname{dom}p$. For each $z\in Z$, choose $p\in G\cap D_z$, then $p\neq\pi_{\operatorname{dom}p}(z)$. Thus $\bigcup G\notin Z$. On the other hand, let $F=\{\alpha_1,\alpha_2,\cdots,\alpha_n\}\in\mathcal{F}$. Choose $p_i\in G$ such that $\alpha_i\in\operatorname{dom}p_i$ for $i=1, 2,\cdots, n$ and then there exists $p\in G$ satisfying $p\leqq p_i$ for $i=1, 2,\cdots, n$. Then $F\subset\operatorname{dom}p$. There exists $z\in Z$ such that $\pi_F(\bigcup G)=\pi_F(z)$. This shows $\bigcup G\in\bar{Z}=Z$, which is a contradiction. The proof is complete.

THEOREM 3-3. *Let Z be a countable space without an isolated point. Then,*

(1) *Z can not be embedded in the product of countably many scattered spaces as a closed subset.*

(2) (MA) *Z can not be embedded in the product of κ-many scattered spaces as a closed subset, where $\kappa<\boldsymbol{c}$.*

PROOF. (1) If Z is embedded in $\prod_{n<\omega}X_n$ as a closed subset, where X_n $(n<\omega)$ are scattered spaces. By Lemma 3-2 there exist an $n\in\omega$ and a non-empty open set U of Z such that the mapping $p_n|U: U\to p_n(U)$ is nowhere dense. It is impossible because the image of a space by a nowhere dense mapping has no isolated point. The proof of the case (2) is similar.

THEOREM 3-4. (1) *If $\boldsymbol{Q}$ is embedded in $\prod_{n<\omega}X_n$ as a closed subset, then at least one factor space X_n contains a copy of $\boldsymbol{Q}$, where X_n $(n<\omega)$ are arbitrary spaces.*

(2) (MA) *If $\boldsymbol{Q}$ is embedded in $\prod_{\alpha<\kappa}X_\alpha$ as a closed subset, then at least one factor space X_α contains a copy of $\boldsymbol{Q}$, where $\kappa<\boldsymbol{c}$ and X_α $(\alpha<\kappa)$ are arbitrary spaces.*

PROOF. (1) If $\boldsymbol{Q}$ is embedded in $\prod_{n<\omega}X_n$ as a closed subset, then by Lemma 3-2 there exist $n<\omega$ and an open subset U of $\boldsymbol{Q}$ such that the mapping $p_n|U$: $U\to p_n(U)$ is nowhere dense. Since U is homeomorphic to $\boldsymbol{Q}$, $p_n(U)$ contains a copy of $\boldsymbol{Q}$ by Lemma 2-1. The proof is complete. The proof of (2) is similar.

In Theorem 3-4, we have concluded that at least one factor space contains a copy of $\boldsymbol{Q}$. It is natural to ask whether the factor space contains a "closed" copy of $\boldsymbol{Q}$? We show that it is impossible.

EXAMPLE 3-5. There exists a space X which does not contain a closed copy of $\boldsymbol{Q}$, but X^2 contains a closed copy of $\boldsymbol{Q}$. Let $X=\boldsymbol{R}$ (=the set of reals) and retopologize X as follows. The set of $\{\{x\in\boldsymbol{R}: q-1/n<x<q+1/n\}: n<\omega\}$ is a neighborhood base of q in X for $q\in\boldsymbol{Q}$ and the set of $\{\{x\in\boldsymbol{R}: p\leqq x<p+1/n\}: n<\omega\}$ is a neighborhood base of p in X for $p\in\boldsymbol{R}\setminus\boldsymbol{Q}$. Clearly X does not contain a closed copy of $\boldsymbol{Q}$ but the set $\{(-q, q): q\in\boldsymbol{Q}\}$ is a closed subset in X^2 which is homeomorphic to $\boldsymbol{Q}$.

REMARK 3-6. Hechler ([4], see [2, §8]) showed that $\boldsymbol{Q}$ is embedded in ${}^{\boldsymbol{d}}\omega$ as a closed subset but not in ${}^{\kappa}\omega$ for any $\kappa<\boldsymbol{d}$, where $\boldsymbol{d}$ is the minimum cardinal of a dominating family of ${}^{\omega}\omega$. Is our Theorem 3-4 (2) provable for $\kappa<\boldsymbol{d}$ within ZFC ?

Appendix.

Here, under the assumption of the continuum hypothesis (CH), we show the existence of a Fréchet space without isolated points which does not contain $\boldsymbol{Q}$ in any generic extension obtained by adjoining Cohen reals. Let P be the poset adjoining a single Cohen real (i.e. $P=\{p\mid p: n\to 2$, for some $n<\omega\}$). Since any subset of ω in a Cohen extension is in some generic extension by P, it suffices to show the existence of a base $\mathscr{D}$ of a space on ω which satisfies the following (1)~(3).

(1) $(\omega, \mathscr{D})$ is a space without isolated points.
(2) $\Vdash_P$"$(\omega, \mathscr{D})$ is a Fréchet space".
(3) $\Vdash_P$"$\boldsymbol{Q}$ is not embeddable in $(\omega, \mathscr{D})$".

In order to guarantee the condition (3), we introduce a certain topological property (say E-property). A space X has *E-property*, if for any convergent sequences $\langle x_k \mid k<\omega\rangle$, $\{\langle y_n^k \mid n<\omega\rangle: k<\omega\}$ of X with $\lim_{n\to\omega} y_n^k=x_k$ $(k<\omega)$, there exists a function $f: \omega\to\omega$ such that $\{y_n^k: k<\omega \ \& \ n<f(k)\}$ does not converge. It is easy to see that $\boldsymbol{Q}$ is not embeddable to any space X with E-property. We shall construct the space $(\omega, \mathscr{D})$ such that

(3)′ $\Vdash_P$"$(\omega, \mathscr{D})$ has E-property".

Henceforth, we assume CH. Let us assume that we can construct $\mathscr{D}_\alpha$ and $\mathscr{E}_\alpha^n$ $(n<\omega, \alpha<\omega_1)$ which satisfy the following (4)~(11).

(4) $\mathscr{D}_\alpha$ is a countable Boolean subalgebra of $P(\omega)$, where $P(\omega)$ is the power

set of ω.

For each $n<\omega$, set $\mathcal{D}_\alpha^n=\{a\in\mathcal{D}_\alpha : n\in a\}$.

(5) $\mathcal{E}_\alpha^n$ is a countable ideal of $P(\omega)$.

(6) $\mathcal{D}_\alpha\subset\mathcal{D}_\beta$ and $\mathcal{E}_\alpha^n\subset\mathcal{E}_\beta^n$ for $n<\omega$, $\alpha<\beta<\omega_1$.

(7) Any nonempty $a\in\mathcal{D}_\alpha$ is infinite.

(8) For any distinct $n, m<\omega$, there exists an $a\in\mathcal{D}_0^n$ such that $n\in a$ and $m\in\omega\setminus a$.

(9) $b\subset a$ (mod. f) for any $a\in\mathcal{D}_\alpha^n$ and $b\in\mathcal{E}_\alpha^n$, where $b\subset a$ (mod. f) means that $b\setminus a$ is finite.

(10) For any $n<\omega$, P-name x and $p\in P$, if $p\Vdash_P$"$x\subset\omega$", then there are $\alpha<\omega_1$ and $q\leqq p$ such that

(10.1) There exists an $a\in\mathcal{D}_\alpha^n$ so that $q\Vdash_P$"$x\cap a$ is finite";

or

(10.2) There exists an infinite $b\in\mathcal{E}_\alpha^n$ so that $q\Vdash_P$"$b\subset x$".

(11) For any $n<\omega$, $p\in P$ and P-names x, f, if p forces

(11.1) $x\subset\omega$ & $f=\langle y_k \mid k\in x\rangle : x\to P(\omega)$;

(11.2) x, y_k are infinite and $x\cap y_k=\emptyset$ for any $k\in x$;

(11.3) $y_k\cap y_l=\emptyset$ for any distinct $k, l\in x$, then there are $\alpha<\omega_1$ and $q\leqq p$ such that

(11.4) There exists an $a\in\mathcal{D}_\alpha^n$ so that $q\Vdash_P$"$x\setminus a$ is infinite"

or

(11.5) There exist an $a\in\mathcal{D}_\alpha^k$ and $k<\omega$ so that $q\Vdash_P$"$k\in x$ & $y_k\setminus a$ is infinite"

or

(11.6) There exists an $a\in\mathcal{D}_\alpha^n$ so that $q\Vdash_P$"$y_k\setminus a\neq\emptyset$ for infinitely many $k\in x$".

Set $\mathcal{D}=\bigcup_{\alpha<\omega_1}\mathcal{D}_\alpha$ and $\mathcal{E}^n=\bigcup_{\alpha<\omega_1}\mathcal{E}_\alpha^n$. Since $\mathcal{D}$ is a Boolean algebra, (8) implies that $(\omega, \mathcal{D})$ is a 0-dimensional Hausdorff space. Moreover, every non-empty element of $\mathcal{D}^n$ is infinite and hence $(\omega, \mathcal{D})$ satisfies (1). It follows from (10) and (11) that $(\omega, \mathcal{D})$ also satisfies (2) and (3)$'$. Hence, it suffices to show the existences of such $\mathcal{D}_\alpha$ and $\mathcal{E}_\alpha^n$ ($n<\omega$, $\alpha<\omega_1$). Let $\mathcal{X}$ be the set of all (n, x, p, f)'s which satisfy the assumptions of (10) or (11). Since CH is assumed, the cardinality of $\mathcal{X}$ is ω_1. Let $(n_\alpha, x_\alpha, p_\alpha, f_\alpha)$ $(\alpha<\omega_1)$ be an enumeration of $\mathcal{X}$. By induction on $\alpha<\omega_1$ we shall construct $\mathcal{D}_\alpha$ and $\mathcal{E}_\alpha^n$ (for $n<\omega_1$) which satisfy (4)$\sim$(9) and (10), (11) in case (n, x, p, f) is $(n_\alpha, x_\alpha, p_\alpha, f_\alpha)$. The following lemma gives each step of the inductive construction.

LEMMA (CH). *Suppose that we are given a system* $(\mathcal{B}, \mathcal{I}^n : n<\omega)$ *where* $\mathcal{B}$ *is a countable Boolean subalgebra of* $P(\omega)$ *and* $\mathcal{I}^n$ *are countable ideals of* $P(\omega)$ *with the following:*

(C0) *For distinct* n *and* m, *there exist* $a\in\mathcal{B}^n$ *and* $b\in\mathcal{B}^m$ *such that* $a\cap b$

$=\emptyset$, *where* $\mathscr{B}^k$ *denotes the set* $\{a\in\mathscr{B}: k\in a\}$;

(C1) *Any nonempty* $a\in\mathscr{B}$ *is infinite;*

(C2) $b\subset a$ (mod. f) *for any* $a\in\mathscr{B}^n$, $b\in\mathscr{I}^n$ $(n<\omega)$.

Then, the following (C3) *and* (C4) *hold.*

(C3) *Suppose that* $n<\omega$, $p\in P$ *and* P-*name* x *satisfy* $p\Vdash$"$x\subset\omega$ & $a\cap x$ *is infinite for any* $a\in\mathscr{B}^n$". *Then, there exist* $q\leqq p$ *and* $b\subset\omega$ *such that*

(C3.1) $(\mathscr{B}, \bar{\mathscr{I}}^i : i<\omega)$ *satisfies* (C0), (C1) *and* (C2);

(C3.2) $q\Vdash b\cap x$ *is infinite;*

where $\bar{\mathscr{I}}^n$ *denotes the ideal generated by* $\mathscr{I}^n\cup\{b\}$ *and* $\bar{\mathscr{I}}^i=\mathscr{I}^i$ *for* $i\neq n$.

(C4) *Suppose that* $n<\omega$, $p\in P$ *and* P-*names* x, f *satisfy*

(C4.1) $\Vdash x\subset\omega$, x *is infinite and* $f=\langle y_k\,|\,k\in x\rangle$;

(C4.2) $\Vdash x\cap y_k=\emptyset$ *and* y_k *is infinite for any* $k\in x$;

(C4.3) $\Vdash y_k\cap y_l=\emptyset$ *for distinct* $k, l\in x$;

(C4.4) $p\Vdash x\subset a$ (mod. f) *for any* $a\in\mathscr{B}^n$;

(C4.5) $p\Vdash y_k\subset a$ (mod. f) *for any* $a\in\mathscr{B}^k$ *and* $k\in x$.

Then there are $q\leqq p$ *and* $n\in a\subset\omega$ *such that*

(C4.6) $(\bar{\mathscr{B}}, \mathscr{I}^i : i<\omega)$ *satisfies* (C0), (C1) *and* (C2);

(C4.7) $q\Vdash y_k\setminus a\neq\emptyset$ *for infinitely many* $k\in x$;

where $\bar{\mathscr{B}}$ *denote the Boolean subalgebra generated by* $\mathscr{B}\cup\{a\}$.

PROOF. (C3) Suppose that $n<\omega$, $p\in P$ and a P-name x satisfy the assumptions of (C3). Let $\langle a_i\,|\,i<\omega\rangle$ be an enumeration of $\mathscr{B}^n$ and $\langle p_i\,|\,i<\omega\rangle$ an enumeration of $\{q\in P: q\leqq p\}$ such that for any $q\leqq p$ there are infinitely many $i<\omega$ such that $q=p_i$. Since $p\Vdash$"$x\cap a$ is infinite for any $a\in\mathscr{B}^n$", by induction on $i<\omega$ we can take $k_i<\omega$ and $q_i\leqq p_i$ so that $k_i<k_{i+1}$ and $k_i\in\bigcap_{j\leq i}a_j$ and $q_i\Vdash k_i\in x$. Set $b=\{k_i : i<\omega\}$. It is easy to see that p and b are the desired ones.

(C4) Suppose that $n<\omega$, $p\in P$ and P-names x, f satisfy the assumptions of (C4). Since there are no problems in the case that there exist $q\leqq p$ and $a\in\mathscr{B}^n$ such that $q\Vdash$"$y_k\setminus a\neq\emptyset$ for infinitely many $k\in x$", we may assume that $p\Vdash$"For any $a\in\mathscr{B}^n$ $\{k\in x: y_k\setminus a\neq\emptyset\}$ is finite".

CLAIM 1. *There exists a* $c\subset\omega$ *such that* $n\notin c$, $p\Vdash$"$c\cap y_k\neq\emptyset$ *for infinitely many* $k\in x$", $c\cap b$ *is finite for any* $b\in\bigcup_{i<\omega}\mathscr{I}^i$ *and* $c\subset a$ (mod. f) *for any* $a\in\mathscr{B}^n$.

PROOF OF CLAIM 1. Since $\langle\mathscr{B}, \mathscr{I}^i : i<\omega\rangle$ satisfy (C0) and (C2) also in the generic extension by P, $p\Vdash$"$y_k\cap b$ is finite for any $\in\mathscr{I}^n$". Hence, we can take a P-name z so that

$p\Vdash$"$z\subset\bigcup_{k\in x}y_k$ & $|z\cap y_k|\leqq 1$ for any $k\in x$",

$p\Vdash$"$z\cap b$ is finite for any $b\in\mathscr{I}^n$",

$p\Vdash$"z is infinite".

Then, $p\Vdash$"$z\subset a$ (mod. f) for any $a\in\mathscr{B}^n$". Similarly as in the proof of (C3), we can take $c\subset\omega$ so that $n\notin c$, c is infinite, $c\cap b$ is finite for any $b\in\mathscr{I}^n$, $c\subset a$

(mod. f) for any $a \in \mathcal{B}^n$ and $p \Vdash$"$c \cap z$ is infinite". This c is the required one. The proof of Claim 1 is complete.

Let c be one as in Claim 1.

FACT [7, 1.1.2 Lemma]. *Let $\mathcal{X}$ and $\mathcal{Y}$ be countable subsets of $P(\omega)$ such that $x \cap y$ is finite for each $x \in \mathcal{X}$ and $y \in \mathcal{Y}$. Then, there exists an $a \subset \omega$ such that $x \subset a$* (mod. f) *for any $x \in \mathcal{X}$ and $a \cap y$ is finite for any $y \in \mathcal{Y}$.*

Applying this fact to $\mathcal{S}^i$ and $\{c\} \cup \{e_j : j<i\} \cup \bigcup_{j>i} \mathcal{S}^j$ inductively, we get the following sequence $\langle e_i \mid i<\omega \rangle$.

CLAIM 2. *There exists a sequence $\langle e_i \mid i<\omega \rangle$ such that $n \in e_n$ and $\langle e_i \mid i<\omega \rangle$ is a partition of $\omega \setminus c$ and $b \subset e_i$* (mod. f) *for any $b \in \mathcal{S}^i$ $(i<\omega)$.*

Let $\langle e_i \mid i<\omega \rangle$ be one as in Claim 2 and define s_i $(i<\omega)$ by: $s_0=\omega \setminus c$ and $s_{i+1}=\bigcup_{j \in s_i} e_j$ for each $i<\omega$. Set $a=\bigcap_{j<\omega} s_j$. Then, the following hold: $n \in a$ and $a \cap c=\emptyset$; $e_i \subset a$ for any $i \in a$; $e_i \subset \omega \setminus a$ for any $i \in \omega \setminus a$.

Now, $q=p$ and a satisfy (C.4.6) and (C.4.7) and the proof of Lemma has been completed.

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