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An invariant of manifold pairs and its applications

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§ 0. Introduction.

Following [3], [6] let $\Theta^{m,n}$ be the set of h-cobordism classes of pairs (S^m, K) consisting of an oriented homotopy n-sphere K embedded in the oriented m-sphere S^m. It forms an abelian group under connected sum of pairs and the inverse element of (S^m, K), denoted by $-(S^m, K)$, is given by reversing both orientations of S^m and K. In case $m-n\geqq 3$ and $n\geqq 5$, $\Theta^{m,n}$ can be regarded as the isotopy classes of such pairs (S^m, K) by the h-cobordism theorem for pairs. Henceforth we will assume $m-n\geqq 3$ and $n\geqq 5$.

The group $\Theta^{m,n}$ is well understood by the work of J. Levine [6]. A result of [6] says that $\Theta^{m,n}$ has a free part of rank one if and only if $n+1\equiv 0 \pmod 4$ and $3(n+1)\geqq 2m$, and is finite otherwise. Moreover Levine's work implicitly says that in case $3n\geqq 2m$, there is a homomorphism called the signature of knots

$$\sigma : \Theta^{m,n} \longrightarrow \boldsymbol{Q}$$

and that

(0.1) the kernel of σ is finite.

When there is a Seifert surface for K, $\sigma(S^m, K)$ is defined as the signature of the Seifert surface. It is easily checked that the value is independent of the choice of a Seifert surface (here we need the assumption $3n\geqq 2m$). Moreover it immediately follows from the definition that the signature of a Seifert surface is additive with respect to connected sum of pairs. Every knot does not have a Seifert surface, but certain times connected sum of it necessarily has a Seifert surface. Hence one can extend the domain of σ to the whole group $\Theta^{m,n}$ by virtue of the additivity property of signature with respect to connected sum.

In this paper we intend to extend the domain of σ to a more general family of pairs (M, F) consisting of a connected, closed, oriented m-dimensional smooth manifold M and a connected closed oriented n-dimensional smooth submanifold F of M. We require this additivity property:

(AP) $$l((M_1, F_1)\#(M_2, F_2)) = l(M_1, F_1)+l(M_2, F_2).$$

A conclusion of this paper is that such an invariant l extending σ can be defined on a family $\bar{\Gamma}^{m,n}$ (see § 1) of pairs (M, F), but Theorem A stated below tells us that there is no such l defined on the whole family of pairs.

We shall exhibit three applications of our invariant l. Another application will be given in [**10**].

For a pair (M, F) we set

$$I(M, F)=\{(S^m, K)\in\Theta^{m,n} \mid (M, F)\#(S^m, K)\cong(M, F)\}$$

where $\cong$ indicates that there is an orientation preserving diffeomorphism of pairs. $I(M, F)$ is a subgroup of $\Theta^{m,n}$. This is a natural extension of the inertia group $I(M)$ of a smooth oriented manifold M to pairs. The inertia group $I(M)$ is always finite (at least if $m\neq 3, 4$) because the group of diffeomorphism classes of oriented homotopy n-spheres is finite [**5**]. In contrast to this, $\Theta^{m,n}$ is often infinite as remarked before. Therefore the following problem arises (see also [**11**]).

PROBLEM. *Is $I(M, F)$ finite?*

Needless to say, this problem has a sense only when $3(n+1)\geqq 2m$ and $n+1\equiv 0 \pmod 4$. The first statement of the following theorem is a direct consequence of (0.1) and (AP), and the latter tells us that one cannot extend σ, preserving the additivity property (AP), to the whole family of pairs.

THEOREM A. (1) *If $(M, F)\in\bar{\Gamma}^{m,n}$ then $I(M, F)$ is finite.*

(2) *There is an example of a pair (M, F) such that $I(M, F)$ is infinite.*

The second application is to splitting of Levine's exact sequence of codimension 3 knots. Levine [**6**] established a long exact sequence relating the group $\Theta^{m,n}$ to well studied homotopy groups. In case $n+1\equiv 0 \pmod 4$, the long exact sequence reduces to a short exact sequence:

$$(*) \qquad 0 \longrightarrow \boldsymbol{Z} \xrightarrow{\partial_3} \Theta^{m,n} \longrightarrow \pi_n(G_{m-n}, SO_{m-n}) \longrightarrow 0$$

(see [**6**, $(3)_k$ of (2.2)]). As far as the author knows, little is known concerning the group extension of this exact sequence (cf. [**7**, § IV], [**8**]).

THEOREM B. *Suppose $n+1\equiv 0 \pmod 4$. Then*

(1) *the composition $l\cdot\partial_3$ is multiplication by* 8,

(2) *if $m-n=3$, then $l(\Theta^{m,n})\subset\boldsymbol{Z}$.*

The following corollary immediately follows from Theorem B.

COROLLARY. *Suppose $n+1\equiv 0 \pmod 4$ and $m-n=3$. Then*

(1) *If $\pi_n(G_3, SO_3)$ is of odd order, then the above exact sequence $(*)$ splits.*

(Ex. $n=7$ see [7, p. 182].)

(2) *The p-component of $\Theta^{m,n}$ is isomorphic to that of $\pi_n(G_3, SO_3)$ if p is an odd prime.*

The final application is to a problem posed by Bredon. In [2, p. 340] he introduced an abelian group Θ_n^m $(0<n<m)$ consisting of L-equivalence classes of smooth involutions T on oriented homotopy m-spheres Σ with oriented fixed point set F of dimension n. The correspondence: $(\Sigma, T, F)\to(\Sigma, F)$ induces a map from Θ_n^m to $\bar{\Gamma}^{m,n}$ identified by L-equivalence relation. It turns out that l is invariant under L-equivalence; so l induces a homomorphism from Θ_n^m to $\boldsymbol{Q}$ provided $3n\geqq 2m$. The involution T_k: $(z_0, \cdots, z_{2n+1})\to(z_0, \cdots, z_{2k}, -z_{2k+1}, \cdots, -z_{2n+1})$ on the Brieskorn manifold defined by

$$W_3^{4n+1}=\{(z_0, \cdots, z_{2n+1})\in \boldsymbol{C}^{2n+2} \mid z_0^3+z_1^2+\cdots+z_{2n+1}^2=0\}\cap S^{4n+3}$$

determines an element of Θ_{4k-1}^{4n+1} if we choose orientations on W_3^{4n+1} and the fixed point set W_3^{4k-1}. Bredon asked if it is of infinite order and observed that this is the case when $k=n$ ([2, p. 341]). We see that $l(W_3^{4n+1}, W_3^{4k-1})$ is non-zero, which implies

THEOREM C. *$(W_3^{4n+1}, T_k, W_3^{4k-1})$ is of infinite order in Θ_{4k-1}^{4n+1} if $3k>2n+1$.*

This paper is organized as follows. We define the invariant l in Section 1 and verify that it agrees with σ on $\Theta^{m,n}$ in Section 2. In Section 3 we establish a product formula. The above three applications are discussed in Sections 4, 5, and 6 respectively.

Throughout this paper we work in the C^∞ category and the following conventions will be used.

CONVENTIONS. (1) A pair (M, F) means that M is a manifold, F is a submanifold of M, and both are closed, connected, and oriented.

(2) Given an oriented manifold W, the boundary ∂W of W will be oriented as follows. Let $(w_1, \cdots, w_m)$ be an orthogonal frame such that the m-form $w_1\wedge\cdots\wedge w_m$ represents the orientation of W and w_1 is outward normal to W. Then we orient ∂W by the $(m-1)$-form $w_2\wedge\cdots\wedge w_m$. In this convention the fundamental (resp. cofundamental) class of $(W, \partial W)$ (resp. ∂W) goes to that of ∂W (resp. $(W, \partial W)$) via the connecting homomorphism in homology (resp. cohomology).

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§ 1. Definition of the invariant l.

As explained in the introduction, a Seifert surface plays a role in the definition of $\sigma: \Theta^{m,n} \to \boldsymbol{Q}$. The definition consists of two steps:

(1) To see when a knot bounds a Seifert surface.

(2) To check that the signature of a Seifert surface is independent of the choice of a Seifert surface.

It is natural to adopt the same definition as σ for a general pair (M^m, F^n). Although the second step does not work without any condition on (M, F), it is not so difficult to seek a sufficient condition on (M, F) so that the second step works. However we encounter a serious problem at the first step. Needless to say every pair (M, F) does not have a Seifert surface. At least the homology class represented by F in $H_n(M; \boldsymbol{Z})$ must be zero if there is a Seifert surface for F. Therefore certain conditions must be imposed on (M, F). But it seems difficult to find an explicit sufficient condition on (M, F) so that a Seifert surface for F exists. We shall explain this in more detail.

When we try to construct a Seifert surface for F in the same way as in the case of a knot, we first assume the normal bundle ν of F is trivial. As usual we identify ν with an open tubular neighborhood of F in M. Hence choosing a trivialization of ν gives a map f from the boundary of $M-\nu$ to S^{m-n-1}. If f extends to a map from $M-\nu$ to S^{m-n-1}, then the inverse image of a regular value in S^{m-n-1} by the extended map gives a Seifert surface for F. This extension problem is not so easy in case $m-n-1 \geqq 2$, because the obstruction group $H^{q+1}(M-\nu, \partial(M-\nu); \pi_q(S^{m-n-1}))$ to extending f is non-trivial in general as so is $\pi_q(S^{m-n-1})$.

To avoid this difficulty we proceed in the following way, which is motivated by the work of Montgomery-Yang [**15**] (see also [**9**]). Suppose the normal bundle of F is trivial and F is null cobordant, say F bounds a compact manifold Y of dimension $n+1$. Then we do "surgery" of M along F using Y. If there is a Seifert surface for F, then the $(n+1)/4$-th L-class of the resulting manifold should be closely related to the signature of the Seifert surface via the signature theorem. Thus we are led to define $l(M, F)$ using the $(n+1)/4$-th L-class. This procedure does not require the existence of a Seifert surface. Philosophically speaking we moved from an internal invariant (i.e. the signature of a Seifert surface) to an external invariant. Our definition involves ambiguity at the surgery step. It turns out that some conditions are necessary to ensure the well-definedness.

Now we shall carry out this idea. Throughout this paper all (co)homology groups will be taken with rational coefficients unless otherwise stated. We consider the family $\Gamma^{m,n}$ of oriented diffeomorphism classes of pairs (M, F)

satisfying these conditions:

(C1) The normal bundle $\nu(F)$ of F is trivial.

(C2) There is an oriented, connected, and compact manifold Y such that $\partial Y=F$ (as oriented manifolds) and the restriction map $H^{4s-1}(Y)\to H^{4s-1}(F)$ is surjective for $2s\leqq m-n$.

(C3) The homology class $[F]$ represented by F is zero in $H_n(M)$.

(C4) If $n+1=4k$, then the k-th L-class $L_k(M)$ of M vanishes in $H^{n+1}(M)$.

Let D^q be the unit disk of $\boldsymbol{R}^q$ equipped with the natural orientation. Let ϕ: $(F\times D^{m-n}, F\times\{0\})\to(M, F)$ be an orientation preserving embedding. We consider the product of M with the unit interval $I=[0, 1]$. Identifying $M\times\{1\}$ with M naturally, we glue $Y\times D^{m-n}$ to $M\times I$ via ϕ. We shall denote the resulting manifold by $\overline{M}(\phi, Y)$ and give it the orientation induced from $Y\times D^{m-n}$. The triad $(\overline{M}(\phi, Y);\ M\times I, Y\times D^{m-n})$ yields a Mayer-Vietoris exact sequence:

$$
\begin{aligned}
(1.1)\qquad & H^n(Y\times D^{m-n})\oplus H^n(M\times I)\longrightarrow H^n(F\times D^{m-n})\xrightarrow{\ \delta\ } H^{n+1}(\overline{M}(\phi, Y))\\
&\longrightarrow \underset{\substack{\| \\ 0}}{H^{n+1}(Y\times D^{m-n})}\oplus H^{n+1}(M\times I).
\end{aligned}
$$

Since $H^*(F\times D^{m-n})$ is isomorphic to $H^*(F)$ via the projection map, we regard the cofundamental class $[F]^*$ of F as an element of $H^n(F\times D^{m-n})$. Note that (C3) is equivalent to the injectivity of the coboundary map δ in (1.1).

LEMMA 1.2. *If $n+1=4k$, then there is a unique rational number $l'(\overline{M}(\phi, Y))$ such that*

$$L_k(\overline{M}(\phi, Y))=l'(\overline{M}(\phi, Y))\delta([F]^*).$$

PROOF. Let i: $M\times I\to\overline{M}(\phi, Y)$ be the inclusion map. By the naturality of characteristic classes ([**14**]) we have

$$i^*L_k(\overline{M}(\phi, Y))=L_k(M\times I).$$

Via the projection map to the first factor $H^*(M)$ is isomorphic to $H^*(M\times I)$ and $L_k(M)$ goes to $L_k(M\times I)$. On the other hand, $L_k(M)$ vanishes by (C4). These show that $i^*L_k(\overline{M}(\phi, Y))$ vanishes. This together with the exactness of (1.1) proves the lemma. Q.E.D.

Although the value $l'(\overline{M}(\phi, Y))$ depends on the choice of Y, we can prove

THEOREM 1.3. *The difference* $\operatorname{Sign} Y-l'(\overline{M}(\phi, Y))$ *is independent of the choices of ϕ and Y, where* $\operatorname{Sign} Y$ *denotes the signature of Y.*

DEFINITION 1.4. We define $l(M, F)=\operatorname{Sign} Y-l'(\overline{M}(\phi, Y))$ in case $n+1$ is divisible by 4 and $l(M, F)=0$ otherwise.

PROOF OF THEOREM 1.3. Let

$$\phi_+, \phi_-: \ (F\times D^{m-n}, F\times\{0\}) \longrightarrow (M, F)$$

be orientation preserving embeddings. Let Y_+ and Y_- be oriented manifolds as in (C2). Take the product of M with the interval $J=[-1, 1]$. As before we identify M with $M\times\{1\}$ (resp. $M\times\{-1\}$) naturally and regard ϕ_+ (resp. ϕ_-) as an embedding to $M\times\{1\}$ (resp. $M\times\{-1\}$). We glue $Y_+\times D^{m-n}$ and $Y_-\times D^{m-n}$ to $M\times J$ via ϕ_+ and ϕ_- respectively. If we cut the resulting manifold, say W, at the level 0, it decomposes into two pieces. They can be identified with $\overline{M}(\phi_+, Y_+)$ and $\overline{M}(\phi_-, Y_-)$ respectively. Thus taking the orientation into account, we obtain W by pasting together $\overline{M}(\phi_+, Y_+)$ and $-\overline{M}(\phi_-, Y_-)$ along $M\times\{0\}=M$. Let V be a closed submanifold of W obtained by glueing Y_+ and Y_- to $F\times J$ via ϕ_+ and ϕ_- respectively. We abbreviate $F\times\{1\}$ (resp. $F\times\{-1\}$) as F_+(resp. F_-). The triples $(W; Y_\pm\times D^{m-n}, M\times J)$ and $(V; Y_\pm, F\times J)$ yield a commutative diagram of Mayer-Vietoris exact sequences:

$$\begin{tikzcd}
H^n(F_\pm\times D^{m-n}) \arrow[r, "\delta_W"] \arrow[d] & H^{n+1}(W) \arrow[r, "\iota^*"] \arrow[d, "j^*"] & \underset{\substack{\| \\ 0}}{H^{n+1}(Y_\pm\times D^{m-n})}\oplus H^{n+1}(M\times J) \arrow[d] \\
H^n(F_\pm) \arrow[r, "\delta_V"] & H^{n+1}(V) \arrow[r] & \underset{\substack{\| \\ 0}}{H^{n+1}(Y_\pm)}\oplus\underset{\substack{\| \\ 0}}{H^{n+1}(F\times J)}
\end{tikzcd} \tag{1.5}$$

where the vertical homomorphisms are induced by the inclusion maps.

LEMMA 1.6. $L_k(W) = l'(\overline{M}(\phi_+, Y_+))\delta_W([F_+]^*)-l'(\overline{M}(\phi_-, Y_-))\delta_W([F_-]^*)$ *where* $[F_\pm]^*$ *are regarded as elements of* $H^n(F_\pm\times D^{m-n})$ *via the projection map to the first factor as before and* δ_W *is the coboundary map in* (1.5).

LEMMA 1.7. *Let* $j: V\to W$ *be the inclusion map. Then*
(1) $j^*L_k(W) = L_k(V)$,
(2) $j^*\delta_W([F_+]^*) = j^*\delta_W([F_-]^*) = [V]^*$.

We shall take these lemmas for granted for the moment and complete the proof of the theorem. We restrict the identity in Lemma 1.6 to V. Then it turns into

$$L_k(V) = (l'(\overline{M}(\phi_+, Y_+))-l'(\overline{M}(\phi_-, Y_-)))[V]^*$$

by Lemma 1.7. On the other hand it follows from the signature theorem and the additivity property of signature ([**1**, p. 588]) that

$$L_k(V)[V] = \mathrm{Sign}\, V = \mathrm{Sign}\, Y_+ - \mathrm{Sign}\, Y_-.$$

These two identities prove the theorem.

PROOF OF LEMMA 1.6. Let $\iota\colon M\times J\to W$ be the inclusion map. By the naturality of characteristic classes and (C4) we have

$$\iota^* L_k(W) = L_k(M\times J) = 0.$$

Hence the exactness of the sequence (1.5) implies that one can express

$$L_k(W) = a_+\delta_W([F_+]^*) + a_-\delta_W([F_-]^*) \tag{1.8}$$

with rational numbers $a_\pm$.

Remember that W is obtained by pasting together $\overline{M}(\phi_+, Y_+)$ and $-\overline{M}(\phi_-, Y_-)$ along M. Restricting (1.8) to $\overline{M}(\phi_+, Y_+)$ yields

$$L_k(\overline{M}(\phi_+, Y_+)) = a_+\delta([F_+]^*)$$

where δ is the coboundary map in (1.1). Since F_+ has the same orientation as F, we conclude

$$a_+ = l'(\overline{M}(\phi_+, Y_+)).$$

Similarly, restricting (1.8) to $\overline{M}(\phi_-, Y_-)$ and taking orientations into account, we conclude

$$a_- = -l'(\overline{M}(\phi_-, Y_-)).$$

These prove Lemma 1.6.

PROOF OF LEMMA 1.7. (1) Since the normal bundle $\nu(V)$ of V in W is of dimension $m-n$, the s-th Pontrjagin class $p_s(\nu(V))$ of $\nu(V)$ vanishes by definition for $2s>m-n$ (see [**14**, p. 174]). Hence it suffices to show $p_s(\nu(V))=0$ for $0<2s \leqq m-n$.

We consider the Mayer-Vietoris exact sequence of the triad obtained by cutting V along $F\times\{0\}=F$. Since the resulting two pieces are homotopy equivalent to $Y_\pm$ respectively, we get an exact sequence

$$H^{4s-1}(Y_+)\oplus H^{4s-1}(Y_-) \xrightarrow{u^*} H^{4s-1}(F) \longrightarrow H^{4s}(V) \xrightarrow{v^*} H^{4s}(Y_+)\oplus H^{4s}(Y_-).$$

By (C2) the above u^* is surjective for $2s\leqq m-n$ and hence the above v^* is injective. On the other hand since $\nu(V)|Y_\pm$ is trivial by our construction, $v^*(p_s(\nu(V)))=0$ for $s>0$ and hence $p_s(\nu(V))=0$ for $0<2s\leqq m-n$ as required.

(2) Consider the diagram (1.5). Since $\delta_V([F_+]^*)=\delta_V([F_-]^*)=[V]^*$ as is well known (see the conventions in the introduction), the assertion follows from the commutativity of (1.5). Q.E.D.

THEOREM 1.9. *Our invariant $l\colon \Gamma^{m,n}\to \boldsymbol{Q}$ satisfies the additivity property* (AP) *in the introduction. Moreover $l(-(M, F))=-l(M, F)$ for $(M, F)\in\Gamma^{m,n}$.*

PROOF. We do boundary connected sum (of pairs) of $M_1\times J$ and $M_2\times J$ at the -1 level. The boundary of the resulting pair is diffeomorphic to the disjoint union of $-((M_1, F_1)\#(M_2, F_2))$, (M_1, F_1), and (M_2, F_2). We do surgery of the same kind as before along $(F_1\times\{-1\})\#(F_2\times\{-1\})$, $F_1\times\{1\}$, and $F_2\times\{1\}$. Then an argument similar to the proof of Lemma 1.6 and to the last part of the proof of Theorem 1.3 verifies the additivity property (AP).

The latter property is clear from the definition of l. Q.E.D.

By virtue of the additivity property (AP) we can well extend the domain of $l: \Gamma^{m,n}\to\boldsymbol{Q}$ to the following semi-group $\bar{\Gamma}^{m,n}$.

DEFINITION 1.10. $\bar{\Gamma}^{m,n}$ is the set of oriented diffeomorphism classes of pairs (M, F) such that certain times connected sum of (M, F) with itself belongs to $\Gamma^{m,n}$.

§ 2. $l=\sigma$ on $\Theta^{m,n}$.

As easily observed $\Theta^{m,n}$ is contained in $\bar{\Gamma}^{m,n}$ if $3n\geqq 2m$. In this section we verify that our invariant $l: \bar{\Gamma}^{m,n}\to\boldsymbol{Q}$ agrees with the signature invariant σ (stated in the introduction) on $\Theta^{m,n}$.

The set of all n-knots in S^m having a Seifert surface forms a subgroup of $\Theta^{m,n}$ of finite index, so it suffices to check their agreement for any such n-knot in S^m. Let (S^m, K) be an n-knot in S^m having a Seifert surface. We push the interior of the Seifert surface into D^{m+1} bounded by S^m to get a pair (D^{m+1}, U^{n+1}) with (S^m, K) as the boundary. Remember that by definition we have

$$\sigma(S^m, K)=\operatorname{Sign} U. \tag{2.1}$$

The following theorem was motivated by this observation and is useful to compute our invariant.

THEOREM 2.2. *Let (M, F) be a pair in $\Gamma^{m,n}$ such that $H^{m-n-1}(F)=0$. Suppose it bounds a pair (L^{m+1}, E^{n+1}) such that*

(1) *$L_k(L)=0$ in $H^{n+1}(L)$ when $n+1=4k$,*

(2) *the normal bundle of E in L is trivial.*

Then $l(M, F)=\operatorname{Sign} E$.

COROLLARY 2.3. *l agrees with σ on $\Theta^{m,n}$ (in case $3n\geqq 2m$).*

PROOF. Let (S^m, K) be an n-knot having a Seifert surface. We note $H^{m-n-1}(K)=0$ because $m-n-1\neq n, 0$. By construction the normal bundle of U in D^{m+1} is trivial, so the pair (D^{m+1}, U) satisfies the condition of Theorem 2.2. It follows from (2.1) and Theorem 2.2 that

$$\sigma(S^m, K)=\operatorname{Sign} U=l(S^m, K)$$

as required. Q.E.D.

The rest of this section is devoted to the proof of Theorem 2.2.

Step 1. Let $\phi: (F\times D^{m-n}, F\times\{0\})\to(M, F)$ be an orientation preserving embedding which extends to an embedding: $(E\times D^{m-n}, E\times\{0\})\to(L, E)$. Let Y be the same as before. The boundary of $\overline{M}(\phi, Y)$ consists of two connected components. One of them is diffeomorphic to M. We shall denote the other one by $M(\phi, Y)$.

Consider the Mayer-Vietoris exact sequences of the triads $(\overline{M}(\phi, Y); M\times I, Y\times D^{m-n})$ and $(M(\phi, Y); M-\mathring{N}, Y\times S^{m-n-1})$ where $\mathring{N}$ denotes the interior of $N=\phi(F\times D^{m,n})$:

$$
(2.4)\qquad
\begin{array}{ccccc}
H^n(M\times I)\oplus H^n(Y\times D^{m-n}) & \longrightarrow & H^n(F\times D^{m-n}) & \xrightarrow{\delta} & H^{n+1}(\overline{M}(\phi, Y)) \\
\downarrow & & \downarrow & & \downarrow \\
H^n(M-\mathring{N})\oplus H^n(Y\times S^{m-n-1}) & \xrightarrow{\Phi} & H^n(F\times S^{m-n-1}) & \xrightarrow{\delta'} & H^{n+1}(M(\phi, Y))
\end{array}
$$

where the vertical maps are restrictions. We will use the same notation $[F]^*$ for the pullbacks of cofundamental class $[F]^*\in H^n(F)$ by the projection maps $F\times D^{m-n}\to F$ and $F\times S^{m-n-1}\to F$. By definition we have

$$L_k(\overline{M}(\phi, Y)) = (\operatorname{Sign} Y - l(M, F))\delta([F]^*).$$

Restricting this to $M(\phi, Y)$ yields

$$L_k(M(\phi, Y)) = (\operatorname{Sign} Y - l(M, F))\delta'([F]^*). \tag{2.5}$$

CLAIM 2.6. *$[F]^*\in H^n(F\times S^{m-n-1})$ is not contained in the image of Φ in* (2.4). *Hence $\delta'([F]^*)$ is non-zero.*

PROOF. Let i_1 (resp. i_2): $F\times S^{m-n-1}\to M-\mathring{N}$ (resp. $Y\times S^{m-n-1}$) be the inclusion map. By definition $\Phi(\alpha, \beta)=i_1^*\alpha-i_2^*\beta$. Let $[F]$ be the homology class of $H_n(F\times S^{m-n-1})$ represented by $F\times\{x_0\}$ ($x_0\in S^{m-n-1}$). We note that

$$i_{1*}[F] = 0 \qquad \text{and} \qquad i_{2*}[F] = 0 .$$

In fact, since we have

$$
\begin{aligned}
H_{n+1}(M, M-\mathring{N}) &\cong H_{n+1}(N, \partial N) && \text{(excision)}\\
&\cong H^{m-n-1}(N) && \text{(Lefschetz duality)}\\
&\cong H^{m-n-1}(F) && \\
&= 0 && \text{(by assumption)},
\end{aligned}
$$

the homomorphism $H_n(M-\mathring{N})\to H_n(M)$ induced by the inclusion map is injective. This and the condition (C3) in § 1 prove the former identity. The latter one is obvious because F is the boundary of Y. Hence for any $\alpha\in H^n(M-\mathring{N})$ and

$\beta \in H^n(Y \times S^{m-n-1})$ we have

$$\langle \Phi(\alpha, \beta), [F] \rangle = \langle i_1^*\alpha - i_2^*\beta, [F] \rangle = \langle \alpha, i_{1*}[F] \rangle - \langle \beta, i_{2*}[F] \rangle = 0$$

where $\langle \ , \ \rangle$ denotes the usual pairing of cohomology and homology. However $\langle [F]^*, [F] \rangle = 1$, in particular non-zero, so the claim follows.

Step 2. Consider the Mayer-Vietoris exaxt sequeces of the triads $(L \cup Y \times D^{m-n}; Y \times D^{m-n}, L)$ and $(E \cup Y; Y, E)$:

$$(2.7)\qquad \begin{tikzcd}
H^n(Y \times D^{m-n}) \oplus H^n(L) \arrow[r, "\Phi_L"] \arrow[d] & H^n(F \times D^{m-n}) \arrow[r, "\delta_L"] \arrow[d] & H^{n+1}(L \cup Y \times D^{m-n}) \arrow[d] \arrow[r] & H^{n+1}(Y \times D^{m-n}) \oplus H^{n+1}(L) = 0 \\
H^n(Y) \oplus H^n(E) \arrow[r, "\Phi_E"] & H^n(F) \arrow[r, "\delta_E"] & H^{n+1}(E \cup Y) \arrow[r] & \underset{\substack{\| \\ 0}}{H^{n+1}(Y)} \oplus \underset{\substack{\| \\ 0}}{H^{n+1}(E)}
\end{tikzcd}$$

where the vertical maps are restrictions. Since the boundaries of E and Y are both F, Φ_E is the zero map. On the other hand the middle vertical map is an isomorphism. It follows from the commutativity of (2.7) that Φ_L is the zero map.

CLAIM 2.8. $L_k(L \cup Y \times D^{m-n}) = (\mathrm{Sign}\, Y - l(M, F))\delta_L([F]^*)$.

PROOF. By the naturality of characteristic classes $L_k(L \cup Y \times D^{m-n})$ goes to $L_k(L)$ via the restriction map to L. But $L_k(L) = 0$ by the assumption, so one can express

$$L_k(L \cup Y \times D^{m-n}) = a_L \delta_L([F]^*)$$

with $a_L \in \boldsymbol{Q}$. We note that $M(\phi, Y)$ is the boundary of $L \cup Y \times D^{m-n}$. Hence restricting the above identity to the boundary component $M(\phi, Y)$, we get

$$L_k(M(\phi, Y)) = a_L \delta'([F]^*).$$

Compare this with (2.5). Then it follows from Claim 2.6 that

$$a_L = \mathrm{Sign}\, Y - l(M, F).$$

Step 3. We restrict the identity of Claim 2.8 to $E \cup Y$. Since ϕ is chosen so that it extends to a trivialization of the normal bundle of E in L, the normal bundle of $E \cup Y$ in $L \cup Y \times D^{m-n}$ is also trivial. Hence $L_k(L \cup Y \times D^{m-n})$ goes to $L_k(E \cup Y)$ via the restriction map to $E \cup Y$. Consequently we get

$$(2.9)\qquad L_k(E \cup Y) = (\mathrm{Sign}\, Y - l(M, F))\delta_E([F]^*).$$

Here we should make mention of an orientation on $E \cup Y$. Remember that

$\partial Y = F$ as oriented manifolds. We give $E \cup Y$ the orientation induced from Y so that the induced orientation on E is different from the original one on E. Hence it follows from the additivity property of signature that

$$\mathrm{Sign}(E \cup Y) = \mathrm{Sign}\, Y - \mathrm{Sign}\, E . \tag{2.10}$$

Moreover, by the definition of δ_E, we have

$$\delta_E([F]^*) = [E \cup Y]^* . \tag{2.11}$$

Now we evaluate both terms of (2.9) on the fundamental class of $E \cup Y$. Then the left term turns into $\mathrm{Sign}\, Y - \mathrm{Sign}\, E$ by the signature theorem and (2.10), and the right term turns into $\mathrm{Sign}\, Y - l(M, F)$ by (2.11). This shows that $l(M, F) = \mathrm{Sign}\, E$, which proves the theorem. Q. E. D.

§ 3. A product formula.

The result of this section will be used in [**10**] but not used in the following sections; so the reader may skip this section.

Given a pair (M, F) and a closed oriented manifold N of dimension r, their product yields a new pair $(M, F) \times N = (M \times N, F \times N)$. Suppose (M, F) belongs to $\Gamma^{m,n}$. Then the product pair does not necessarily belong to $\Gamma^{m+r,n+r}$. In fact, taking product preserves the conditions (C1)-(C3) but not (C4). A necessary and sufficient condition for the product pair to belong to $\Gamma^{m+r,n+r}$ is

$$L_u(M \times N) = 0$$

where $4u = n + r + 1$. By the product formula of L-class we have

$$L_u(M \times N) = \sum_{0 \leq u-j \leq r/4} L_j(M) \times L_{u-j}(N) .$$

Hence a sufficient condition for the product pair to belong to $\Gamma^{m+r,n+r}$ is

$$L_j(M) = 0 \qquad \text{for} \quad n+1 \leqq 4j \leqq n+r+1 . \tag{3.1}$$

If the product pair belongs to $\Gamma^{m+r,n+r}$, then it is natural to ask how the value $l((M, F) \times N)$ can be described in terms of $l(M, F)$ and N. The following theorem answers this question.

THEOREM 3.2 (Product formula). *Let (M, F) be a pair in $\Gamma^{m,n}$ and N a closed, connected, and oriented manifold of dimension r. Suppose M satisfies the condition* (3.1). *Then we have*

$$l((M, F) \times N) = l(M, F)\, \mathrm{Sign}\, N .$$

REMARK 3.3. This identity makes sense only when $n+r+1$ is divisible by 4, because otherwise the values in both sides vanish by the definition of l and signature.

PROOF OF THEOREM 3.2. Let ϕ and Y be the same as in the previous sections. The Mayer-Vietoris exact sequence of the triad $(\overline{M}(\phi, Y);\ Y\times D^{m-n}, M\times I)$ implies that the restriction map $i^*\colon H^q(\overline{M}(\phi, Y))\to H^q(M\times I)$ is an isomorphism for $q>n+1$ (cf. (1.1)). This together with (3.1) shows

$$L_j(\overline{M}(\phi, Y))=0 \qquad \text{for} \quad n+1<4j\leqq n+r+1. \tag{3.4}$$

For the product pair $(M, F)\times N$ we choose the product embedding $\phi\times\mathrm{id}\colon (F\times D^{m-n}, F\times\{0\})\times N\to(M, F)\times N$ and take $Y\times N$ as an oriented manifold bounded by $F\times N$. Then the manifold obtained by doing surgery of $M\times N$ (in the sense of §1) using $\phi\times\mathrm{id}$ and $Y\times N$ is exactly $\overline{M}(\phi, Y)\times N$. Hence it follows from the definition of l that

$$L_u(\overline{M}(\phi, Y)\times N)=(\operatorname{Sign} Y\times N-l((M, F)\times N))\delta([F\times N]^*). \tag{3.5}$$

On the other hand since $L_i(N)\in H^{4i}(N)=0$ for $4i>r$, we have

$$L_u(\overline{M}(\phi, Y)\times N)=\sum_{n+1\leqq 4j\leqq n+r+1} L_j(\overline{M}(\phi, Y))\times L_{u-j}(N) \tag{3.6}$$

by the product formula of L-class (remember that $4u=n+r+1$). We distinguish two cases.

Case 1. The case where $n+1\neq 0 \pmod 4$. In this case $l(M, F)$ vanishes by definition. Hence it suffices to verify

$$l((M, F)\times N)=0. \tag{3.7}$$

Since $n+1\neq 0 \pmod 4$, (3.4) and (3.6) imply that $L_u(\overline{M}(\phi, Y)\times N)$ vanishes. Hence we have $l((M, F)\times N)=\operatorname{Sign} Y\times N$ by (3.5). Here

$$\operatorname{Sign} Y\times N=\operatorname{Sign} Y\ \operatorname{Sign} N \tag{3.8}$$

and $\operatorname{Sign} Y=0$ as $\dim Y=n+1\not\equiv 0 \pmod 4$. These verify (3.7).

Case 2. The case where $n+1\equiv 0 \pmod 4$. In this case $r\equiv 0 \pmod 4$ as $n+r+1\equiv 0 \pmod 4$. We set $n+1=4k$ and $r=4s$. Then we have

$$L_u(\overline{M}(\phi, Y)\times N)=L_k(\overline{M}(\phi, Y))\times L_s(N) \tag{3.9}$$

by (3.4) and (3.6). On the other hand we have

$$L_k(\overline{M}(\phi, Y))=(\operatorname{Sign} Y-l(M, F))\delta([F]^*)$$

by the definition of $l(M, F)$ and

$$L_s(N)=\operatorname{Sign} N[N]^*$$

by the signature theorem. Substituting these for the right hand side of (3.9) and using a well known formula $(\delta[F]^*)\times[N]^*=\delta([F]^*\times[N]^*)=\delta([F\times N]^*)$ (see [**17**, p. 250]), we get

$$L_u(\overline{M}(\phi, Y)\times N) = (\operatorname{Sign} Y - l(M, F)) \operatorname{Sign} N\delta([F\times N]^*).$$

Comparing this with (3.5) and using the fact (3.8), we get the desired formula.

Q. E. D.

§ 4. Proof of Theorem A.

As indicated in the introduction the statement (1) of Theorem A immediately follows from the fact (0.1) and the additivity property (AP) of l. As for the statement (2) the following theorem, which is an immediate consequence of [**10**] or [**12**], gives many desired examples.

THEOREM 4.1 ([**10**], [**12**]). *Let N be a closed, connected, and oriented manifold of dimension n. Then*

$$I(N\times \boldsymbol{C}P^q, N\times\{x_0\})\otimes \boldsymbol{Q} = \Theta^{n+2q,n}\otimes \boldsymbol{Q} \qquad \text{for} \quad q\geqq 2$$

where x_0 is a point of $\boldsymbol{C}P^q$.

We shall give another such example. Let $\Sigma^{m,n}$ be the subset of $\Theta^{m,n}$ consisting of knots diffeomorphic to S^n with trivial normal bundle. Note that $\Sigma^{m,n}$ is a subgroup of $\Theta^{m,n}$ of finite index.

THEOREM 4.2. *$I(S^n\times S^{m-n}, S^n\times\{y_0\})$ contains $\Sigma^{m,n}$, where y_0 is a point of S^{m-n}. In particular*

$$I(S^n\times S^{m-n}, S^n\times\{y_0\})\otimes \boldsymbol{Q} = \Theta^{m,n}\otimes \boldsymbol{Q}.$$

PROOF. Let (S^m, K) be an element of $\Sigma^{m,n}$ and consider the connected sum $(S^n\times S^{m-n}, S^n\times\{y_0\})\#(S^m, K)$. Look at the subset $(S^n\times\{y_0\})\#K\cup\{x_0\}\times S^{m-n}$ ($x_0\in S^n$), which is exactly the wedge sum of S^n and S^m as K is diffeomorphic to S^n. As easily checked, the complement of an open regular neighborhood of the subset is contractible and hence diffeomorphic to the standard m-disk as $m\geqq 5$ (remember that we always assume $m\geqq n+3\geqq 8$). Therefore one concludes

$$(S^n\times S^{m-n}, S^n\times\{y_0\})\#(S^m, K)\cong(S^n\times S^{m-n}, S^n\times\{y_0\})\#\Sigma \tag{4.3}$$

where Σ is a homotopy m-sphere and the latter connected sum is done away from the submanifold S^n.

On the other hand the ambient manifold must be diffeomorphic to $S^n\times S^{m-n}$ because it is the connected sum of $S^n\times S^{m-n}$ and S^m by our construction. These mean that Σ belongs to the inertia group of $S^n\times S^{m-n}$. But the group is trivial ([**16**]), so Σ must be the standard sphere. This together with (4.3) proves the lemma. Q. E. D.

These examples show that we cannot drop the condition (C3) in § 1. The author does not know whether the other conditions are essentially necessary.

§ 5. Proof of Theorem B.

As stated in the introduction, there is a Levine's exact sequence:

$$0 \longrightarrow \boldsymbol{Z} \xrightarrow{\partial_3} \Theta^{m,n} \longrightarrow \pi_n(G_{m-n}, SO_{m-n}) \longrightarrow 0$$

where $n+1\equiv 0 \pmod 4$. By definition $\partial_3(1)$ is an n-knot in S^m having a Seifert surface with signature 8. Hence the statement (1) of Theorem B follows from Corollary 2.3.

In the following we will assume $m-n=3$ and prove the statement (2) of Theorem B. Let (S^m, K) be an element of $\Theta^{m,n}$. According to Levine [7, p. 171] every codimension 3 knot has the trivial normal bundle; so there is an embedding ϕ: $K\times D^3\to S^m$ giving a trivialization of the normal bundle of K in S^m. Since K is a homotopy sphere, it bounds a compact oriented manifold Y. The manifold $\bar{S}^m(\phi, Y)$ obtained by doing surgery of S^m using ϕ and Y (see § 1) has two boundary components. One of them is S^m. The other one is $S^m(\phi, Y)$ in the terminology of § 2. We shall denote $S^m(\phi, Y)$ by X.

As observed in § 2 (see (2.5))

$$L_k(X) = (\mathrm{Sign}\, Y - l(S^m, K))\delta'([K]^*) \tag{5.1}$$

where δ'; $H^n(K\times S^2)\to H^{n+1}(X)$ is the coboundary map as before. Consider the Mayer-Vietoris exact sequence of the triad $(X;\ S^m-\phi(K\times \mathring{D}^3),\ Y\times S^2)$:

$$H^{q-1}(K\times S^2) \xrightarrow{\delta'} H^q(X) \xrightarrow{(u_1^*,\ u_2^*)} H^q(S^m-\phi(K\times \mathring{D}^3))\oplus H^q(Y\times S^2) \tag{5.2}$$

where u_1: $S^m-\phi(K\times \mathring{D}^3)\to X$ and u_2: $Y\times S^2\to X$ are both the inclusion maps.

LEMMA 5.3. *u_2^*: $H^2(X)\to H^2(Y\times S^2)$ is an isomorphism.*

PROOF. Consider the exact sequence of the pair $(X, Y\times S^2)$:

$$H^2(X, Y\times S^2) \longrightarrow H^2(X) \xrightarrow{u_2^*} H^2(Y\times S^2) \longrightarrow H^3(X, Y\times S^2).$$

Here we have isomorphisms

$$\begin{aligned} H^q(X, Y\times S^2) &\cong H^q(S^m-\phi(K\times \mathring{D}^3),\ \phi(K\times S^2)) && \text{(excision)} \\ &\cong H_{m-q}(S^m-\phi(K\times \mathring{D}^3)) && \text{(Lefschetz duality)} \\ &\cong H^{q-1}(\phi(K\times \mathring{D}^3)) && \text{(Alexander duality)} \\ &\cong H^{q-1}(K). \end{aligned}$$

Therefore

$$H^2(X, Y\times S^2) = H^3(X, Y\times S^2) = 0$$

as K is a homotopy n-sphere and $n\geqq 5$. This together with the above exact sequence proves the lemma. Q. E. D.

By the above lemma there is a unique element x of $H^2(X)$ such that $u_2^*x=[S^2]^*$ where $[S^2]^*$ is the pullback of the cofundamental class of S^2 by the projection map from $Y\times S^2$ to S^2. Clearly $u_2^*x^2=0$. On the other hand $S^m-\phi(K\times\mathring{D}^3)$ is a cohomology 2-sphere by the Alexander duality; so $u_1^*x^2=0$. Since $H^3(K\times S^2)=0$ as K is a homotopy n-sphere and $n\geqq 5$, it follows from the exactness of (5.2) that $x^2=0$ and hence

$$x^s = 0 \qquad \text{for} \quad s\geqq 2. \tag{5.4}$$

By the coboundary formula (see [**17**, p. 252]) we have

$$x\cup\delta'([K]^*) = \delta'([S^2]^*\cup[K]^*) = \delta'([K\times S^2]^*) = [X]^*.$$

Hence taking the cup product of (5.1) with x, we get

$$x\cup L_k(X) = (\mathrm{Sign}\, Y - l(S^m, K))[X]^*.$$

Therefore it suffices to prove the following integrality

$$\langle x\cup L_k(X), [X]\rangle \in \boldsymbol{Z} \tag{5.5}$$

as $\mathrm{Sign}\, Y$ is an integer. By (5.4) $x=\tanh x$. Hence the value in (5.5) agrees with the signature of the codimension 2 closed submanifold of X representing the cycle dual to x (see [**4**, Chap. 2, § 9]). This shows (5.5) and completes the proof of Theorem B.

§ 6. Proof of Theorem C.

Let (Σ_i, T_i, F_i) $(i=1, 2)$ denote an oriented homotopy m-sphere Σ_i with a smooth involution T_i fixing an oriented n-dimensional manifold F_i (in fact, F_i is a $\boldsymbol{Z}_2$ homology sphere by Smith theorem). We shall say that (Σ_1, T_1, F_1) and (Σ_2, T_2, F_2) are L-equivalent if there exists a smooth involution on $\Sigma_1\times I$ whose ends are equivariantly diffeomorphic to (Σ_1, T_1, F_1) and $(-\Sigma_2, T_2, -F_2)$ as oriented manifolds respectively. We denote by Θ_n^m the set of L-equivalence classes of all oriented homotopy m-spheres with a smooth involution fixing an oriented n-dimensional manifold. Bredon [**2**, pp. 339-340] observed that Θ_n^m forms an abelian group under equivariant connected sum provided $0<n<m$ and $m\geqq 5$.

We consider the subset $\bar{\Theta}_n^m$ of Θ_n^m consisting of those elements which have a representative whose fixed point set has trivial normal bundle. Clearly it is a subgroup of Θ_n^m. Let (Σ, T, F) be a representative of an element of $\bar{\Theta}_n^m$. Since the normal bundle of F is trivial and F is a $\boldsymbol{Z}_2$ homology n-sphere, the pair (Σ, F) belongs to the family $\bar{\Gamma}^{m,n}$ if $3n \geqq 2m$ and $n \geqq 5$. Hence we can assign the rational number $l(\Sigma, F)$ to (Σ, T, F).

By an argument similar to the proof of Theorem 1.3 one can see that the value $l(\Sigma, F)$ is invariant under L-equivalence. Therefore the correspondence: $(\Sigma, T, F) \rightarrow l(\Sigma, F)$ induces a map: $\bar{\Theta}_n^m \rightarrow \boldsymbol{Q}$, denoted again by l. It is obviously a homomorphism.

Now we proceed to the proof of Theorem C. We first note that we may replace the zero term in the equation defining W_3^{4n+1} (see the introduction) by a small non-zero number ε because those $\boldsymbol{Z}_2$ manifolds are equivariantly diffeomorphic to each other. Then the pair (W_3^{4n+1}, W_3^{4k-1}) bounds a pair (V_3^{4n+2}, V_3^{4k}) defined by

$$V_3^{4n+2} = \{(z_0, \cdots, z_{2n+1}) \in \boldsymbol{C}^{2n+2} \mid z_0^3 + z_1^2 + \cdots + z_{2n+1}^2 = \varepsilon\} \cap D^{4n+4}$$
$$V_3^{4k} = V_3^{4n+2} \cap \{(z_0, \cdots, z_{2n+1}) \in \boldsymbol{C}^{2n+2} \mid z_{2k+1} = \cdots = z_{2n+1} = 0\}.$$

As is well known, V_3^{4n+2} is homotopy equivalent to $S^{2n+1} \vee S^{2n+1}$, the normal bundle of V_3^{4k} in V_3^{4n+2} is trivial, and Sign $V_3^{4k} = \pm 2$ (see [**13**]). Hence it follows from Theorem 2.2 that

$$l(W_3^{4n+1}, T_k, W_3^{4k-1}) = l(W_3^{4n+1}, W_3^{4k-1}) = \operatorname{Sign} V_3^{4k} = \pm 2$$

which verifies Theorem C.

References

[1] M.F. Atiyah and I.M. Singer, The index of elliptic operators. III, Ann. of Math., **87** (1968), 546-604.

[2] G.E. Bredon, Introduction to Compact Transformation Groups, Academic Press, **46**, New York, 1972.

[3] A. Haefliger, Knotted $(4k-1)$-spheres in $6k$-spaces, Ann. of Math., **75** (1962), 452-466.

[4] F. Hirzebruch, Topological Methods in Algebraic Geometry, Springer, 1966.

[5] M. Kervaire and J. Milnor, Groups of homotopy spheres: I, Ann. of Math., **77** (1963), 504-537.

[6] J. Levine, A classification of differentiable knots, Ann. of Math., **82** (1965), 15-50.

[7] J. Levine, Semi-free circle actions on spheres, Invent. Math., **22** (1973), 161-186.

[8] M. Masuda, Generators of certain groups of semi-free S^1 actions on spheres and splitting of codimension 3 knot exact sequences, Michigan Math. J., to appear.

[9] M. Masuda, Smooth group actions on sphere bundles over spheres and on Brieskorn manifolds, Proc. Amer. Math. Soc., **92** (1984), 119-124.

[10] M. Masuda, A product formula for connected sum, Proc. of Transformation Groups, Lecture Notes in Math., **1375**, Springer, pp. 231-239.

[11] M. Masuda and M. Sakuma, Knotting codimension 2 submanifolds locally, Enseign. Math., to appear.

[12] M. Masuda and R. Schultz, Equivariant inertia groups and rational invariants of non-free actions, in preparation.

[13] J. W. Milnor, Singular Points of Complex Hypersurfaces, Ann. of Math. Studies, **61**, Princeton, 1968.

[14] J. W. Milnor and J. D. Stasheff, Characteristic Classes, Ann. of Math. Studies, **76**, Princeton, 1974.

[15] D. Montgomery and C. T. Yang, Differentiable actions on homotopy seven spheres, Trans. Amer. Math. Soc., **122** (1966), 480-498.

[16] R. Schultz, Smooth structures on $S^p \times S^q$, Ann. of Math., **90** (1969), 187-198.

[17] E. H. Spanier, Algebraic Topology, MacGraw-Hill, 1966.

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