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Delta-unknotting operation and the second coefficient of the Conway polynomial

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§1. Introduction.

In this paper, we study oriented tame links in the oriented 3-sphere S^3. A Δ-unknotting operation is a local move on an oriented link diagram as indicated in Figure 1.1.

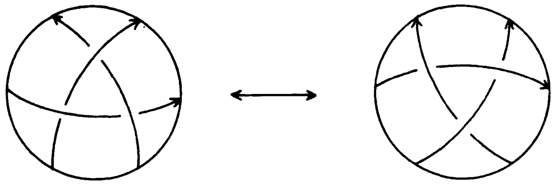


Figure 1.1. Δ-unknotting operation.

In [8], H. Murakami and Y. Nakanishi introduced this notion and proved that every knot can be transformed into a trivial knot by a finite number of Δ-unknotting operations. Let K and K' be oriented knots in S^3. The Δ-Gordian distance from K to K', denoted by $d_G^\Delta(K, K')$, is the minimum number of Δ-unknotting operations which are necessary to deform a diagram of K into that of K'. The Δ-unknotting number of K, denoted by $u^\Delta(K)$, is the Δ-Gordian distance from K to a trivial knot. Then they showed the congruences $d_G^\Delta(K, K') \equiv \mathrm{Arf}(K)-\mathrm{Arf}(K') \pmod 2$ and $u^\Delta(K) \equiv \mathrm{Arf}(K) \pmod 2$ in [8], where $\mathrm{Arf}(K)$ is the Arf invariant of a knot K. Let $a_i(L)$ be the i-th coefficient of the Conway polynomial $\nabla_L(z)$ of a link L. It is known that $a_i(L)$ has a relation to the Casson's invariant ([1], [3]). For the definition and fundamental properties of the Conway polynomial, we refer to [4]. In this paper, we show the following:

THEOREM 1.1. *Let K and K' be two knots with $d_G^\Delta(K, K')=1$. Then, we have*

$$|a_2(K)-a_2(K')| = 1 .$$

As an immediate consequence of Theorem 1.1, we have the following:

COROLLARY 1.2. *For any two knots K and K', the difference $d_G^\Delta(K, K')-|a_2(K)-a_2(K')|$ is a non-negative even integer. In particular the difference $u^\Delta(K)$*

$-|a_2(K)|$ *is also a non-negative even integer.*

Since $a_2(K) \equiv \mathrm{Arf}(K) \pmod 2$ ([**4**]), Corollary 1.2 extends to Murakami and Nakanishi's congruences. For the signatures $\sigma(K)$, $\sigma(K')$ of knots K, K' ([**9**]), Murakami and Nakanishi also observed the inequalities $d_G^{\Delta}(K, K') \geqq (1/2)|\sigma(K)-\sigma(K')|$ and $u^{\Delta}(K) \geqq (1/2)|\sigma(K)|$. By their own congruences and inequalities, they determined $d_G^{\Delta}(3_1, 5_1)$ and $u^{\Delta}(K)$ ($K=3_1$, 4_1, 5_1, 5_2, 6_1, 6_2 and 6_3). Using these inequalities and Theorem 1.1, we shall determine Δ-unknotting numbers of prime knots of $\leqq 8$ crossings, and Δ-Gordian distances between any two of twist knots.

§ 2. Proof and examples.

PROOF OF THEOREM. Considering a skein tree indicated in Figure 2.1, we obtain

$$\nabla_{K_{11}}(z)-\nabla_K(z) = z\nabla_{K_{12}}(z), \qquad \nabla_{K_{11}}(z)-\nabla_{K'}(z) = z\nabla_{K_{23}}(z),$$
$$\nabla_{K_{12}}(z)-\nabla_{K_{21}}(z) = z\nabla_{K_{22}}(z), \qquad \nabla_{K_{23}}(z)-\nabla_{K_{31}}(z) = z\nabla_{K_{32}}(z).$$

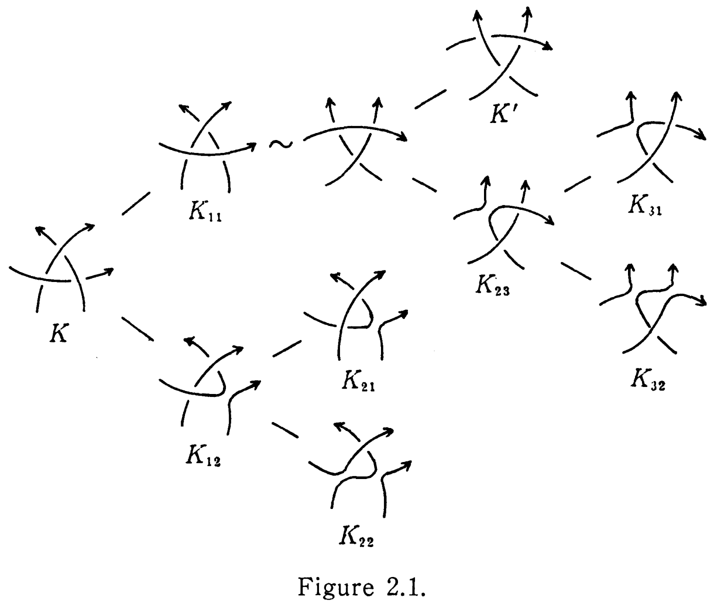


Figure 2.1.

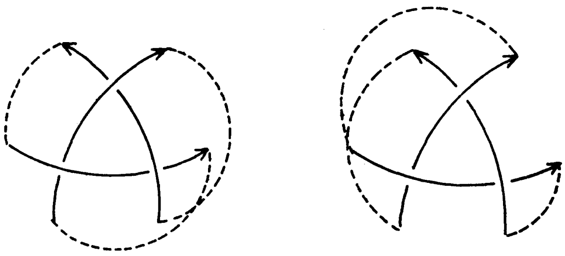
Since $K_{21} \sim K_{31}$, we obtain

$$\nabla_{K'}(z)-\nabla_K(z) = z^2(\nabla_{K_{22}}(z)-\nabla_{K_{32}}(z)).$$

Hence

$$a_2(K')-a_2(K) = a_0(K_{22})-a_0(K_{32}).$$

Since K is a knot, we may consider two cases as indicated in Figure 2.2 (where dotted lines denote the connecting relations).



(i) (ii)

Figure 2.2.

In the case (i), K_{22} has one component (i.e. a knot) and K_{32} has three components. Hence $a_0(K_{22})=1$, $a_0(K_{32})=0$. In the case (ii), we have $a_0(K_{22})=0$, $a_0(K_{32})=1$ by a similar argument. So we have the conclusion, completing the proof of Theorem 1.1.

Here are some examples.

EXAMPLE 2.1. Let $T(n)$, $T(m)$ be two twist knots as in Figure 2.3. Then,

$$d_G^{\Delta}(T(n), T(m)) = \begin{cases} \dfrac{|n-m|}{2} & \text{if } n+m=\text{even}, \\ \dfrac{n+m+1}{2} & \text{if } n+m=\text{odd}. \end{cases}$$

In particular,

$$u^{\Delta}(T(n)) = \begin{cases} \dfrac{n+1}{2} & \text{if } n=\text{odd}, \\ \dfrac{n}{2} & \text{if } n=\text{even}. \end{cases}$$

To see this, note that

$$\nabla_{T(n)}(z) = \begin{cases} 1+\dfrac{n+1}{2}z^2 & \text{if } n=\text{odd}, \\ 1-\dfrac{n}{2}z^2 & \text{if } n=\text{even}. \end{cases}$$

When both n and m are odd, by Corollary 1.2, we have

$$d_G^{\Delta}(T(n), T(m)) \geqq \frac{|n-m|}{2}.$$

On the other hand, we can actually transform $T(n)$ into $T(m)$ by $|n-m|/2$ times of Δ-unknotting operations (see Figure 2.4).

Therefore $d_G^{\Delta}(T(n), T(m))=|n-m|/2$. The other cases can be also obtained by a similar method.

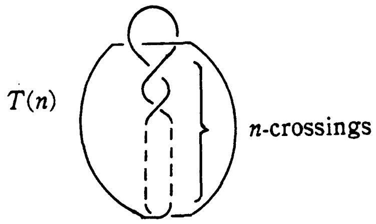


Figure 2.3.

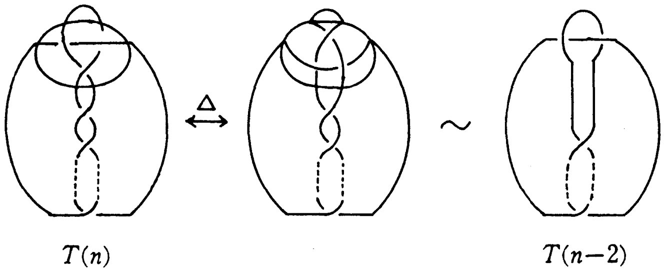


Figure 2.4.

EXAMPLE 2.2. For the diagrams of these prime knots, we refer to [10].

Table 2.2. Δ-unknotting numbers.

K	$(1/2)\lvert\sigma(K)\rvert$	$a_2(K)$	$u^{\Delta}(K)$	K	$(1/2)\lvert\sigma(K)\rvert$	$a_2(K)$	$u^{\Delta}(K)$
0_1	0	0	0	8_4	1	−3	3
3_1	1	1	1	8_5	2	−1	3
4_1	0	−1	1	8_6	1	−2	2
5_1	2	3	3	8_7	1	2	2
5_2	1	2	2	8_8	0	2	2
6_1	0	−2	2	8_9	0	−2	2
6_2	1	−1	1	8_{10}	1	3	3
6_3	0	1	1	8_{11}	1	−1	1
7_1	3	6	6	8_{12}	0	−3	3
7_2	1	3	3	8_{13}	0	1	1
7_3	2	5	5	8_{14}	1	0	2
7_4	1	4	4	8_{15}	2	4	4
7_5	2	4	4	8_{16}	1	1	1
7_6	1	1	1	8_{17}	0	−1	1
7_7	0	−1	1	8_{18}	0	1	1
8_1	0	−3	3	8_{19}	3	5	5
8_2	2	0	2	8_{20}	0	2	2
8_3	0	−4	4	8_{21}	1	0	2

EXAMPLE 2.3. In this Example 2.3, we don't deal with Δ-Gordian distances between the mirror images of them.

Table 2.3. Δ-Gordian distances.

K	3_1	4_1	5_1	5_2	6_1	6_2	6_3
3_1	0	2	2	1	3	2	2
4_1		0	4	3	1	2	2
5_1			0	1	5	4	2 or 4
5_2				0	4	3	1 or 3
6_1					0	1	3
6_2						0	2

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