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Some note on Gevrey hypoellipticity and solvability on torus

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1. Introduction.

In this paper, we are concerned with the hypoellipticity and the solvability in a Gevrey class. Let G^{σ} be a Gevrey class of order σ, $1 \leqq \sigma \leqq \infty$. For a domain Ω in $\boldsymbol{R}^n$ and a differential operator P on Ω, P is said to be (G^{σ}-) hypoelliptic in Ω if, for any subdomain $\Omega' \subset \Omega$ and any distribution $u \in D'(\Omega')$ such that $Pu \in G^{\sigma}(\Omega')$ it follows that $u \in G^{\sigma}(\Omega')$. We say that P is (G^{σ}-) globally hypoelliptic in Ω if, for any $u \in D'(\Omega)$ such that $Pu \in G^{\sigma}(\Omega)$ it follows that $u \in G^{\sigma}(\Omega)$. Though hypoelliptic operators are globally hypoelliptic, there are considerable gaps between these notions. In fact, the hyperbolic operator $(\partial/\partial x_1 - \tau\partial/\partial x_2)^2$ on $\boldsymbol{T}^2 = \boldsymbol{R}^2/2\pi\boldsymbol{Z}^2$, $x = (x_1, x_2) \in \boldsymbol{T}^2$ is globally hypoelliptic for some real value τ (cf. Greenfield-Wallach [**4**]).

In Section 2, we discuss solvability, hypoellipticity and non hypoellipticity in G^{σ} for operators satisfying a resonance type condition ($\mathcal{R}$). In fact, we give equations which are G^{σ}-globally hypoelliptic for σ near 1, and that are not G^{σ}-globally hypoelliptic for large σ. We note that such examples are known for local hypoellipticity. But, our example does not follow from such examples because global hypoellipticity does not necessarily imply local hypoellipticity.

In Section 3, we study the resonance type condition ($\mathcal{R}$) in Section 2. By Greenfield's example, we immediately see that ($\mathcal{R}$) is not necessary for the global hypoellipticity, in general. Instead of ($\mathcal{R}$), we introduce a so-called Siegel condition in G^{σ} (cf. (3.2)), which is weaker than ($\mathcal{R}$). We consider equations treated in [**16**], namely, equations satisfying the condition (A.1) which follows. Then, the Siegel condition is necessary and sufficient for G^{σ}-global hypoellipticity. We also give an example such that we cannot drop (A.1), in general (cf. Remark 3.3). We also note that hyperbolic equations as well as elliptic equations may satisfy the condition (3.2).

The last section is devoted to the study of the solvability of semilinear equations. As applications of the arguments in §§3 and 4, we show the existence of periodic solutions for hyperbolic equations whose ratio of periods is not rational (cf. Example 4.2.).

2. Solvability and hypoellipticity under resonance type restriction.

We consider the following equation on the torus $\boldsymbol{T}^2=\boldsymbol{R}^2/2\pi\boldsymbol{Z}^2$,

$$Pu=(D_x+ia(x)D_y)^2u-(aD_y+b)u=f(x,y), \tag{2.1}$$

where $D_x=i^{-1}\partial_x=i^{-1}\partial/\partial x$, $a, b\in\boldsymbol{C}$, $(x, y)\in\boldsymbol{T}^2$, and where $a(x)$ is a real-valued real analytic 2π-periodic function on $\boldsymbol{R}$.

We assume that $a(x)$ does not change its sign and $a(x)$ is not identically equal to 0, and without loss of generality we demand

$$a(x)\leqq 0, \qquad x\in\boldsymbol{R}. \tag{2.2}$$

We set $\omega=\int_0^{2\pi}a(r)dr$, $k(\eta)=a\eta+b$, $\lambda(\eta)=k(\eta)^{1/2}$, where we take the branch of $\lambda(\eta)$ such that $\operatorname{Im}\lambda(\eta)\leqq 0$. Then we assume

$$(\mathscr{R}) \qquad \omega\eta\pm 2\pi i\lambda(\eta)\notin 2\pi i\boldsymbol{Z},\ \forall\eta\in\boldsymbol{Z}, \quad\text{and}\quad a\eta+b\neq 0 \quad\text{for all } \eta\in\boldsymbol{Z}.$$

For a domain $\Omega\subset\boldsymbol{T}^2$ and $1\leqq\sigma\leqq\infty$, we define $G^\sigma(\Omega)$ the set of all smooth functions f in Ω such that, for any compact set $K\subset\Omega$, there exists $C>0$ so that

$$\sup_{(x,y)\in K}|\partial_{x,y}^\alpha f(x, y)|\leqq C^{|\alpha|+1}(\alpha!)^\sigma,$$

for all $\alpha\in\boldsymbol{Z}_+^2$, where $\boldsymbol{Z}_+$ is the set of nonnegative integers. Here, if $\sigma=\infty$ we put $G^\sigma(\Omega)=C^\infty(\Omega)$. If we set $\hat{f}_\eta=\int_{\boldsymbol{T}^2}f(z)e^{-i\eta z}dz$, $\eta\in\boldsymbol{Z}^2$, then the above condition is equivalent to

$$\sup_\eta|\hat{f}_\eta e^{C|\eta|^{1/\sigma}}|\leqq C^{-1},$$

with another constant $C>0$. For $\sigma>1$, we denote by $G^\sigma(\Omega)'$ the space of σ-ultradistributions (cf. [6]). We say that L is $G^\sigma(\boldsymbol{T}^2)$-solvable if, for any $f\in G^\sigma(\boldsymbol{T}^2)$, the equation $Lu=f(x)$ has a solution u in $G^\sigma(\boldsymbol{T}^2)$. We determine the integer k by

$$2k=\max\{j;\ \exists x\in\boldsymbol{T},\ \partial_x^l a(x)=0,\ 0\leqq l\leqq j-1,\ \partial_x^j a(x)\neq 0\},$$

and we set $\sigma_0=2+2/(2k-1)$. Then we have the following

THEOREM 2.1. *Suppose that the conditions* (2.2), ($\mathscr{R}$), *and* $1\leqq\sigma<\sigma_0$ *are satisfied. Then, the operator* P *is* G^σ*-globally hypoelliptic and* $G^\sigma(\boldsymbol{T}^2)$*-solvable with a trivial kernel* $\ker\{P;\ G^\sigma(\boldsymbol{T}^2)'\to G^\sigma(\boldsymbol{T}^2)'\}=\{0\}$.

PROOF. We shall prove the solvability. Suppose that u solves (2.1) with $f \in G^{\sigma}(\boldsymbol{T}^2)$, $\sigma \geqq 1$. Performing the discrete partial Fourier transformation with respect to y in the equation (2.1), we get

$$((D_x + ia(x)\eta)^2 - k(\eta))\hat{u}(x, \eta) = \hat{f}(x, \eta), \qquad \eta \in \boldsymbol{Z}. \tag{2.3}$$

From the well-known formulas in the theory of ordinary differential equations, we can write formally, for each $\eta \in \boldsymbol{Z}$,

$$\begin{aligned} \hat{u}(x, \eta) = &-\int_{\delta_\eta}^{x} e^{A(x, s, \eta)} \frac{\sin((x-s)\lambda(\eta))}{\lambda(\eta)} \hat{f}(s, \eta) ds \\ &+ B(\eta) e^{A(x, \delta_\eta, \eta) + ix\lambda(\eta)} + C(\eta) e^{A(x, \delta_\eta, \eta) - ix\lambda(\eta)} \end{aligned} \tag{2.4}$$

where $\delta_\eta = \pi(1 - \mathrm{sign}(\eta))$, $A(x, s, \eta) = \eta \int_s^x a(r) dr$. Here, $\mathrm{sign}(\eta) = 1$ (if $\eta > 0$); $=0$ (if $\eta = 0$); $=-1$ (if $\eta < 0$).

In order to obtain 2π-periodicity of $\hat{u}(x, \eta)$ $(\eta \in \boldsymbol{Z})$ in x, we need two conditions

$$\hat{u}(2\pi, \eta) - \hat{u}(0, \eta) = 0, \qquad \hat{u}_x(2\pi, \eta) - \hat{u}_x(0, \eta) = 0.$$

Let us assume, for the moment, $\eta > 0$. Then, by (2.4) and the periodicity of $a(x)$ we have the system of equations for $B(\eta)$ and $C(\eta)$,

$$\begin{aligned} \alpha_+(\eta) B(\eta) + \alpha_-(\eta) C(\eta) &= F_1(\eta), \\ \beta_+(\eta) B(\eta) + \beta_-(\eta) C(\eta) &= F_2(\eta) + F_1(\eta) \eta a(2\pi), \end{aligned} \tag{2.5}$$

where

$$\alpha_\pm(\eta) = e^{\omega\eta \pm 2\pi i \lambda(\eta)} - 1, \quad \beta_\pm(\eta) = \pm i\lambda(\eta) \alpha_\pm(\eta), \quad \omega = A(2\pi, 0, \eta)\eta^{-1},$$

$$F_1(\eta) = (\lambda(\eta))^{-1} \int_0^{2\pi} e^{A(2\pi, s, \eta)} \sin((2\pi - s)\lambda(\eta)) f(s, \eta) ds,$$

$$F_2(\eta) = \int_0^{2\pi} e^{A(2\pi, s, \eta)} \cos((2\pi - s)\lambda(\eta)) f(s, \eta) ds.$$

It follows from $(\mathscr{R})$ that (2.5) has a unique solution

$$\begin{aligned} B(\eta) &= \frac{i\lambda(\eta) F_1(\eta) + F_2(\eta) + F_1(\eta) \eta a(2\pi)}{2\beta_+(\eta)}, \\ C(\eta) &= \frac{-i\lambda(\eta) F_1(\eta) + F_2(\eta) + F_1(\eta) \eta a(2\pi)}{2\beta_-(\eta)}. \end{aligned} \tag{2.6}$$

We need to estimate the growth of $B(\eta)$ and $C(\eta)$: If $f \in G^{\sigma}(\boldsymbol{T}^2)$, $1 \leqq \sigma \leqq \sigma_0$, then we have, for some $M > 0$

$$|B(\eta)| \leqq M^{-1} e^{-M|\eta|^{1/\sigma}}, \qquad |C(\eta)| \leqq M^{-1} e^{-M|\eta|^{1/\sigma}}, \tag{2.7}$$

for all $\eta \in \boldsymbol{Z}$.

By $(\mathscr{R})$, the absolute values of the denominators, $|\beta_\pm(\eta)|$ of (2.6) are bounded from below by some positive constant. On the other hand, since $f\in G^{\sigma}(\boldsymbol{T}^2)$ we have, for $\hat{f}(s, \eta)=\int_0^{2\pi} f(s, y)e^{-i\eta y}dy$,

$$|\hat{f}(s, \eta)| \leqq K^{-1}e^{-K|\eta|^{1/\sigma}} \tag{2.8}$$

for all $\eta\in\boldsymbol{Z}$, and all $0\leqq s\leqq 2\pi$, where $K>0$ is independent of s.

It follows from $|\lambda(\eta)|\sim|\eta|^{1/2}$ as $|\eta|\to\infty$ that

$$|e^{A(2\pi, s, \eta)}\sin((2\pi-s)\lambda(\eta))| \leqq C\exp\Big(|2\pi-s|C_1|\eta|^{1/2}+|\eta|\int_s^{2\pi} a(r)dr\Big). \tag{2.9}$$

If $a(r)$ does not vanish at $r=2\pi$, then it follows from (2.2) that the right-hand side of (2.9) is estimated by $C\exp(-\varepsilon|2\pi-s||\eta|)$ for some $\varepsilon>0$, if $|\eta|$ is sufficiently large. Hence, by (2.6) and (2.8) we get (2.7).

On the other hand, if $a(r)$ vanishes at $r=2\pi$ exactly of order $2k$, then the right-hand side of (2.9) is bounded by

$$C\exp(|2\pi-s|C_1|\eta|^{1/2}-|2\pi-s|^{2k+1}C_3|\eta|) \tag{2.10}$$

for some $C_3>0$, if η is sufficiently large. If s and η satisfy $|2\pi-s|^{2k}|\eta|^{1/2}>L$ for some large $L>0$, we see that the term (2.10) is bounded by $C\exp(-\varepsilon|2\pi-s||\eta|^{1/2})$ for some $\varepsilon>0$. If otherwise, we have $|2\pi-s|^{2k}|\eta|^{1/2}\leqq L$. Hence, the term (2.10) is estimated by $C\exp(C_1'|\eta|^{1/2(1-1/2k)})$ for some $C_1'>0$ independent of η. These estimates, together with (2.6), (2.8), (2.9), and the definition of σ_0 show (2.7). We note that one deals similarly in case $\eta<0$. Hence, (2.7) is valid for all $\eta\in\boldsymbol{Z}$. We also note that the same arguments apply to the integrand of the first term in the right-hand side of (2.4). We have the same estimate as (2.7).

It follows from (2.4) and (2.7) that the formal solution $\hat{u}(x, \eta)$, $x\in\boldsymbol{T}$, $\eta\in\boldsymbol{Z}$ is a discrete partial Fourier transform of the G^{σ}-function $u(x, y)=\sum_{\eta\in Z}\hat{u}(x, \eta)e^{iy\eta}$. This proves the solvability, and, in fact, G^{σ}-hypoellipticity of P, because the adjoint operator P^* is of the same type. □

REMARK 2.2. (a) We note that the conditions (2.2) and ($\mathfrak{R}$) in Theorem 2.7 are superfluous for the G^{σ}-global hypoellipticity of P. Indeed, we can replace them by $b\neq 0$. In this case, we replace σ_0 by $\sigma_0=2+2/(k-1)$, where k is the maximal vanishing order of $a(x)$. This follows from the arguments of [**3**] for the local hypoellipticity. We note that (2.2) and ($\mathfrak{R}$) are necessary in order to assure the solvability.

(b) We point out that, in the case $f\in G^{\sigma}(\boldsymbol{T}^2)'$ one obtains in the same way a solution $u\in G^{\sigma}(\boldsymbol{T}^2)'$. In fact, we have the estimate, $\max\{|B(\eta)|, |C(\eta)|\}\leqq L_{\varepsilon}e^{\varepsilon|\eta|^{1/\sigma}}$, $\forall\varepsilon>0$ in the line of considerations in [**3**].

(c) We note that the previous assertion is valid for operators of the type

$$P=(D_x+ia(x)D_y)^m+\sum_{j=0}^{m-1}B_j(x, D_y)(D_x+ia(x)D_y)^{m-1-j}$$

where $B_j(x, D_y)$ are analytic differential operators of order j, $0\leqq j\leqq m-1$, as in [**3**], and where $a(x)$ does not change the sign.

Next, we consider the relation between global hypoellipticity and the Gevrey index σ. Let us consider the operator

$$P_0=(D_x+ia(x)D_y)^2-b(x)D_y\,, \tag{2.11}$$

where $a(x)\geqq 0$ is a 2π-periodic real-valued real analytic function on $\boldsymbol{R}$ having zero of exact order $2k$ at $x=0$. Here, $b(x)$ is given by, for $|x|\leqq\pi$

$$b(x)=-(cx^{2\mu-1}+id)^2 \quad \text{for } |x|<1/4, \qquad \operatorname{supp} b(x)\subseteqq[-1/2, 1/2]\,,$$

and periodically extended on $\boldsymbol{R}$, where $\mu\in\boldsymbol{N}$, $2\mu\leqq k$, $c>0$, and $d\in\boldsymbol{R}/\{0\}$. Then we have

THEOREM 2.3. *The operator* (2.11) *is $G^\sigma(\boldsymbol{T}^2)$-hypoelliptic for $1\leqq\sigma\leqq\sigma_0=2+2/(2k-1)$, but is not $G^\sigma(\boldsymbol{T}^2)$-hypoelliptic if $\sigma_1\leqq\sigma\leqq\infty$, $\sigma_1=2+4\mu/\{2(k-2\mu)+1\}>\sigma_0$.*

REMARK 2.4. We note that the operators (2.11) are the first examples of global (non) hypoelliptic operators with such properties. The class of operators, G^σ-hypoelliptic for σ close to 1, that are not G^σ-hypoelliptic for $\sigma\gg 1$ locally in $\boldsymbol{R}^2$ is known. (See Theorem 2.3 in [**3**], L. Rodino [**8**], L. Cattabriga-L. Rodino-L. Zanghivati [**2**], T. Okaji [**7**], as well as S. Baouendi-F. Treves [**1**] for two analytic vector fields.)

PROOF OF THEOREM 2.3. The hypoellipticity follows from Remark 2.2 and the standard arguments. According to the classical results on asymptotics for ordinary differential equations (cf. [**13**]), there exists a solution $v(x, \eta)$, $\eta\in\boldsymbol{Z}_+$ to the ordinary differential equation,

$$P_\eta v\equiv(D_x+ia(x)\eta)^2 v-b(x)\eta v=0\,. \tag{2.12}$$

$v(x, \eta)$ has the form, denoting $\alpha(x)=\int_0^x a(s)ds$,

$$v(x, \eta)=\exp(\alpha(x)\eta-C(2\mu)^{-1}x^{2\mu}\eta^{1/2}-idx\eta^{1/2})q(x, \eta)\,, \quad \eta\in\boldsymbol{Z}_+,\ |x|\leqq 1/4, \tag{2.13}$$

where $q(x, \eta)\sim\sum_{j=0}^\infty q_{-j}(x)\eta^{-j/2}$ is an analytic classical symbol in $\eta^{1/2}$ such that, for some constant $R>0$,

$$|\partial_x^l q(x, \eta)|\leqq R^{l+1}l!\,, \qquad l=0, 1, 2, \cdots, \eta>1\,, \tag{2.14}$$

with $q_{-j}(x)$ verifying transport equations, $q''_{-j}(x)+\theta(x)q_{-j}(x)=q''_{-j+1}(x)$, $q_{-j}(0)=\delta_{j0}$

(δ_{j0} stands for the Kronecker symbol), $j=0, 1, \cdots, q_1:=0$, $\theta(x)$ being a real analytic function expressed through $b(x)$. One obtains the estimates (2.14) in a standard way, solving consecutively the transport equations and taking into account the analyticity of $\theta(x)$.

Evidently, $\alpha(x)=x^{2k+1}\alpha_0(x)$ with $m_1\leqq\alpha_0(x)\leqq m_2$ for some $m_1>0$ and $m_2>0$. We choose $X(x)\in G^{1+\varepsilon}$, $0<\varepsilon\ll 1$, and $\phi(x)\in G^{1+\varepsilon}(\boldsymbol{R})$ so that

$$(2.15)\qquad \begin{gathered} \operatorname{supp} X \subseteqq(-\infty, \delta_1], \quad X=1 \text{ on } (-\infty, \delta_2], \quad 0<\delta_2<\delta_1\ll 1\,, \\ \operatorname{supp}\phi \subseteqq [-1/2, 1/2], \quad \phi=1 \text{ on } [-1/4, 1/4]\,. \end{gathered}$$

We put

$$(2.16)\qquad \omega(x, \eta)=\exp(\alpha(x)\eta-Cx^{2\mu}(2\mu)^{-1}\eta^{1/2}-idx\eta^{1/2})q(x, \eta)\phi(x)X(x\eta^{1/\theta}),$$

where $\theta=2(2(k-\mu)+1)$. Then $W(x, y)=\sum_{\eta\in Z_+}\omega(x, \eta)e^{iy\eta}$ satisfies $PW\in G^{\sigma_1}(\boldsymbol{T}^2)$, while $(0, 0)\in C^\infty$-sing supp W, i.e., W is not smooth at $x=y=0$. Indeed, it follows from the assumption $C>0$ that W is a distribution on $\boldsymbol{T}^2$. The condition $q(0, 0)\neq 0$ and the definition of ϕ and X implies that $(0, 0)\in C^\infty$-sing supp W. In order to show that $PW\in G^{\sigma_1}$ we note, by (2.12), (2.13), and (2.16) that

$$(2.17)\qquad \begin{aligned} PW &= \sum_{\eta\in Z_+} P_\eta(v(x, \eta)\phi(x)X(x\eta^{1/\theta}))e^{iy\eta} \\ &= \sum_\eta (D_x+ia(x)\eta)v(x, \eta)(D_x+ia(x)\eta)(\phi(x)X(x\eta^{1/\theta}))e^{iy\eta} \\ &\quad +\sum_\eta v(x, \eta)(D_x+ia(x)\eta)^2(\phi(x)X(x\eta^{1/\theta}))e^{iy\eta}\equiv F\,. \end{aligned}$$

We estimate the exponential factor of $v(x, \eta)$. It follows from (2.13) and $\mu\leqq k$ that,

$$(2.18)\qquad \begin{aligned} \alpha(x)\eta-(C/2\mu)x^{2\mu}\eta^{1/2} &\leqq m_2x^{2k+1}\eta-(C/2\mu)x^{2\mu}\eta^{1/2} \\ &= x^{2\mu}\eta^{1/2}(m_2x^{2(k-\mu)+1}\eta^{1/2}-C/2\mu)\,. \end{aligned}$$

It follows from the definition of $X(x\eta^{1/\theta})$ and (2.15) that $x^{2(k-\mu)+1}\eta^{1/2}\leqq\delta_1^{\theta/2}$. By taking δ_1 sufficiently small, the right-hand side of (2.18) is bounded by $-C(4\mu)^{-1}x^{2\mu}\eta^{1/2}$. On the other hand, $P_\eta(v\phi X)$ does not vanish only if $\phi\not\equiv 1$ or $X\not\equiv 1$. This implies that the term $-C(4\mu)^{-1}x^{2\mu}\eta^{1/2}$ is bounded by $-K\eta^{1/\sigma_1}$ for some $K>0$. Since all terms in F are in $G^{1+\varepsilon}$ with respect to x, and since the differentiations with respect to x and y yield only some powers of η it follows that $F\in G^{\sigma_1}$. □

3. Necessary and sufficient conditions.

In Section 2 we have shown G^σ-hypoellipticity and G^σ-nonhypoellipticity under the resonance condition $(\mathcal{R})$. In this section, we consider the necessity of $(\mathcal{R})$.

Let $m \geqq 1$ be an integer, and let $p_m(\eta)$ be a polynomial of degree m, $p_m(\eta) = \sum_{|\alpha|=m} a_\alpha \eta^\alpha$ with $a_\alpha \in \boldsymbol{C}$, and let $q_\beta(\eta)$ be a function on $\boldsymbol{Z}^d$ such that, for some $C>0$, $|q_\beta(\eta)| \leqq C\langle\eta\rangle^{m-1}$ for all $\eta \in \boldsymbol{Z}^d$. We denote by $q_\beta(D)$ the pseudo-differential operator with the symbol $q_\beta(\eta)$. We consider the following operator on $\boldsymbol{T}^d$:

$$P(x, D) = \sum_{|\alpha|=m} a_\alpha D^\alpha + \sum_{|\beta| \leqq m-1} b_\beta(x) q_\beta(D), \qquad b_\beta \in G^s(\boldsymbol{T}^d). \tag{3.1}$$

We expand $b_\beta(x)$ into Fourier series, $b_\beta(x) = \sum_\gamma b_{\beta,\gamma} e^{i\gamma x}$, and we define Γ_P as the smallest closed convex cone with apex at the origin which contains all γ such that, $b_{\beta,\gamma} \neq 0$ for some β. If $b_\beta = 0$ for all β, then we set $\Gamma_P = \{0\}$. We set $p(\eta) = p_m(\eta) + \sum_{|\beta| \leqq m-1} b_{\beta,0} q_\beta(\eta)$. Our major premise is

(A.1) Γ_P is a proper cone, that is, Γ_P contains no ray.

Moreover, we assume

(A.2) For every $\xi \in \boldsymbol{R}^d \setminus \{0\}$ such that $p_m(\xi) = 0$, we have $\theta \cdot \nabla_\xi p_m(\xi) \neq 0$ for all $\theta \in \Gamma_P \setminus \{0\}$.

THEOREM 3.1. *Suppose* (A.1) *and* (A.2). *Then, P is G^s-globally hypoelliptic, if and only if a Siegel condition*

$$\lim_{\eta \to \infty,\, \eta \in Z^d} |\eta|^{-1/s} \log |p(\eta)| = 0 \tag{3.2}$$

is satisfied.

REMARK 3.2. (a) For the operator P in (2.1) ($\mathfrak{R}$) implies (3.2). Indeed, we have, $p(\eta) = (\eta_1 + i\omega\eta_2)^2 - k(\eta_2)$, $\eta = (\eta_1, \eta_2)$. Since $\omega \neq 0$ by ($\mathfrak{R}$), we have (3.2). We note that P satisfies (A.2), but does not satisfy (A.1).

(b) (A.2) is too strong for the proof of Theorem 3.1. Indeed, if $d=2$ we can replace (A.2) by (A.2)$'$; $p_m(\xi) \neq 0$ for $\forall \xi \in \Gamma_P \setminus \{0\}$. In the general case, $d \geqq 3$, we assume: For every $\theta \in \Gamma_P$ and $\xi \in \boldsymbol{R}^d$, $|\xi| = 1$, and $p_m(\xi) = 0$ we have $L_\xi(\theta) \neq 0$. Here $L_\xi(\theta)$ is the localization of $p_m(\eta)$ at $\eta = \xi$ defined by, for ξ, $\theta \in \boldsymbol{R}^d$,

$$p_m(\xi + t\theta) = L_\xi(\theta) t^q + O(t^{q+1}), \tag{3.3}$$

where $q = q(\xi)$ is a nonnegative integer, and $L_\xi(\theta) \not\equiv 0$. Furthermore, we assume the regularity on the roots of $p(\eta)$. We set $S(\eta, t) = t^m p(t^{-1}\eta)$, and take a vector θ such that $p_m(\theta) \neq 0$ and write $\eta = \zeta_1\theta + \zeta'$. We factor $S(\zeta_1\theta + \zeta', t)$ as a polynomial of ζ_1:

$$S(\zeta_1\theta + \zeta', t) = c \prod_{j=1}^{j_0} (\zeta_1 - \lambda_j(\zeta', t))^{m_j}. \tag{3.4}$$

Then, we assume that the roots $\lambda_j(\zeta', t)$ $(j = 1, \cdots, j_0)$ are smooth with respect to ζ' and t. Under these assumptions, Theorem 3.1 is valid for s such that,

$1 \leqq s < q/(q-1)$, where $q = \max_\xi q(\xi)$. Finally, we note that (A.2) is equivalent to say that p is simple characteristic and microhyperbolic (cf. [**5**]).

REMARK 3.3. (a) The assumption (A.1) is essential in Theorem 3.1. In fact, the Mathieu operator $M = D_1^2 + 2\cos x_1$ on $\boldsymbol{T}^2$ satisfies the generalized (A.2)′ in Remark 3.2 and does not satisfy (A.1) and (3.2). Nevertheless, we can show the global $G^\sigma(\boldsymbol{T}^2)$-hypoellipticity of M, $(1 \leqq \sigma \leqq \infty)$.

(b) By modifying the proof of Theorem 3.1, we can prove the solvability of (3.1) under (A.1) and (A.2)′. In fact, $P(D)u = f$ is uniquely solvable in $G^s(\boldsymbol{T}^d)$, if and only if $p(\eta) \neq 0$ for all $\eta \in \boldsymbol{Z}^d$, and (3.2) is satisfied.

Preliminary lemmas. For $\varepsilon > 0$ and $\xi \in \boldsymbol{R}^d$, we define an ε-conical neighborhood $C(\xi;\varepsilon)$ of ξ by $C(\xi;\varepsilon) = \{\eta \in \boldsymbol{R}^d ; \|\eta/\|\eta\| - \xi\| < \varepsilon\}$, where $\|\cdot\|$ denotes the usual l^2-norm in $\boldsymbol{R}^d$. Then we have

LEMMA 3.4. *Let Γ be a closed convex cone with apex at the origin, and let Σ be a closed set on the sphere $\|\xi\| = 1$ such that $\Sigma \cap \pm\Gamma = \emptyset$. Then, there exists $c > 0$ depending only on Σ and Γ such that $\|\theta\|/\|\zeta\| \leqq 16\varepsilon c$ for every $\xi \in \Sigma$, every small $\varepsilon > 0$ and every $\zeta (\neq 0)$ and $\zeta + \theta (\theta \in \pm\Gamma)$ in $C(\xi;\varepsilon)$. Moreover, if ζ is not necessarily in $C(\xi;\varepsilon)$, then it follows that $\|\zeta + \theta\| \leqq c\|\zeta\|$ for $\forall \theta \in \pm\Gamma$.*

This lemma is a slight modification of Lemma 4.1 of [**15**]. So we omit the proof.

LEMMA 3.5. a) *For $a \geqq 0$, $b \geqq 0$, and $s \geqq 1$ we have $a^{1/s} + b^{1/s} \geqq (a+b)^{1/s}$.*

b) *Let $\tau \in \boldsymbol{R}$. For $\zeta \in \boldsymbol{Z}^d$ $(d \geqq 2)$, we set $\langle\zeta\rangle = 1 + |\zeta|$. Then we have that $\langle\delta\rangle^\tau \langle\eta\rangle^{-\tau} \leqq \langle\delta - \eta\rangle^{|\tau|}$ for all δ, $\eta \in \boldsymbol{Z}^d$.*

PROOF. a) is proved by elementary computations. In order to prove (b), let us suppose $|\delta| \geqq |\eta|$, $\tau > 0$. Then

$$\langle\delta\rangle^\tau \langle\eta\rangle^{-\tau} = \left(\frac{1+|\delta|}{1+|\eta|}\right)^\tau \leqq (|\delta| - |\eta| + 1)^\tau \leqq (|\delta - \eta| + 1)^\tau = \langle\delta - \eta\rangle^{|\tau|}.$$

The other cases will be proved similarly. □

PROOF OF THE SUFFICIENCY OF THEOREM 3.1. The proof is done by the similar method as in the proof of the sufficiency of Theorem 2.1 in [**16**]. Indeed, we use Gevrey estimate in place of C^∞-estimate. Hence, in order to avoid the repetition we give a rough sketch of the proof. Throughout the proof, we set $\sigma = 1/s$.

We take a closed convex proper cone, Γ with apex at the origin such that the interior of Γ contains $\Gamma_P \setminus \{0\}$, and that (A.2) is still valid, if we replace Γ_P by Γ. We set $\tilde{\Gamma} = \Gamma \cap \boldsymbol{Z}^d$, and we take $e \in -\tilde{\Gamma} \setminus \{0\}$ and $\eta \in \tilde{\Gamma}$. We define $\chi_n(\delta)$ $(n = 0, 1, 2, \cdots, \infty)$ by

(3.5) $\chi_n(\delta)=1$ if $\delta\in\eta_0-\tilde{\Gamma}$ and $e\cdot\delta<n+e\cdot\eta_0$; $=0$ if otherwise,

where the dots denote the usual inner products in $\boldsymbol{R}^d$. We denote by $\chi_n(D)$ the pseudodifferential operator with the symbol $\chi_n(\delta)$. We note that $\chi_\infty\to 1$ on $\boldsymbol{Z}^d$ when $\eta\to\infty$ in $\tilde{\Gamma}$.

For $\tau\in\boldsymbol{R}$, and $u(x)=\sum_\delta u_\delta e^{i\delta x}\in D'(\boldsymbol{T}^d)$, we define the norm $\|u\|_\tau$ of u by

(3.6) $$\|u\|_\tau=\sup_{\delta\in Z^d}\exp(\tau|\delta|^\sigma)|u_\delta|,$$

if the right-hand side is finite. We define $\tilde{\varphi}(\delta)$ by $\tilde{\varphi}(\delta)=1$ if δ satisfies $p(\delta)=0$; $\tilde{\varphi}(\delta)=0$, if otherwise. Since $p(\delta)$ vanishes only for a finite number of δ's in $\boldsymbol{Z}^d$, by (2.3), $\tilde{\varphi}$ has a compact support. We take κ so that $\|b_\beta\|_{2\kappa}<\infty$ for all β.

We want to show the following estimates; for any $\varepsilon>0$, there exist constants C_1, C_2 and N such that, if $u\in D'$, $\|Pu\|_{\kappa+3\varepsilon}<\infty$ then

(*) $$\|\chi_\infty(D)(1-\chi_n(D))u\|_\kappa\leqq C_1\|\chi_\infty(D)(1-\chi_n(D))Pu\|_\kappa,$$

(**) $$\|\chi_\infty(D)u\|_\kappa\leqq C_1\|Pu\|_{\kappa+3\varepsilon}+C_2\|\tilde{\varphi}u\|_{\kappa+3\varepsilon},$$

for all $n\geqq N_0$. Here, the constants C_1 and C_2 are independent of $\eta\in\Gamma$. If we can prove these estimates, then, by letting $\eta_0\to\infty$, $\eta_0\in\tilde{\Gamma}$ in (**) and by noting that $\chi_\infty\to 1$, we have that $\|u\|_k<\infty$, if $\|Pu\|_{\kappa+3\varepsilon}<\infty$. This proves the assertion.

The proofs of (*) and (**) are done by modifying the methods of the proof of the corresponding two estimates in the proof of Theorem 2.1 of [**16**], where we replace all C^∞-estimates with G^σ-estimates. This will be done by use of Lemma 3.5. □

In order to prove the necessity, we prepare a lemma. For $\eta'\in\boldsymbol{Z}^d$, and Γ and a vector e given in the definition of $\chi_n(\delta)$, we define $\psi_n(\delta)$ $(n=1, 2, \cdots, \infty)$ by $\psi_n(\delta)=1$ if $\delta\in\eta'+\Gamma$ and $-e\cdot(\delta-\eta')<n$; $\psi_n(\delta)=0$ if otherwise. Let $p(\eta)$ be as in Theorem 3.1, and let $P_0(D)$ be the corresponding pseudodifferential operator. We write $P=P_0(D)+P_1(x, D)$. Then we have

LEMMA 3.6. *Suppose* (A.1) *and* (A.2). *Then we have*

a) *If $p(\eta)\neq 0$ for $\forall\eta\in\eta'+\tilde{\Gamma}$, then, for any $f\in G^\sigma$ there exists a solution u of the equation $P(\psi_\infty u)=\psi_\infty f$ given by*

(3.7) $$\psi_\infty u=\sum_{\nu=0}^{\infty}Q^\nu P_0^{-1}\psi_\infty f, \qquad Q=-P_0^{-1}P_1.$$

Every Fourier coefficient of $\psi_\infty u$ is determined by a finite number of terms in the right-hand side of (3.7).

b) *If $p(\eta')=0$ and $p(\eta)\neq 0$ for $\forall\eta\in\eta'+\tilde{\Gamma}$, then the smooth solution of the equation $P(\psi_\infty u)=0$ is given by $\psi_\infty u=\sum_{\nu=0}^\infty Q^\nu\psi_1 u$.*

PROOF. We have $P_1(\psi_\infty u)=\psi_\infty(1-\psi_1)P_1(\psi_\infty u)$. Since P_0^{-1} exists on $\psi_\infty C^\infty$, it follows from the equation $P\psi_\infty u=\psi_\infty f$ that $P_0^{-1}\psi_\infty f=\psi_\infty u-Q\psi_\infty u$. Hence, if we can show the convergence of the sum in (3.7) we have the assertion, because $Q^\nu\psi_\infty g=\psi_\infty(1-\psi_\nu)Q^\nu\psi_\infty g$. By (A.2) with $\xi=\theta$, we see that $p_m(\xi)\neq 0$ in $\Gamma_P\setminus\{0\}$. Hence it follows that $p_m(\eta)\neq 0$ if $(1-\psi_n(\eta))\psi_\infty(\eta)\neq 0$ for sufficiently large n. Noting that Q is at most of order -1 we have, for some $\sigma>0$, $\tau<1$ and every $k\geqq 1$, $\|Q^{n+k}g\|_\sigma\leqq\tau^k\|Q^n g\|_\sigma$ with $g=P_0^{-1}\psi_\infty f$, which proves the convergence.

In order to prove b), we consider the equation $P(\psi_\infty u)=0$. We set $\psi=\psi_\infty(1-\psi_1)$. Since $P_1(\psi_\infty u)=P_1(\psi_1 u)+P_1\psi u$, it follows that $P_0\psi u+P_1\psi u=-P_1(\psi_1 u)$. Noting that P_0 is invertible on $\psi D'$ we have that $(I-Q)(\psi u)=Q\psi_1 u$. Then the solution is obtained by iteration. □

PROOF OF THE NECESSITY OF THEOREM 3.1. We divide the proof into three steps.

Step 1. Let us suppose that (3.2) is not satisfied. First, we suppose that $p(\eta)=0$ for infinitely many $\eta\in \boldsymbol{Z}^d$. We want to show that, for any ζ with $p(\zeta)=0$, $p(\eta)\neq 0$ for sufficiently large $\eta\in\zeta+\Gamma_P$, where $\tilde\Gamma_P=\Gamma_P\cap \boldsymbol{Z}^d$. If otherwise, there exist distinct $\eta_n\in\zeta+\Gamma_P$, such that $p(\eta_n)=0$, $\eta_n/\|\eta_n\|\to\exists\xi\in\Gamma_P$, to yield $p_m(\xi)=0$, a contradiction to (A.2).

We take $\eta_1\in\zeta+\tilde\Gamma_P$ such that, it attains the minimum of $e\cdot(\eta-\zeta)$ among $\eta\in\zeta+\tilde\Gamma_P$, $p(\eta)=0$, where e is given in the definition of $\chi_n(\delta)$. If $p(\eta_1+\eta')=0$ for some $\eta'\in\tilde\Gamma_P\setminus\{0\}$, then, we have that $\eta_1+\eta'-\zeta\in\tilde\Gamma_P+\tilde\Gamma_P\subset\tilde\Gamma_P$, and that $e\cdot(\eta_1+\eta'-\zeta)<e\cdot(\eta_1-\zeta)$ by definition, which contradicts to the choice of η_1. Hence we have $p(\eta_1+\eta)\neq 0$ for all $\eta\in\Gamma_P\setminus\{0\}$. Repeating this argument, we can choose $\eta_k\in \boldsymbol{Z}^d$ in such a way that, for $k=1, 2, \cdots$,

$$p(\eta_k)=0,\quad p(\eta_k+\eta)\neq 0\qquad \text{for } \forall\eta\in\tilde\Gamma_P\setminus\{0\}. \tag{3.8}$$

Without loss of generality, we may assume that $\eta_k/\|\eta_k\|\to\exists\xi\notin\pm\Gamma_P$, $\|\xi\|=1$.

Step 2. By (3.8) and Lemma 3.6, there exist $u(x;\eta_k)\in G^s$ $(k=1, 2, \cdots)$ such that $Pu(x;\eta_k)=0$. We set $a_k=\|u(\cdot,\eta_k)\|_0$. In view of (3.6), we can find $\zeta_k\in\eta_k+\Gamma_P$ such that a_k is equal to the absolute value of the Fourier coefficient u_{ζ_k} of $u(x;\eta_k)$. If $\{|\zeta_k|\}_k$ is bounded, then there exists $\zeta\in \boldsymbol{Z}^d$ such that $\zeta=\eta_k+\theta_k$ for infinitely many $\theta_k\in\Gamma_P$. Since $\eta_k/\|\eta_k\|\to\xi$ it follows that $\theta_k/\|\theta_k\|\to-\xi\in\Gamma_P$, which is impossible. Therefore, we may assume that $|\zeta_1|<|\zeta_2|<\cdots$, $\zeta_k/|\zeta_k|\to\exists\xi'$ as $k\to\infty$.

We want to show that $\xi'\notin\pm\Gamma_P$. We write $\zeta_k=\eta_k+\theta_k$ for some $\theta_k\in\Gamma_P$. Since η_k is in a small conical neighborhood of $\xi\notin\pm\Gamma_P$, it follows that $\xi'\notin-\Gamma_P$. Suppose $\xi'\in\Gamma_P$. Without loss of generality, we may assume that $\zeta_k\neq\eta_k$. Then we write

$$u(x\,;\eta_k)=\sum_0^N Q^\nu\phi_1 u+\sum_{\nu=N+1}^\infty Q^\nu\phi_1 u\equiv\phi_1 u+QA_1+R,\ R=\sum_{\nu=N+1}^\infty Q^\nu\phi_1 u\,.$$

We take N so large that $\langle\phi_\infty u,\ e^{i\zeta_k\cdot}\rangle=\langle A_1,\ e^{i\zeta_k\cdot}\rangle$, by Lemma 3.6. We note that Q is elliptic in the Fourier support of $Q^N\phi_\infty u$. It follows that $\|\Lambda^\varepsilon R\|=\|\Lambda^\varepsilon QQ^N\phi_\infty u\|\leqq K_1 a_k$ for some $K_1>0$ independent of k, where $\Lambda=\langle\eta\rangle$. On the other hand, it follows from $\|A_1\|=a_k$ and $\xi'\in\Gamma_P$ that, $\|\Lambda^\varepsilon QA_1\|\leqq K_2 a_k$ for some $K_2>0$ independent of k. Hence, we have $\|\Lambda^\varepsilon(\phi_\infty u-\phi_1 u)\|=\langle\zeta_k\rangle^\varepsilon a_k\leqq(K_1+K_2)a_k$, a contradiction to the definition of ζ_k.

Since $\{\eta_k\}$ is in a conical neighborhood of ξ, $\xi\notin\pm\Gamma_P$, it follows from Lemma 3.4 that the number of η_k's contained in $\zeta_1-\Gamma_P$ is at most of polynomial order of $|\zeta_1|$. We thus find an integer $k(2)$ such that $\eta_k\notin\zeta_1-\Gamma_P$ for $k\geqq k(2)$. Moreover, we may assume that $\zeta_{k(2)}\notin\eta_1+\Gamma_P$. Indeed, if otherwise, then we have $\zeta_k=\eta_1+\theta_k$ for $\exists\theta_k\in\Gamma_P$. Since $\zeta_k/|\zeta_k|\to\xi'$ it follows that $\theta_k/|\theta_k|\to\xi'\in\bar{\Gamma}_P=\Gamma_P$, a contradiction to a choice of ξ'. In the same way, we can choose a sequence $\{k(n)\}$, $k(1)=1$ so that $\eta_k\notin\zeta_{k(n)}-\Gamma_P$ for $k\geqq k(n+1)$, $\zeta_{k(n)}\notin\eta_{k(\nu)}+\Gamma_P$ for $\nu=1, 2, \cdots, n-1$.

We define

$$v(x)\equiv\sum_\zeta v_\zeta e^{i\zeta x}=\sum_{n=1}^\infty u(x\,;\eta_{k(n)})a_{k(n)}^{-1}\,. \tag{3.9}$$

Since $\|u(\cdot\,;\eta_k)\|_0 a_k^{-1}=1$, $|v_\zeta|$ is bounded by the number of η_k's contained in $\zeta-\Gamma_P$. It is of some power of $|\zeta|$, by Lemma 3.4, which implies $v\in D'$. We have $Pv=0$, by definition. Moreover, since $\zeta_{k(n)}\notin\eta_{k(\nu)}+\Gamma_P$ for $n\neq\nu$ it follows that $|v_{\zeta_{k(\nu)}}|=1$, $\nu=1, 2, \cdots$, that is $v\notin C^\infty$.

Step 3. Now, let us assume that there exists $\eta_k\in\boldsymbol{Z}^d$ and $\rho>0$ such that $0<|p(\eta_k)|\leqq\exp(-\rho\|\eta_k\|^{1/s})$, $\|\eta_k\|\geqq k$ $(k=1, 2, \cdots)$. We may assume that $\eta_k/\|\eta_k\|\to\xi\notin\pm\Gamma_P$, and that $p(\eta+\eta_k)\neq 0$ for $\forall\eta\in\Gamma_P$.

By Lemma 3.6, there exists a solution, $u(x\,;\eta_k)=\sum u_\eta e^{i\eta x}\in G^s$ of $Pu=f$, $f=\delta_{\eta-\eta_k,0}e^{i\eta x}$. We define a_k and a subsequence $\{k(n)\}$ as in Step 2. Since $u_{\eta_k}=1/p(\eta_k)$, we have that $a_k\geqq|p(\eta_k)|^{-1}\geqq\exp(\rho\|\eta_k\|^{1/s})$. Then, the functions $v(x)$ and $f(x)$ given by (3.9) and $f(x)=\sum_{n=1}^\infty a_{k(n)}^{-1}\exp(i\eta_{k(n)}x)$ satisfy $Pv=f$, $v\in D'\setminus C^\infty$, $f\in G^s$, a contradiction. □

4. Solvability of nonlinear equations on compact manifolds.

Let Ω be an n-dimensional compact closed real analytic Riemannian manifold, and let P be an analytic differential operator of order m. We suppose that there exist $s>n/2$, $0<\delta<m$, and a left inverse Q of P such that

$$Q: H^s(\Omega)\longrightarrow H^{s+m-\delta}(\Omega)\,, \tag{4.1}$$

where $H^\sigma(\Omega)$ denotes the Sobolev space of order σ. In order to have analytic hypoellipticity, we demand that there exist positive numbers s_0 and γ such that

$$\|Qu\|_{s+m-\delta} \leqq \gamma^s \sum_{j=0}^{s} j!\|u\|_{s-j}, \quad s \in \boldsymbol{Z}_+, \quad s \geqq s_0. \tag{4.2}$$

Obviously, (4.2) implies the analyticity of Qf if f is analytic.

Consider the following semilinear equation on M,

$$Lu = Pu + g(u) = f, \tag{4.3}$$

where $g(u) \in C^l(\boldsymbol{R}, \boldsymbol{R})$, $l=[s]+1$, and

$$g(u) = O(u^2), \qquad \text{for } |u| \ll 1. \tag{4.4}$$

THEOREM 4.1. *Suppose* (4.1) *and* (4.4). *Then, there exists an* $\varepsilon>0$ *such that, for every* $f \in H^s(M)$, $\|f\|_s<\varepsilon$ (4.3) *has a solution* $u \in H^{s+m-\delta}(M)$. *Furthermore, if* (4.2) *holds then we can choose a solution* u *to be real-valued* C^∞ (*resp. real analytic*) *function on* M *if* f *and* g *are* C^∞ (*resp. real analytic*).

PROOF. Evidently, we can reduce the equation (4.3) to

$$u = K(u)+F, \qquad F = Qf, \qquad K(u) = -Q(g(u)). \tag{4.5}$$

We use the classical iteration scheme

$$u_k = K(u_{k-1})+F, \qquad k = 1, 2, \cdots, \quad u_0 = F. \tag{4.6}$$

We want to show that $\exists c_0>0$ verifying with $N=s+m-\delta$

$$\|u_k\|_N \leqq c_0, \qquad k = 0, 1, \cdots, \tag{4.7}$$

if $\|f\|_s<\varepsilon$ for sufficiently small $\varepsilon>0$. Indeed, (4.7) with $k=0$ follows from (4.2). Now, let us assume that $\|u_j\|_N \leqq 2q\varepsilon = c_0 \leqq 1$, $0 \leqq j \leqq k$. Since $\|F\|_N \leqq q\varepsilon$, q being the norm of Q, we have, by induction

$$\begin{aligned} \|u_{j+1}\|_N &\leqq \|K(u_j)\|_N + q\varepsilon \leqq C\|g(u_j)\|_N + q\varepsilon \\ &\leqq c_1(\|u_j\|_N)^2 + q\varepsilon \leqq c_1 4q^2\varepsilon^2 + q\varepsilon \leqq 2q\varepsilon, \end{aligned}$$

if ε is small enough. We have used the fact that (4.4) and the smallness of u imply $\|g(u)\|_N \leqq c(\|u\|_N)^2$ for some $c>0$.

It follows from (4.4) and (4.5) that, writing simply $\|\cdot\|$ instead of $\|\cdot\|_N$,

$$\|u_{k+1}-u_k\| \leqq c(\|u_k-u_{k-1}\|)(\|u_k\|+\|u\|_{k-1}) \leqq 4q\varepsilon\|u_k-u_{k-1}\|.$$

By taking $\varepsilon<(8qc)^{-1}$, we see that the contraction method holds for the integral equation (4.6). Clearly, the solution of (4.6) $u \in H^r(M)$ solves the semilinear equation (4.3) as well. If (4.2) is valid for all $s>n/2$, then $f \in C^\infty(M)$ implies $F \in C^\infty(M)$. Moreover, if $u \in H^s(M)$ then $Q(g(u)) \in H^{s+m-\delta}$. By repeating the same arguments, we can show that $u \in C^\infty(M)$.

If (4.2) holds, one estimates H^s norm of u—the solution of (4.6) in the next way

$$\|u\|_s \leqq R^{s+1}s!, \qquad s=0, 1, 2, \cdots, \tag{4.8}$$

where the constant $R>0$ depends on γ and c, c appears in the inequalities: $\|f\|_s$, $\|g\|_s \leqq c^{s+1}s!$, $s=0, 1, \cdots$. Since these estimates imply analyticity of u, we have proved Theorem 4.1. $\square$

EXAMPLE 4.1. Let P be as in Theorem 2.1, $a=0$. Then (4.2) holds. The proof is an easier version of the considerations from [**3**; Theorem 4.1].

EXAMPLE 4.2. Let us consider the existence of periodic solutions for (4.3) with P given by,

$$P=\prod_{j=1}^{4}(\partial_{x_1}-\tau_j\partial_{x_2}) \qquad \text{on } \boldsymbol{T}^2. \tag{4.9}$$

If all τ_j are rational, and if the order of P is two, then the existence of periodic solutions is proved in [**12**] by variational methods. But, it does not work well, if some τ_j is irrational. Here, we apply Theorem 4.1 for operators of order 4 with irrational τ_j.

For the sake of simplicity, we assume that τ_j ($j=1, \cdots, 4$) are real irrational, distinct, and are algebraic numbers of degree 2, i.e. $\pm\sqrt{a_j/b_j}$ for some integers a_j and b_j. By the elementary theory of numbers, τ_j satisfy, for some $c>0$ independent of η, $|\eta_1-\tau_j\eta_2|>c(|\eta_1|+|\eta_2|)^{-1}$ for all $\eta\in\boldsymbol{Z}^2$, $j=1, 2$. This implies (4.1) and (4.2). By Theorem 4.1, (4.3) has a periodic solution. We note that the same argument works, for operators with variable coefficients which have constant coefficients principal parts.

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