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Structure and duality of $\mathscr{D}$-modules related to KP hierarchy

By Toshikazu MIYAJIMA, Atsushi NAKAYASHIKI and Kanehisa TAKASAKI*

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1. Introduction.

Nonlinear integrable systems, as pointed out by Sato [**10**, **11**], may be understood as "deformations" of $\mathscr{D}$-modules. For the case of the KP hierarchy, such a $\mathscr{D}$-module is given by

$$\mathscr{M}_0=\mathscr{D}_0 W,$$

where W is a pseudo-differential operator of the form

$$W=1+\sum_{m=1}^{\infty} w_m(t, x)\partial^{-m}, \qquad \partial=\partial/\partial x,$$

$\mathscr{D}_0$ is a ring of ordinary differential operators in x, and $w_m(t, x)$ are unknown functions of the KP hierarchy [8, **9**] that depend on an infinite number of "time variables" $t=(t_1, t_2, \cdots)$. In algebraic approaches [8, **9**, **6**], it has been customary to consider "formal" (or "formally regular") solutions, i.e., solutions for which

$$w_m(t, x)\in \boldsymbol{C}[[t, x]].$$

In that setting, $\mathscr{D}_0$ is given by

$$\mathscr{D}_0=\{P;\ \exists m,\ P=\sum_{n=0}^{m} p_n\partial^n,\ p_n\in \boldsymbol{C}[[t, x]]\}.$$

This is a subring of the ring

$$\mathscr{E}_0=\{P;\ \exists m,\ P=\sum_{n=-\infty}^{m} p_n\partial^n,\ p_n\in \boldsymbol{C}[[t, x]]\}$$

of pseudo-differential operators. $\mathscr{M}_0$ is thus a free left $\mathscr{D}_0$-submodule of $\mathscr{E}_0$. With these notions, Sato [**10**, **11**] presented an abstract formulation of the KP hierarchy and its relation to the "universal Grassmannian manifold" (UGM) [8, **9**].

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The relation to UGM becomes quite manifest if one considers the underlying $\boldsymbol{C}[[t, x]]$-module structure of $\mathcal{M}_0$. As a $\boldsymbol{C}[[t, x]]$-module, $\mathcal{M}_0$ is also a free module, but of infinite rank, and has a special generator system of the form

$$\mathcal{M}_0 = \bigoplus_{i\geqq 0} \boldsymbol{C}[[t, x]]W_i, \qquad W_i = \partial^i - \sum_{j<0} w_{ij}\partial^j .$$

The coefficients w_{ij} can be identified with a set of affine coordinates on UGM. The KP hierarchy then becomes a set of evolutionary differential equations on w_{ij}. Further, there is a "dual" description that uses the right $\mathcal{D}_0$-module

$$\mathcal{M}_0^* = W^{-1}\mathcal{D}_0$$

with a similar $\boldsymbol{C}[[t, x]]$-generator system of the form

$$\mathcal{M}_0^* = \bigoplus_{j<0} W_j^* \boldsymbol{C}[[t, x]], \qquad W_j^* = \partial^{-j-1} + \sum_{i\geqq 0} \partial^{-i-1} w_{ij} .$$

Sato [**10**, **11**] explains these two types of representations as a kind of cohomological duality.

These affine coordinates w_{ij}, however, can cover only a subset of UGM called the "open Schubert cell." On boundaries of this open subset, these coordinates as well as the coefficients w_n of W have singularities (poles). It is known [8, **9**] that these poles are of very special type, coming from zeros of the so called "τ function." The τ function is one of the so called Plücker coordinates. The Plücker coordinates ξ_S, where S run over an index set introduced later in this paper, are not true functions of UGM but sections of a line bundle over UGM. Their quotients however give rational functions on UGM. If the denominator is fixed to be ξ_S and another index S' varies over a collection of indices (which is determined by S), the quotients $\xi_{S'}/\xi_S$ give affine coordinates over the open subset

$$U_S \underset{\text{def}}{=} \{\xi_S \neq 0\}$$

of UGM. Varying S now, one can thereby obtain an affine open covering of UGM. The previous coordinates w_{ij} are also obtained in that way with a special choice of S, written $S_\emptyset$; the denominator, $\xi_{S_\emptyset}$ can be identified with the τ function. $\mathcal{M}_0$ and $\mathcal{M}_0^*$ are thus, in a sense, defined on $U_{S_\emptyset}$. What $\mathcal{D}$-modules are then sitting on U_S for order S's? This is a question that motivates the present paper.

To answer this question, we reinterpret UGM as a "differential-algebraic variety" in the sense of Kolchin [**4**] (Section 2). More precisely, we shall introduce a set of differential rings $\mathcal{A}_S$ with a single derivation ∂, where S runs over the same set of indices as mentioned above. As a commutative ring, $\mathcal{A}_S$ is nothing but the coordinate ring of U_S (in other words, $U_S = \operatorname{Spec} \mathcal{A}_S$). These affine subsets are then "glued together" to form UGM, and $\mathcal{A}_S$'s give rise to the structure sheaf $\mathcal{A}$ of UGM in the usual sense. The derivation ∂ is introduced to be consistent with this gluing, hence the ringed space (UGM, $\mathcal{A}$) defines a "differential-algebraic variety," the structure sheaf $\mathcal{A}$ being a sheaf of differential rings. Besides the commutative structure sheaf $\mathcal{A}$, this differential-algebraic variety also carries the sheaves $\mathcal{D}$ and $\mathcal{E}$ of noncommutative rings of differential and pseudo-differential operators. By definition, the rings $\mathcal{D}_S$ and $\mathcal{E}_S$ of sections over U_S are comprised of differential and pseudo-differential operators of the form $\sum a_n \partial^n$ with coefficients in $\mathcal{A}_S$, and they glue together to become sheaves over UGM.

We then construct left and right $\mathcal{D}_S$-submodules $\mathcal{M}_S$ and $\mathcal{M}_S^*$ of $\mathcal{E}_S$ and show that they are also consistent with the gluing of U_S's (Section 3). Thus actually we have two sheaves $\mathcal{M}$ and $\mathcal{M}^*$ of $\mathcal{D}$-modules over the differential algebraic variety (UGM, $\mathcal{A}$). Sections 4 and 5 are devoted to the study of cohomological properties of these sheaves. If $S=S_\emptyset$, $\mathcal{M}_S$ and $\mathcal{M}_S^*$ become free modules with a single generator. (If $\mathcal{A}_S$ is replaced by $\boldsymbol{C}[[t, x]]$, this exactly reproduces the structure of $\mathcal{M}_0$ and $\mathcal{M}_0^*$.) For other S's, $\mathcal{M}_S$ and $\mathcal{M}_S^*$ are no longer free, but we show that they are projective $\mathcal{D}$-modules (Section 4) and dual to each other (Section 5). These seem to answer the question addressed above.

In Section 6, we give a differential-algebraic interpretation of the KP hierarchy itself in the language of $\mathcal{A}_S$. Actually, we simply add an infinite set of derivations, $(\partial_1, \partial_2, \cdots)$ to $\mathcal{A}_S$ and show again their consistency with the gluing of U_S. The KP hierarchy thus gives rise to yet another structure of differential-algebraic variety on UGM.

Our results show that $\mathcal{M}_S$ and $\mathcal{M}_S^*$ are of very special nature; the projectivity and duality imply that they are still close to the free case. It seems likely that these results can be extended to the case where $\mathcal{D}$ is comprised of more than one derivations and where $\mathcal{D}$-modules are, in some sense, "deformations" of a free module generated by a single element. Remarkably, several "higher dimensional" nonlinear integrable systems are known to be related with such free $\mathcal{D}$-modules (and, possibly, their nontrivial deformations). This fact was discovered by one of the present authors [**14**] and recently extended to a more general case by Ohyama [**7**].

We do not know what implications these results have in the theory of the KP hierarchy. Nevertheless, the interpretation of UGM as a differential-algebraic variety, along with the construction of the sheaves $\mathcal{M}$ and $\mathcal{M}^*$, seems to provide new material to the algebraic standpoint of Sato [**10-12**] on differential equations as well as that of Kolchin [**4**]. It should be also noted that as opposed to the usual situation, we are now dealing with very abstract rings

$\mathcal{A}_S$ rather than the ring $\boldsymbol{C}[[t, x]]$ of formal power series. Such a possibility seems to have been overlooked.

2. Universal Grassmannian manifold as differential algebraic variety.

The universal Grassmannian manifold (UGM) is an infinite dimensional extension of ordinary, finite dimensional Grassmannian manifolds. Several different descriptions are known [**8**-**13**], most of which are set-theoretical and recognize UGM as a set of vector subspaces included in a fixed vector space. A more axiomatic and functiorial definition is presented by Kashiwara [**3**] in the language of scheme theory.

From the standpoint of algebraic geometry, UGM should be understood as a collection of affine open sets glued together by holomorphic maps. We now give such a construction of UGM. In the finite dimensional case, this is a well known construction.

We now consider the set of all strictly increasing sequences S of integers of the form

$$S=(s_0, s_1, s_2, \cdots), \qquad s_i \in \boldsymbol{Z}, \qquad s_0<s_1<s_2<\cdots \tag{2.1}$$

under the condition

$$s_i = i \qquad \text{for all but a finite number of } i\text{'s}.$$

Let S^c denote the complement $\boldsymbol{Z}\setminus S$ with a similar strictly increasing numbering.

$$S^c=(\cdots, s_{-2}, s_{-1}), \qquad s_i \in \boldsymbol{Z}, \qquad \cdots <s_{-2}<s_{-1}. \tag{2.2}$$

A graphical representation of such a sequence S, or the pair (S^c, S), is given by "Maya diagrams" [**9**, **10**]. A Maya diagram consists of an infinite chain of boxes, which are numbered by all integers from the left (negative) side to the right (positive) side and colored either "black" (if the box lies in S^c) or "white" (if the box lies in S) (see Fig. 1). These are also in one-to-one correspondence with the set of all Young diagrams [8, **9**]. In the literature of the KP hierarchy [8-**10**], Young diagrams or Maya diagrams have been used as an index set of the Schubert cells of UGM. In our construction below, they arise as a index set of affine open covering of UGM.

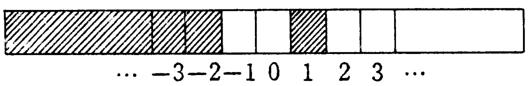

Figure 1. Maya diagram for $S=(-1, 0, 2, 3, \cdots)$

For each S as above, we now introduce a polynomial ring $\mathcal{A}_S$ with an infinite number of variables as:

$$\mathcal{A}_S \underset{\text{def}}{=} \boldsymbol{C}[w_{S,ij} \quad (i \in S,\ j \in S^c)]\,. \tag{2.3}$$

This represents the coordinate ring of an affine space with coordinates $w_{S,ij}$. We now have to specify how to glue together these affine spaces. To this end, it is convenient to introduce infinite matrices

$$\begin{aligned} \eta_S &= (\eta_{S,ij};\ i \in S,\ j \in \boldsymbol{Z})\,, \\ \eta_S^* &= (\eta_{S,ij}^*;\ i \in \boldsymbol{Z},\ j \in S^c)\,, \end{aligned} \tag{2.4}$$

with matrix elements given by

$$\begin{aligned} \eta_{S,ij} &\underset{\text{def}}{=} \begin{cases} -w_{S,ij} & \text{if } i \in S,\ j \in S^c\,, \\ \delta_{ij} & \text{if } i,\ j \in S\,, \end{cases} \\ \eta_{S,ij}^* &\underset{\text{def}}{=} \begin{cases} \delta_{ij} & \text{if } i,\ j \in S^c\,, \\ w_{S,ij} & \text{if } i \in S,\ j \in S^c\,, \end{cases} \end{aligned} \tag{2.5}$$

where δ_{ij} is the Kronecker delta. In the usual set-theoretical interpretation, the row vectors of η_S span a vector subspace in a given big vector space, which is the corresponding point of UGM. Similarly, η_S^* is to represent a collection of column vectors in the dual vector space, and gives a "dual" representation. These two matrices are connected by the relation

$$\eta_S \eta_S^* = 0\,, \tag{2.6}$$

which determines each other. Note that η_S^* corresponds to the ξ-matrix of in the literature [8-10]. The rule of gluing between $\mathcal{A}_S$ and $\mathcal{A}_{S'}$ is given by the algebraic relation

$$\eta_S = h_{SS'}\eta_{S'}\,, \tag{2.7}$$

where

$$h_{SS'} \underset{\text{def}}{=} (\eta_{S,ij};\ i \in S,\ j \in S')\,. \tag{2.8}$$

Since $h_{SS'}$ differs from the identity matrix by at most a finite number of rows, one can define its determinant as

$$\det h_{SS'} \underset{\text{def}}{=} \lim_{N \to \infty} \det(h_{SS'}\,;\ i,\ j \geqq -N)\,. \tag{2.9}$$

(The finite determinants on the right hand side does not depend on N if N is sufficiently large, hence the limit is meaningful.) Determinant (2.9) becomes a non-vanishing rational function of $w_{S,ij}$. Equation (2.7) defines a birational relation between two sets of variables $w_{S,ij}$ and $w_{S',ij}$, which is holomorphic if the determinants do not vanish. More precisely, (2.7) gives rise to an isomorphism

$$g_{SS'}: \mathcal{A}_S[\det h_{SS'}{}^{-1}] \xrightarrow{\sim} \mathcal{A}_{S'}[\det h_{S'S}{}^{-1}]. \tag{2.10}$$

Because of the obvious relations

$$h_{SS'}{}^{-1} = h_{S'S},$$
$$h_{SS'}h_{S'S''} = h_{SS''},$$

the collection of ring isomorphisms $g_{SS'}$ defines a consistent gluing of the affine spaces U_S represented by $\mathcal{A}_S$. This gives a realization of UGM as an infinite dimensional algebraic variety. Actually, the above gluing rule can be restated in terms of η_S^* as:

$$\eta_S^* = \eta_{S'}^* h_{S'S}^*, \tag{2.11}$$

where

$$h_{S'S}^* \underset{\text{def}}{=} (h_{S,ij}^*;\ i \in S'^c,\ j \in S^c). \tag{2.12}$$

We now define a derivation ∂ on $\mathcal{A}_S$ by

$$\begin{aligned}\partial(w_{S,ij}) \underset{\text{def}}{=} &\ \theta_{S^c}(i+1)\delta_{i+1,j} + \theta_S(i+1)w_{S,i+1,j} \\ &- \theta_{S^c}(j-1)w_{S,i,j-1} - \sum_{k=0}^{m} w_{S,i,i_k-1}w_{S,i_k,j},\end{aligned} \tag{2.13}$$

where θ_S and θ_{S^c} are the characteristic functions of S and S^c, i.e., for any subset X of $\boldsymbol{Z}$,

$$\theta_X(i) \underset{\text{def}}{=} \begin{cases} 1 & \text{if } i \in X, \\ 0 & \text{if } i \notin X. \end{cases}$$

PROPOSITION (2.14). *This definition of ∂ is consistent with the gluing isomorphism $g_{SS'}$. More precisely, if ∂_S denotes the derivation on $\mathcal{A}_S$ defined by* (2.13), *∂_S and $\partial_{S'}$ obey the relation*

$$\partial_S \circ g_{SS'} = g_{SS'} \circ \partial_{S'}.$$

PROOF. Let us note that, in terms of the matrix η_S, the action of ∂ can be rewritten

$$\partial(\eta_S) = \boldsymbol{B}_S\eta_S - \eta_S\Lambda, \tag{2.15}$$
$$\boldsymbol{B}_S \underset{\text{def}}{=} (\eta_{S,i,j-1};\ i, j \in S),$$

where Λ denotes the infinite shift matrix

$$\Lambda \underset{\text{def}}{=} (\delta_{i,j-1};\ i, j \in \boldsymbol{Z}). \tag{2.16}$$

In the dual form, this is also equivalent to the following.

$$\partial(\eta_S^*) = \Lambda\eta_S^* - \eta_S^* \boldsymbol{A}_S\,, \qquad (2.17)$$
$$\boldsymbol{A}_S \underset{\text{def}}{=} (\eta_{S,i+1,j}^*;\ i,\ j\in S)\,.$$

The proposition asserts that the same equations are also satisfied by $\eta_{S'}$ and $\eta_{S'}^*$ for any other S'. To see this, insert (2.7) into (2.15). This results in the equation

$$\partial(\eta_{S'}) = \boldsymbol{B}_{S'}\eta_{S'} - \eta_{S'}\Lambda \qquad (2.15')$$

where $\boldsymbol{B}_{S'}$ is given by

$$\boldsymbol{B}_{S'} = h_{SS'}^{-1}\boldsymbol{B}_{S'}h_{SS'} - h_{SS'}^{-1}\partial(h_{SS'})^{-1}\,.$$

Actually, one does not have to do further calculations for the right hand side; extracting the $S'\times S'$ part of both hand sides of (2.15′) gives the relation

$$0 = \boldsymbol{B}_{S'} - (\eta_{S',i,j-1};\ i,\ j\in S')\,,$$

which determines $\boldsymbol{B}_{S'}$ just as in (2.15) with S replaced by S'. The equation for $\eta_{S'}^*$ can be derived in the same way. Q.E.D.

This result says a geometric fact that ∂ gives a globally defined vector field on UGM. Our UGM thus becomes a "differential-algebraic variety" whose structure sheaf $\mathcal{A}$ is associated with the differential algebras $(\mathcal{A}_S, \partial)$.

One can define the rings $\mathcal{D}_S$ and $\mathcal{E}_S$ of differential and pseudo-differential operators

$$\mathcal{D}_S \underset{\text{def}}{=} \{P;\ \exists m,\ P=\sum_{n=0}^{m} p_n\partial^n,\ p_n\in\mathcal{A}_S\}, \qquad (2.18)$$

$$\mathcal{E}_S \underset{\text{def}}{=} \{P;\ \exists m,\ P=\sum_{n=-\infty}^{m} p_n\partial^n,\ p_n\in\mathcal{A}_S\}, \qquad (2.19)$$

with the same summation and multiplication rules as the ordinary ones [8, **10**, **12**];

$$\sum p_n\partial^n + \sum q_n\partial^n \underset{\text{def}}{=} \sum(p_n+q_n)\partial^n\,,$$
$$(\sum p_i\partial^i)\cdot(\sum q_j\partial^j) \underset{\text{def}}{=} \sum r_n\partial^n\,,$$
$$r_n \underset{\text{def}}{=} \sum_{i+j-k=n}\binom{i}{k}p_i\partial^k(q_j)\,.$$

Let $(\ )_\pm$ denote the projection onto the components of the direct sum decomposition (as an $\mathcal{A}_S$-module)

$$\mathcal{E}_S = \mathcal{D}_S \oplus \mathcal{E}_S^{(-1)}, \qquad \mathcal{E}_S^{(-1)} \underset{\text{def}}{=} \{\sum p_n\partial^n;\ p_n = 0\,\forall n\geqq 0\} \qquad (2.20)$$

In other words,

$$(\partial^n)_+ = 0 \quad (n<0), \quad = \partial^n \qquad (n \geqq 0),$$
$$(\partial^n)_- = \partial^n \quad (n<0), \quad = 0 \qquad (n \geqq 0).$$

It should be noted that the commutative ring $\mathcal{A}_S$ now has two distinct roles. In one hand, it is a commutative subalgebra of $\mathcal{D}_S$ or of $\mathcal{E}_S$; on the other hand, it is a differential ring on which $\mathcal{D}_S$ acts. To distinguish notationally, we write Pf or $P\cdot f$ $(P \in \mathcal{D}_S,\ f \in \mathcal{A}_S)$ to mean the multiplication of differential operators, whereas let $P(f)$ denote the action of P on f.

3. $\mathcal{D}_S$-modules $\mathcal{M}_S$ and $\mathcal{M}_S^*$.

Let $\mathcal{M}_S$ be the following left free $\mathcal{A}_S$-submodule of $\mathcal{E}_S$.

$$\text{(3.1)} \qquad \mathcal{M}_S \underset{\text{def}}{=} \bigoplus_{i \in S} \mathcal{A}_S W_{S,i},$$
$$W_{S,i} \underset{\text{def}}{=} \partial^i - \sum_{j \in S^c} w_{S,ij} \partial^j \qquad (i \in S).$$

Actually, this becomes a $\mathcal{D}_S$-module. To specify its structure, we write S as a disjoint union of finite intervals and a semi-infinite interval as:

$$\text{(3.2)} \qquad S = [i_0, j_0] \cup \cdots \cup [i_{m-1}, j_{m-1}] \cup [i_m, \infty),$$
$$i_0 < j_0 < i_1 \leqq \cdots \leqq i_{m-1} < j_{m-1} \leqq i_m,$$
$$n \underset{\text{def}}{=} \sum_{l=0}^{m-1} (i_l - j_l + 1),$$

where $[i_l, j_l]$, for $l=0, \cdots, m-1$, denotes the sequence of integers from i_l to j_l, and $[i_m, \infty]$ all integers greater than or equal to i_m (see Fig. 2).

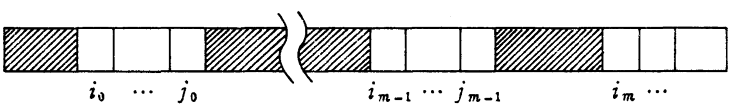

Figure 2. Maya diagram associated with S

PROPOSITION (3.3). 1) *The generators $W_{S,i}$ satisfy the relations*

$$\text{(3.3.1)} \qquad \partial \cdot W_{S,i} = \theta_S(i+1) W_{S,i+1} - \sum_{l=0}^{m} w_{S,i,i_l-1} W_{S,i_l} \qquad (\forall i \in S).$$

2) *$\mathcal{M}_S$ is closed under the left multiplication of ∂, hence becomes a left $\mathcal{D}_S$-module.*

3) *$\mathcal{M}_S$ is generated over $\mathcal{D}_S$ by a finite number of elements:*

$$\text{(3.3.3)} \qquad \mathcal{M}_S = \sum_{l=0}^{m} \mathcal{D}_S W_{S,i_l}.$$

(*Actually, this is a minimal generator system; see Appendix.*)

PROOF. According to the general composition rule, $\partial\cdot W_{S,i}$ can be written

$$\partial\cdot W_{S,i}=\partial^{i+1}-\sum_{j\in S^c}w_{S,ij}\partial^{j+1}-\sum_{j\in S^c}\partial(w_{S,ij})\partial^j .$$

The first term on the right hand side can be further rewritten:

$$\begin{aligned}\partial^{i+1}&=\theta_S(i+1)\partial^{i+1}+\theta_{S^c}(i+1)\partial^{i+1}\\&=\theta_S(i+1)W_{S,i+1}+\sum_{j\in S^c}(\theta_{S^c}(i+1)w_{S,i+1,j}+\theta_S(i+1)\delta_{i+1,j})\partial^j .\end{aligned}$$

and ∂^{j+1} ($j\in S^c$) has a similar expression. Inserting these formulas and gathering up various terms appropriately, one arrives at the following expression of $\partial\cdot W_{S,i}$.

$$\begin{aligned}\partial\cdot W_{S,i}=\ &\theta_S(i+1)W_{S,i+1}-\sum_{j\in S^c}\theta_S(j+1)w_{ij}W_{S,j+1}\\&+\sum_{j\in S^c}[\theta_{S^c}(i+1)\delta_{i+1,j}+\theta_S(i+1)w_{S,i+1,j}-\theta_{S^c}(j-1)w_{S,i,j-1}\\&-\sum_{l\in S^c}\theta_S(l+1)w_{S,il}w_{S,l+1,j}-\partial(w_{S,ij})] .\end{aligned}$$

The last sum over $j\in S^c$ disappears due to the definition of ∂. One thus obtains (3.3.1). This readily implies the second statement of the proposition. The last part is a consequence of (3.3.1) and a simple induction argument. Q.E.D.

EXAMPLE. Let us consider the case for $S=(-1,0,2,3,\cdots)$. The Maya diagram is shown in Fig. 1. The $\mathcal{D}_S$-module $\mathcal{M}_S$ is now given by

$$\mathcal{M}_S=\mathcal{A}_SW_{-1}\oplus\mathcal{A}_SW_{S,0}\bigoplus_{k\geq 2}\mathcal{A}_SW_{S,k} ,\tag{3.4}$$

$$W_{S,i}=\partial^i-w_{S,i,1}\partial-\sum_{j\leq -2}w_{S,ij}\partial^j .$$

The derivation ∂ acts on the generators of $\mathcal{M}_S$ as:

$$\begin{aligned}&\partial\cdot W_{S,-1}=W_{S,0}-w_{S,-1,-2}W_{-1}-w_{S,-1,1}W_{S,2} ,\\&\partial\cdot W_0=-w_{S,0,-2}W_{S,-1}-w_{S,0,1}W_{S,2} ,\\&\partial\cdot W_{S,k}=W_{S,k+1}-w_{S,k,-2}W_{S,-1}-w_{S,k,1}W_{S,2}\qquad(\forall k\geqq 2) .\end{aligned}\tag{3.4}$$

$\mathcal{M}_S$ is generated by two elements $W_{S,-1}$ and $W_{S,2}$

$$\mathcal{M}_S=\mathcal{D}_SW_{S,-1}+\mathcal{D}_SW_{S,2}\tag{3.5}$$

with a $\mathcal{D}_S$-linear relation of the form

$$(\partial^2+\partial\cdot w_{-1,-2}+w_{0,-2})W_1+(\partial\cdot w_{-1,1}+w_{0,1})W_2=0 .\tag{3.6}$$

Now let $\mathcal{M}_S^*$ be the right $\mathcal{A}_S$-submodule $\mathcal{M}_S^*$ of $\mathcal{E}_S$ given by

(3.7)
$$\mathcal{M}_S^* \underset{\text{def}}{=} \sum_{j\in S^c} W_{S,j}^* \mathcal{A}_S ,$$
$$W_{S,j}^* \underset{\text{def}}{=} \partial^{-j-1} + \sum_{i\in S} \partial^{-i-1}\cdot w_{S,ij} .$$

The index set S^c can be written as a disjoint union of finite intervals and a semi-infinite interval like the expression of S in (2.1) (see Fig. 3):

(3.8)
$$S^c = (-\infty, i_m^*]\cup[j_{m-1}^*, i_{m-1}^*]\cup\cdots\cup[j_0^*, i_0^*] ,$$
$$i_m^* \leqq j_{m-1}^* < i_{m-1}^* < \cdots \leqq i_1^* < j_0^* < i_0^* ,$$
$$n^* \underset{\text{def}}{=} \sum_{l=0}^{m-1} (i_l^* - j_l^* + 1) .$$

The endpoints of the intervals are related with the previous expression as:

(3.9)
$$i_m^* = i_0 - 1,\ i_{m-1}^* = i_1 - 1,\ \cdots,\ i_0^* = i_m - 1 ,$$
$$j_{m-1}^* = j_0 + 1,\ j_{m-2}^* = j_1 + 1,\ \cdots,\ j_0^* = i_{m-1} + 1 ,$$

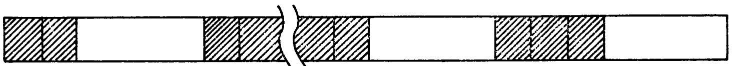

Figure 3. Maya diagram in "dual" form

We now have an analogue of Proposition (3.3).

PROPOSITION (3.10). 1) *The $\mathcal{A}_S$-generators $W_{S,j}^*$ satisfy the relations*

(3.10.1)
$$W_{S,j}^*\partial = W_{S,j-1}^*\theta_{S^c}(j-1) + \sum_{l=0}^{m} W_{S,i_l-1}^* w_{S,i_l,j} \qquad (j\in S^c) .$$

2) *$\mathcal{M}_S^*$ becomes a right $\mathcal{D}_S$-module.*

3) *As a $\mathcal{D}_S$-module, $\mathcal{M}_S^*$ is generated by a finite number of elements as*

(3.10.3)
$$\mathcal{M}_S^* = \sum_{0\leq l\leq m} \mathcal{D}_S W_{S,i_l-1}^* .$$

(*These generators give a minimal $\mathcal{D}_S$-generator system.*)

Let us consider the case of

(3.11)
$$S = S_\emptyset \underset{\text{def}}{=} (0, 1, 2, \cdots) .$$

To simplify notations, we write $\mathcal{A}_\emptyset$, $\mathcal{D}_\emptyset$. etc. rather than $\mathcal{A}_{S_\emptyset}$, $\mathcal{D}_{S_\emptyset}$, $\cdots$.

PROPOSITION (3.12). 1) *$\mathcal{M}_\emptyset$ and $\mathcal{M}_\emptyset^*$ are cyclic $\mathcal{D}_\emptyset$-modules, i.e., generated by a single element:*

(3.12.1)
$$\mathcal{M}_\emptyset = \mathcal{D}_\emptyset W_{\emptyset,0} ,$$
$$\mathcal{M}_\emptyset^* = W_{\emptyset,-1}^* \mathcal{D}_\emptyset .$$

2) *$W_{\emptyset,i}$ and $W^*_{\emptyset,j}$ can be written explicitly:*

$$W_{\emptyset,i}=(\partial^i\cdot W_{\emptyset,0}{}^{-1})_+W_{\emptyset,0}, \tag{3.12.2}$$

$$W^*_{\emptyset,j}=W^*_{\emptyset,-1}(W^*_{\emptyset,-1}{}^{-1}\partial^{-j-1})_+ .$$

3) *$W_{\emptyset,0}$ and $W^*_{\emptyset,-1}$ are connected by the relation*

$$W_{\emptyset,0}W^*_{\emptyset,-1}=1 . \tag{3.12.3}$$

PROOF. The first statement is obvious from the previous propositions. To see (3.12.2), let us note the obvious relation

$$(\partial^i\cdot W_{\emptyset,0}{}^{-1})_+W_{\emptyset,0}=\partial^i-(\partial^i\cdot W_{\emptyset,0}{}^{-1})_-W_{\emptyset,0},$$

which shows that this operator takes such a form as

$$=\partial^i-\sum_{j<0}a_{ij}\partial^j, \qquad a_{ij}\in\mathcal{A}_\emptyset .$$

Such an element of $\mathcal{M}_S$ is unique and should be given by $W_{\emptyset,i}$. In much the same way, one can check the expression for $W^*_{\emptyset,j}$, completing the proof of (3.12.2). Equation (3.12.3) can be proven as follows. Recall that $W_{\emptyset,i}$ satisfy the relation

$$\partial\cdot W_{\emptyset,i-1}=W_{\emptyset,i}-w_{\emptyset,i-1,-1}W_{\emptyset,0} .$$

This gives rise to a recursion formula for $W_{\emptyset,i}W_{\emptyset,0}{}^{-1}$:

$$W_{\emptyset,i}W_{\emptyset,0}{}^{-1}=\partial\cdot W_{\emptyset,i-1}W_{\emptyset,0}{}^{-1}+w_{\emptyset,i-1,-1} .$$

Solving this and using (3.12.2), one has the relation

$$(\partial^i\cdot W_{\emptyset,0}{}^{-1})_+=W_{\emptyset,i}W_{\emptyset,0}{}^{-1}=\partial^i+\sum_{k=0}^{i-1}\partial^{i-1-k}\cdot w_{\emptyset,k,-1} .$$

Since this is valid for all $i\geqq 0$,

$$W_{\emptyset,0}{}^{-1}=1+\sum_{k\geqq 0}\partial^{-k-1}\cdot w_{\emptyset,k,-1}=W^*_{\emptyset,-1} .$$

Q.E.D.

COROLLARY (3.13). *For any i and j, $W_{\emptyset,i}W^*_{\emptyset,j}$ is a differential operator.*

The situation for $S=S_\emptyset$ is thus exactly the same as $\mathcal{M}_0$ and $\mathcal{M}_0^*$ in Section 1, though the differential algebras are different.

Remarkably, the last result can be generalized.

PROPOSITION (3.14). *For any S, $i\in S$ and $j\in S^c$, $W_{S,i}W^*_{S,j}$ is a differential operator.*

PROOF. Let $\mathcal{K}$ denote the quotient field of $\mathcal{A}_S$. We do not specify the suffix S for $\mathcal{K}$, because all $\mathcal{A}_S$ are now connected by an invertible transformation, their quotient fields all being isomorphic. $\mathcal{K}$ is now a differential field with a single derivation ∂. Let $\mathcal{D}_{\mathcal{K}}$ and $\mathcal{E}_{\mathcal{K}}$ denote the associated rings of differential and pseudo-differential operators. They give a ring extension of $\mathcal{D}_S$ and $\mathcal{E}_S$. We now notice that linear relations (2.7) and (2.11) give rise to the following relations of operators:

$$W_{S,i} = \sum_{j\in S'} h_{SS',ij} W_{S',j}, \tag{3.15}$$

$$W_{S,j}^* = \sum_{i\in S'^c} W_{S',i}^* h_{S'S,ij}^* .$$

Applying these relations to the case of $S'=S_g$, one finds that each of the operator products $W_{S,i}W_{S,j}^*$ is linear combination of similar operator products for S_g with coefficients in $\mathcal{K}$. (The matrix elements of h_{S,S_g} and $h_{S_g,S}^*$ are included in $\mathcal{K}$.) As we have seen above, $W_{S_g,i}W_{S_g,j}^*$ are differential operators. Therefore $W_{S,i}W_{S,j}^*$ are also differential operators, though their coefficients belong to $\mathcal{K}$. Actually, they are members of $\mathcal{E}_S$ from the construction, hence turn out to be in $\mathcal{D}_S$. Q. E. D.

REMARK. It is immediate from the above observation that $\mathcal{M}_S$ and $\mathcal{M}_S^*$ become cyclic (actually, isomorphic to $\mathcal{D}_{\mathcal{K}}W_{S_g,0}$ and $W_{S_g,-1}^*\mathcal{D}_{\mathcal{K}}$) if their ring of coefficients are extended from $\mathcal{D}_S$ to $\mathcal{D}_{\mathcal{K}}$. This is exactly the situation considered in Ref. 10.

4. Vanishing of cohomology and projectivity.

This and the next sections deal with cohomological properties of the modules $\mathcal{M}_S$ and $\mathcal{M}_S^*$ with S fixed. Let us suppress the suffix "S" for simplicity of notation.

In view of Proposition (3.3), we now take a finite set of generators of $\mathcal{M}$ as:

$$\mathcal{M} = \sum_{i\in S,\, i\leq i_m} \mathcal{D} W_i . \tag{4.1}$$

For the definition of i_m, see (3.2). (One may also take the minimal generator system as presented therein; see Appendix for such a treatment.) Our task below is to construct a free resolution of this module.

DEFINITION (4.2). We now introduce the following matrices of differential operators [cf. Proposition (3.14)]. Note that M is an $n\times(n+1)$ matrix, N an $(n+1)\times n$ matrix, and U an $(n+1)\times(n+1)$ matrix, n being the number defined in (3.2).

$$M \underset{\text{def}}{=} (M_{ij};\, i\in S,\, i<i_m,\, j\in S,\, j\leqq i_m), \tag{4.2.1}$$

$$M_{ij} \underset{\text{def}}{=} \delta_{ij}\partial - \theta_S(i+1)\delta_{i+1,j} + \theta_{S^c}(j-1)w_{i,j-1}.$$

$$N \underset{\text{def}}{=} (N_{ij}\,;\, i\in S,\ i\leqq i_m,\ j\in S,\ j<i_m), \tag{4.2.2}$$

$$N_{ij} \underset{\text{def}}{=} -(W_i\partial^{-j-1})_+ .$$

$$U \underset{\text{def}}{=} (U_{ij}\,;\, i,\ j\in S,\ i,\ j\leqq i_m), \tag{4.2.3}$$

$$U_{ij} \underset{\text{def}}{=} \theta_{S^c}(j-1)W_iW_{j-1}^* .$$

Proposition (3.3) implies the following relations among the generators W_i in (4.1).

$$\sum_{j\in S,\, j\leq i_m} M_{ij}W_j = 0 \qquad (i\in S,\ i<i_m). \tag{4.3}$$

The following lemma plays a key role in this and the next sections.

LEMMA (4.4). *M and N obey the relations*

$$MN = 1_n . \tag{4.4.1}$$

$$NM = 1_{n+1} - U , \tag{4.4.2}$$

where 1_n, in general, denotes the $n\times n$ unit matrix.

PROOF. We first prove (4.4.1). From (4.3), obviously,

$$\sum_{j\in S,\, j\leq i_m} (M_{ij}W_j\partial^{-k-1})_+ = 0, \qquad \forall k\in \boldsymbol{Z}.$$

In view of the definition of M_{ij}, on the other hand, the left hand side can be rewritten

$$\sum_{j\in S,\, j\leq i_m} (M_{ij}W_j\partial^{-k-1})_+ = (\partial\cdot W_i\partial^{-k-1})_+ - \partial\cdot(W_i\partial^{-k-1})_+ - \sum_{k\in S,\, k\leq i_m} M_{ij}N_{jk} .$$

We notice here that for any pseudo-differential operator $P=\sum p_n\partial^n$,

$$(\partial\cdot P)_+ - \partial\cdot(P)_+ = p_{-1} .$$

Applying this formula to $W_i\partial^{-k-1}$, one finds that

$$(\partial\cdot W_i\partial^{-k-1})_+ - \partial\cdot(W_i\partial^{-k-1})_+ = \delta_{ik} .$$

One can thus deduce that

$$\sum_{j\in S,\, j\leq i_m} M_{ij}N_{jk} = \delta_{ik}$$

for $i,\ k\in S,\ i,\ k<i_m$. This is exactly (4.4.1). One can derive (4.4.2) by similar calculations: First, from the definition of M_{ij} and N_{jk},

$$\sum_{j\in S,\, j<i_m} N_{ij}M_{jk} = -(W_i\partial^{-k-1})_+\partial + \sum_{j\in S,\, j<i_m} (W_i\partial^{-j-1})_+\theta_S(j+1)\delta_{j+1,k}$$
$$- \sum_{j\in S,\, j<i_m} (W_i\partial^{-j-1})_+\theta_{S^c}(k-1)w_{j,k-1} \underset{\text{def}}{=} \mathrm{I} + \mathrm{II} + \mathrm{III} .$$

The second part on the right hand side is equal to $(W_i\partial^{-k})_+$ if $k-1\in S$ and vanishes if $k-1\in S^c$. Hence

$$\mathrm{II} = \theta_S(k-1)(W_i\partial^{-k})_+ .$$

The third part can be gathered up to give the following contribution:

$$\mathrm{III} = \theta_{S^c}(k-1)(W_i \sum_{j\in S, j<i_m} \partial^{-j-1}\cdot w_{j,k-1})_+ .$$

Since i is now bounded by i_m, W_i is of order less than or equal to i_m. The range of j on the right hand side may be extended to all S because ∂^{-j-1} for $j\geqq i_m$ have no contribution to $(\cdots)_+$. Therefore

$$\mathrm{III} = \theta_{S^c}(k-1)(W_i \sum_{j\in S}\partial^{-j-1}\cdot w_{j,k-1})_+ = \theta_{S^c}(k-1)(W_i(W^*_{k-1}-\partial^{-k})) .$$

To summarize,

$$\begin{aligned}\mathrm{I}+\mathrm{II}+\mathrm{III} &= -(W_i\partial^{-k-1})_+\partial+(W_i\partial^{-k})_+-\theta_{S^c}(k-1)(W_iW^*_{k-1})_+ \\ &= \delta_{ik}-\theta_{S^c}(k-1)(W_iW^*_{k-1})_+ .\end{aligned}$$

This implies (4.4.2). Q.E.D.

PROPOSITION (4.5). *$\mathcal{M}$ has a free resolution of the form*

$$0 \longrightarrow \mathcal{D}^n \xrightarrow{\alpha} \mathcal{D}^{n+1} \xrightarrow{\beta} \mathcal{M} \longrightarrow 0 \quad \text{(exact)},$$

where $\mathcal{D}^n$ and $\mathcal{D}^{n+1}$ are the free $\mathcal{D}$-modules of row vectors of differential operators, and α and β are left $\mathcal{D}$-homomorphisms given by

$$\begin{aligned}\alpha : Q. = (Q_j;\, j\in S,\, j<i_m) &\longmapsto (\sum_{i\in S, i<i_m} Q_iM_{ij};\, j\in S,\, j\leqq i_m) , \\ \beta : P. = (P_j;\, j\in S,\, j\leqq i_m) &\longmapsto \sum_{j\in S, j\leqq i_m} P_jW_j .\end{aligned}$$

PROOF. Since the first n columns of M is a matrix of differential operators of the form $\partial\cdot 1_n+$(order 0), α is an injective homomorphism. From (4.1) and (4.3), β is surjective and its composition with β vanishes. It remains to prove that $\operatorname{Ker}\beta$ is equal to $\operatorname{Im}\alpha$. This readily follows from (4.4.2) and the construction of M, N and U: Suppose that $P.=(P_j)$ is in the kernel. Obviously,

$$\sum_i P_iU_{ij} = 0, \qquad \forall j\in S,\ j\leqq i_m$$

Therefore from (4.4.2),

$$P_k = \sum_i\sum_j P_iN_{ij}M_{jk} ,$$

hence $P.$ is in the image of α. Q.E.D.

COROLLARY (4.6). *As a left $\mathcal{D}$-module,*

$$\mathcal{M} \cong \mathcal{D}^{n+1} / \sum_{i \in S, i < i_m} \mathcal{D} M_{i\cdot} ,$$

where $M_{i\cdot}$ *are row vectors given by*

$$M_{i\cdot} \underset{\mathrm{def}}{=} (M_{ij} ; j \in S, j \leqq i_m) .$$

From the above free resolution, in particular, one finds that

$$\operatorname{Ext}^p_{\mathcal{D}}(\mathcal{M}, \mathcal{D}) = 0 \qquad \text{for } p \geqq 2 . \tag{4.7}$$

At first sight, this result looks almost obvious in view of the basic knowledge on $\mathcal{D}$-modules on complex manifolds [2]. Actually, this is not the case. Since $\mathcal{A}$ is a polynomial ring of an infinite number of variables, one can rather prove that

$$\mathrm{gl.dim}\, \mathcal{D} = \infty,$$

which means that there is no uniform upper bound for the range of p with non-vanishing $\operatorname{Ext}^p_{\mathcal{D}}$ if one allows an arbitrary $\mathcal{D}$-module. Further, ordinary methods [2] are more or less rely upon dimensional induction and cannot be applied for the present case. The above result is due to some very special nature of $\mathcal{M}$.

One can actually prove the following even stronger result.

PROPOSITION (4.8).

$$\operatorname{Ext}^1_{\mathcal{D}}(\mathcal{M}, \mathcal{D}) = 0 .$$

PROOF. From the free resolution of (4.5),

$$\operatorname{Ext}^1_{\mathcal{D}}(\mathcal{M}, \mathcal{D}) \cong \mathcal{D}^n / \sum_{j \in S, j \leqq i_m} M_{\cdot j} \mathcal{D} ,$$

where $M_{\cdot j}$ is the following column vector

$$M_{\cdot j} \underset{\mathrm{def}}{=} (M_{ij} ; i \in S, i < i_m) .$$

On the other hand, (4.4.1) implies $\sum M_{\cdot j} \mathcal{D} = \mathcal{D}^n$. Q.E.D.

From these results, we arrive at our first theorem.

THEOREM (4.9). *For any* S, $\mathcal{M}_S$ *is a projective* $\mathcal{D}_S$*-module.*

PROOF. One can obtain a long exact sequence from the short exact sequence of Proposition (4.5) by the contravariant functor $\operatorname{Ext}^i_{\mathcal{D}}(\cdot, \mathcal{D})$. Because of the above vanishing of cohomology, this long exact sequence can be reduced to the short exact sequence

$$0 \longleftarrow \mathcal{D}^n \overset{\alpha^*}{\longleftarrow} \mathcal{D}^{n+1} \overset{\beta^*}{\longleftarrow} \operatorname{Hom}_{\mathcal{D}}(\mathcal{M}, \mathcal{D}) \longleftarrow 0 ,$$

which splits because the first and second terms are free modules. $\operatorname{Hom}_{\mathcal{D}}(\mathcal{M}, \mathcal{D})$

thus turns out to be a projective right $\mathcal{D}$-module. We now have the following commutative diagram, whose both rows are exact.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{D}^n & \xrightarrow{\alpha} & \mathcal{D}^{n+1} & \xrightarrow{\beta} & \mathcal{M} & \longrightarrow & 0 \\
 & & \cong\downarrow & & \cong\downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{D}^n & \xrightarrow{\alpha^{**}} & \mathcal{D}^{n+1} & \xrightarrow{\beta^{**}} & \mathrm{Hom}_{\mathcal{D}}(\mathrm{Hom}_{\mathcal{D}}(\mathcal{M}, \mathcal{D}), \mathcal{D}) & \longrightarrow & 0
\end{array}$$

Therefore the canonical homomorphism $\mathcal{M}\to\mathrm{Hom}_{\mathcal{D}}(\mathrm{Hom}_{\mathcal{D}}(\mathcal{M}, \mathcal{D}), \mathcal{D})$ is actually an isomorphism, and since $\mathrm{Hom}_{\mathcal{D}}(\mathcal{M}, \mathcal{D})$ is projective, $\mathcal{M}$ also becomes a projective $\mathcal{D}$-module. Q.E.D.

5. Duality of $\mathcal{D}$-modules.

In the previous section, the key relations in Lemma (4.4) are used to construct a special free resolution of $\mathcal{M}$. We now use the same relations to deduce a different exact sequence.

As a counterpart of (4.1), we now take a finite set of generators of $\mathcal{M}^*$ as:

$$\mathcal{M}^* = \sum_{j\in S^c,\, j\geq i_m^*} W_j^*\mathcal{D}\,. \tag{5.1}$$

LEMMA (5.2). *There is an isomorphism*

$$\varphi^* : \mathcal{M}^* \xrightarrow{\sim} \sum_{j\in S^c,\, j\geq i_m^*} V_{\cdot j}\mathcal{D} \qquad (\subset \mathcal{D}^{n+1})$$

of right $\mathcal{D}$-modules given by

$$\varphi^*\Big(\sum_{j\in S^c,\, j\geq i_m^*} W_j^* P_j\Big) = \sum_{j\in S^c,\, j\geq i_m^*} V_{\cdot j} P_j\,,$$

where $\mathcal{D}^{n+1}$ is now considered as a set of column vectors of differential operators, and $V_{\cdot j}$ is an element of $\mathcal{D}^{n+1}$ given by

$$V_{\cdot j} \underset{\mathrm{def}}{=} (V_{ij}\,;\, i\in S,\ i\leqq i_m)\,,$$

$$V_{ij} \underset{\mathrm{def}}{=} W_i W_j^*\,.$$

PROOF. φ^* is well-defined from the construction; if $\sum W_j^* P_j=0$ for some $P_j\in\mathcal{D}$, then $\sum V_{ij}P_j=\sum W_i W_j^* P_j=0$. The surjectivity of φ is also obvious. To prove the injectivity, suppose that $\varphi(\sum W_j P_j)=0$. This means that $\sum W_i W_j^* P_j=0$ for $i\in S$, $i<i_m$. Since $\mathcal{E}$ contains no zero-divisor, it follows that $\sum W_j^* P_j=0$. Q.E.D.

PROPOSITION (5.3). *The following sequence of right $\mathcal{D}$-modules is exact.*

$$0 \longleftarrow \mathcal{D}^{n} \xleftarrow{\psi^*} \mathcal{D}^{n+1} \xleftarrow{\varphi^*} \mathcal{M}^* \longleftarrow 0,$$

where φ^* is the homomorphism induced by the φ^* of (5.2), *and ψ^* is the homomorphism that sends $(n+1)$-column vectors of differential operators to n-column vectors as:*

$$\psi^*: P. = (P_i;\ i\in S,\ i\leqq i_m) \longmapsto (\sum_j M_{ij}P_j;\ i\in S,\ i<i_m).$$

PROOF. Step i) φ^* is injective [Lemma (5.2)].

Step ii) The surjectivity of ψ^*:—For any $Q.=(Q_i;\ i\in S,\ i<i_m)\in\mathcal{D}^n$, let $P.=(P_i;\ i\in S,\ i\leqq i_m)$ be an element of $\mathcal{D}^{n+1}$ given by

$$P_i = \sum_{j\in S,\, j\leq i_m} N_{ij}Q_j$$

with some $(P_j)\in\mathcal{D}^{n+1}$. Because of (4.4.1), this turns out to satisfy the relation

$$Q_i = \sum_j M_{ij}P_j,$$

hence $Q.\in\operatorname{Im}\psi^*$.

Step iii) $\operatorname{Im}\varphi^*\subset\operatorname{Ker}\psi^*$:—From the construction,

$$\psi^*\circ\varphi^*(W_k) = \sum_{j\in S,\, j\leq i_m} M_{ij}V_{jk},$$

and since $V_{jk}=W_jW_k^*$, the right hand side vanishes.

Step iv) $\operatorname{Ker}\psi^*\subset\operatorname{Im}\varphi^*$:—Suppose that $P.=(P_i;\ i\in S,\ i\leqq i_m)$ is in $\operatorname{Ker}\psi^*$, i.e.,

$$\sum_{k\in S,\, k\leq i_m} M_{jk}P_k = 0 \qquad \text{for } j\in S,\ j<i_m.$$

Notice here that (4.4.2) implies the relation

$$\sum_{j\in S,\, j<j_m} N_{ij}M_{jk} = \delta_{ik} - U_{ik}.$$

Multiplying the previous relation by N_{ij}, summing over j and using this identity, one readily finds that

$$P_i = \sum_{k\in S,\, k<i_m} U_{ik}P_k = \sum_{l\in S^c,\, l\geq i_m^*} \theta_S(l+1)V_{il}P_{l+1}.$$

This means that $P.\in\operatorname{Im}\varphi^*$. Q.E.D.

REMARK. In a fully analogous way, one can obtain an exact sequence of the form

(5.4) $$0 \longrightarrow \mathcal{M} \xrightarrow{\varphi} \mathcal{D}^{n^*+1} \xrightarrow{\psi} \mathcal{D}^{n^*} \longrightarrow 0.$$

where $\mathcal{D}^{n^*}$ and $\mathcal{D}^{n^*+1}$ are, now, free $\mathcal{D}$-modules of row vectors, and φ and ψ are left $\mathcal{D}$-homomorphisms, n^* being defined in (3.8). Just as the first exact sequence in the proof of Theorem (4.9), this exact sequence also splits and leads, again, to the conclusion that $\mathcal{M}$ is a projective $\mathcal{D}$-module. Since we have the duality theorem below, $\mathcal{M}^*$ also turns out to be a projective $\mathcal{D}$-module.

Our second theorem is concerned with duality.

THEOREM (5.5). *For any S, $\mathcal{M}_S$ and $\mathcal{M}_S^*$ are dual, i.e.,*

$$\mathcal{M}_S^* \cong \mathrm{Hom}_{\mathcal{D}_S}(\mathcal{M}_S, \mathcal{D}_S),$$
$$\mathcal{M}_S \cong \mathrm{Hom}_{\mathcal{D}_S}(\mathcal{M}_S^*, \mathcal{D}_S).$$

PROOF. We only prove the first isomorphism; the second one is proven in a parallel way. The exact sequence of (4.5) gives rise to a long exact sequence of the form

$$\cdots \longleftarrow \mathrm{Ext}^1_{\mathcal{D}}(\mathcal{M}, \mathcal{D}) = 0 \longleftarrow \mathcal{D}^n \xleftarrow{\alpha^*} \mathcal{D}^{n+1} \xleftarrow{\beta^*} \mathrm{Hom}_{\mathcal{D}}(\mathcal{M}, \mathcal{D}) \longleftarrow 0\,.$$

From this and the exact sequence of (5.3), one can obtain the following commutative diagram of homomorphisms of right $\mathcal{D}$-modules, whose rows are both exact.

$$\begin{array}{ccccccccc}
0 & \longleftarrow & \mathcal{D}^n & \xleftarrow{\psi^*} & \mathcal{D}^{n+1} & \xleftarrow{\varphi^*} & \mathcal{M}^* & \longleftarrow 0 & \text{(exact)} \\
 & & \cong\downarrow & & \cong\downarrow & & \rho\downarrow & & \\
0 & \longleftarrow & \mathcal{D}^n & \xleftarrow{\alpha^*} & \mathcal{D}^{n+1} & \xleftarrow{\beta^*} & \mathrm{Hom}_{\mathcal{D}}(\mathcal{M}, \mathcal{D}) & \longleftarrow 0 & \text{(exact)}
\end{array}$$

where ρ is the canonical homomorphism that maps W_j^* to an element of $\mathrm{Hom}_{\mathcal{D}}(\mathcal{M}, \mathcal{D})$ as:

$$\rho(W_j^*): W_i \longmapsto V_{ij}\,.$$

The commutativity is a consequence of the construction and (4.4). The so called "five lemma" now deduces the first isomorphism. Q.E.D.

6. KP hierarchy as differential-algebraic structure on UGM.

The ordinary expression of the KP hierarchy, as mentioned in Section 1, is based upon the $\mathcal{D}$-module structure for $S=S_\emptyset$. We now extend it to other coordinates associated with a general S. (Actually, the following results can further be generalized to a hierarchy associated with an infinite dimensional flag manifold [**5**].)

Let us briefly review the case of $S=S_\emptyset$ within the context of the present paper. The KP hierarchy now means a differential algebra with the same commutative algebra $\mathcal{A}=\mathcal{A}_\emptyset$ but with an addional set of derivations ∂_n $(n=1,2,\cdots)$. This may be called the "KP differential algebra." These derivations, by definition, act on the generators $w_{\emptyset,ij}$ as:

$$\partial_n(w_{\varnothing,ij}) = w_{\varnothing,i+n,j} - w_{\varnothing,i,j-n} - \sum_{k=-n}^{-1} w_{\varnothing,ik} w_{\varnothing,k+n,j}\,. \tag{6.1}$$

It is not hard to see [14] that this is indeed equivalent to the system of equations

$$\partial_n(W) \underset{\text{def}}{=} \sum_{m=1}^{\infty} \partial_n(w_m)\partial^{-m} = B_n W - W\partial^n\,, \qquad n=1, 2, \cdots, \tag{6.2}$$

which is one of various equivalent expressions of the KP hierarchy. It should be, however, noted that (6.2) is now a definition of the derivations ∂_n rather than equations for unknown functions.

The new derivations ∂_n can be extended to all other $\mathcal{A}_S$'s. To see this, it is convenient to start from the matrix form

$$\begin{aligned} \partial_n(\eta_\varnothing) &= \boldsymbol{B}_{\varnothing,n}\eta_\varnothing - \eta_\varnothing \Lambda^n\,, \\ \boldsymbol{B}_{\varnothing,n} &\underset{\text{def}}{=} (\eta_{\varnothing,i,j-n}\,;\, i,\, j>0)\,, \end{aligned} \tag{6.3}$$

where Λ is the matrix defined in (2.1). The "dual" matrix $\eta^*_\varnothing$ gives an equivalent expression of (6.3):

$$\begin{aligned} \partial_n(\eta^*_\varnothing) &= \Lambda^n \eta^*_\varnothing - \eta^*_\varnothing \boldsymbol{A}_{\varnothing,n}\,, \\ \boldsymbol{A}_{\varnothing,n} &\underset{\text{def}}{=} (\eta^*_{\varnothing,i+n,j}\,;\, i,\, j<0)\,. \end{aligned} \tag{6.4}$$

These equations give rise to the equations

$$\begin{aligned} \partial_n(\eta_S) &= \boldsymbol{B}_{S,n}\eta_S - \eta_S \Lambda^n\,, \\ \partial_n(\eta^*_S) &= \Lambda^n \eta^*_S - \eta^*_S \boldsymbol{A}_{S,n}\,, \end{aligned} \tag{6.5}$$

for the transformed matrices

$$\eta_S = h_{S,S_\varnothing}\eta_\varnothing, \qquad \eta^*_S = \eta^*_\varnothing h^*_{S_\varnothing,S}\,. \tag{6.6}$$

One can determine the coefficients $\boldsymbol{A}_{S,n}$ and $\boldsymbol{B}_{S,n}$ just as in the proof of Proposition (2.14):

$$\begin{aligned} \boldsymbol{A}_{S,n} &= (\eta^*_{S,i+n,j}\,;\, i,\, j\in S^c)\,, \\ \boldsymbol{B}_{S,n} &= (\eta_{S,i,j-n}\,;\, i,\, j\in S)\,. \end{aligned} \tag{6.7}$$

Having obtained (6.5), one can change the point of view, and now understand (6.5) as the definition of new derivations ∂_n on $\mathcal{A}_S$. Proposition (2.14) can be now generalized as follows.

PROPOSITION (6.8). *The derivations defined by* (6.5) *are consistent with the gluing homomorphisms* $g_{SS'}$. *More precisely, if* $\partial_{S,n}$ *denotes the derivation on* $\mathcal{A}_S$ *defined by* (6.5), $\partial_{S,n}$ *and* $\partial_{S',n}$ *obey the relation*

$$\partial_{S,n}\circ g_{SS'} = g_{SS'}\circ\partial_{S',n}.$$

This fact again allows a geometric intepretation that ∂_n, like ∂, are globally defined vector fields on UGM. Our UGM thus acquires a new structure of "differential-algebraic variety," which now represents the KP hierarchy itself.

Let us rewrite (6.5) in terms of $\mathcal{M}_S$ and $\mathcal{M}_S^*$. Note that the generators $W_{S,i}$ and $W_{S,j}^*$ of these $\mathcal{D}_S$-modules are linked with η_S and η_S^* by the simple relation

$$W_{S,i} = \sum_j \eta_{S,ij}\partial^j, \qquad W_{S,j}^* = \sum_i \partial^{-i-1}\eta_{S,ij}^*. \tag{6.9}$$

From this fact, one can readily give an equivalent expression of (6.5) in terms of these generators. As in (6.2), we define

$$\partial_n(P) \underset{\text{def}}{=} \sum \partial_n(p_i)\partial^i \qquad \text{for } P=\sum p_i\partial^i \in \mathcal{E}_S.$$

PROPOSITION (6.10). *In terms of $W_{S,i}$ and $W_{S,j}^*$,* (6.5) *can be rewritten*

$$\partial_n(W_{S,i}) = \sum_{j\in S} \boldsymbol{B}_{S,n,ij}W_{S,j} - W_{S,i}\partial^n, \tag{6.10.1}$$

$$\partial_n(W_{S,j}^*) = \partial^n W_{S,j}^* - \sum_{i\in S^c} W_{S,i}^*\boldsymbol{A}_{S,n,ij}, \tag{6.10.2}$$

where $\boldsymbol{A}_{S,n,ij}$ and $\boldsymbol{B}_{S,n,ij}$ stand for the (i, j)-component of $\boldsymbol{A}_{S,n}$ and $\boldsymbol{B}_{S,n}$. The summation on the right hand side actually contains only a finite number of non-vanishing terms.

This result can be restated in a more intrinsic way as:

$$\partial_n(W)+W\partial^n \in \mathcal{M}_S \qquad \text{for } W\in\mathcal{M}_S, \tag{6.11.1}$$

$$\partial_n(W^*)-\partial^n W^* \in \mathcal{M}_S^* \qquad \text{for } W^*\in\mathcal{M}_S^*. \tag{6.11.2}$$

One can derive from (6.10) these relations; conversely, starting from (6.11), one can reproduce (6.10) just as we have derived (6.7) from (6.5) alone. These two expressions are thus entirely equivalent. The latter expression is manifestly invariant under the action

$$W \longmapsto PW, \qquad W^* \longmapsto W^*P$$

of elements P of $\mathcal{D}_S$, hence more suitable from the point of view of $\mathcal{D}$-modules. Selecting a special generator system of $\mathcal{M}_S$ and $\mathcal{M}_S^*$ ammonts to selecting a special matrices like η_S and η_S^* among various "equivalent frames" [8-10].

If $S=S_\emptyset$, (6.11) reproduces Sato's intrinsic formulation of the KP hierarchy [**10**, **11**]. Our $\mathcal{D}_S$-modules thus give a natural extension of Sato's formulation of the KP hierarchy, now taking into account the gluing structure of UGM. Another point to be noted is that we have constructed such a global object without using any actual realization of solutions (like the ring $\boldsymbol{C}[[t, x]]$ of formal power series); we just reinterpret the structure sheaf of UGM as a sheaf of differential rings.

Appendix. Minimal generators of $\mathcal{D}$-modules.

We here present several results on the minimal generators W_{i_k} of $\mathcal{M}_S$. Proofs are omitted because they are parallel to the previous case. The results presented below can be readily extended to $\mathcal{M}_S^*$. We again suppress the suffix "S" in the following.

A counterpart of (4.3) for the minimal generators is given by

$$\sum_{k=0}^{m} \tilde{M}_{lk} W_{i_k} = 0 \qquad (0 \leqq l \leqq m-1), \tag{A.1}$$

where

$$\tilde{M}_{lk} \underset{\text{def}}{=} (\partial^{j_l+1} W^*_{i_{k-1}})_+ . \tag{A.2}$$

An associated free resolution of $\mathcal{M}$ is given by

$$0 \longrightarrow \mathcal{D}^m \xrightarrow{\tilde{\alpha}} \mathcal{D}^{m+1} \xrightarrow{\tilde{\beta}} \mathcal{M} \longrightarrow 0 \tag{A.3}$$

where

$$\tilde{\alpha} : Q. = (Q_l\,;\, 0 \leqq l \leqq m-1) \longmapsto \Big(\sum_{l=0}^{m-1} Q_l \tilde{M}_{lk}\,;\, 0 \leqq k \leqq m\Big), \tag{A.4}$$

$$\tilde{\beta} : P. = (P_k\,;\, 0 \leqq k \leqq m) \longmapsto \sum_{k=0}^{m} P_k W_{i_k} .$$

Let $\tilde{M}$ denote the matrix

$$\tilde{M} \underset{\text{def}}{=} (\tilde{M}_{lk}\,;\ 0 \leqq l \leqq m-1,\ 0 \leqq k \leqq m) . \tag{A.5}$$

Likewise,

$$\tilde{N} \underset{\text{def}}{=} (\tilde{N}_{lk}\,;\ 0 \leqq l \leqq m,\ 0 \leqq k \leqq m-1), \qquad \tilde{N}_{lk} \underset{\text{def}}{=} -(W_{i_l} \partial^{-j_k-1})_+ . \tag{A.6}$$

One then has a variation of Proposition (4.4) as follows.

PROPOSITION (A.7). *$\tilde{M}$ and $\tilde{N}$ obey the relations*

$$\tilde{M}\tilde{N} = X^{-1}, \tag{A.7.1}$$

$$\tilde{N}X\tilde{M} = 1_{m+1} - \tilde{U}, \tag{A.7.2}$$

where

$$X \underset{\text{def}}{=} (X_{lk}\,;\ 0 \leqq l,\ k \leqq m-1), \qquad X_{lk} \underset{\text{def}}{=} \delta_{lk} - \delta_{l,k+1}\partial^{j_l - j_k}$$

$$\tilde{U} \underset{\text{def}}{=} (\tilde{U}_{lk}\,;\ 0 \leqq l,\ k \leqq m), \qquad \tilde{U}_{lk} \underset{\text{def}}{=} W_{i_l} W^*_{i_k - 1}\,.$$

COROLLARY (A.8). *W_{i_k} and $W^*_{i_k-1}$ give a "partition of unity":*

$$\sum_{k=0}^{m} W^*_{i_k-1} W_{i_k} = 1\,.$$

From these results, one can prove that W_{i_k}'s and $W^*_{i_k-1}$'s indeed give minimal generator systems of $\mathcal{M}$ and $\mathcal{M}^*$.

The following is an analogue of (5.2).

(A.9) $$\mathcal{M}^* \cong \mathcal{D}^{m+1}\tilde{U} \subset \mathcal{D}^{m+1}\,,$$

This induces an exact sequence of the form [cf. (5.3)]

(A.10) $$0 \longrightarrow \mathcal{M}^* \xrightarrow{\tilde{\varphi}} \mathcal{D}^{m+1} \xrightarrow{\tilde{\psi}} \mathcal{D}^m \longrightarrow 0\,,$$

which splits:

(A.11) $$\mathcal{D}^{m+1} \cong \mathcal{M}^* \oplus \mathcal{D}^m\,.$$

From Corollary (A.8), one can further derive the following expression of W_i's and W^*_j's. This gives an extension of (3.12.2) to a general S.

PROPOSITION (A.12). *W_i and W^*_j for general $i \in S$ and $j \in S^c$ can be written*

$$W_i = \sum_{k=0}^{m} (\partial^i W^*_{i_k-1})_+ W_{i_k}\,,$$

$$W^*_j = \sum_{k=0}^{m} W^*_{i_k-1} (W_{i_k} \partial^{-j-1})_+\,.$$

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Toshikazu MIYAJIMA
Department of Mathematics
Nagoya University
Chikusa-ku, Nagoya 465
Japan

Atsushi NAKAYASHIKI
Research Institute for Mathematical Sciences
Kyoto University
Sakyo-ku, Kyoto 606
Japan

Kanehisa TAKASAKI
Department of Mathematics
College of General Education
Kyoto University
Sakyo-ku, Kyoto 606
Japan