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On special values of Selberg type zeta functions on $SU(1, q+1)$

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§ 0. Introduction.

There is mystery in the arithmetic nature of the special values of $\zeta(s)=\sum_{n=1}^{\infty}n^{-s}$ at the odd integers greater than one.

It is widely believed that the special values of the Dedekind zeta function $\zeta_K(s)$ of an algebraic number field K at the positive integer m is written in the form

$$\zeta_K(m) = R\cdot P\cdot A$$

where $R=\mathrm{vol}(\Gamma\backslash \boldsymbol{R}^r)$ is the (higher) regulator with $r=\mathrm{ord}_{s=1-m}\zeta_K(s)$ and $\Gamma\subset\boldsymbol{R}^r$ a $\boldsymbol{Z}$-lattice, P is the period and A is an algebraic number called the algebraic part of the special value $\zeta_K(m)$. A typical example is the residue formula at $s=1$, that is, $\zeta_K(s)$ has a simple pole at $s=1$ and

$$\operatorname*{Res}_{s=1}\zeta_K(s) = R(K)\cdot P\cdot A$$

where $R(K)=\mathrm{vol}(U_K\backslash\boldsymbol{R}^{r_1+r_2-1})$ is the usual regulator of K with $r_1+r_2-1=\mathrm{ord}_{s=0}\zeta_K(s)$, $P=2^{r_1}(2\pi)^{r_2}$, and $A=h/(w\sqrt{|D|})$. Here U_K is the unit group of the maximal order of K, r_1 (resp. r_2) is the number of the real (resp. complex) places of K, h is the class number of K, w is the number of the roots of unity contained in K, and D is the absolute discriminant of K.

In this paper, we will show that special values of Selberg zeta functions are also written as a product of "regulator" and "period".

In § 1, we will recall basic facts on the irreducible unitary representations of the special unitary group $SU(1, q+1)$ of signature $(1, q+1)$ $(q>0)$. The unitary dual of a real rank one semi-simple Lie group is determined by [**BSB**]. We will recall a result of Kraljevic [**Kr**] in which we can find a parametrization of the irreducible unitary representations of $SU(1, q+1)$ and the irreducible decomposition of them restricted to a maximal compact subgroup K of $SU(1, q+1)$. We will give a connection between the Harish-Chandra parametrization of square-integrable representations of $SU(1, q+1)$ and the parametrization of Kraljevic.

In § 2, we will prove Paley-Wiener theorem for spherical Fourier transform on $SU(1, q+1)$ in the case of one-dimensional K-type. It is a reformulation of Paley-Wiener theorem in [**Tr**] or in [**Wk1**]. Such a reformulation is possible thanks to the detailed parametrization of Kraljevic [**Kr**].

In § 3, we will define a Selberg type zeta function on $SU(1, q+1)$ with one-dimensional K-type. The proofs may be omitted because they are well-known ([**Wk1, 2**]).

In § 4, we will give, using the trace formula, a formula which expresses the special values of the Selberg type zeta functions as a product of two factors R and P (Theorem 4.4 or Theorem 4.5). Analogous formulae are given by several authors D'Hoker-Phong [**DP**], Fried [**Fr**] or Voros [**Vr**]. The purpose of this paper is to give a new interpretation of the formula.

In § 5, we will treat Dedekind zeta functions. The purpose is to give, using the explicit formula, a formula which expresses the residue of the Dedekind zeta function at $s=1$ as a product of two factors R and P (Theorem 5.4). Comparing with the classical residue formula cited above, it is suggested that the factors R and P correspond to the factors $R(K)h/(w\sqrt{|D|})$ and $2^{r_1}(2\pi)^{r_2}$ respectively. In other words, R is the regulator and P is the period.

Selberg [**Sl**] defined a "Selberg zeta function" in order to solve the following crossword puzzle;

Crossword Puzzle

Dedekind zeta function	?
explicit formula	Selberg's trace formula

There exist a lot of analogy between Dedekind zeta functions and Selberg zeta functions. We can find a clear correspondence between the arguments in § 4 and in § 5. Such a correspondence of arguments suggests that the factors R and P in Theorem 4.4 or Theorem 4.5 play the role of the regulator and the period respectively for special values of Selberg type zeta functions (see § 6 for details).

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NOTATIONS. For any topological space X, we will denote by $C_c(X)$ the $\boldsymbol{C}$-vector space of the $\boldsymbol{C}$-valued continuous functions on X with compact support. For any ring R, we will denote by $M_{m,n}(R)$ the R-module of the matrices with m rows and n columns whose entries are in R. The transposed matrix of $g \in M_{m,n}(R)$ is denoted by ${}^t g \in M_{n,m}(R)$. Put $M_n(R)=M_{n,n}(R)$. We will denote by $\mathrm{diag}(a_1, \cdots, a_n)$ the diagonal matrix with diagonal elements $a_1, \cdots, a_n$. The unit matrix of size n is denoted by I_n. Put $g^*={}^t\bar{g}$ for any $g\in M_{m,n}(\boldsymbol{C})$ where $\bar{g}$ denote the complex conjugation of g.

§ 1. Preliminaries.

1) Special unitary group of signature $(1, q+1)$. Let $G=SU(1, q+1)$ be the special unitary group of signature $(1, q+1)$ with $q>0$. The group G is defined by

$$G=\{g\in SL(q+2, \boldsymbol{C})\,|\,g^*Jg=J\}$$

where $g^*={}^t\bar{g}$ and $J=\begin{pmatrix} 0 & 0 & 1 \\ 0 & -I_q & 0 \\ 1 & 0 & 0 \end{pmatrix}$. The connected semi-simple Lie group G has the Iwasawa decomposition $G=K\cdot A\cdot N$ where

$$K=\{g\in G\,|\,g^*g=1\}=\left\{\begin{pmatrix} a & b & c \\ d & e & -d \\ c & -b & a \end{pmatrix}\in G\right\}$$

is a maximal compact subgroup of G and

$$A=\left\{\begin{pmatrix} a & 0 & 0 \\ 0 & I_q & 0 \\ 0 & 0 & a^{-1} \end{pmatrix}\in G\,|\,0<a\in\boldsymbol{R}\right\}$$

$$N=\left\{\begin{pmatrix} 1 & b & (1/2)bb^*+\sqrt{-1}t \\ 0 & I_q & b^* \\ 0 & 0 & 1 \end{pmatrix}\in G\,|\,b\in\boldsymbol{C}^q,\ t\in\boldsymbol{R}\right\}.$$

Let $\mathfrak{g}=\mathrm{Lie}(G)=\{X\in sl(q+2, \boldsymbol{C})\,|\,X^*J+JX=0\}$ be the Lie algebra of G, and $\mathfrak{g}_C=\mathfrak{g}\otimes_{\boldsymbol{R}}\boldsymbol{C}$ the complexification of $\mathfrak{g}$. Choose a Cartan subalgebra $\mathfrak{B}=\{$diagonal matrix$\in\mathfrak{g}\}$ of $\mathfrak{g}$. The root system of $(\mathfrak{g}_C, \mathfrak{B}_C)$ is $\sum=\{\lambda_i-\lambda_j\,|\,i, j=0, 1, \cdots, q+1, i\neq j\}$. Here we define $\lambda_i\in\mathfrak{B}_C^*$ by $\lambda_i(H)=a_i$ for $H=\mathrm{diag.}(a_0, a_1, \cdots, a_{q+1})\in\mathfrak{B}_C$. The ordering of $\sum$ is defined by the fundamental root system $\{\lambda_j-\lambda_{j+1}\,|\,j=0, 1, \cdots, q\}$. Put $P_+=\{0<\lambda\in\sum\,|\,\lambda(\mathcal{A})\neq 0\}$ where $\mathcal{A}=\mathrm{Lie}(A)\subset\mathfrak{g}$ is the Lie algebra of A. Put $\rho=(1/2)\sum_{0<\lambda\in\Sigma}\delta$. Any element $x\in G$ has the unique expression $x=\kappa(x)\cdot\exp H(x)\cdot n(x)$ with $\kappa(x)\in K$, $H(x)\in\mathcal{A}$, and $n(x)\in N$. Normalize the Haar measures $d_K(k)$ and $d_N(n)$ on K and N so that

$$\int_K d_K(k)=1,\qquad \int_N \exp(-2\rho H(n^*))d_N(n)=1$$

respectively. Define a Haar measure $d_A(a)$ on A by

$$d_A\begin{pmatrix} a & 0 & 0 \\ 0 & I_q & 0 \\ 0 & 0 & a^{-1} \end{pmatrix}=a^{-1}\cdot da.$$

The Haar measure d_G on G is normalized so that $d_G(x)=\exp(2\rho H(x))\cdot d_N(n)\cdot d_A(a)\cdot d_K(k)$ for $x=kan\in G=K\cdot A\cdot N$.

2) **Square-integrable representations of** $SU(1, q+1)$. Take a compact Cartan subalgebra

$$\mathcal{C}=\left\{\begin{pmatrix} a & 0 & c \\ 0 & e & 0 \\ c & 0 & a \end{pmatrix}\in \mathcal{G}\,\middle|\, e=\operatorname{diag}(e_1, \cdots, e_q)\right\}$$

of $\mathcal{G}$. Define a $\boldsymbol{C}$-linear form $\nu_j\in\mathcal{C}_{\boldsymbol{C}}^*$ on $\mathcal{C}_{\boldsymbol{C}}$ by

$$\nu_0\begin{pmatrix} a & 0 & c \\ 0 & e & 0 \\ c & 0 & a \end{pmatrix}=a-c, \qquad \nu_{q+1}\begin{pmatrix} a & 0 & c \\ 0 & e & 0 \\ c & 0 & a \end{pmatrix}=a+c$$

$$\nu_j\begin{pmatrix} a & 0 & c \\ 0 & e & 0 \\ c & 0 & a \end{pmatrix}=e_j \quad (1\leqq j\leqq q,\ e=\operatorname{diag}(e_1, \cdots, e_q)).$$

Then the root system of $(\mathcal{G}_{\boldsymbol{C}}, \mathcal{C}_{\boldsymbol{C}})$ is

$$\Delta=\{\pm(\nu_i-\nu_j)\,|\,0\leqq i<j\leqq q+1\}$$

and

$$\Psi=\{\nu_j-\nu_{j+1}\,|\,0\leqq j\leqq q\}$$

is a fundamental root system of Δ. The set of the compact roots and the set of the non-compact roots are

$$\Delta_K=\{\pm(\nu_i-\nu_j)\,|\,0\leqq i<j\leqq q\}$$
$$\Delta_n=\{\pm(\nu_j-\nu_{q+1})\,|\,0\leqq j\leqq q\}$$

respectively. We have a relation $\nu_0+\nu_1+\cdots+\nu_{q+1}=0$, and $\{\nu_0, \nu_1, \cdots, \nu_q\}$ is a $\boldsymbol{C}$-basis of $\mathcal{C}_{\boldsymbol{C}}^*$. Then the Weyl group W_Δ of Δ is a subgroup of $GL_{\boldsymbol{C}}(\mathcal{C}_{\boldsymbol{C}}^*)$ generated by the permutations of $\{\nu_0, \nu_1, \cdots, \nu_{q+1}\}$. We denote by W_K the subgroup of W_Δ generated by the reflection corresponding to the compact roots. Then W_K is a subgroup of $GL_{\boldsymbol{C}}(\mathcal{C}_{\boldsymbol{C}}^*)$ generated by the permutations of $\{\nu_0, \nu_1, \cdots, \nu_q\}$.

Put $T=\exp\mathcal{C}$ which is a connected closed subgroup of K. The exponential mapping is a surjective group homomorphism from $\mathcal{C}$ to T. Let Λ be the $\boldsymbol{Z}$-lattice in $\sqrt{-1}\mathcal{C}^*\subset\mathcal{C}_{\boldsymbol{C}}^*$ corresponding to the character group of T via the exponential mapping. Then

$$\Lambda=\left\{\sum_{j=0}^{q} m_j\nu_j\,\middle|\, m_j\in\boldsymbol{Z}\right\},$$

and the character ξ_λ of T corresponding to $\lambda\in\Lambda$ is defined by $\xi_\lambda(\exp X)=\exp\lambda(X)$ for all $X\in\mathcal{C}$. Let Λ' be the subset of Λ consisting of the $\lambda\in\Lambda$ such that $w(\lambda)\neq\lambda$ for all non-trivial $w\in W_{\Delta}$. Then

$$\Lambda^{+}=\left\{\sum_{j=0}^{q} m_j\nu_j \mid 0\neq m_j\in \boldsymbol{Z},\ m_0>m_1>\cdots>m_q\right\}$$

is a complete set of the representatives of $W_K\backslash\Lambda'$.

Now we will write down the Harish-Chandra parametrization. We denote by $\hat{G}_d$ the set of unitary equivalent classes of the square-integrable irreducible unitary representations of G. Then there exists a bijection (the Harish-Chandra parametrization) $\lambda\mapsto\pi_\lambda$ from Λ^{+} to $\hat{G}_d$. Here π_λ is a square-integrable irreducible unitary representation of G whose character is

$$\Theta=\varepsilon(\lambda)\Delta_T^{-1}\sum_{w\in W_K}\det(w)\xi_{w(\lambda)}$$

on T' the regular elements of T. Here $\varepsilon(\lambda)=\prod_{\alpha\in\Delta+(\Psi)}(\alpha,\lambda)$ and

$$\Delta_T(t)=\xi_\rho(t)\prod_{\alpha\in\Delta+(\Psi)}(1-\xi_\alpha(t)^{-1})$$

with $\rho=(1/2)\sum_{\alpha\in\Delta+(\Psi)}\alpha=\sum_{j=0}^{q}(q+1-j)\nu_j\in\Lambda$. With the Haar measure on G normalized as above, the formal degree of $\pi_\lambda\in\hat{G}_d$ with $\lambda=\sum_{j=0}^{q}\nu_j\in\Lambda^{+}$ is

$$d_\lambda=\frac{\pi^{q+1}}{2^q q!}\times(2\pi)^{-(q+1)}\prod_{0\leq i<j\leq q}\frac{m_i-m_j}{|i-j|}\times\prod_{j=0}^{q}|m_j|\,. \tag{1.1}$$

By the result of [**HS**], the representation $\pi_\lambda\in\hat{G}_d$ with $\lambda=\sum_{j=0}^{q}m_j\nu_j\in\Lambda^{+}$ is integrable if and only if $|m_j|>q+1$ for all $j=0, 1, \cdots, q$.

The compact group K is isomorphic to the unitary group $U(q+1)$ via

$$\begin{pmatrix} a & b & c\\ d & e & -d\\ c & -b & a\end{pmatrix}\longmapsto\begin{pmatrix} a-c & \sqrt{2}b\\ \sqrt{2}d & e\end{pmatrix}.$$

We have $\det\begin{pmatrix} a-c & \sqrt{2}b\\ \sqrt{2}d & e\end{pmatrix}=(a+c)^{-1}$ for $\begin{pmatrix} a & b & c\\ d & e & -d\\ c & -b & a\end{pmatrix}\in K$. Put

$$M^{+}=\left\{\sum_{j=0}^{q} m_j\nu_j \mid m_j\in\boldsymbol{Z},\ m_0\geqq m_1\geqq\cdots\geqq m_q\right\}.$$

Take a $\mu=\sum_{j=0}^{q}m_j\nu_j\in M^{+}$. Let δ be an irreducible representation of $U(q+1)$ corresponding to the Young diagram

1	2	$\cdots$		$\cdots$		$\cdots$	l_1
1	2	$\cdots$		$\cdots$	l_2		
		$\cdots$		$\cdots$			
1	2	$\cdots$	l_q				

with $l_j=m_{j-1}-m_q$. Let δ_μ be an irreducible representation of K defined by

$$\delta_\mu\begin{pmatrix} a & b & c \\ d & e & -d \\ c & -b & a \end{pmatrix}=(a+c)^{-m_q}\delta\begin{pmatrix} a-c & \sqrt{2}b \\ \sqrt{2}d & e \end{pmatrix}.$$

Then $\delta_\mu\in\hat{K}$ is the irreducible representation of highest weight μ, and $\mu\to\delta_\mu$ is a bijection from M^+ to $\hat{K}$.

Now we will recall the K-type theorem for square-integrable representations of G. Take a $\lambda=\sum_{j=0}^q m_j\nu_j\in\Lambda^+$. Put

$$\Delta^+(\lambda)=\{\alpha\in\Delta\,|\,(\alpha,\lambda)>0\}$$

and

$$\rho_c=\frac{1}{2}\sum_{\alpha\in\Delta^+(\lambda)\cap\Delta_K}\alpha,\quad \rho_n=\frac{1}{2}\sum_{\alpha\in\Delta^+(\lambda)\cap\Delta_n}\alpha.$$

Then $\mu_0=\lambda-\rho_c+\rho_n\in M^+$, and the K-type $\delta_{\mu_0}\in\hat{K}$ has the following properties;

1) the multiplicity of δ_{μ_0} in $\pi_\lambda|_K$ is equal to one,

2) $\pi_\lambda|_K$ does not contain any $\delta_\mu\in\hat{K}$ of $\mu=\mu_0-A$ with a non-empty sum A of roots in $\Delta^+(\lambda)\cap\Delta_n$.

On the other hand, we have the following proposition;

PROPOSITION 1.1. *Up to infinitesimal equivalences, π_λ is the unique representation of G whose restriction to K contains $\delta_{\mu_0}\in\hat{K}$, and does not contain any $\delta_\mu\in\hat{K}$ of $\mu=\mu_0-A$ with a non-empty sum A of roots in $\Delta^+(\lambda)\cap\Delta_n$.*

3) **Unitary dual of** $SU(1, q+1)$. We will recall a result of Kraljevic [**Kr**].

For complex numbers α and β, we put

$$\alpha\geqq\beta \text{ if and only if } 0\leqq\alpha-\beta\in\boldsymbol{Z}$$
$$\alpha\succ\beta \text{ if and only if } 0<\alpha-\beta\in\boldsymbol{Z}.$$

Let $\boldsymbol{C}^n_>$ be the set of the n-tuples $(l_1, l_2, \cdots, l_n)\in\boldsymbol{C}^n$ of complex numbers such that $l_1\geqq l_2\geqq\cdots\geqq l_n$. For any integer $0\leqq t\leqq q+1$, let $\boldsymbol{C}^{q+1}_{t,>}$ be the set of $(r_1, \cdots, r_{q+1})\in\boldsymbol{C}^{q+1}$ such that

$$r_1\geqq r_2\geqq\cdots\geqq r_t\succ r_{t+1}\geqq\cdots\geqq r_{q+1}.$$

In particular, $\boldsymbol{C}^{q+1}_{0,>}=\boldsymbol{C}^{q+1}_{q+1,>}=\boldsymbol{C}^{q+1}_>$. For any integers $0\leqq u<v\leqq q+1$, let $\boldsymbol{R}^{q+2}_{u,v,>}$ be the set of $(s_1, \cdots, s_{q+2})\in\boldsymbol{R}^{q+2}$ such that

$$s_1\geqq\cdots\geqq s_u\succ s_{u+1}\geqq\cdots\geqq s_{v+1}\succ s_{v+2}\geqq\cdots\geqq s_{q+2}$$
$$s_1+\cdots+s_{q+2}=u+v-(q+1).$$

For any $p\in\boldsymbol{C}^q_>$, let $\Gamma(p)$ be the set of $l\in\boldsymbol{C}^{q+1}_>$ such that

$$l_1\geqq p_1\geqq l_2\geqq p_2\geqq\cdots\geqq l_q\geqq p_q\geqq l_{q+1}.$$

For any $r\in C_{t,>}^{q+1}$, let $\Gamma_{t}(r)$ be the set of $l\in C_{>}^{q+1}$ such that

$$l_{1}\geqq r_{1}\geqq l_{2}\geqq r_{2}\geqq\cdots\geqq l_{t}\geqq r_{t}$$
$$r_{t+1}\geqq l_{t+1}\geqq r_{t+2}\geqq\cdots\geqq l_{q}\geqq r_{q+1}\geqq l_{q+1}.$$

In particular

$$\Gamma_{0}(r)=\{l\in C_{>}^{q+1}|r_{1}\geqq l_{1}\geqq r_{2}\geqq l_{2}\geqq\cdots\geqq r_{q+1}\geqq l_{q+1}\}$$

for $r\in C_{0,>}^{q+1}$ and

$$\Gamma_{q+1}(r)=\{l\in C_{>}^{q+1}|l_{1}\geqq r_{1}\geqq l_{2}\geqq r_{2}\geqq\cdots\geqq l_{q+1}\geqq r_{q+1}\}$$

for $r\in C_{q+1,>}^{q+1}$. For any $s\in R_{u,v,>}^{q+2}$, let $\Gamma_{u,v}(s)$ be the set of $l\in C_{>}^{q+1}$ such that

$$l_{1}\geqq s_{1}\geqq l_{2}\geqq s_{2}\geqq\cdots\geqq l_{u}\geqq s_{u}$$
$$s_{u+1}\geqq l_{u+1}\geqq s_{u+2}\geqq\cdots\geqq l_{v}\geqq s_{v+1}$$
$$s_{v+2}\geqq l_{v+1}\geqq s_{v+3}\geqq\cdots\geqq l_{q}\geqq s_{q+2}\geqq l_{q+1}.$$

That is $\Gamma_{u,v}(s)=\Gamma_{u}(r)\cap\Gamma_{v}(r')$ with

$$r=(s_{1},\cdots,s_{v},s_{v+2},\cdots,s_{q+2})\in C_{u,>}^{q+1}$$
$$r'=(s_{1},\cdots,s_{u},s_{u+2},\cdots,s_{q+2})\in C_{v,>}^{q+1}.$$

The elements of $((q+2)^{-1}Z)_{>}^{q+1}$ correspond bijectively to the elements of M^{+} by the mapping

$$l=(l_{1},\cdots,l_{q+1})\longmapsto\mu=\sum_{j=0}^{q}(l_{j}+\langle l\rangle)\nu_{j}$$

where $\langle l\rangle=l_{1}+\cdots+l_{q+1}$. For any $l=(l_{1},\cdots,l_{q+1})\in((q+2)^{-1}Z)_{>}^{q+1}$, put $\delta_{l}=\delta_{\mu}\in\hat{K}$ with $\mu=\sum_{j=0}^{q}(l_{j}+\langle l\rangle)\nu_{j}\in M^{+}$.

For an irreducible unitary representation π of G, let $\Gamma(\pi)$ be the set of $l\in((q+2)^{-1}Z)_{>}^{q+1}$ such that $\delta_{l}\in\hat{K}$ is contained in $\pi|_{K}$. For all $l\in\Gamma(\pi)$, the multiplicity of $\delta_{l}\in\hat{K}$ in $\pi|_{K}$ is equal to one. We will denote by χ_{π} the infinitesimal character of π. Then we have a proposition [**Kr**, Theorem 9.2]

PROPOSITION 1.2. *Irreducible unitary representations π_{1} and π_{2} of G is infinitesimally equivalent if and only if*

$$\Gamma(\pi_{1})=\Gamma(\pi_{2})\quad\text{and}\quad\chi_{\pi_{1}}(\Omega)=\chi_{\pi_{2}}(\Omega)$$

where Ω is the Casimir operator on G.

Let U_{1} be the set of pairs $(p, \pm\lambda)$ with $p\in((q+2)^{-1}Z)_{>}^{q}$ and $\lambda\in C$ such that $0<\lambda_{p}^{2}-\lambda^{2}\in R$. Here

$$\lambda_p = \mathrm{Min}\{0 \leqq r \in \boldsymbol{R} \mid \{\pm r\} \cap K(p) \neq \emptyset\}$$
$$K(p) = \bigcap_{j=0}^{q} \{s_j(p) + 2t \mid 0 \neq t \in \boldsymbol{Z}\}$$
$$s_j(p) = \langle p \rangle + q + 1 + 2(p_j - j).$$

Let U_2 be the set of pairs (t, r) of an integer $0 \leqq t \leqq q+1$ and $r \in ((q+2)^{-1}\boldsymbol{Z})^{q+1}_{t,>}$ such that one of the following three conditions

i) $r_j + \langle r \rangle = t + j - (q+1)$ for some $1 \leqq j \leqq t$,

ii) $r_j + \langle r \rangle = t + j - (q+1) - 1$ for some $t < j \leqq q+1$,

iii) $r_t + \langle r \rangle > 2t - (q+1) > r_{t+1} + \langle r \rangle$,

and a condition

$$r_t + \langle r \rangle - j_1(r) \geqq t - (q+1) > r_{t+1} + \langle r \rangle - j_2(r)$$

are fulfilled. Here $r_0 = \infty$, $r_{q+2} = -\infty$ and

$$j_1(r) = \mathrm{Max}\{1 < j \leqq t \mid r_{j-1} > r_j\}$$
$$j_2(r) = \mathrm{Min}\{t < j \leqq q+1 \mid r_j > r_{j+1}\}.$$

Let U_3 be the set of triple (u, v, s) of integers $0 \leqq u < v \leqq q+1$ and $s \in ((q+2)^{-1}\boldsymbol{Z})^{q+2}_{u,v,>}$ such that $s_{u+1} = \cdots = s_{v+1}$.

For any $\tau = (p, \pm\lambda) \in U_1$, there exists a $\pi^\tau \in \hat{G}$ such that

$$\Gamma(\pi^\tau) = \Gamma(p)$$
$$\chi_{\pi^\tau}(\Omega) = \frac{1}{4(q+2)}\left\{\lambda^2 - (q+1)^2 + 2\Big(\sum_{j=1}^{q} p_j^2 + \sum_{1 \leqq i \leqq j \leqq q} p_i p_j + \Sigma j = 1^q (q+1-2j) p_j\Big)\right\}.$$

For any $\tau = (t, r) \in U_2$, there exists a $\pi^\tau \in \hat{G}$ such that

$$\Gamma(\pi^\tau) = \Gamma_t(r)$$
$$\chi_{\pi^\tau}(\Omega) = \frac{1}{q+2}\Big\{\sum_{1 \leqq i \leqq j \leqq q+1} r_i r_j + \sum_{1 \leqq j \leqq t} (q+1-t-j) r_j$$
$$+ \sum_{t < j \leqq q+1} (q+2-t-j) r_j - t(q+1-t)\Big\}.$$

For any $\tau = (u, v, s) \in U_3$, there exists a $\pi^\tau \in \hat{G}$ such that

$$\Gamma(\pi^\tau) = \Gamma_{u,v}(s)$$
$$\chi_{\pi^\tau}(\Omega) = \frac{1}{4(q+2)}\Big\{2\Big(\sum_{j=1}^{q+2} s_j^2 + \sum_{1 \leqq i \leqq j \leqq q+2} s_i s_j\Big)$$
$$+4\Big(\sum_{1 \leqq j \leqq u} j s_j + \sum_{u < j \leqq v+1} (j-1) s_j + \sum_{v+1 < j \leqq q+2} (j-2) s_j\Big)$$
$$+(q+1-(u+v))^2 - 4uv\Big\}.$$

By Proposition 1.2, $\pi^{\tau}\in\hat{G}$ is uniquely determined for all $\tau\in U=U_1\cup U_2\cup U_3$. Then we have [**Kr**, Th. 10.3, Prop. 11.4]

PROPOSITION 1.3. *The correspondence* $\tau\mapsto\pi^{\tau}$ *gives a bijection from* $U=U_1\cup U_2\cup U_3$ *to* $\hat{G}$.

Let U_d be the subset of U_2 consisting of $(t, r)\in U_2$ such that

$$\begin{cases} r_j+\langle r\rangle > j+t-(q+1) & \text{for } 1\leqq j\leqq t \\ r_j+\langle r\rangle < j+t-(q+2) & \text{for } t<j\leqq q+1. \end{cases}$$

Then we have

PROPOSITION 1.4. *The correspondence* $\tau\mapsto\pi^{\tau}$ *gives a bijection from* U_d *to* $\hat{G}_d$. $\tau=(t, r)\in U_d$ *corresponds to* $\pi^{\tau}\in\hat{G}_d$ *of Harish-Chandra parameter*

$$\sum_{0\leq j<t}(r_{j+1}+\langle r\rangle+q-t-j)\nu_j+\sum_{t\leq j\leq q}(r_{j+1}+\langle r\rangle+q+1-t-j)\nu_j\in\Lambda^+$$

PROOF. Take a $\lambda=\sum_{j=0}^{q}m_j\nu_j\in\Lambda^+$ such that $m_{t-1}>0>m_t(0\leqq t\leqq q+1)$. Then

$$\Delta^+(\lambda)\cap\Delta_K=\{\nu_i-\nu_j\,|\,0\leqq i<j\leqq q\}$$
$$\Delta^+(\lambda)\cap\Delta_n=\{\nu_j-\nu_{q+1}\,|\,0\leqq j<t\}\cup\{\nu_{q+1}-\nu_j\,|\,t\leqq j\leqq q\},$$

and

$$\rho_c=\sum_{j=0}^{q}\left(\frac{q}{2}-j\right)\nu_j,\qquad \rho_n=\left(t-\frac{q}{2}\right)\sum_{j=0}^{t-1}\nu_j+\left(t-\frac{q}{2}-1\right)\sum_{j=t}^{q}\nu_j.$$

On the other hand, for any $\mu=\sum_{j=0}^{q}(l_{j+1}+\langle l\rangle)\nu_j\in M^+$ with $l\in((q+2)^{-1}\boldsymbol{Z})_{>}^{q+1}$, we have

$$\mu-(\nu_j-\nu_{q+1})=\sum_{j=0}^{q}(l'_{j+1}+\langle l'\rangle)\nu_j$$
$$\mu-(\nu_{q+1}-\nu_j)=\sum_{j=0}^{q}(l''_{j+1}+\langle l''\rangle)\nu_j$$

where $l'=l-\varepsilon_{j+1}$ and $l''=l+\varepsilon_{j+1}$. Here $\varepsilon_{j+1}=(0,\cdots,0,1,0,\cdots,0)$ is a row vector with only one 1 appears at $j+1$-th component. Then Proposition 1.1 shows that $\pi_{\lambda}\in\hat{G}_d$ has the K-types $\Gamma(\pi_{\lambda})=\Gamma_t(r)$ with $r\in((q+2)^{-1}\boldsymbol{Z})_{t,>}^{q+1}$ such that

$$\lambda-\rho_c+\rho_n=\sum_{j=0}^{q}(r_{j+1}+\langle r\rangle)\nu_j.$$

Then the proof is complete. ■

§2. Paley-Wiener theorem.

Take an integer $r\in\boldsymbol{Z}$ and fix a K-type δ_r defined by

$$\delta_r\begin{pmatrix} a & b & c \\ d & e & -d \\ c & -b & a \end{pmatrix}=(a+c)^{-r}.$$

Using the notations of §1, we have $\delta_r=\delta_\mu=\delta_l$ with

$$\mu=\sum_{j=0}^{q} r\nu_j\in M^+,\quad\text{and}\quad l=\left(\frac{r}{q+2},\cdots,\frac{r}{q+2}\right)\in\left(\frac{1}{q+2}\boldsymbol{Z}\right)_>^{q+1}.$$

We denote by $\hat{G}(\delta_r)$ the subset of $\hat{G}$ consisting of $\pi\in\hat{G}$ such that the K-type δ_r is contained in $\pi|_K$. Put

$$U_1(l)=\{(p,\pm\lambda)\in U_1 \mid l\in\Gamma(p)\}$$
$$U_2(l)=\{(t,s)\in U_2 \mid l\in\Gamma_t(s)\}$$
$$U_3(l)=\{(u,v,s)\in U_3 \mid l\in U_{u,v}(s)\}$$

and $U(l)=U_1(l)\cup U_2(l)\cup U_3(l)$. Then, by Proposition 1.3, the set $\hat{G}(\delta_r)$ is paramentrized by $U(l)$. We have

PROPOSITION 2.1.

1) *$U_1(l)$ consists of the pairs $(p,\pm\lambda)$ such that*

$$p=\left(\frac{r}{q+2},\cdots,\frac{r}{q+2}\right)\in\left(\frac{1}{q+2}\boldsymbol{Z}\right)_>^{q},\quad\text{and}\quad 0<\lambda_p^2-\lambda^2\in\boldsymbol{R}$$

where

$$\lambda_p=\begin{cases} q+1-|r| & \text{if } |r|<q+1 \\ 0 & \text{if } |r|\geqq q+1,\ r\equiv q+1(2) \\ 1 & \text{if } |r|\geqq q+1,\ r\not\equiv q+1(2), \end{cases}$$

2) *$U_2(l)$ consists of the pairs (t,s) such that*

i) *if $r>0$, then $t=q+1$ and*

$$s_1=\cdots=s_q=\frac{r}{q+2},\quad s_{q+1}-\frac{r}{q+2}\in\boldsymbol{Z}$$

$$\frac{r}{q+2}\geqq s_{q+1}\geqq \mathrm{Min}\left\{\frac{r}{q+2},-\frac{1}{2}\left(\frac{qr}{q+2}-(q+1)\right)\right\},$$

ii) *if $r<0$, then $t=0$ and*

$$s_2=\cdots=s_{q+1}=\frac{r}{q+2},\quad s_1-\frac{r}{q+2}\in\boldsymbol{Z}$$

$$\frac{r}{q+2}\leqq s_1\leqq \mathrm{Max}\left\{\frac{r}{q+2},-\frac{1}{2}\left(\frac{qr}{q+2}+q+1\right)\right\},$$

iii) *if $r=0$, then $U_2(l)=\emptyset$.*

3)

$$U_3(l)=\begin{cases} \{(0,q+1,s) \mid s=(0,\cdots,0)\} & \text{if } r=0 \\ \emptyset & \text{if } r\neq 0. \end{cases}$$

In the case of $r=0$, *the trivial one-dimensional representation of* G *corresponds to the unique element of* $U_3(l)$.

PROOF.

1) If $(p, \pm\lambda)\in U_1(l)$, we have

$$p=\left(\frac{r}{q+2}, \cdots, \frac{r}{q+2}\right)$$

$$K(p)=\{n\in \boldsymbol{Z} \mid n\equiv r+q+1(2),\ |n-r|\geqq q+1\}.$$

Then

$$\lambda_p=\begin{cases} q+1-|r| & \text{if } |r|<q+1 \\ 0 & \text{if } |r|\geqq q+1,\ r\equiv q+1(2) \\ 1 & \text{if } |r|\geqq q+1,\ r\not\equiv q+1(2). \end{cases}$$

2) If $(t, s)\in U_2(l)$, we have two cases

i) $t=q+1,\ s_1=\cdots=s_q=\dfrac{r}{q+2},\ s_{q+1}\leqq\dfrac{r}{q+2}$

ii) $t=0,\ s_2=\cdots=s_{q+1}=\dfrac{r}{q+2},\ s_1\geqq\dfrac{r}{q+2}$.

In case i), we have

$$s_{q+1}\geqq-\frac{1}{2}\left(\frac{qr}{q+2}-(q+1)\right) \quad \text{or} \quad s_{q+1}=j-\frac{q+1}{q+2}r \quad \text{for some } j=1, 2, \cdots, q$$

and

$$s_{q+1}=\frac{r}{q+2}>0 \quad \text{or} \quad -\frac{1}{2}\left(\frac{qr}{q+2}-(q+1)\right)\leqq s_{q+1}<\frac{r}{q+2}.$$

These conditions are equivalent to

$$\frac{r}{q+2}\geqq s_{q+1}\geqq \operatorname{Min}\left\{\frac{r}{q+2}, -\frac{1}{2}\left(\frac{qr}{q+2}-(q+2)\right)\right\}, \quad r>0.$$

The case ii) is treated similarly.

3) Obvious. ■

Any spherical function Ψ on G of K-type δ_r and $\Psi(1)=1$ is written in the form

$$\Psi_{\lambda, r}(x)=\int_K \delta_r(k)\overline{\delta_r(\kappa(x^{-1}k))}\exp(-(\lambda+\rho)H(x^{-1}k))d_K(k) \tag{2.1}$$

with $\lambda\in\mathcal{A}_C^*$ where $\rho=(1/2)\sum_{0<\lambda\in\Sigma}\lambda=\sum_{j=0}^{q}(q+1-j)\lambda_j$ ([**Wr**, vol. 2, p. 42]). We have $\Psi_{\lambda, r}=\Psi_{\lambda', r}$ if and only if $\lambda=\pm\lambda'$. Put $H_0=\mathrm{diag}(1, 0, \cdots, 0, 1)$ which is a base of $\mathcal{A}$. The dual space $\mathcal{A}_C^*$ is identified with $\boldsymbol{C}$ via $\lambda\mapsto\lambda(H_0)$. Then the spherical function $\Psi_{\lambda, r}$ is parametrized by $\lambda\in\boldsymbol{C}$ modulo the multiplication by $\{\pm 1\}$. Using the notations of [**Kr**], the spherical function $\Psi_{\lambda, r}$ is the spherical function of $\pi_{p,\lambda}$ with $p=(r/(q+2), \cdots, r/(q+2))$.

Take a $\pi\in\hat{G}(\delta_r)$ with a representation space H. The multiplicity of δ_r in $\pi|_K$ is equal to one, and there exists a unit vector $u\in H$ such that $\pi(k)u=\delta_r(k)u$ for all $k\in K$. Put $\Psi_{\pi,r}(x)=(\pi(x)u, u)$ for $x\in G$. Then $\Psi_{\pi,r}$ is a spherical function on G of K-type δ_r and $\Psi_{\pi,r}(1)=1$. Two representations π and $\pi'\in\hat{G}(\delta_r)$ are unitarily equivalent if and only if $\Psi_{\pi,r}=\Psi_{\pi',r}$[**Gd**]. For any $\pi\in\hat{G}(\delta_r)$, there exists a $\lambda\in\mathcal{A}_C^*=\boldsymbol{C}$ such that $\Psi_{\pi,r}=\Psi_{\lambda,r}$. Then the set $\hat{G}(\delta_r)$, or $U(l)=U_1(l)\cup U_2(l)\cup U_3(l)$, is identified with a subset $V(r)=V_1(r)\cup V_2(r)\cup V_3(r)$ of $\boldsymbol{C}$ modulo $\{\pm 1\}$ where $U_j(l)$ corresponds to $V_j(r)$.

PROPOSITION 2.2.

1)

$$V_1(r)=\{\lambda\in\boldsymbol{C}\,|\,0<\lambda_p^2-\lambda^2\in\boldsymbol{R},\ -\pi<\arg(\lambda)\leqq\pi\}$$

where

$$\lambda_r=\begin{cases} q+1-|r| & \text{if } |r|<q+1 \\ 0 & \text{if } |r|\geqq q+1,\ r\equiv q+1(2) \\ 1 & \text{if } |r|\geqq q+1,\ r\not\equiv q+1(2), \end{cases}$$

2)

$$V_2(r)=\begin{cases} \emptyset & \text{if } r=0 \\ \left\{m\in\boldsymbol{Z}\,\middle|\,\begin{matrix} m\equiv r+q+1(2) \\ |r|-(q+1)\geqq m\geqq \mathrm{Min}\{|r|-(q+1),\ 0\}\end{matrix}\right\} & \text{if } r\neq 0. \end{cases}$$

3)

$$V_3(r)=\begin{cases} \{q+1\} & \text{if } r=0 \\ \emptyset & \text{if } r\neq 0. \end{cases}$$

PROOF. We will use the notations and the terminologies of [**Kr**].

1) If $\tau=(p, \pm\lambda)\in U_1(l)$, then $\Gamma(\pi^\tau)=\Gamma(p)$ is λ-connected and so $\pi^\tau=\pi^{p,\lambda}$ infinitesimally. Then $\Psi_{\pi^\tau,r}=\Psi_{\lambda,r}$.

2) If $\tau=(0, s)\in U_2(l)$, then $\Gamma(\pi^\tau)=\Gamma_0(s)=\Gamma_-^\lambda(p)$ is a λ-connected component of $\Gamma(p)$ where

$$p=\left(\frac{r}{q+2}, \cdots, \frac{r}{q+2}\right)\in\left(\frac{1}{q+2}\boldsymbol{Z}\right)_>^q,$$

$$\lambda=2s_1+\frac{qr}{q+2}+q+1.$$

In this case, we have $r<0$,

$$r+q+1\leqq\lambda\leqq\mathrm{Max}\{r+q+1,\ 0\},$$

and $s_1-r/(q+2)\in\boldsymbol{Z}$ if and only if $\lambda\in\boldsymbol{Z}$ such that $\lambda\equiv r+q+1(2)$. If $\tau=(q+1, s)\in U_2(l)$, then $\Gamma(\pi^\tau)=\Gamma_{q+1}(s)=\Gamma_+^\lambda(p)$ is a λ-connected component of $\Gamma(p)$ where

$$p=\Big(\frac{r}{q+2}, \cdots, \frac{r}{q+2}\Big)\in\Big(\frac{1}{q+2}\boldsymbol{Z}\Big)_{>}^{q},$$

$$\lambda=2s_{q+1}+\frac{qr}{q+2}-(q+1).$$

In this case, we have $r>0$,

$$r-(q+1)\geqq\lambda\geqq\mathrm{Min}\{r-(q+1),\ 0\}$$

and $s_{q+1}-r/(q+2)\in\boldsymbol{Z}$ if and only if $\lambda\in\boldsymbol{Z}$ such that $\lambda\equiv r+q+1(2)$. In any cases, π^{τ} is a subquotient of $\pi^{p,\lambda}$ and we have $\Psi_{\pi^{\tau},r}=\Psi_{\lambda,r}$.

3) $U_3(l)$ is non-empty only if $r=0$. In this case, the unique element of $U_3(l)$ corresponds to the trivial one-dimensional representation of G. On the other hand, we have $\Psi_{q+1,0}=1$. ■

The correspondence from $V(r)$ to $U(l)$ is given by

1) on $V_1(r)$

$$\lambda\longmapsto\Big(\Big(\frac{r}{q+2}, \cdots, \frac{r}{q+2}\Big),\ \pm\lambda\Big),$$

2) on $V_2(r)$

$$m\longmapsto\Big(q+1,\ \Big(\frac{r}{q+2}, \cdots, \frac{r}{q+2}, \frac{1}{2}\Big(m-\frac{qr}{q+2}+q+1\Big)\Big)\Big) \quad \text{if } r>0$$

$$m\longmapsto\Big(0,\ \Big(-\frac{1}{2}\Big(m+\frac{qr}{q+2}+q+1\Big), \frac{r}{q+2}, \cdots, \frac{r}{q+2}\Big)\Big) \quad \text{if } r<0.$$

Thus we have identified the set $\hat{G}(\delta_r)$ with a subset $V(r)$ of $\boldsymbol{C}$. Let χ_{π} be the infinitesimal character of $\pi\in\hat{G}(\delta_r)\subset\boldsymbol{C}$. Then we have

$$\chi_{\pi}(\Omega)=\frac{1}{4(q+2)}\Big\{\pi^2+\frac{q}{q+2}r^2-(q+1)^2\Big\} \tag{2.2}$$

where Ω is the Casimir operator on G.

Now $\hat{G}_d(\delta_r)=\hat{G}_d\cap\hat{G}(\delta_r)$ is identified with a subset $V_d(r)$ of $V(r)\subset\boldsymbol{C}$. Proposition 1.4 gives the following.

PROPOSITION 2.3.

$$V_d(r)=\begin{cases}\varnothing & \textit{if } |r|\leqq q+1\\ \{m\in\boldsymbol{Z}\,|\,m\equiv r+q+1(2),\ |r|-(q+1)\geqq m>0\} & \textit{if } |r|>q+1.\end{cases}$$

If $|r|>q+1$, *then* $m\in V_d(r)$ *corresponds to* $\pi_m\in\hat{G}_d$ *of Harish-Chandra parameter*

$$\sum_{0\leq j<q}\Big\{\frac{1}{2}(m+r+q+1)-(j+1)\Big\}\nu_j+m\nu_q\in\Lambda^+ \quad \textit{if } r>0$$

$$-m\nu_0+\sum_{0<j\leq q}\Big\{\frac{1}{2}(-m+r+q+1)-j\Big\}\nu_j\in\Lambda^+ \quad \textit{if } r<0.$$

$\pi_m \in \hat{G}_d$ *is integrable if and only if* $|m| > q+1$.

Now we will prove Paley-Wiener theorem for the spherical Fourier transform with respect to $\Psi_{\lambda, r}$. It is a reformulation of a result of [**Wk1**, p. 603].

Let $\theta X = -X^*$ be the Cartan involution on $\mathcal{G}$ associated with (G, K). The real Lie algebra $\mathcal{G}$ has the structure of Eucledian space with the norm $|X|_\theta^2 = -B_{\mathcal{G}}(X, \theta X)$ where $B_{\mathcal{G}}(X, Y) = 2(q+1)\mathrm{tr}(XY)$ is the Killing form of $\mathcal{G}$. Put $\mathcal{P} = \{X \in \mathcal{G} \mid \theta X = -X\}$. Then any $x \in G$ has the unique decomposition $x = k \cdot \exp X(x)$ with $k \in K$ and $X(x) \in \mathcal{P}$, and we put $\sigma(x) = |X(x)|_\theta$. We denote by $\mathcal{C}^1(G)$ the complex vector space consisting of the complex valued C^∞-functions f on G such that

$$\nu_{D,t}(f) = \sup_{x \in G}(1+\sigma(x))^t \omega_0(x)^{-2} |(D \cdot f)(x)| < \infty$$

for all $0 \leqq t \in \boldsymbol{R}$ and all left G-invariant differential operator D on G. Here we put

$$\omega_0(x) = \int_K \exp(-2\rho H(xk)) d_K(k). \tag{2.3}$$

Then $\mathcal{C}^1(G)$ is a subspace of $L^1(G)$ and it is a Frechet space with respect to the system of semi-norm $\{\nu_{D,t} \mid 0 \leqq t \in \boldsymbol{R}, D\}$. We denote by $\mathcal{C}^1(G, \delta_r)$ the complex vector space consisting to the functions $f \in \mathcal{C}^1(G)$ such that $f(xk) = f(kx) = \delta_r(k) f(x)$ for all $k \in K$.

Put $V_{q+1} = \{\lambda \in \boldsymbol{C} \mid |\mathrm{Re}(\lambda)| \leqq q+1\}$. Then, by the identification $\hat{G}(\delta_r) = V_1(r) \cup V_2(r) \cup V_3(r)$ given above, we have $V_{q+1} \cap \hat{G}_d(\delta_r) = \{\pi \in \hat{G}_d(\delta_r) \text{ not integrable}\}$.

Let $\mathcal{F}(r)$ be the complex vector space consisting of the complex valued functions ϕ on $V_{q+1} \cup \hat{G}_d(\delta_r)$ which satisfies the following conditions;

1) ϕ is holomorphic on $|\mathrm{Re}(\lambda)| < q+1$,
2) $\sup_{|\mathrm{Re}(\lambda)| < q+1} (1+|\lambda|)^t |\phi^{(n)}(\lambda)| < \infty$ for all $0 \leqq t \in \boldsymbol{R}$ and $0 \leqq n \in \boldsymbol{Z}$,
3) $\phi(-\lambda) = \phi(\lambda)$ for all $\lambda \in V_{q+1}$.

For any $f \in \mathcal{C}^1(G, \delta_r)$, the spherical Fourier transform $\hat{f}$ of f is defined by

$$\hat{f}(\lambda) = \int_G f(x) \cdot \Psi_{\lambda, r}(x) d_G(x) \tag{2.4}$$

with the identification $\mathcal{A}_C^* = \boldsymbol{C}$ via the mapping $\lambda \mapsto \lambda(\mathrm{diag.}(1, 0, \cdots, 0, -1))$.

Then the Paley-Wiener theorem for the spherical Fourier transform is

THEOREM 2.4. *The mapping* $f \mapsto \hat{f}$ *is a bijection from* $\mathcal{C}^1(G, \delta_r)$ *to* $\mathcal{F}(r)$.

PROOF. We will use the notations of [**Wr**, p. 603], but n and q in his paper must be $q+1$ and r respectively in our situation. Put $\tau = \tau_r$. The set $\sqrt{-1} \cdot V^\tau$ coincides with $\hat{G}_d(\tau)$ in our notation, if $\hat{G}_d(\tau)$ is identified with a subset of $\boldsymbol{C}$ as above. Exactly speaking, these two sets coincide modulo the multiplication by $\{\pm 1\}$, but the corresponding representations of G is the same.

Then the set V_0^τ corresponds to the set $\{\pi \in \hat{G}_d(\tau): \text{not integrable}\} = V_{q+1} \cap \hat{G}_d(\tau)$ in our notation. If $|r| > q+1$, the linear relation

$$F_f(\lambda) = \sum_{j \in \sqrt{-1} \cdot V_0^\tau} \Theta_{\omega(\lambda)}(\alpha_j) \cdot F_A(\tau^M, \sqrt{-1} \cdot j)$$

is trivial because $\Theta_{\omega(\lambda)}(\alpha_j) = 0$ for all $j \in \sqrt{-1} \cdot V_0^\tau$ if $\omega(\lambda) \notin \hat{G}_d(\tau)$ and

$$\Theta_{\omega(\lambda)}(\alpha_j) = \int_G \alpha_j(x^{-1}) \cdot \phi^\tau(\sqrt{-1} \cdot k, x) dx = \delta_{jk}$$

by the definition of α_j if $\omega(\lambda) \in \hat{G}_d(\tau)$ corresponds to $k \in V_0^\tau$. Then the Proposition 3.1 of [**Wr**] is simplified to our theorem. ∎

§ 3. Selberg type zeta function.

The Selberg type zeta function with non-trivial K-type is considered by [**Sc**] and by [**Wk1, 2**]. In this section, we will recall and simplify results of [**Wk1**].

Let Γ be a discrete torsion-free subgroup of G such that $\Gamma \backslash G$ is compact. Let (χ, V) be a finite dimensional unitary representation of Γ. Take an integer $r \in \boldsymbol{Z}$ and fix a K-type δ_r defined by $\delta_r \begin{pmatrix} a & b & c \\ d & e & -d \\ c & -b & a \end{pmatrix} = (a+c)^{-r}$.

Take a non-compact Cartan subgroup $J_h = \{\text{diag.}(a_0, a_1, \cdots, a_{q+1}) \in G\}$ of G. Under the assumption on Γ, any $\Gamma \ni \gamma \neq 1$ is hyperbolic and G-conjugate to an element $h(\gamma) = \text{diag.}(a_0(\gamma), a_1(\gamma), \cdots, a_{q+1}(\gamma))$ of J_h with $|a_0(\gamma)| > 1$, and the centralizer Γ_γ of γ in Γ is a cyclic group. A Γ-conjugacy class $\{\gamma\}_\Gamma \neq \{1\}$ is called primitive hyperbolic if Γ_γ is generated by γ. We will denote by P_Γ the set of the primitive hyperbolic Γ-conjugacy classes.

For any $\alpha \in P_+$, define a (non-unitary) character ξ_α of J_h by $\text{Ad}(x) X_\alpha = \xi_\alpha(h) X_\alpha$ with a root vector X_α of α. If $\alpha = \lambda_i - \lambda_j \in P_+$ then $\xi_\alpha(h) = a_i a_j^{-1}$ for $h = \text{diag.}(a_0, a_1, \cdots, a_{q+1}) \in J_h$. We denote by $\langle P_+ \rangle$ the set of the linear combination $\sum_{\alpha \in P_+} m_\alpha \cdot \alpha \in \mathcal{B}_C^*$ with non-negative integers m_α. Put $\xi_\lambda = \prod_{\alpha \in P_+} \xi_\alpha^{m_\alpha}$ for $\lambda = \sum_{\alpha \in P_+} m_\alpha \cdot \alpha \in \langle P_+ \rangle$. For any $\lambda \in \langle P_+ \rangle$, we denote by $n(\lambda)$ the number of the ways in which λ is expressed in the form $\lambda = \sum_{\alpha \in P_+} m_\alpha \cdot \alpha$ with non-negative integers m_α.

The Selberg zeta function $Z_{\Gamma, r}(\chi, s)$ with respect to (Γ, χ) with K-type δ_r is defined by the infinite product

$$(3.1) \quad Z_{\Gamma, r}(\chi, s) = \prod_{\{\gamma\}_\Gamma \in P_\Gamma} \prod_{\lambda \in \langle P_+ \rangle} \det \left\{ 1 - \chi(\gamma) \xi_\lambda(h(\gamma)^{-1}) \left(\frac{a_0(\gamma)}{|a_0(\gamma)|} \right)^r |a_0(\gamma)|^{-(s+q+1)} \right\}^{n(\lambda)}$$

which converges absolutely for $\text{Re}(s) > \text{Max}\{q+1, |r| - (q+1)\}$.

Using the trace formula, the log-derivative $Z'_{\Gamma, r}(\chi, s)/Z_{\Gamma, r}(\chi, s)$ of $Z_{\Gamma, r}(\chi, s)$ is meromorphically continued to the whole s-plane, and its poles are all simple whose locations and the residues are given by the following table;

CASE 1; $|r| \geqq q+1$

pole	residue
(A) $0 \neq \lambda \in \boldsymbol{C}$ s.t. $\lambda^2 - \lambda_r^2 < 0$	$m(\pm\lambda, \mathrm{Ind}_\Gamma^G \chi)$ $(-\pi < \arg(\pm\lambda) \leqq \pi)$
(B) 0	$2 \cdot m(0, \mathrm{Ind}_\Gamma^G \chi)$
$-n \in \boldsymbol{Z}$ s.t. $1 < n \leqq \|r\| - (q+1)$, $n \equiv r+q+1 (2)$	$2 \cdot m(n, \mathrm{Ind}_\Gamma^G \chi)$
(C) $-n \in \boldsymbol{Z}$ s.t. $n \geqq \|r\| + (q+1)$, $n \equiv r+q+1 (2)$	$(-1)^q \cdot 2 \cdot m(n)$
± 1 (only if $r \equiv q(2)$)	$m(1, \mathrm{Ind}_\Gamma^G \chi) \mp \dim \chi \cdot \mathrm{vol}(\Gamma \backslash G) \cdot d_\pi$

CASE 2; $0 < |r| < q+1$. Adding to (A), (B), and (C) of CASE 1,

pole	residue
$\pm(q+1-\|r\|)$	$m(q+1-\|r\|, \mathrm{Ind}_\Gamma^G \chi)$

CASE 3; $r=0$. Adding to (A), (B), and (C) of CASE 1

pole	residue
$-(q+1)$	$(-1)^q \cdot 2 \cdot m(q+1) + m(\boldsymbol{1}_\Gamma, \chi)$
$q+1$	$m(\boldsymbol{1}_\Gamma, \chi)$

Here we use the identification $\hat{G}(\delta_r) = V_1(r) \cup V_2(r) \cup V_3(r) \subset \boldsymbol{C}$ given in §2. We use also the following notations; $\mathrm{Ind}_\Gamma^G \chi$ denote the unitary representation of G induced from χ, and $m(\pi, \mathrm{Ind}_\Gamma^G \chi)$ denote the multiplicity of π in $\mathrm{Ind}_\Gamma^G \chi$. The trivial one-dimensional representation of Γ is denoted by $\boldsymbol{1}_\Gamma$. For any integer n such that $n \geqq |r|+q+1$ and $n \equiv |r|+q+1 (\mathrm{mod}\, 2)$, put $m(n) = \dim \chi \cdot \mathrm{vol}(\Gamma \backslash G) \cdot d_\lambda$ with the formal degree d_λ of $\pi_\lambda \in \hat{G}_d$ such that $\lambda = n\nu_0 + \sum_{j=1}^q (2^{-1}(n+|r|+q+1)-j)\nu_j \in \Lambda^+$. The explicit formula of d_λ is given in §1 and $m(n)$ is a polynomial function of n with rational coefficients. We have $m(n) = m(\pi_\lambda, \mathrm{Ind}_\Gamma^G \chi)$ if $n > q+1-|r|$ by [**HP**], and $m(n) \in \boldsymbol{Z}$ for all $n \in \boldsymbol{Z}$. The last statement is proved as follows; the set $X = \{n \in \boldsymbol{Z} \mid n > q+1-|r|\}$ is dense in the p-adic integer ring $\boldsymbol{Z}_p$ for any prime number p. On the other hand, $m(n)$ is a $\boldsymbol{Q}_p$-valued polynomial function on $\boldsymbol{Z}_p$ such that $m(X) \subset \boldsymbol{Z} \subset \boldsymbol{Z}_p$. Then $m(\boldsymbol{Z}_p) \subset \boldsymbol{Z}_p$ for all p. Hence $m(\boldsymbol{Z}) \subset \boldsymbol{Z}_p \cap \boldsymbol{Q}$ for all p, and we have $m(\boldsymbol{Z}) \subset \boldsymbol{Z}$.

Then $Z_{\Gamma, r}(\chi, s)$ has a meromorphic continuation to the whole s-plane, and the locations and the orders of the zeros or poles of $Z_{\Gamma, r}(\chi, s)$ are given by the table above. Also deduced by the trace formula is the functional equation

$$Z_{\Gamma, r}(\chi, -s) = Z_{\Gamma, r}(\chi, s)\cdot\exp\left\{\dim\chi\cdot\mathrm{vol}(\Gamma\backslash G)\cdot\int_0^s \mu_r(\sqrt{-1}s)ds\right\} \tag{3.2}$$

where

$$\mu_r(t) = \frac{\pi}{(2^q q!)^2}\frac{t}{2}\prod_{j=1}^{q}\left\{\left(\frac{t}{2}\right)^2+\left(\frac{r+q+1}{2}-j\right)^2\right\}\times\begin{cases}\tanh\left(\frac{\pi}{2}t\right) & \text{if } r\equiv q (\mathrm{mod}\, 2)\\ \coth\left(\frac{\pi}{2}t\right) & \text{if } r\not\equiv q (\mathrm{mod}\, 2)\end{cases} \tag{3.3}$$

is the weight function of the Planchrel measure.

§ 4. Special values of the Selberg type zeta functions.

Fix the data (Γ, χ) and the K-type δ_r given in § 3. We will start with a lemma;

LEMMA 4.1. *For any real number n, the integral*

$$J_n(s) = (2\pi)^{-1}\{4(q+2)\}^{-s}\int_0^\infty (x^2+n^2)^{-s}\mu_r(x)dx \tag{4.1}$$

converges absolutely for $\mathrm{Re}(s)>q+1$, *and has a meromorphic continuation to the whole s-plane which is holomorphic except for the possible simple poles at* $s=1, 2, \cdots, q+1$.

PROOF. The weight function $\mu_r(x)$ is an odd polynomial function of x times tanh $(\pi t/2)$ or coth $(\pi t/2)$, and an elementary calculation gives the formula

$$\int_0^\infty (x^2+n^2)^{-s}x^{2m+1}\tanh(\pi x)dx$$
$$=\frac{\pi}{2}\prod_{j=0}^{m}\frac{m!}{(m-j)!}\prod_{k=1}^{j+1}(s-k)^{-1}\int_0^\infty (x^2+n^2)^{j+1-s}x^{2(m-j)}\,\mathrm{sech}^2(\pi x)dx$$

and

$$\int_0^\infty (x^2+n^2)^{-s}x^{2m+1}\coth(\pi x)dx$$
$$=\int_0^1 (x^2+n^2)^{-s}x^{2m+1}\coth(\pi x)dx+\frac{1}{2}\coth(\pi)\sum_{j=0}^{m}\frac{m!}{(m-j)!}(1+n^2)^{j+1-s}\prod_{k=1}^{j+1}(s-k)^{-1}$$
$$+\frac{\pi}{2}\sum_{j=0}^{m}\frac{m!}{(m-j)!}\prod_{k=1}^{j+1}(s-k)^{-1}\int_1^\infty (x^2+n^2)^{j+1-s}x^{2(m-j)}\,\mathrm{cosech}^2(\pi x)dx$$

and the proof of the lemma is completed. ■

Put $n_0=\mathrm{Max}\{q+1, |r|-(q+1)\}$ and $u_0=\{n^2+qr^2/(q+2)-(q+1)^2\}/\{4(q+2)\}$. Let Ω be the Casimir operator of $\mathfrak{g}_C$, and u a real number such that $u \geqq u_0$. We denote by $\Delta_{r,u}$ the differential operator $\Omega-u$ acting on the δ_r-isotypic component of $\mathrm{Ind}_\Gamma^G\chi$. Then $\chi_\pi(\Omega-u)\leqq 0$ for any $\pi\in\hat{G}(\delta_r)$ (χ_π is the infinitesimal character of π), and we put

$$T(s, \Delta_{r,u}) = \sum_\pi m(\pi, \mathrm{Ind}_\Gamma^G\chi)\cdot|\chi_\pi(\Omega-u)|^{-s} \tag{4.2}$$

where $\sum_\pi$ is the summation over the $\pi\in\hat{G}(\delta_r)$ such that $\chi_\pi(\Omega-u)\neq 0$. Then we have

PROPOSITION 4.2. *The Dirichlet series $T(s, \Delta_{r,u})$ converges absolutely for $\mathrm{Re}(s)>q+1$ and has a meromorphic continuation to the whole s-plane which is holomorphic except for the possible simple poles at $s=1, 2, \cdots, q+1$.*

DEFINITION 4.3. We put $\det\Delta_{r,u}=\exp(-T'(0, \Delta_{r,u}))$ which is called the functional determinant of the operator $\Delta_{r,u}$.

Then we have the following theorems;

THEOREM 4.4. *Take an integer $n>n_0$ and put $u=\{n^2+qr^2/(q+2)-(q+1)^2\}/\{4(q+2)\}$. Then $Z_{\Gamma,r}(\chi, n)=R\cdot P$ with*

$$P=\exp\{J_n'(0)\cdot\dim\chi\cdot\mathrm{vol}(\Gamma\backslash G)\}$$
$$R=(\det\Delta_{r,u})\prod_{\pi\in\hat{G}_d(\delta_r)}|\chi_\pi(\Omega-u)|^{-\dim\chi\cdot\mathrm{vol}(\Gamma\backslash G)\cdot d_\pi}$$

and

THEOREM 4.5. *Suppose $r=0$. Then $Z_{\Gamma,r}(\chi, s)$ has a zero of order $m(\mathbf{1}_\Gamma, \chi)$ at $s=q+1$, and we have*

$$Z_{\Gamma,r}(\chi, s)=R\cdot P\cdot(s-q-1)^{m(\mathbf{1}_\Gamma,\chi)}+[\text{terms of degree}>m(\mathbf{1}_{\Gamma,\chi})]$$

with

$$P=(2q+2)^{m(\mathbf{1}_\Gamma,\chi)}\exp\{J_{q+1}'(0)\cdot\dim\chi\cdot\mathrm{vol}(\Gamma\backslash G)\}$$
$$R=\det\Delta_{0,0}.$$

The rest of this section is devoted to the proof of these results. We will imitate the arguments of [**Fr**].

The trace formula

$$\sum_{\pi\in\hat{G}(\delta_r)} m(\pi, \mathrm{Ind}_\Gamma^G\chi)\cdot\hat{f}(\pi)$$
$$=\dim\chi\cdot\mathrm{vol}(\Gamma\backslash G)\cdot f(1)$$
$$+\sum_{\{\gamma\}_\Gamma\neq\{1\}}\mathrm{tr}\,\chi(\gamma)\frac{\log|a_0(\gamma)|}{(\Gamma_\gamma:\langle\gamma\rangle)}D(h(\gamma))\Big(\frac{a_0(\gamma)}{|a_0(\gamma)|}\Big)^r F_r\begin{pmatrix}|a_0(\gamma)| & 0 & 0\\ 0 & I_q & 0\\ 0 & 0 & |a_0(\gamma)|^{-1}\end{pmatrix}$$

is valid for all $f\in C^1(G,\delta_r)$ (c.f. [**GW**], [**Mt**]). Here $\hat{f}(\pi)=\int_G f(x)\psi_{\pi,r}(x)d_G(x)$ is the spherical Fourier transform of f with respect to the spherical function $\Psi_{\pi,r}(x)$ cited in §2, $\sum_{\{\gamma\}\Gamma\neq\{1\}}$ is the summation over the Γ-conjugacy classes $\{\gamma\}_\Gamma\neq\{1\}$,

$$D(h)=\exp(-\rho H(h))\prod_{\alpha\in P_+}|1-\xi_\alpha(h)^{-1}|^{-1}$$

for $h\in J_h$, and

$$F_f(a)=\exp(-\rho H(a))\int_N f(na)d_N(n)$$

for $a\in A$ is the Abel transform of f.

We will take, as a test function f for the trace formula, a solution of the heat equation $\Omega f=\partial f/\partial t$ on G. We have $(\partial/\partial t)\hat{f}(\pi)=\widehat{\Omega f}(\pi)=\chi_\pi(\Omega)\hat{f}(\pi)$ for all $\pi\in\hat{G}(\delta_r)$, and hence, $\hat{f}(\pi)=C(\pi)\cdot\exp(\chi_\pi(\Omega)t)$. By the formula (2.2) of the infinitesimal character $\chi_\pi(\Omega)$ and by the Paley-Wiener theorem (Theorem 2.4), there exists a function $f_t\in C^1(G,\delta_r)$ such that $\hat{f}_t(\pi)=\exp(\chi_\pi(\Omega)t)$ for all $\pi\in\hat{G}(\delta_r)$ and all $t>0$. Then we have

$$\begin{aligned}f_t(1)=&\sum_{\pi\in\hat{G}_d(\delta_r)}d_\pi\cdot\exp(\chi_\pi(\Omega)t)\\&+(2\pi)^{-1}\int_0^\infty\exp\left[-\frac{1}{4(q+2)}\left\{x^2+(q+1)^2-\frac{q}{q+2}r^2\right\}t\right]\mu_r(x)dx\end{aligned}$$

by the Planchrel formula, and

$$\begin{aligned}&F_{f_t}\begin{pmatrix}a&0&0\\0&I_q&0\\0&0&a^{-1}\end{pmatrix}\qquad(a>0,\ t>0)\\&=\sqrt{\frac{q+2}{\pi t}}\cdot\exp\left[\frac{1}{4(q+2)}\left\{\frac{q}{q+2}r^2-(q+1)^2\right\}t-(q+2)(\log a)^2\frac{1}{t}\right].\end{aligned}$$

By virtue of the results of [**HP**], we have $m(\pi,\mathrm{Ind}_\Gamma^G\chi)=\dim\chi\cdot\mathrm{vol}(\Gamma\backslash G)\cdot d_\pi$ for all $\pi\in\hat{G}_d(\delta_r)$ such that $|\pi|>1$ with respect to the identification $\hat{G}(\delta_r)\hookrightarrow\boldsymbol{C}$ given in §2.

Proof of Proposition 4.2. Let u be a real number such that $u\geqq u_0$, and put $u=\{n^2+qr^2/(q+2)-(q+1)^2\}/\{4(q+2)\}$ with $n\geqq n_0$. Applying the trace formula to the function $f_t\in C^1(G,\delta_r)$, we have

$$H_u^1(t)+H_u^0=I_n(t)+G_n(t)\tag{4.3}$$

for $t>0$ where

$$H_u^1(t) = \sum_{\substack{\pi\in\hat{G}(\delta_\Gamma)-\hat{G}_d(\delta_\Gamma) \\ \chi_\pi(\Omega-u)\neq 0}} m(\pi, \mathrm{Ind}_\Gamma^G\chi)\cdot\exp(\chi_\pi(\Omega-u)t)$$
$$+ \sum_{\substack{\pi\in\hat{G}_d(\delta_\Gamma) \\ |\pi|=1, \chi_\pi(\Omega-u)\neq 0}} (m(\pi, \mathrm{Ind}_\Gamma^G\chi)-\dim\chi\cdot\mathrm{vol}(\Gamma\backslash G)\cdot d_\pi)\exp(\chi_\pi(\Omega-u)t)$$
$$H_u^0 = \sum_{\substack{\pi\in\hat{G}(\delta_\Gamma) \\ \chi_\pi(\Omega-u)=0}} m(\pi, \mathrm{Ind}_\Gamma^G\chi)$$
$$I_n(t)=\dim\chi\cdot\mathrm{vol}(\Gamma\backslash G)(2\pi)^{-1}\int_0^\infty \exp\left\{-\frac{1}{4(q+2)}(x^2+n^2)t\right\}\mu_r(x)dx$$

and

$$G_n(t) = \sum_{\{\gamma\}_\Gamma\neq\{1\}} \mathrm{tr}\chi(\gamma)\frac{\log|a_0(\gamma)|}{(\Gamma_\gamma:\langle\gamma\rangle)}\cdot D(h(\gamma))\cdot\left(\frac{a_0(\gamma)}{|a_0(\gamma)|}\right)^r\left(\frac{q+2}{\pi t}\right)^{1/2}$$
$$\times\exp\left\{-\frac{n^2}{4(q+2)}t+(q+2)(\log|a_0(\gamma)|)^2t^{-1}\right\}.$$

Let $\chi_{(0,1]}$(resp.$\chi_{(1,\infty)}$) be the characteristic function of $(0, 1]$(resp.$(1, \infty)$), and put

$$A_u(t) = H_u^1(t)+H_u^0\cdot\chi_{(0,1]}(t)-I_n(t) = G_n(t)-H_u^0\cdot\chi_{(1,\infty)}(t).$$

Then $A_u(t)$ decays exponentially as $t\to+0$ or $t\to+\infty$, and the Mellin transform

$$\tilde{A}_u(s) = \int_0^\infty A_u(t)t^{s-1}dt$$

is an entire function of s (we will denote by $\tilde{F}(s)$ the Mellin transform $\int_0^\infty F(t)t^{s-1}dt$ of the function $F(t)$). We have $H_u^1(t)=O(t^{-(q+1)})$ as $t\to+0$ by (4.3), and the Mellin transform $\tilde{H}_u^1(s)$ converges absolutely for $\mathrm{Re}(s)>q+1$. On the other hand, we have

$$\tilde{H}_u^1(s) = \tilde{A}_u(s)-H_u^0\cdot s^{-1}+\tilde{I}_n(s) \tag{4.4}$$

for $\mathrm{Re}(s)>q+1$ with

$$\tilde{I}_n(s) = \dim\chi\cdot\mathrm{vol}(\Gamma\backslash G)\cdot\Gamma(s)\cdot J_n(s)$$

and $\tilde{H}_u^1(s)$ has a meromorphic continuation to the whole s- plane which is holomorphic except for the possible poles at $s=0, 1, \cdots, q+1$ by Lemma 4.1. We have

$$\tilde{H}_u^1(s) = \Gamma(s)\Big(T(s, \Delta_{r,u})-\dim\chi\cdot\mathrm{vol}(\Gamma\backslash G)\sum_{\pi\in\hat{G}_d(\delta_\Gamma)} d_\pi\cdot|\chi_\pi(\Omega-u)|^{-s}\Big)$$

and the proof of the Proposition 4.2 is complete. ∎

The log-derivative $\Phi_{\Gamma,r}(\chi, s)=Z'_{\Gamma,r}(\chi, s)/Z_{\Gamma,r}(\chi, s)$ of the Selberg zeta function $Z_{\Gamma,r}(\chi, s)$ is

$$\Phi_{\Gamma,r}(\chi, s)=\sum_{\{\gamma\}_\Gamma\neq\{1\}} \mathrm{tr}\chi(\gamma)\frac{\log|a_0(\gamma)|}{(\Gamma_\gamma:\langle\gamma\rangle)}D(h(\gamma))\left(\frac{a_0(\gamma)}{|a_0(\gamma)|}\right)^r|a_0(\gamma)|^{-s}$$

for $\mathrm{Re}(s)>n_0$, and we have

$$(4.5)\quad \tilde{G}_n(s)=\Gamma(1-s)^{-1}2\sqrt{q+2}\int_0^\infty \left\{x\left(x+\frac{n}{\sqrt{q+2}}\right)\right\}^{-s}\Phi_{\Gamma,r}(\chi,\ 2\sqrt{q+2}x+n)dx$$

for $\mathrm{Re}(s)<1$ by the formula

$$(4.6)\quad \begin{aligned}&\int_0^\infty (4\pi t)^{-1/2}\exp\{-(c^2t+d^2t^{-1})\}\cdot t^{s-1}dt\\ &=\Gamma(1-s)^{-1}\int_0^\infty \exp\{-d(x+c/2)\}\cdot\{x(x+c)\}^{-s}dx\end{aligned}$$

for $\mathrm{Re}(s)<1$, $d>0$, and $c>0$([**Fr**]). On the other hand, we have

$$(4.7)\qquad \tilde{G}_n(s)=\tilde{A}_u(s)-H_u^0\cdot s^{-1}$$

for $\mathrm{Re}(s)<0$ and $\tilde{G}_n(s)$ has a meromorphic continuation to the whole s-plane with unique simple pole at $s=0$ of residue H_u^0. Combining (4.4) and (4.7), we have $\tilde{H}_u^1(s)=\tilde{G}_n(s)+\tilde{I}_n(s)$.

Proof of Theorem 4.4. Let n be an integer such that $n>n_0$, and put $u=\{n^2+qr^2/(q+2)-(q+1)^2\}/\{4(q+2)\}>u_0$. In this case, $H_u^0=0$ and $\exp(-\tilde{G}_n(0))=Z_{\Gamma,r}(\chi,\ n)$ by (4.5). On the other hand, we have

$$T(0,\ \Delta_{r,u})=\dim\chi\cdot\mathrm{vol}(\Gamma\backslash G)(J_n(0)+\sum_{\pi\in\hat{G}_d(\delta_r)}d_\pi)$$

$$\Gamma(s)\cdot T(s,\ \Delta_{r,u})=\dim\chi\cdot\mathrm{vol}(\Gamma\backslash G)\cdot\Gamma(s)(J_n(s)+\sum_{\pi\in\hat{G}_d(\delta_r)}d_\pi|\chi_\pi(\Omega-u)|^{-s})+\tilde{G}_n(s).$$

Hence

$$\begin{aligned}T'(0,\ \Delta_{r,u})&=\lim_{s\to 0}\Gamma(s)(T(s,\ \Delta_{r,u})-T(0,\ \Delta_{r,u}))\\ &=\dim\chi\cdot\mathrm{vol}(\Gamma\backslash G)(J_n'(0)-\sum_{\pi\in\hat{G}_d(\delta_r)}d_\pi\cdot\log|\chi_\pi(\Omega-u)|)+\tilde{G}_n(0).\end{aligned}$$

The proof of Theorem 4.4 is complete. ■

Proof of Theorem 4.5. Suppose $r=0$. Put $u=u_0=0$ and $n=n_0=q+1$. In this case, $\chi_\pi(\Omega)=0$ for $\pi\in\hat{G}_{\delta_r}$ if and only if $\pi=\mathbf{1}_G$, and we have $H_u^0=m(\mathbf{1}_G,\ \mathrm{Ind}_\Gamma^G\chi)=m(\mathbf{1}_\Gamma,\ \chi)$ by the Frobenius reciprocity law. Then we have

$$\begin{aligned}T'(0,\ \Delta_{0,0})&=\lim_{s\to 0}(T(s,\ \Delta_{0,0})-T(0,\ \Delta_{0,0}))\\ &=\lim_{s\to 0}(\tilde{G}_n(s)+\Gamma(s)m(\mathbf{1}_\Gamma,\ \chi)+\dim\chi\cdot\mathrm{vol}(\Gamma\backslash G)\cdot J_n'(0).\end{aligned}$$

On the other hand, similar to [**Fr**], we can show that

$$\begin{aligned}&\int_0^\infty\{x(x+2q+2)\}^{-s}\Phi_{\Gamma,r}(\chi,\ x+q+1)dx\\ &=-m(\mathbf{1}_\Gamma,\ \chi)\cdot s^{-1}+m(\mathbf{1}_\Gamma,\ \chi)\log\varepsilon(\varepsilon+2q+2)-\log Z_{\Gamma,r}(\chi,\ \varepsilon+q+1)+o(\varepsilon)+o(s)\end{aligned}$$

as $\varepsilon\to 0$ and $s\to 0$. Then, putting $Z_{\Gamma,r}(\chi,\ s+q+1)=s^{m(1_\Gamma,\chi)}\cdot F(s)$ where $F(s)$ is holomorphic at $s=0$ and $F(0)\neq 0$, we have

$$\lim_{s\to 0}(\tilde{G}_n(s)+\Gamma(s)m(\mathbf{1}_\Gamma,\chi)) = m(\mathbf{1}_\Gamma,\chi)\cdot\log(2q+2)-\log F(0)$$

and the proof of Theorem 4.5 is complete. ■

§5. Dedekind zeta functions.

Let K be a finite algebraic number field and

$$\zeta_K(s) = \prod_{v<\infty}(1-N(v)^{-s})^{-1} \qquad (\mathrm{Re}(s)>1)$$

the Dedekind zeta function of K. Then we have the explicit formula ([**Wl**])

$$\begin{aligned}\sum_\omega \Phi(\omega) &= F(0)\log|D|+2\int_{-\infty}^{\infty}F(x)\cosh(x/2)dx \\ &\quad -\sum_{v<\infty}\sum_{n=1}^{\infty}N(v)^{-n/2}\log N(v)\cdot\{F(\log N(v^n))+F(-\log N(v^n))\} \\ &\quad +\sum_{v|\infty}\frac{1}{\pi}\int_{-\infty}^{\infty}\Phi\Big(\frac{1}{2}+\sqrt{-1}x\Big)\cdot\mathrm{Re}\Big(\frac{\Gamma_v'}{\Gamma_v}\Big(\frac{1}{2}+\sqrt{-1}x\Big)\Big)dx\end{aligned} \tag{5.1}$$

where F is a suitable test function and

$$\Phi(s) = \int_{-\infty}^{\infty}F(x)\exp((s-1/2)x)dx, \tag{5.2}$$

$\sum_\omega$ is the summation with multiplicity over the zeros ω of $\zeta_K(s)$ such that $0<\mathrm{Re}(\omega)<1$, D is the absolute discriminant of K, $\sum_{v<\infty}$(resp. $\sum_{v|\infty}$) is the summation over the finite places (resp. infinite places) of K, and $\Gamma_v(s)=\pi^{-s/2}\Gamma(s/2)$ if v is real, $\Gamma_v=(2\pi)^{1-s}\Gamma(s)$ if v is complex. Using the explicit formula, we will prove the following results;

LEMMA 5.1. *The integral*

$$J_v(s) = \frac{1}{\pi}\int_0^\infty\Big(\frac{1}{4}+x^2\Big)^{-s}\mathrm{Re}\Big(\frac{\Gamma_v'}{\Gamma_v}\Big(\frac{1}{2}+\sqrt{-1}x\Big)\Big)dx \tag{5.3}$$

converges absolutely for $\mathrm{Re}(s)>1/2$, *and has a meromorphic continuation to the whole s-plane which is holomorphic except for the possible double poles at* $s=1/2-j(0\leqq j\in\boldsymbol{Z})$.

PROPOSITION 5.2. *The infinite series*

$$T(s,\Delta_K) = \frac{1}{2}\sum_{\omega=1/2+\sqrt{-1}u}\Big(\frac{1}{4}+u^2\Big)^{-s} \tag{5.4}$$

converges absolutely for $\mathrm{Re}(s)>1/2$, *and has a meromorphic continuation to the whole s-plane which is holomorphic except for the possible double poles at* $s=1/2-j(0\leqq j\in\boldsymbol{Z})$.

In Proposition 5.2, we used the following notations; $\sum_{\omega=1/2+\sqrt{-1}u}$ is the summation with multiplicity over the zeros $\omega=1/2+\sqrt{-1}u$ of $\zeta_K(s)$ such that $0<\mathrm{Re}(\omega)<1$. Note that $\mathrm{Re}(1/4+u^2)>0$ for $\omega=1/2+\sqrt{-1}\in\boldsymbol{C}$ such that $0<\mathrm{Re}(\omega)<1$, and we will put $(1/4+u^2)^{-s}=\exp(-s\cdot\log(1/4+u^2))$ where $\log(1/4+u^2)=\log|1/4+u^2|+\sqrt{-1}\arg(1/4+u^2)$ with $-\pi<\arg(1/4+u^2)<\pi$.

DEFINITION 5.3. We put $\det\Delta_K=\exp(-T'(0, \Delta_K))$.

THEOREM 5.4. *We have an equality*

$$2^{r_1}(2\pi)^{r_2}R(K)\frac{h}{w}|D|^{-1/4}=(\exp J_1'(0))^{r_1}(\exp J_2'(0))^{r_2}\det\Delta_K$$

where $J_1(s)=J_{real}(s)$ *and* $J_2(s)=J_{complex}(s)$. *The other notations are defined in* § 0.

REMARK. Applying Theorem 5.4 to some special fields, we have $\exp J_1'(0)=(\det\Delta_{\boldsymbol{Q}})^{-1}$ and $\exp J_2'(0)=2^{-3/2}\cdot\pi(\det\Delta_{\boldsymbol{Q}(\sqrt{-1})})^{-1}$.

The rest of this section is devoted to the proof of these results.

Applying the explicit formula to the function

$$F_t(x)=(4\pi\cdot t)^{-1/2}\exp\{-(x/2)^2t^{-1}\} \qquad (x\in\boldsymbol{R},\ t>0) \tag{5.5}$$

which is the fundamental solution of the heat equation $\partial^2F/\partial x^2=\partial F/\partial t$, we have

$$H(t)=I(t)+2+G(t)+\sum_{v|\infty}I_v(t) \tag{5.6}$$

where

$$H(t)=\sum_{\omega=1/2+\sqrt{-1}u}\exp\left\{-\left(\frac{1}{4}+u^2\right)t\right\}$$

$$G(t)=-2\sum_{v<\infty}\sum_{n=1}^{\infty}N(v)^{-n/2}\log N(v)\cdot(4\pi\cdot t)^{-1/2}\exp\left[-\frac{1}{4}\{t+(\log N(v))^2t^{-1}\}\right]$$

$$I(t)=\log|D|\cdot(4\pi\cdot t)^{-1/2}\exp(-t/4)$$

$$I_v(t)=\frac{2}{\pi}\int_0^\infty\exp\left\{-\left(\frac{1}{4}+x^2\right)t\right\}\cdot\mathrm{Re}\left(\frac{\Gamma_v'}{\Gamma_v}\left(\frac{1}{2}+\sqrt{-1}x\right)\right)dx.$$

Let $\chi_{(0,1]}$ (resp. $\chi_{(1,\infty)}$) be the characteristic function of $(0, 1]$ (resp. $(1, \infty)$), and put

$$A(t)=H(t)-2\chi_{(0,1]}(t)-I(t)-\sum_{v|\infty}I_v(t)=G(t)+2\chi_{(0,\infty)}(t). \tag{5.7}$$

Then $A(t)$ decays exponentially as $t\to+0$ or $t\to+\infty$, and the Mellin transform $\tilde{A}(s)=\int_0^\infty A(t)t^{s-1}dt$ is an entire function of s. The following lemma is easily proved;

LEMMA 5.5. *Let F be a C^∞-function on* $[0, 1]$. *Then the integral*

$$\int_0^1 F(t)t^{s-1}dt$$

converges absolutely for $\mathrm{Re}(s)>0$, *and has a meromorphic continuation to the whole s-plane which is holomorphic except for the possible poles at* $s=-j(0\leqq j\in\boldsymbol{Z})$.

PROOF OF LEMMA 5.1. We have

$$\mathrm{Re}\Big(\frac{\Gamma_v'}{\Gamma_v}\Big(\frac{1}{2}+\sqrt{-1}x\Big)\Big)=a_v+\log x+\sum_{n=1}^{N}a_v(n)x^{-2n}+R_N(x)$$
$$|R_N(x)|\leqq M_n\cdot x^{-(2N+1)}$$

for $x\geqq 1/2$ with suitable constants a_v, $a_v(n)$, and M_n ([**Mg**, p. 18]). Then $I_v(t)=O(t^{-1/2})$ as $t\to+0$ and the Mellin transform $\tilde{I}_v(s)=2\Gamma(s)J_v(s)$ converges absolutely for $\mathrm{Re}(s)>1/2$. Decompose $\tilde{I}_v(s)$ into four terms;

$$\begin{aligned}\tilde{I}_v(s)&=\int_1^\infty I_v(t)t^{s-1}dt\\&+\int_0^1 t^{s-1}\Big(\int_0^1\exp\{-(x^2+1/4)t\}\cdot\mathrm{Re}\Big(\frac{\Gamma_v'}{\Gamma_v}\Big(\frac{1}{2}+\sqrt{-1}x\Big)dx\Big)dt\\&+\int_0^1 t^{s-1}\Big(\int_1^\infty\exp\{-(x^2+1/4)\}\cdot B_v(x)dx\Big)dt\\&+\int_0^1 t^{s-1}\Big(\int_1^\infty\exp\{-(x^2+1/4)\}\cdot(a_v+\log x)dx\Big)dt\end{aligned}\tag{5.8}$$

with $B_v(x)=\mathrm{Re}(\Gamma_v'/\Gamma_v(1/2+\sqrt{-1}x))-a_v-\log x$. The first term is an entire function of s. The second term has a meromorphic continuation to the whole s-plane which is holomorphic except for the simple poles at $s=-j(0\leqq j\in\boldsymbol{Z})$ by Lemma 5.5. The third term is equal to

$$\begin{aligned}&\alpha_v s^{-1}+b_v s^{-1}\int_0^1 t^s\Big(\int_1^\infty\exp\{-(x^2+1/4)\}dx\Big)dt\\&\quad+s^{-1}\int_0^1 t^s\Big(\int_1^\infty C_v(x)\exp\{-(x^2+1/4)\}dx\Big)dt\end{aligned}$$

with

$$\alpha_v=\int_1^\infty B_v(x)\exp\{-(x^2+1/4)\}dx$$
$$B_v(x)(x^2+1/4)=b_v+C_v(x).$$

Then the third term of (5.8) has a meromorphic continuation to the whole s-plane which is holomorphic except for the simple poles at $s=-j(0\leqq j\in\boldsymbol{Z})$ by Lemma 5.5. We have

$$\int_1^\infty(a_v+\log x)\exp(-x^2t)dx=c_vt^{-1/2}\log t+d_vt^{-1/2}$$

with suitable constants c_v and d_v, and

$$\begin{aligned}&\int_0^1 t^{s-3/2}\exp(-t/4)\log t\,dt=e^{-1/4}\cdot(s-1/2)^{-2}\\&+\frac{1}{4}(s-1/2)^{-1}\int_0^1 t^{s-1/2}\exp(-t/4)\cdot\log t\,dt-\frac{1}{4}(s-1/2)^{-2}\int_0^1 t^{s-1/2}\exp(-t/4)dt.\end{aligned}$$

Then, by Lemma 5.5, the fourth term of (5.8) has a meromorphic continuation to the whole s-plane which is holomorphic except for the double poles at $s=1/2-j(0\leqq j\in \boldsymbol{Z})$. The proof of Lemma 5.1 is complete. ■

PROOF OF PROPOSITION 5.2. We have $H(t)+O(t^{-1/2})$ as $t\to+0$ and $I_v(t)=O(t^{-1/2})$ as $t\to+0$. Then the Mellin transform $\tilde{H}(s)=2\Gamma(s)T(s, \Delta_K)$ converges absolutely for $\mathrm{Re}(s)>1/2$. On the other hand, by (5.7), we have

$$\tilde{H}(s)=\tilde{A}(s)+2s^{-1}+\tilde{I}(s)+2\Gamma(s)\sum_{v\mid\infty}J_v(s) \tag{5.9}$$

with $\tilde{I}(s)=(4\sqrt{\pi})^{-1}\log|D|\cdot\Gamma(s-1/2)$, and the proof of Proposition 5.2 is complete. ■

PROOF OF THEOREM 5.4. By (5.6), $G(t)$ is bounded as $t\to+\infty$ and exponentially decays as $t\to+0$. Then the Mellin transform $\tilde{G}(s)$ converges absolutely for $\mathrm{Re}(s)<0$ and $\tilde{G}(s)=\tilde{A}(s)+2s^{-1}$ by (5.7). Hence $\tilde{G}(s)$ has a meromorphic continuation to the whole s-plane with unique simple pole at $s=0$ with residue 2. On the other hand, we have

$$\frac{\zeta_K'}{\zeta_K}(s)=-\sum_{v<\infty}\sum_{n=1}^{\infty}\log N(v^n)\cdot N(v)^{-ns}$$

for $\mathrm{Re}(s)>1$ and

$$\tilde{G}(s)=2\Gamma(1-s)^{-1}\int_0^\infty\frac{\zeta_K'}{\zeta_K}(x+1)\cdot\{x(x+1)\}dx$$

for $\mathrm{Re}(s)<0$ here we use the formula (4.6). We have

$$2\Gamma(s)T(s, \Delta_K)=\tilde{G}(s)+\tilde{I}(s)+2\Gamma(s)\sum_{v\mid\infty}J_v(s)$$
$$T(0, \Delta_K)=1+\sum_{v\mid\infty}J_v(0)$$

by (5.9), and then

$$\begin{aligned}T'(0, \Delta_K)&=\lim_{s\to 0}\Gamma(s)(T(s, \Delta_K)-T(0, \Delta_K))\\&=\lim_{s\to 0}\Big(\frac{1}{2}\tilde{G}(s)-\Gamma(s)\Big)+\frac{1}{2}\tilde{I}(0)+\sum_{v\mid\infty}J_v'(0)\end{aligned}$$

and $(1/2)\tilde{I}(0)=-(1/4)\log|D|$. Similar to the case of the Selberg zeta function, putting $\zeta_K(s+1)=s^{-1}f(s)$ with $f(0)=\mathrm{Res}_{s=1}\zeta_K(s)$, we can show that

$$\int_0^\infty\frac{\zeta_K'}{\zeta_K}(x+1)\cdot\{x(x+1)\}^{-s}dx=s^{-1}-\log f(0)+o(s)$$

as $s\to 0$. Then

$$\lim_{s\to 0}\Big(\frac{1}{2}G(s)-\Gamma(s)\Big)=-\log f(0)$$

and we have

$$T'(0, \Delta_K) = -\log(\mathrm{Res}_{s=1}\zeta_K(s)) - \frac{1}{4}\log|D| + \sum_{v|\infty} J_v'(0).$$

Being combined with the classical residue formula

$$\mathrm{Res}_{s=1}\zeta_K(s) = 2^{r_1}(2\pi)^{r_2}R(K)\frac{h}{w}|D|^{-1/2},$$

the proof of the Theorem 5.4 is complete. ■

§6. Concluding remarks.

1) The factors $2^{r_1}(2\pi)^{r_2}$ and $(\exp J_1'(0))^{r_1}(\exp J_2'(0))^{r_2}$ in Theorem 5.4 depends only on the structure of $K\otimes_Q \boldsymbol{R}=\boldsymbol{R}^{r_1}\times \boldsymbol{C}^{r_2}$. On the other hand, the factors $R(K)h/w|D|^{-1/4}$ and $\det\Delta_K$ in Theorem 5.4 depends deeply on the arithmetic number field K itself. In this sense, the factors $2^{r_1}(2\pi)^{r_2}$ and $R(K)h/w|D|^{-1/4}$ correspond to the factors $(\exp J_1'(0))^{r_1}(\exp J_2'(0))^{r_2}$ and $\det\Delta_K$ respectively. In other word, $(\exp J_2'(0))^{r_1}(\exp J_2'(0))^{r_2}$ is the period and $\det\Delta_K$ is the regulator for the residue of Dedekind zeta function at $s=1$.

Now we have a completed crossword puzzle;

Crossword Puzzle

Dedekind zeta function	Selberg zeta function
explicit formula	Selberg's trace formula

In the proof of Theorem 5.4, we used, as the test function for the explicit formula, the fundamental solution of the heat equation $\partial^2F/\partial x^2=\partial F/\partial t$. On the other hand, in the proof of Theorem 4.4 or Theorem 4.5, we used, as the test function for the trace formula, the fundamental solution of the heat equation $\Omega F=\partial F/\partial t$ on G. The functions $T(s, \Delta_{r,0})$ and $T(s, \Delta_K)$ are defined as summations over non-trivial zeros of Selberg zeta function $Z_{\Gamma,r}(\chi, s)$ and Dedekind zeta function $\zeta_K(s)$ respectively. Thus we have quite parallel arguments for Dedekind zeta functions and Selberg type zeta functions. It suggests that the factors $R=\det\Delta_{0,0}$ and $P=(2q+2)^{m(1_\Gamma,\chi)}\exp(J_{q+1}'(0)\cdot\dim\chi\cdot\mathrm{vol}(\Gamma\backslash G))$ in Theorem 4.5 correspond to the factors $\det\Delta_K$ and $(\exp J_1'(0))^{r_1}(\exp J_2'(0))^{r_2}$ in Theorem 5.4 respectively. There correspondence suggests that the factors P and R in Theorem 4.4 or Theorem 4.5 play the role of the period and the regulator for the special values of the Selberg zeta functions.

2) We will consider the Selberg zeta function of $G=SL(2, \boldsymbol{R})$. In this case, $K=SO(2, \boldsymbol{R})$ is a maximal compact subgroup of G. All the K-type is one-dimensional, that is, $\delta_r\begin{pmatrix} a & -c \\ c & a\end{pmatrix}=(a+\sqrt{-1}c)^r$ with $r\in\boldsymbol{Z}$. Let Γ be a discrete torsion-free subgroup of G such that $\Gamma\backslash G$ is compact, and χ is a finite

dimensional unitary representation of Γ. Then the Selberg zeta function of G with respect to (Γ, χ) with K-type δ_r is defined by

$$\zeta_{\Gamma,r}(\chi, s)=\prod_{\{\gamma\}_\Gamma}\prod_{n=0}^{\infty}\det\left\{1-\chi(\gamma)\left(\frac{a(\gamma)}{|a(\gamma)|}\right)^r a(\gamma)^{-2(s+n)}\right\}.$$

Here $\prod_{\{\gamma\}_\Gamma}$ is the product over the primitive hyperbolic Γ-conjugacy classes of Γ. By the conditions torsion-free and co-compact, any elements $1\neq\gamma\in\Gamma$ is hyperbolic, and it is G-conjugate to $\begin{pmatrix} a(\gamma) & 0 \\ 0 & a(\gamma)^{-1}\end{pmatrix}\in G$ such that $|a(\gamma)|>1$. We have $\zeta_{\Gamma,r}(\chi, s)=\zeta_{\Gamma,r'}(\chi, s)$ if $r\equiv r'(\mathrm{mod}\,2)$, and $\zeta_{\Gamma,0}(\chi, s)$ is the function originally considered by [Sl]. The arguments in §4 with slight modification holds in the case of $G=SL(2, \boldsymbol{R})$. In fact, we have $\zeta_{\Gamma,r}(\chi, s)=Z_{\Gamma,r}(\chi, 2s-1)$ where $Z_{\Gamma,r}(\chi, s)$ is the function (3.1) formally specialized $q=0$. Let g is the genus of the Riemann surface $\Gamma\backslash G/K$ with which we have $\mathrm{vol}(\Gamma\backslash G)=2g-2$. Then we have

THEOREM 6.1. *Take an integer* $n>\mathrm{Max}\{2, |r|\}$ *and put* $u=n(n-2)/8$. *Then* $\zeta_{\Gamma,r}(\chi, n/2)=R\cdot P$ *with*

$$R=\det(\Delta_{r,u})\prod_{\substack{1<m\leq r\\ m\equiv r\,(2)}}\{(n+m-2)(n-m)\}^{-(m-1)(g-1)\dim\chi}$$
$$P=\exp\{J_n'(0)\cdot\dim\chi\cdot(2g-2)\}.$$

THEOREM 6.2. *The function* $\zeta_{\Gamma,0}(\chi, s)$ *has a zero at* $s=1$ *of order* $m(\mathbf{1}_\Gamma, \chi)$, *and we have*

$$\zeta_{\Gamma,r}(\chi, s)=R\cdot P\cdot(s-1)^{m(\mathbf{1}_\Gamma,\chi)}+[\textit{terms of degree}>m(\mathbf{1}_\Gamma, \chi)]$$

with

$$R=\det\Delta_{0,0}$$
$$P=4^{m(\mathbf{1}_\Gamma,\chi)}\exp\{J_1'(0)\cdot\dim\chi\cdot(2g-2)\}.$$

As shown by [Vg], the zeta function $\zeta_{\Gamma,r}(\chi, s)$ has the "Γ-factor". The double Γ-function of Bernes is defied by

$$\Gamma_2(s+1)^{-1}=(2\pi)^{s/2}\exp[-\{s(s+1)-\gamma s^2\}/2]\prod_{n=1}^{\infty}(1+s/n)^n\exp\{-s+s^2/(2n)\}$$

where γ is Euler's constant. The double Γ-function $\Gamma_2(s)$ has the following properties;

$$\log\frac{\Gamma_2(1-s)}{\Gamma_2(1+s)}=s\cdot\log(2\pi)-\int_0^s\pi x\cdot\cot(\pi x)dx,$$
$$\Gamma_2(s+1)=\Gamma(s)^{-1}\Gamma_2(s).$$

Define that Γ-factor $\zeta_{\Gamma,r}^{(\infty)}(\chi, s)$ for $\zeta_{\Gamma,r}(\chi, s)$ by

$$\zeta_{\Gamma,r}^{\infty}(\chi, s)=\begin{cases}\{(2\pi)^s\Gamma_2(s)\Gamma_2(s+1)\}^{\dim\chi\cdot(2g-2)}, & \text{if } r\equiv 0\,(\mathrm{mod}\,2)\\ \{(2\pi)^s\Gamma_2(s+1/2)^2\}^{\dim\chi\cdot(2g-2)}, & \text{if } r\not\equiv 0\,(\mathrm{mod}\,2)\end{cases}$$

Put $\zeta^*_{\Gamma,\tau}(\chi, s)=\zeta^{(\infty)}_{\Gamma,\tau}(\chi, s)\cdot\zeta_{\Gamma,\tau}(\chi, s)$. Then the functional equation of $\zeta_{\Gamma,\tau}(\chi, s)$ is written in the symmetric form

$$\zeta^*_{\Gamma,\tau}(\chi, 1-s)=\zeta^*_{\Gamma,\tau}(\chi, s).$$

Let $n>1$ be an integer such that $n\equiv r\pmod 2$. Then $1-n/2$ is a pole of the "Γ-factor" $\zeta^{(\infty)}_{\Gamma,\tau}(\chi, s)$, and $n/2$ is not a critical point of $\zeta_{\Gamma,\tau}(\chi, s)$ in the sense of [**Dl**]. So it is very natural that the "regulator" R appears in the special value $\zeta_{\Gamma,\tau}(\chi, n/2)=Z_{\Gamma,\tau}(\chi, n-1)$ of the Selberg zeta function in Theorem 6.1 or Theorem 6.2.

3) As considered in 1), the factor $\det\Delta_K$ in Theorem 5.4 corresponds to $R(K)h/w|D|^{-1/4}$ in which appears the algebraic part $h/w|D|^{-1/4}$ of $\mathrm{Res}_{s=1}\zeta_K(s)$. On the other hand, the algebraic part of the special values of the Selberg zeta functions are still in mystery.

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