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On Keen's moduli inequality in two generator Möbius groups

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§1. Introduction and statement of results.

In general, it is difficult to determine whether a two generator Möbius group is Kleinian or not, or is discrete or not. In [1], for the purpose of studying one dimensional Teichmüller spaces, Keen obtained a moduli inequality which assures some two generator Möbius groups are Kleinian. To state her theorem, we need some notation. Let A and B be Möbius transformations and let $G=\langle A, B\rangle$ be the group generated by A and B. By the well known isomorphism between the Möbius group and $PSL(2, \boldsymbol{C})$, we put

$$x=\operatorname{trace}(A), \quad y=\operatorname{trace}(B) \quad \text{and} \quad z=\operatorname{trace}(AB).$$

The groups we are interested in this article are those which satisfy the following.

$$x^2+y^2+z^2=xyz, \tag{1}$$

$$z>2 \quad \text{and} \tag{2}$$

$$|x|>2 \quad \text{and} \quad |y|>2. \tag{3}$$

For those groups Keen showed the following.

THEOREM 1 ([1]). *If the moduli triple* (x, y, z) *satisfies* (1), (2), (3) *and the inequality*

$$|z\operatorname{Im}(x)-2\operatorname{Im}(y)| < 2|\operatorname{Re}(x)|, \tag{4}$$

then the group $G=\langle A, B\rangle$ *is Kleinian.*

On the other hand, we showed the following.

THEOREM 2 ([3]). *Let* $U=\begin{pmatrix}\alpha & 0\\ 0 & \beta\end{pmatrix}$ *and* $V=\begin{pmatrix}a & b\\ c & d\end{pmatrix}$, $bc\neq 0$, *be loxodromic elements of* $PSL(2, \boldsymbol{C})$ *such that* $UVU^{-1}V^{-1}$ *is parabolic. If, for each integer* n, *the inequality*

$$\frac{|\alpha^n a|+|\beta^n d|}{|\alpha^n a+\beta^n d|} < \frac{|\alpha|+|\beta|}{|\alpha-\beta|} \tag{5}$$

holds, then $G'=\langle U, V\rangle$ *is Kleinian.*

Restricting to the case in which $\mathrm{trace}(U)=z>2$ and putting $\mathrm{trace}(V)=y$ and $x=\mathrm{trace}(UV^{-1})$, we see that the triple (x, y, z) satisfies (1), (2) and (3). In this article we compare Theorem 1 with Theorem 2 in this situation. That is, we compare (4) with (5). As a by-product of this comparison, we obtain the following improvement of Theorem 1, which is also an improvement of Theorem 2 in the case $\alpha+\beta>2$.

THEOREM 3. *If the moduli triple* (x, y, z) *satisfying* (1), (2), (3) *also satisfies*

$$\mathrm{Im}(x)\,\mathrm{Im}(y)<0 \tag{6}$$

$$x\bar{y}+\bar{x}y>2z \tag{7}$$

$$2|x|^2>|x\bar{y}-\bar{x}y| \tag{8}$$

$$2|y|^2>|x\bar{y}-\bar{x}y|, \tag{9}$$

then the group $G=\langle A, B\rangle$ *is Kleinian.*

Our comparison of Theorems 1 and 2 relys on the comparison of the two different normalizations. We summarize each of these two normalizations in §2 and §3, respectively. After an observation of some special cases in §4, we compare Theorems 1 and 2 in §5. In §6 we prove Theorem 3.

REMARK 1. Equality (1) is a special case considered in [**1**], but also the most interesting case. Equality (1) implies that the commutator $A^{-1}B^{-1}AB$ is parabolic. Instead of (1), the equality $x^2+y^2+z^2=xyz+4-k$ is studied there, where $k=2-\mathrm{trace}(A^{-1}B^{-1}AB)$. Our statement of Theorem 1 is for the case $k=4$ and it is a main assertion in [**1**] that $\Omega(G)/G$ is a pair of once punctured tori, where $\Omega(G)$ is the region of discontinuity of G.

REMARK 2. In appearance, inequality (4) is not symmetric with respect to x and y. Later we will rewrite (4) so that it is symmetric in x and y. (See (29) in §5.)

REMARK 3. Inequality (6) comes up from our normalization generators.

§2. Normalization I.

In this section we recall the normalization in [**1**]. (See also [**4**].) Throughout this article we assume that the moduli triple (x, y, z) satisfies (1), (2) and (3). By conjugation in $SL(2, \boldsymbol{C})$ we normalize A and B such that

$$A=\frac{1}{2}\begin{pmatrix} x & b_1 \\ c_1 & x \end{pmatrix} \quad \text{and} \quad B=\frac{1}{2}\begin{pmatrix} y & b_2 \\ c_2 & y \end{pmatrix}$$

and that the fixed point of $A^{-1}B^{-1}AB$ is -1. One computes the matrix AB and obtains

$$AB=\frac{1}{4}\begin{pmatrix} xy+b_1c_2 & b_2x+b_1y \\ c_2x+c_1y & xy+c_1b_2 \end{pmatrix}.$$

Since trace $(AB)=z$, we have

$$b_1c_2+c_1b_2=4z-2xy\,. \tag{10}$$

One computes

$$\begin{aligned}(b_1c_2-c_1b_2)^2 &= (b_1c_2+c_1b_2)^2-4b_1c_1b_2c_2 \\ &= (4z-2xy)^2-4(x^2-4)(y^2-4) \\ &= 16(z^2-xyz+x^2+y^2-4) = -64\,.\end{aligned}$$

Hence we obtain $b_1c_2-c_1b_2=\pm 8i$, where $i=\sqrt{-1}$. We choose a sign such that

$$b_1c_2-c_1b_2=8i\,. \tag{11}$$

Now we can write down b_1, c_1, b_2 and c_2 by the moduli (x, y, z).

LEMMA 1. *$b_1=x-\frac{2}{z}(y-ix)$, $c_1=x-\frac{2}{z}(y+ix)$,*

$b_2=y-\frac{2}{z}(x+iy)$ and $c_2=y-\frac{2}{z}(x-iy)$.

PROOF. From (10) and (11) we obtain $b_1c_2=2z-xy+4i$ so that

$$\frac{1}{4}(xy+b_1c_2)=\frac{1}{2}(z+2i)\,.$$

Likewise, we obtain

$$\frac{1}{4}(xy+c_1b_2)=\frac{1}{2}(z-2i)\,.$$

Therefore we see that, for some X and Y,

$$AB=\frac{1}{2}\begin{pmatrix} z+2i & X \\ Y & z-2i \end{pmatrix}.$$

Comparing two matrices AB and BA, we see that

$$BA=\frac{1}{2}\begin{pmatrix} z-2i & X \\ Y & z+2i \end{pmatrix}.$$

Our normalization that the fixed point of $A^{-1}B^{-1}AB$ is -1 implies that $AB(-1)=BA(-1)$. So we obtain easily

$$X+Y=2z\,.$$

On the other hand, computing the determinant of AB, we have

$$XY = z^2.$$

Therefore we have $X=Y=z$ and

$$AB = \frac{1}{2}\begin{pmatrix} z+2i & z \\ z & z-2i \end{pmatrix}. \tag{12}$$

From the equation

$$4AB = \begin{pmatrix} xy+b_1c_2 & b_2x+b_1y \\ c_2x+c_1y & xy+c_1b_2 \end{pmatrix} = \begin{pmatrix} 2z+4i & 2z \\ 2z & 2z-4i \end{pmatrix}$$

one derives

$$xy^2+b_1c_2y = 2(z+2i)y \quad \text{and} \quad b_2c_2x+b_1c_2y = 2zc_2.$$

Noting $y^2-b_2c_2=4$, one obtains from these equations

$$c_2 = y-\frac{2}{z}(x-iy).$$

Similar computations yield the remaining three equations.

Thus we showed that *the moduli triple* (x, y, z) *and a normalization with the choice* (11) *determine* A *and* B *uniquely and so* G, *too.*

Let p_{AB} and q_{AB} be the repelling and the attractive fixed points of AB, respectively. Then

$$p_{AB} = -\frac{\sqrt{z^2-4}}{z}+\frac{2}{z}i \quad \text{and} \quad q_{AB} = \frac{\sqrt{z^2-4}}{z}+\frac{2}{z}i. \tag{13}$$

Let K be the circle passing through three points p_{AB}, q_{AB} and $(x+2)/c_1$. Then it is shown in [1] that K passes through more three points $(x-2)/c_1$ and $-(y\pm 2)/c_2$ and that the center c_K of K lies on the imaginary axis of the complex plane, perhaps at the point at infinity. If c_1 is not real, then c_K does not lie at infinity. Then, from the equation of equidistance of two points from the center

$$|(x+2)/c_1-c_K| = |(x-2)/c_1-c_K|,$$

we have

$$c_K = \frac{x+\bar{x}}{c_1-\bar{c}_1}. \tag{14}$$

Denoting by r_K the radius of K, we can state that the geometric form of Keen's inequality (4) is

$$|c_K| > r_K. \tag{15}$$

For completeness, we derive (4) from (15).

LEMMA 2. *Inequality* (4) *is equivalent to* (15).

PROOF. First we shall show that (15) is equivalent to

$$(b_1-\bar{b}_1)(c_1-\bar{c}_1)>0. \tag{16}$$

Since r_K is the distance between c_K and $(x+2)/c_1$, (15) is equivalent to

$$|x+\bar{x}||c_1|>|x\bar{c}_1+\bar{x}c_1-2(c_1-\bar{c}_1)|.$$

The square of the right hand side of the above is

$$\begin{aligned}(x\bar{c}_1+\bar{x}c_1)^2-4(c_1-\bar{c}_1)^2 &= (x^2-4)\bar{c}_1^2+(\bar{x}^2-4)c_1^2+2(|x|^2+4)|c_1|^2 \\ &= (b_1\bar{c}_1+\bar{b}_1c_1+8+2|x|^2)|c_1|^2,\end{aligned}$$

so (15) is equivalent to

$$x^2+\bar{x}^2>b_1\bar{c}_1+\bar{b}_1c_1+8.$$

Making use of the equations $x^2-b_1c_1=4$ and $\bar{x}^2-\bar{b}_1\bar{c}_1=4$, one easily obtains (16). Thus we have shown that (15) and (16) are equivalent. By Lemma 1 we have

$$b_1-\bar{b}_1=2i\Big(\mathrm{Im}(x)-\frac{2}{z}\mathrm{Im}(y)+\frac{2}{z}\mathrm{Re}(x)\Big)$$

and

$$c_1-\bar{c}_1=2i\Big(\mathrm{Im}(x)-\frac{2}{z}\mathrm{Im}(y)-\frac{2}{z}\mathrm{Re}(x)\Big).$$

Therefore (16) is written as

$$\Big(\mathrm{Im}(x)-\frac{2}{z}\mathrm{Im}(y)\Big)^2-\Big(\frac{2}{z}\mathrm{Re}(x)\Big)^2<0.$$

This is equivalent to (4).

By considering the normal subgroup, $H=\langle AB, BA, A^{-1}B, BA^{-1}\rangle$, of G of index 2, the following is shown under this normalization.

THEOREM 4 ([**4**]). *If the moduli triple* (x, y, z) *satisfies* (1), (2), (3) *and* $x=\bar{y}$, *then* G *is Kleinian.*

§ 3. Normalization II.

In this section we recall some results and normalizations in [**3**]. Let (x, y, z) be the moduli triple satisfying (1), (2) and (3). Let U and V be elements of $SL(2, \boldsymbol{C})$ such that $\mathrm{trace}(U)=z$ and $\mathrm{trace}(V)=y$ and that they satisfy (1) with $x=\mathrm{trace}(UV^{-1})$. Conjugating by a Möbius transformation, we normalize U and V so that

$$U=\begin{pmatrix}\alpha & 0\\ 0 & \beta\end{pmatrix} \quad\text{and}\quad V=\begin{pmatrix}a & b\\ c & d\end{pmatrix},$$

where $\alpha\beta=1$, $\alpha>\beta$ *and* $ad-bc=1$. We further conjugate U and V by $\begin{pmatrix}k & 0\\ 0 & 1/k\end{pmatrix}$ with some complex number k so that α, β, a and d are invariant and the following holds:

$$c=\frac{\alpha+\beta}{\alpha-\beta}. \tag{17}$$

We note the following.

LEMMA 3. *Equation* (1) *is equivalent to saying that the commutator* $UVU^{-1}V^{-1}$ *is parabolic. It is also equivalent to*

$$ad=c^2=\frac{z^2}{z^2-4}=\frac{(\alpha+\beta)^2}{(\alpha-\beta)^2}. \tag{18}$$

PROOF. First we note that the second and the third equalities of (18) are clear from (17). Substituting $x=\beta a+\alpha d$, $y=a+d$ and $z=\alpha+\beta$ into (1), we see that (1) is equivalent to $(\alpha+\beta)^2=(\alpha-\beta)^2 ad$. It follows that (1) is equivalent to (18). One computes

$$\begin{aligned}\operatorname{trace}(UVU^{-1}V^{-1}) &= 2ad-(\alpha^2+\beta^2)bc\\ &= -(\alpha-\beta)^2 ad+(\alpha+\beta)^2-2.\end{aligned}$$

It follows that (18) is equivalent to $\operatorname{trace}(UVU^{-1}V^{-1})=-2$ which is equivalent to $UVU^{-1}V^{-1}$ being parabolic.

As a special case of Theorem 2, the following is shown.

PROPOSITION 5 ([**3**]). *Let U and V be in Theorem 2. Let C_1 and C_2 be the circles such that*

$$\begin{aligned}C_1&=\{w\in \boldsymbol{C}\mid |w-a/c|=1/|d|\} \quad \textit{and}\\ C_2&=\{w\in \boldsymbol{C}\mid |w+d/c|=1/|a|\}.\end{aligned} \tag{19}$$

If C_1 and C_2 are separated by the imaginary axis, then $G'=\langle U, V\rangle$ is Kleinian.

Our comparison is made by Proposition 5 and it will be shown in § 5 that Theorem 1 and Proposition 5 are equivalent. Our improvement of Theorem 2 in our case is made in an altered form of it. (See also Proposition 6′ in § 6.) To state this we need one more piece of notation. Put

$$p_1=\frac{a}{c}+\frac{1}{d} \quad\text{and}\quad p_2=\frac{a}{c}-\frac{1}{d} \tag{20}$$

and denote by R the ring domain bounded by the circles

$$R_i=\{w\in \boldsymbol{C}\mid |w|=|p_i|\} \qquad (i=1, 2). \tag{21}$$

Now we can state Theorem 2 in our case in the following form.

PROPOSITION 6 ([**3**]). *Let U and V be as in Theorem 2 and let U be hyperbolic. Let n be the integer such that $U^n(C_2)\cap R\neq\emptyset$ and $U^i(C_2)\cap R=\emptyset$ ($i\neq n$,*

$n+1)$. *If* C_1 *intersects neither* $U^n(C_2)$ *nor* $U^{n+1}(C_2)$, *then* $G'=\langle U, V\rangle$ *is Kleinian.*

PROOF. The geometric meaning of (5) is that, for each n, $U^n(C_2)$ does not meet C_1. Since C_1 is contained in the closure of the ring domain R, it suffices to check this for those $U^n(C_2)$ which meet R. The condition of the proposition is just for such $U^n(C_2)$'s.

§ 4. Special cases.

In this section we shall dispose of some special cases in order to ease our discussion in the succesive sections. We write x and y as the sum of their real and imaginary parts so that

$$x=x_1+ix_2 \quad \text{and} \quad y=y_1+iy_2\,.$$

Then, separating the real and the imaginary parts of (1), we have

$$x_1^2-x_2^2+y_1^2-y_2^2+z^2=(x_1y_1-x_2y_2)z \tag{22}$$

and

$$2(x_1x_2+y_1y_2)=(x_1y_2+x_2y_1)z\,. \tag{23}$$

PROPOSITION 7. *If* $x_1=y_1=0$, *then the group* $G=\langle A, B\rangle$ *is Kleinian.*

PROOF. By Lemma 1 we have $b_1=-2x_2/z+i(x_2-2y_2/z)$ and $c_1=2x_2/z+i(x_2-2y_2/z)$ so that $b_1=-\bar{c}_1$. We also have $b_2=-\bar{c}_2$. It follows that

$$A=\frac{1}{2}\begin{pmatrix} ix_2 & -\bar{c}_1 \\ c_1 & ix_2 \end{pmatrix} \quad \text{and} \quad B=\frac{1}{2}\begin{pmatrix} iy_2 & -\bar{c}_2 \\ c_2 & iy_2 \end{pmatrix}.$$

It is easy to see that both A and B leave invariant the unit circle $\{w\in \boldsymbol{C}\,|\,|w|=1\}$. Therefore the limit set of G is contained in the unit circle. Hence G is Kleinian.

Because of this proposition we henceforth assume

$$x_1\neq 0 \quad \text{or} \quad y_1\neq 0\,. \tag{A0}$$

Next we consider the case in which either x or y is real.

PROPOSITION 8 (LEMMA A.2 in [2]). *Let* $\langle X, Y\rangle\subset PSL(2, \boldsymbol{C})$ *be a free group such that* $\mathrm{trace}(XYX^{-1}Y^{-1})=-2$. *If* $\mathrm{trace}(X)$ *and* $\mathrm{trace}(Y)$ *are both real, then* $\langle X, Y\rangle$ *is a quasi-Fuchsian group.*

Putting $X=AB$ and $Y=A$ or B, we see by Lemma 3 that $\mathrm{trace}(XYX^{-1}Y^{-1})=-2$. If either x or y is real, then Proposition 8 implies that $G=\langle A, B\rangle=\langle AB, A\rangle=\langle AB, B\rangle$ is Kleinian. Henceforth we assume

$$x_2\neq 0 \quad \text{and} \quad y_2\neq 0\,. \tag{A1}$$

If $x_1=0$ or $y_1=0$, say $x_1=0$, then we have by (23) and (A0) that $y_2=x_2z/2$. Substituting this into (22), we have $y_1^2+((z/2)^2-1)x_2^2+z^2=0$. In view of (2) this equation does not hold. Hence we may assume hereafter

$$\text{(A2)} \qquad x_1 \neq 0 \quad \text{and} \quad y_1 \neq 0 .$$

For the case in which $|x|=|y|$ we have

LEMMA 4. *If* $|x|=|y|$, *then* $y=\bar{x}$.

PROOF. First we shall show that $|x|=|y|$ implies

$$x_1x_2+y_1y_2=x_1y_2+x_2y_1=0 .$$

Assume to the contrary that $x_1y_2+x_2y_1\neq 0$. Then by (23) we have $x_1x_2+y_1y_2 \neq 0$. We also have

$$z=\frac{2(x_1x_2+y_1y_2)}{x_1y_2+x_2y_1}. \tag{24}$$

Substituting (24) into (22) and factoring, we have

$$4(x_1x_2+y_1y_2)^2=(x_1y_2+x_2y_1)(x_2y_1-x_1y_2)(|x|^2-|y|^2). \tag{25}$$

The assumption $|x|=|y|$ and (25) imply $x_1x_2+y_1y_2=0$, a contradiction. Hence we have shown that $x_1y_2+x_2y_1=0$. By (23) we have $x_1x_2+y_1y_2=0$. Thus we have the desired equalities.

Now, adding $2(x_1x_2+y_1y_2)=0$ to $|x|^2=|y|^2$, we have

$$(x_1+x_2-y_1+y_2)(x_1+x_2+y_1-y_2)=0 .$$

If $x_1+x_2-y_1+y_2=0$ (resp. $x_1+x_2+y_1-y_2=0$), then we have from $x_1x_2+y_1y_2=0$ the following.

$$(x_1+y_2)(x_2+y_2)=0 \qquad (\text{resp. } (x_1+y_1)(x_2+y_1)=0).$$

There are four cases to consider.

CASE I. $y_1=x_1+x_2+y_2$ and $y_2=-x_1$: In this case we have $y=y_1+iy_2=(x_1+x_2-x_1)+(-x_1)i=-i(x_1+ix_2)=-ix$. Then (1) becomes $z^2=-ix^2z$. Hence we have $z=-ix^2=2x_1x_2-i(x_1^2-x_2^2)$. By (2) we have $x_1=x_2$ so that $x=(1+i)x_1$. It follows that $y=-ix=(1-i)x_1=\bar{x}$.

CASE II. $y_1=x_1+x_2+y_2$ and $y_2=-x_2$: In this case we have $y=(x_1+x_2-x_2)+i(-x_2)=x_1-ix_2=\bar{x}$.

CASE III. $y_2=x_1+x_2+y_1$ and $y_1=-x_1$: In this case we have $y=-x_1+(x_1+x_2-x_1)i=-x_1+ix_2=-\bar{x}$. Then (1) becomes $x^2+\bar{x}^2+z^2=-|x|^2z$. Substituting $x=x_1+ix_2$ into this equation, we have

$$(z+2)x_1^2+(z-2)x_2^2+z^2=0 .$$

But (2) implies that the left hand side of this is positive. Hence this case does not occur.

CASE IV. $y_2=x_1+x_2+y_1$ and $y_1=-x_2$: In this case we have $y=-x_2+i(x_1+x_2-x_2)=ix$. Then (1) becomes $z^2=ix^2z$. Hence we have $z=ix^2=-2x_1x_2+i(x_1^2-x_2^2)$. By (2) we have $x_2=-x_1$ so that $x=(1-i)x_1$ and $y=ix=(1+i)x_1=\bar{x}$.

Thus we have shown that $y=\bar{x}$ in all cases and the proof of the lemma is completed.

By Theorem 4 and Lemma 4 we can omit the case in which $|x|=|y|$, so we assume hereafter

(A3) $$|x| \neq |y|.$$

It is not hard to see from (23), (A1) and (A2) that if $x_1y_2+x_2y_1=0$, then $|x|=|y|$. So we assume hereafter

(A4) $$x_1y_2+x_2y_1 \neq 0.$$

We note that (A4) and (23) imply that (24) and (25) are available. For later use we prove two lemmas.

LEMMA 5. $(x_1^2-y_1^2)(x_2^2-y_2^2)>0$.

PROOF. By (A4) we see that $xy\neq\bar{x}\bar{y}$. So from (1) and $\bar{x}^2+\bar{y}^2+z^2=\bar{x}\bar{y}z$ we have $z=(x^2+y^2-\bar{x}^2-\bar{y}^2)/(xy-\bar{x}\bar{y})$. From this equality one computes and obtains

$$z+2 = 2(x_1+y_1)(x_2+y_2)/(x_1y_2+x_2y_1)$$

and

$$z-2 = 2(x_1-y_1)(x_2-y_2)/(x_1y_2+x_2y_1).$$

Hence we have

(26) $$z^2-4 = 4(x_1^2-y_1^2)(x_2^2-y_2^2)/(x_1y_2+x_2y_1)^2.$$

By (2) we have Lemma 5.

LEMMA 6. $x_1y_1>0$.

PROOF. In view of (25) there are several cases to consider. First we assume $|x|>|y|$. If $x_1y_2+x_2y_1>0$ (resp. <0), then by (24) we have $x_1x_2+y_1y_2>0$ (resp. <0). By Lemma 5 we have $|x_1x_2|>|y_1y_2|$. It follows that $x_1x_2>0$ (resp. <0). By (25) we have $x_2y_1-x_1y_2>0$ (resp. <0) so that $x_2y_1>0$ (resp. <0). Therefore we obtain $x_1x_2^2y_1>0$. Hence $x_1y_1>0$. Next we assume $|x|<|y|$. If $x_1y_2+x_2y_1>0$ (resp. <0), then by (24) we have $x_1x_2+y_1y_2>0$ (resp. <0). By our assumption $|x|<|y|$ and by Lemma 5 we have $|y_1y_2|>|x_1x_2|$. It follows that $y_1y_2>0$ (resp. <0). By (25) we have $x_2y_1-x_1y_2<0$ (resp. >0)

so that we have $x_1y_2>0$ (resp. <0). Hence we obtain $x_1y_1y_2^2>0$ so that $x_1y_1>0$. Thus we have shown in all cases that $x_1y_1>0$.

§ 5. Comparison.

In this section we shall show that Theorem 1 and Proposition 5, which is a special case of Theorem 2, are equivalent. As mentioned in § 4, we are assuming (A1)~(A4) and so (24) and (25) are available. Recall that $V=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $a+d=y$ under normalization II. Write a and d as the sum of their real and imaginary parts so that

$$a=a_1+ia_2, \quad \text{and} \quad d=d_1+id_2 .$$

LEMMA 7. $a_1d_1>0$.

PROOF. By (A2) and (18) we have i) $a_1+d_1\neq 0$, ii) $a_1d_1-a_2d_2=c^2$ and iii) $a_1d_2+a_2d_1=0$. If $a_1=0$, then by i) we have $d_1\neq 0$. Then iii) implies $a_2=0$ so that $a=0$. But $ad=c^2\neq 0$, it does not occur. Hence $a_1\neq 0$. If $a_2=0$, then by iii) we have $d_2=0$. Then $y_2=a_2+d_2=0$. This contradicts (A1). Hence $a_2\neq 0$. From ii) and iii) we then have

$$d_2=(a_1d_1-c^2)/a_2=-a_2d_1/a_1 ,$$

so that we have $|a|^2d_1=c^2a_1$. Since $c^2>0$, this implies that a_1 and d_1 must have the same sign.

LEMMA 8. *The condition of Proposition* 5 *is equivalent to*

$$\frac{z}{2}\left(c-\frac{1}{c}\right)>\left|\frac{d-\bar{d}}{d+\bar{d}}\right|. \tag{27}$$

PROOF. By Lemma 7 and (17) we see that the real parts of the centers a/c and $-d/c$ of the circles C_1 and C_2, respectively, have different signs. Moreover we see from (18) and (19) that C_1 and the positive multiple $|a/d|$ of C_2 are symmetric with respect to the imaginary axis. Hence the condition of Proposition 5 that C_1 and C_2 are separated by the imaginary axis is equivalent to saying that if l_1 and l_2 are lines passing through 0 and tangent to C_1 at t_1 and at t_2, respectively, then the real parts of t_1 and t_2 have the same sign. A calculation shows that

$$t_1=\frac{1}{d}\left(c-\frac{1}{c}+\frac{2}{z}i\right) \quad \text{and} \quad t_2=\frac{1}{d}\left(c-\frac{1}{c}-\frac{2}{z}i\right). \tag{28}$$

Hence we have

$$\mathrm{Re}(t_1) = ((c-1/c)(d+d)-2i(d-\bar{d})/z)/|d|^2$$

and

$$\mathrm{Re}(t_2) = ((c-1/c)(d+\bar{d})+2i(d-\bar{d})/z)/|d|^2 .$$

So the condition $\mathrm{Re}(t_1)\,\mathrm{Re}(t_2)>0$ is equivalent to

$$(c-1/c)^2(d+\bar{d})^2+4(d-\bar{d})^2/z^2 > 0 .$$

Noting $(d-\bar{d})^2=-|d-\bar{d}|^2$, we obtain (27).

To compare (4) and (27) we need to rewrite (27) in x and y.

LEMMA 9. $\dfrac{z}{2}\left(c-\dfrac{1}{c}\right) = |x_1y_2+x_2y_1|/\sqrt{(x_1^2-y_1^2)(x_2^2-y_2^2)}$.

PROOF. By (26) we have

$$(z/2)^2-1 = (x_1^2-y_1^2)(x_2^2-y_2^2)/(x_1y_2+x_2y_1)^2$$

and by (17) we have

$$\frac{z}{2}\left(c-\frac{1}{c}\right) = 2/(\alpha-\beta) = 1/\sqrt{(z/2)^2-1} .$$

Hence the lemma follows.

LEMMA 10. $\left|\dfrac{d-\bar{d}}{d+\bar{d}}\right| = \sqrt{(x_1^2-y_1^2)/(x_2^2-y_2^2)}$.

PROOF. We know that a and d are the solutions of the equation $X^2-(a+d)X+ad=X^2-yX+c^2=0$. One checks that $(y\pm(y_2Y+iy_1/Y))/2$ satisfy the equation, where we put $Y=\sqrt{(x_1^2-y_1^2)/(x_2^2-y_2^2)}$. Hence we have

$$(d-\bar{d})/(d+\bar{d}) = i(y_2\pm y_1/Y)/(y_1\pm y_2Y) = \pm iY .$$

Thus we have Lemma 10.

It follows from Lemmas 9 and 10 that (27) can be written as

$$|x_1y_2+x_2y_1| > |x_2^2-y_2^2| . \tag{29}$$

LEMMA 11. *Keen's inequality* (4) *is equivalent to* (29).

PROOF. Substituting (24) into (4), we have

$$\left|\frac{x_1x_2+y_1y_2}{x_1y_2+x_2y_1}x_2-y_2\right| < |x_1| .$$

Making use of (A2), one can reduce this to (29) easily.

Thus we have completed our comparison with the conclusion that *Theorem* 1 *and Proposition* 5 *are equivalent.*

§6. Improvement and the proof of Theorem 3.

In this section we shall improve Proposition 6 under normalization I yielding a proof of Theorem 3. To compare two normalizations I and II we introduce a map. Let

$$T(w)=q_{AB}\frac{w-t_1}{w-t_2}, \tag{30}$$

where q_{AB} is in (13) and t_1, t_2 are in (28). The following lemmas explain the map T.

LEMMA 12. *$T(0)=p_{AB}$, $T(\infty)=q_{AB}$, $T(p_1)=1$ and $T(p_2)=-1$, where p_{AB} is in* (13) *and p_1, p_2 are in* (20).

PROOF. We note that by (17) and (18) we can write (13) in the following form:

$$p_{AB}=-1/c+2i/z \quad \text{and} \quad q_{AB}=1/c+2i/z\,.$$

To show $T(0)=p_{AB}$ it suffices to show

$$(1/c+2i/z)(c-1/c+2i/z)=(-1/c+2i/z)(c-1/c-2i/z)\,.$$

Making use of the identity

$$c^2=z^2/(z^2-4) \quad \text{or} \quad 1/c^2+4/z^2=1\,, \tag{31}$$

one can check the above equality. The second is clear. Since $p_1=(c+1)/d$, to see $T(p_1)=1$ it suffices to show

$$(1/c+2i/z)(1+1/c-2i/z)=1+1/c+2i/z\,.$$

It is easy to see that this is equivalent to (31), so we have the third. The same computation shows that $T(p_2)=-1$ with $p_2=(c-1)/d$ in place of p_1.

LEMMA 13. *$TUT^{-1}=AB$.*

PROOF. Both TUT^{-1} and AB have the same fixed points p_{AB} and q_{AB} and map -1 to 1, so they are identical.

We denote by D_{AB} and D'_{AB} the isometric circles of AB and $(AB)^{-1}$, respectively. By (12) we see that the centers of D_{AB} and D'_{AB} are $-1+2i/z$ and $1+2i/z$, respectively. Since their radii are identical and equal $2/z$, they are tangent to the real axis at -1 and at 1, respectively. Denoting by $\boldsymbol{R}$ the real axis, we have the following.

LEMMA 14. *$T(C_1)=\boldsymbol{R}$, $T(R_1)=D'_{AB}$ and $T(R_2)=D_{AB}$, where C_1 is in* (19) *and R_1, R_2 are in* (21).

PROOF. Recall that p_1 and p_2 are points on C_1. Lemma 12 tells us that $T(p_1)$ and $T(p_2)$ lie on the real axis. Also recall that t_1 is the point at which l_1 and C_1 are tangent. Hence t_1 is the third point on C_1 and $T(t_1)=0$ by (30). Therefore we have $T(C_1)=\boldsymbol{R}$. Next we show $T(R_1)=D'_{AB}$. Since C_1 and R_1 are tangent at p_1, Lemma 12 tells us that $T(R_1)$ is a circle tangent to $\boldsymbol{R}$ at -1. By (20) and (21) we see that $-p_1$ lies on R_1. Hence it suffices to show that $T(-p_1)$ lies on D'_{AB}. One computes and obtains $T(-p_1)=(z^2-4+4zi)/(z^2+4)$. It is easy to see that this point lies on D'_{AB}. Thus we have shown that $T(R_1)=D'_{AB}$. A similar argument and computation give us $T(R_2)=D_{AB}$.

Next we shall show that the generators A and B are conjugate to UV^{-1} and to V by T, respectively. In order to show this we prove two lemmas.

LEMMA 15. *The fixed points of TVT^{-1} are symmetric with respect to the origin.*

PROOF. Writing the matrices of T and T^{-1} in $GL(2, \boldsymbol{C})$ as

$$\begin{pmatrix} q_{AB} & -q_{AB}t_1 \\ 1 & -t_2 \end{pmatrix} \text{ and } \begin{pmatrix} t_2 & -q_{AB}t_1 \\ 1 & -q_{AB} \end{pmatrix},$$

respectively, one computes that the matrix of TVT^{-1} is

$$\begin{pmatrix} q_{AB}(at_2-dt_1-ct_1t_2+b) & * \\ * & q_{AB}(-at_1+dt_2+ct_1t_2-b) \end{pmatrix}.$$

Hence it suffices to show that

$$at_2-dt_1-ct_1t_2+b = -at_1+dt_2+ct_1t_2-b\,.$$

This is equivalent to

$$(a-d)(t_1+t_2)-2ct_1t_2+2b = 0\,.$$

Making use of the equalities $t_1+t_2=2(c-1/c)/d$, $t_1t_2=(c^2-1)/d^2$, $b=(c^2-1)/c$ and $d=c^2/a$, which follow easily from (28), (17) and (18), and of (31), one checks this easily.

LEMMA 16. *The fixed points of $T(UV^{-1})T^{-1}$ are symmetric with respect to the origin.*

PROOF. The idea of the proof is the same as that of Lemma 15. Noting that the matrix of UV^{-1} is $\begin{pmatrix} \alpha d & -\alpha b \\ -\beta c & \beta a \end{pmatrix}$, one computes the matrix of $TUV^{-1}T^{-1}$; the $1-1$ element is $q_{AB}(-\beta at_1+\alpha dt_2+\beta ct_1t_2-\alpha b)$ and the $2-2$ element is $q_{AB}(\beta at_2-\alpha dt_1-\beta ct_1t_2+\alpha b)$. Hence it suffices to show that

$$-\beta at_1+\alpha dt_2+\beta ct_1t_2-\alpha b = \beta at_2-\alpha dt_1-\beta ct_1t_2+\alpha b$$

or

$$(\beta a-\alpha d)(t_1+t_2)-2\beta ct_1t_2+2\alpha b=0.$$

Making use of the five equalities in the proof of Lemma 15, one checks this easily.

By [3] or by a straightforward computation we see that p_2 is the fixed point of the parabolic element $VU^{-1}V^{-1}U$. Lemma 12 tells us that the fixed point of $T(VU^{-1}V^{-1}U)T^{-1}$ is at -1. Since $T(UV^{-1})^{-1}V^{-1}(UV^{-1})VT^{-1}=T(VU^{-1}V^{-1}U)T^{-1}$, we see by Lemmas 15 and 16 that $T(UV^{-1})T^{-1}$ and TVT^{-1} have the same normalization as A and B. Lemma 13 says that our choice (17) in normalization II is compatible with (11) in normalization I. Therefore, by the uniqueness of A and B in normalization I we have the following.

PROPOSITION 9. *$T(UV^{-1})T^{-1}=A$ and $TVT^{-1}=B$.*

Now we see by Proposition 9 that, under normalization I, Proposition 6 assumes the following form.

PROPOSITION 6′. *Assume that $A(\boldsymbol{R})\cap D'_{AB}\neq\emptyset$ and $B^{-1}(\boldsymbol{R})\cap D_{AB}\neq\emptyset$. If each of $A(\boldsymbol{R})$ and $B^{-1}(\boldsymbol{R})$ does not meet $\boldsymbol{R}$, then $G=\langle A, B\rangle$ is Kleinian.*

Here we show that the first assumption of Proposition 6′ is equivalent to (6).

LEMMA 17. *(6) is equivalent to the conditions*

$$A(\boldsymbol{R})\cap D'_{AB}\neq\emptyset \quad \text{and} \quad B^{-1}(\boldsymbol{R})\cap D_{AB}\neq\emptyset.$$

PROOF. The condition that $A(\boldsymbol{R})\cap D'_{AB}\neq\emptyset$ and $B^{-1}(\boldsymbol{R})\cap D_{AB}\neq\emptyset$ is equivalent to that $C_2\cap R_2\neq\emptyset$ and $U(C_2)\cap R_1\neq\emptyset$. These are equivalent to the condition that the absolute values of the centers of C_2 and $U(C_2)$ satisfy the following:

$$|d|/c<|a|/c<\alpha^2|d|/c.$$

It is also equivalent to the conditions that $|a|-|d|>0$ and $\beta|a|-\alpha|d|<0$. Writing $a=|a|e^{i\theta}$ and $d=|d|e^{-i\theta}$, ($\theta\neq 0$, π by (A1)), we see that

$$\operatorname{Im}(\operatorname{trace}(V))=\operatorname{Im}(a+d)=(|a|-|d|)\sin\theta$$

and

$$\operatorname{Im}(\operatorname{trace}(VU^{-1}))=\operatorname{Im}(\beta a+\alpha d)=(\beta|a|-\alpha|d|)\sin\theta.$$

Since $x_2=\operatorname{Im}(\operatorname{trace}(VU^{-1}))$ and $y_2=\operatorname{Im}(\operatorname{trace}(V))$, it follows that the condition is equivalent to $x_2y_2<0$.

We set $A(\boldsymbol{R})=D_{AB^{-1}}$, $A^{-1}(\boldsymbol{R})=D'_{A^{-1}B}$, $B(\boldsymbol{R})=D'_{AB^{-1}}$ and $B^{-1}(\boldsymbol{R})=D_{A^{-1}B}$. Then $D'_{A^{-1}B}=A^{-1}B(D_{A^{-1}B})$ and $D'_{AB^{-1}}=BA^{-1}(D_{BA^{-1}})$. By the normalization of A and B we see that $D_{A^{-1}B}$ and $D'_{BA^{-1}}$ (resp. $D'_{BA^{-1}}$ and $D_{A^{-1}B}$) are symmetric with respect to the origin.

PROPOSITION 10. *Under the assumption of Proposition 6', the exterior of eight circles D_{AB}, D'_{AB}, D_{BA}, D'_{BA}, $D_{A^{-1}B}$, $D'_{A^{-1}B}$, $D_{BA^{-1}}$ and $D'_{BA^{-1}}$ is a fundamental domain for the normal subgroup $H=\langle AB, BA, A^{-1}B, BA^{-1}\rangle$ of G.*

PROOF. The assumption of Proposition 6' tells us that the exterior of the eight circles consists of two polygonal regions, one is bounded and contains the origin and the other is unbounded. It is not hard to check the cycle condition of the Poincaré theorem. It follows that it is a fundamental domain for H. It is easy to see that the quotient of the region of discontinuity $\Omega(H)$ by H is a pair of twice punctured surfaces of genus 2.

The proof of Proposition 10 does not need the condition that both $D_{A^{-1}B}$ and $D_{BA^{-1}}$ do not intersect the real axis, $\boldsymbol{R}$, but only the fact, which the condition implies, that they do not intersect four circles lying in the lower half plane. So we have the following.

PROPOSITION 11. *Let G be a group satisfying* (1), (2), (3) *and* (6). *If the eight conditions below are satisfied, then the exterior of eight circles in Proposition* 10 *is a fundamental domain for H and so G is Kleinian*:

i) $D_{A^{-1}B}\cap D_{BA}=\emptyset$ ii) $D_{A^{-1}B}\cap D'_{A^{-1}B}=\emptyset$ iii) $D_{A^{-1}B}\cap D'_{BA^{-1}}=\emptyset$ iv) $D_{A^{-1}B}\cap D'_{BA}=\emptyset$ v) $D_{BA^{-1}}\cap D_{BA}=\emptyset$ vi) $D_{BA^{-1}}\cap D'_{A^{-1}B}=\emptyset$ vii) $D_{BA^{-1}}\cap D'_{BA^{-1}}=\emptyset$ viii) $D_{BA^{-1}}\cap D'_{BA}=\emptyset$.

We shall write these eight conditions in terms of moduli (x, y, z) and then compare them with the conditions of Theorem 3. Before to do this we need two preparatory lemmas.

LEMMA 18. *The centers and the radii of $D_{A^{-1}B}$ and $D_{BA^{-1}}$ are*

$$(|y|^2-b_2\bar{c}_2)/(y\bar{c}_2-\bar{y}c_2) \quad \textit{and} \quad -(|x|^2-b_1\bar{c}_1)/(x\bar{c}_1-\bar{x}c_1)$$

$$4/|y\bar{c}_2-\bar{y}c_2| \quad \textit{and} \quad 4/|x\bar{c}_1-\bar{x}c_1|,$$

respectively.

PROOF. Since $D_{A^{-1}B}=B^{-1}(\boldsymbol{R})$, it passes through $B^{-1}(0)=-b_2/y$, $B^{-1}(\infty)=-y/c_2$ and $B^{-1}(1)=(y-b_2)/(y-c_2)$. Elementary geometry gives us the values for the center and the radius of $D_{A^{-1}B}$ as stated in the lemma. The statement for $D_{BA^{-1}}$ follows similarly.

By Lemma 1 we have easily the following.

LEMMA 19.

$$|x|^2-b_1\bar{c}_1=\frac{2}{z}(x\bar{y}+\bar{x}y-2i|x|^2)+\frac{4}{z^2}(|x|^2-|y|^2-i(x\bar{y}+\bar{x}y))$$

$$|y|^2-b_2\bar{c}_2=\frac{2}{z}(x\bar{y}+\bar{x}y+2i|y|^2)-\frac{4}{z^2}(|x|^2-|y|^2+i(x\bar{y}-\bar{x}y))$$

$$x\bar{b}_1-\bar{x}b_1=-\frac{2i}{z}(2|x|^2-i(x\bar{y}-\bar{x}y))$$

$$x\bar{c}_1-\bar{x}c_1=\frac{2i}{z}(2|x|^2+i(x\bar{y}-\bar{x}y))$$

$$y\bar{b}_2-\bar{y}b_2=\frac{2i}{z}(2|y|^2-i(x\bar{y}-\bar{x}y))$$

$$y\bar{c}_2-\bar{y}c_2=-\frac{2i}{z}(2|y|^2+i(x\bar{y}-\bar{x}y)).$$

Now we shall rewrite the eight conditions. We shall begin with iii).

iii) Since $D_{A^{-1}B}$ and $D'_{BA^{-1}}$ are symmetric with respect to the origin, Lemma 18 tells us that the condition is equivalent to

$$||y|^2-b_2\bar{c}_2|>4.$$

Making use of the equality $16=(y^2-b_2c_2)(\bar{y}^2-\bar{b}_2\bar{c}_2)$, one reduces the inequality $(|y|^2-b_2\bar{c}_2)(|y|^2-\bar{b}_2c_2)>16$, which is equivalent to the above, to

$$(y\bar{b}_2-\bar{y}b_2)(y\bar{c}_2-\bar{y}c_2)>0.$$

By Lemma 19 one reduces this to

$$4|y|^4+(x\bar{y}-\bar{x}y)^2>0.$$

This is equivalent to

$$2|y|^2>|x\bar{y}-\bar{x}y|. \tag{9}$$

vi) By a computation similar to iii) we can rewrite the condition as

$$2|x|^2>|x\bar{y}-\bar{x}y|. \tag{8}$$

i) The center and the radius of D_{BA} are $-1-2i/z$ and $2/z$, respectively. It follows from Lemma 18 that the condition is equivalent to

$$|(|y|^2-b_2\bar{c}_2)/(y\bar{c}_2-\bar{y}c_2)+1+2i/z|>4/|y\bar{c}_2-\bar{y}c_2|+2/z.$$

By Lemma 19 one eliminates b_2, c_2 and obtains

$$|zx\bar{y}-|y|^2+3|x|^2-2i\bar{x}y|>z^2+2|x|^2+i(x\bar{y}-\bar{x}y).$$

Squaring both sides, making use of the identity $z^4=(x^2+y^2-xyz)(\bar{x}^2+\bar{y}^2-\bar{x}\bar{y}z)$ and dividing by $2|x|^2+i(x\bar{y}-\bar{x}y)$, one obtains

$$z(x\bar{y}+\bar{x}y)>z^2-|x|^2+|y|^2. \tag{32}$$

viii) By a computation similar to i) we can rewrite the condition as

$$z(x\bar{y}+\bar{x}y)>z^2+|x|^2-|y|^2. \tag{33}$$

We note that two conditions i) and viii) are written together as

$$z(x\bar{y}+\bar{x}y)>z^2+||x|^2-|y|^2|. \tag{34}$$

iv) Since the center of $D_{A^{-1}B}$ lies in the left half plane $\{w\in\boldsymbol{C}\mid\mathrm{Re}(w)<0\}$ and since D_{AB} and D'_{AB} are symmetric with respect to the imaginary axis, the condition is included in i).

v) By a similar reasoning to iv) we see that the condition is included in viii).

Before rewriting ii) and vii), we show the following.

LEMMA 20.

$$\begin{aligned}&-(x\bar{c}_1-\bar{x}c_1)(y\bar{b}_2-\bar{y}b_2)-(y\bar{c}_2-\bar{y}c_2)(x\bar{b}_1-\bar{x}b_1)-32\\&=(|x|^2-b_1\bar{c}_1)(|y|^2-\bar{b}_2c_2)+(|x|^2-\bar{b}_1c_1)(|y|^2-b_2\bar{c}_2).\end{aligned}$$

PROOF. Since $D_{A^{-1}B}$ and $D_{BA^{-1}}$ are tangent externally, we have by Lemma 18

$$\begin{aligned}&|(|x|^2-b_1\bar{c}_1)/(x\bar{c}_1-\bar{x}c_1)+(|y|^2-b_2\bar{c}_2)/(y\bar{c}_2-\bar{y}c_2)|\\&=4/|x\bar{c}_1-\bar{x}c_1|+4/|y\bar{c}_2-\bar{y}c_2|.\end{aligned}$$

Squaring both sides, making use of the identities

$$\begin{aligned}16&=(x^2-b_1c_1)(\bar{x}^2-\bar{b}_1\bar{c}_1)\\&=|x|^4-(\bar{b}_1\bar{c}_1x^2+b_1c_1\bar{x}^2)+|b_1c_1|^2\\16&=|y|^4-(\bar{b}_2\bar{c}_2y^2+b_2c_2\bar{y}^2)+|b_2c_2|^2\\|x\bar{c}_1-\bar{x}c_1|\,|y\bar{c}_2-\bar{y}c_2|&=(x\bar{c}_1-\bar{x}c_1)(y\bar{c}_2-\bar{y}c_2)\end{aligned}$$

and dividing by $(x\bar{c}_1-\bar{x}c_1)(y\bar{c}_2-\bar{y}c_2)$, one obtains the desired equality.

ii) By Lemma 18 we see that the condition is equivalent to

$$\begin{aligned}&|(|y|^2-b_2\bar{c}_2)/(y\bar{c}_2-\bar{y}c_2)-(|x|^2-b_1\bar{c}_1)/(x\bar{c}_1-\bar{x}c_1)|\\&>4/|x\bar{c}_1-\bar{x}c_1|+4/|y\bar{c}_2-\bar{y}c_2|.\end{aligned}$$

This is equivalent to

$$\begin{aligned}&|(|y|^2-b_2\bar{c}_2)/(y\bar{c}_2-\bar{y}c_2)-(|x|^2-b_1\bar{c}_1)/(x\bar{c}_1-\bar{x}c_1)|\\&>|(|y|^2-b_2\bar{c}_2)/(y\bar{c}_2-\bar{y}c_2)+(|x|^2-b_1\bar{c}_1)/(x\bar{c}_1-\bar{x}c)|,\end{aligned}$$

because $D_{A^{-1}B}$ and $D_{BA^{-1}}$ are tangent externally. It is easy to see that this is equivalent to

$$(|y|^2-b_2\bar{c}_2)(|x|^2-\bar{b}_1c_1)+(|x|^2-b_1\bar{c}_1)(|y|^2-\bar{b}_2c_2)>0.$$

By Lemmas 19 and 20 we obtain

$$(2|x|^2+i(x\bar{y}-\bar{x}y))(2|y|^2-i(x\bar{y}-\bar{x}y))$$
$$+(2|y|^2+i(x\bar{y}-\bar{x}y))(2|x|^2-i(x\bar{y}-\bar{x}y))>8z^2.$$

This is equivalent to

(35) $$(x\bar{y}+\bar{x}y)^2>4z^2.$$

By (34) and (35) we obtain

(7) $$x\bar{y}+\bar{x}y>2z.$$

vii) By symmetry it is clear that the condition is equivalent to (35) in ii).

Thus we have shown that, *under the conditions* (1), (2), (3) *and* (6), *the eight conditions are equivalent to four conditions* (7), (8), (9) *and* (34).

Now we shall prove Theorem 3. To prove the theorem it suffices to show that (34) follows from other conditions. If we could show

$$z^2>||x|^2-|y|^2|,$$

then (34) would follow from (7). By (24) and (25) we see that the desired inequality is equivalent to

$$|x_1y_2-x_2y_1|>|x_1y_2+x_2y_1|.$$

Since $x_1y_1>0$ by Lemma 6 and since $x_2y_2<0$ by (6), it holds except for the cases in §4, in which G is Kleinian. Thus we have shown that (34) follows from others and have completed the proof of Theorem 3.

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