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Factors generated by direct sums of II_1 factors

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Introduction.

In 1983 Jones introduced in [3] the concept of an index for a pair of type II_1 factors, called Jones index nowadays, and he showed the importance of such indices. With this as a momentum, the interests of research in the theory of operator algebras have been gradually extended from a single factor to a pair of factors. Thereafter Pimsner-Popa [8] gave an important relation between the index and the relative entropy for a pair of finite von Neumann algebras and showed that if $N\subset M$ is a pair of II_1 factors with finite index, then there exists a certain orthonormal basis of M over N. In the case of type III factors, Kosaki [4] defined an index depending on a conditional expectation and, on the other hand, Longo [5] gave another definition by using the canonical endomorphism. And in the case of C^*-algebras, Watatani [12] defined an index by using a quasi-basis.

However it is not easy to calculate explicitly the index even for a pair of II_1 factors only from the definition. For this reason, useful index formulas are expected. So far, Pimsner-Popa [8], Wenzl [13] and Ocneanu [7] gave index formulas respectively. Wenzl's formula is applicable only for pairs of approximately finite dimensional (=AFD) II_1 factors. In this paper we give a new index formula, that is the extension of Wenzl's one, and its application, for a pair of II_1 factors which are not necessarily AFD.

We treat a pair of II_1 factors arising from two increasing sequences of finite direct sums of II_1 factors. Let us explain more exactly, denote the sequences by $\{M_n\}_{n\in N}$ and $\{N_n\}_{n\in N}$, and assume that the diagram

$$\text{(A)}\qquad \begin{array}{ccc} M_n & \subset & M_{n+1} \\ \cup & & \cup \\ N_n & \subset & N_{n+1} \end{array}$$

is a commuting square for any n. Set $M=(\bigcup_n M_n)''$ and $N=(\bigcup_n N_n)''$. If the inclusion relations $N_n\subset N_{n+1}$, $M_n\subset M_{n+1}$ and $N_n\subset M_n$ are periodic, then M and N are found to be II_1 factors. For such a pair $N\subset M$ we give an index formula.

THEOREM 2.3. *Let* $\{M_n\}_{n\in\mathbf{N}}$ *and* $\{N_n\}_{n\in\mathbf{N}}$ *be increasing sequences of finite direct sums of* II_1 *factors such that for any* $n\in\mathbf{N}$ *the diagram* (A) *is a commuting square. Set* $M=(\bigcup_n M_n)''$ *and* $N=(\bigcup_n N_n)''$.

(1) *Assume* M *and* N *are* II_1 *factors, and* $[M:N]<\infty$. *Then*

$$[M:N]=\lim_n \langle \vec{t}_n, \vec{f}_n\rangle,$$

where $\vec{f}_n$ *is a vector defined by the inclusion* $N_n\subset M_n$, $\vec{t}_n$ *is the trace vector of* N_n *and* $\langle\cdot,\cdot\rangle$ *is the standard inner product.*

(2) *If the periodicity condition holds, then there exists* $n_0\in\mathbf{N}$ *such that*

$$[M:N]=\langle \vec{t}_n, \vec{f}_n\rangle=[M_n:N_n] \quad \text{for } n\geqq n_0.$$

Here, for $[M_n:N_n]$, we follow Goodman-de la Harpe-Jones' definition [2] of index for $N_n\subset M_n$ of von Neumann algebras which are direct sums of II_1 factors.

This formula is applicable even in case that M_n and N_n are finite direct sums of finite type I factors.

Furthermore we give an evaluation of dimension of the relative commutant $N'\cap M$.

THEOREM 2.4. *Let* $\{M_n\}_{n\in\mathbf{N}}$ *and* $\{N_n\}_{n\in\mathbf{N}}$ *be as in Theorem* 2.3 *and set* $M=(\bigcup_n M_n)''$ *and* $N=(\bigcup_n N_n)''$. *Let* $\{p_{n,i}\}_{i=1}^{m_n}$ *be the minimal central projections of* N_n. *Suppose that* $N\subset M$ *is a pair of* II_1 *factors with finite index and there exists a constant* $c>0$ *such that* $\mathrm{tr}(p_{n,i})>c$ *for all* i *and* n.

Then for any nonzero projection $p\in N_n$, *the following inequality holds:*

$$\dim(N'\cap M)\leqq \dim(N_n'\cap M_n)_p.$$

Next we give an application of our index formula. Starting from an irreducible pair of II_1 factors $A_{-1}\subset A_0$, we construct two increasing sequences of finite direct sums of II_1 factors $\{M_n\}_{n\in\mathbf{N}}$ and $\{N_n\}_{n\in\mathbf{N}}$ by using the basic construction. In detail, let $A_{-1}\subset A_0\subset A_1=\langle A_0, e_1\rangle\subset A_2=\langle A_1, e_2\rangle\subset\cdots$ be a sequence of II_1 factors and $e_i=e_{A_{i-2}}$ be a projection obtained from the basic construction, and define $M_j=A_j$ for $j\geqq 0$ and $N_0=A_{-1}$, $N_i=(A_{-1}\cup\{e_1,\cdots,e_i\})''$ for $i\geqq 1$. Then $M=(\bigcup_n M_n)''$ and $N=(\bigcup_n N_n)''$ are II_1 factors and we calculate the index $[M:N]$ by means of our formula.

THEOREM 3.4. *Let* $A_{-1}\subset A_0$ *be an irreducible pair of* II_1 *factors with index* λ *and construct* $\{M_n\}_n$ *and* $\{N_n\}_n$ *as above.*

(1) $\{M_n\}_n$ *and* $\{N_n\}_n$ *satisfy the lower boundedness condition, in the sense of Condition II of section* 2, *if and only if the index* $\lambda<4$.

(2) *The index* $[M:N]$ *is given by*

$$[M:N]=\begin{cases} \dfrac{k}{4\sin^2(\pi/k)} & if\ \lambda<4, \\ \infty & if\ \lambda\geqq 4, \end{cases}$$

where k is an integer such that $\lambda=4\cos^2(\pi/k)$.

This paper consists of three sections. In §1, we prepare the notations concerning finite direct sums of II_1 factors and review certain properties of traces on the relative commutant. In §2, we give an index formula and an evaluation of dimension of the relative commutant. In §3, we apply our index formula to a certain kind of concrete pairs of II_1 factors.

§1. Preliminaries.

In this section, we review the some notations and terminologies in [**2**] which we need below.

1.1. Let $M=\bigoplus_{j=1}^m M_j$ be a finite direct sum of II_1 factors and q_j the minimal central projection corresponding to M_j. Since the normalized normal trace on II_1 factor is unique, a trace on M (denoted by tr) is specified by a numerical vector $(\mathrm{tr}(q_i))_{i=1,\dots,m}$ called the trace vector.

Let $N=\bigoplus_{i=1}^n N_i\subset M$ be another finite direct sum of II_1 factors having the same identity and p_i the corresponding minimal central projection. We assume that the trace on N is the restriction of the trace on M. The trace vector for M (resp. N) is denoted by $\vec{s}$ (resp. $\vec{t}$).

Now we define two matrices representing the inclusion relation $N\subset M$, one is the index matrix and another is the trace matrix. The index matrix $\Lambda_N^M=(\lambda_{ij})$ is given by

$$\lambda_{ij}=\begin{cases} [p_iq_jMp_iq_j : p_iq_jNp_iq_j]^{1/2} & \text{if } p_iq_j\neq 0, \\ 0 & \text{if } p_iq_j=0, \end{cases}$$

and the trace matrix $T_N^M=(t_{ij})$ is given by $t_{ij}=\mathrm{tr}_{q_jM}(p_iq_j)$, tr_{q_jM} being the normalized trace on q_jM. The following properties (1.1)$\sim$(1.4) come from the very definitions.

(1.1) $\lambda_{ij}\in\{0\}\cup\{2\cos(\pi/n)\,;\ n\geqq 3\}\cup[2,\infty]$

(1.2) Trace matrix T_N^M is column-stochastic, i.e., $t_{ij}\geqq 0$ and $\sum_i t_{ij}=1$ for all j.

(1.3) The equality $\vec{t}=T_N^M\vec{s}$ holds.

(1.4) If $N\subset M\subset L$ are finite direct sums of II_1 factors, then $T_N^L=T_N^M T_M^L$.

1.2. We suppose that N is of finite index in M in the sense of [**2**], i.e., there is a faithful representation π of M on a Hilbert space such that the commutant $\pi(N)'$ is finite. Then the algebra $\langle M, e_N\rangle$ obtained from the basic construction for $N\subset M$ is a finite direct sum of II_1 factors and the corresponding minimal central projections are $Jq_1J, \cdots, Jq_mJ$, where J is the canonical conjugation on $L^2(M, \mathrm{tr})$.

As is shown in [2], the index matrix and the trace matrix for $M\subset\langle M, e_N\rangle$ have the following properties (1.5)$\sim$(1.7).

(1.5) $\Lambda_M^{\langle M, e_N\rangle} = (\Lambda_N^M)^t$

(1.6) $T_M^{\langle M, e_N\rangle} = \tilde{T}_N^M F_N^M$,

where $(\tilde{T}_N^M)_{ji} = \begin{cases} \dfrac{\lambda_{ij}^2}{t_{ij}} & p_iq_j \neq 0, \\ 0 & p_iq_j = 0, \end{cases}$ $\quad F_N^M = \mathrm{diag}(\varphi_1, \cdots, \varphi_n)$, $\varphi_i = (\Sigma_j(\tilde{T}_N^M)_{ij})^{-1}$.

(1.7) For any trace Tr on $\langle M, e_N\rangle$, $\mathrm{Tr}(e_N J p_i J) = \varphi_i \mathrm{Tr}(J p_i J)$.

The index $[M:N]$ is defined as follows,

(1.8) $[M:N] = r(\tilde{T}_N^M T_N^M)$, where $r(T)$ is the spectral radius of T.

1.3. We conclude this section by recalling the trace on the relative commutant.

Let $M_0\subset M_1$ be an irreducible pair, that is $M_0'\cap M_1=\boldsymbol{C}$, of II_1 factors with finite index. By the basic construction, we obtain a tower of II_1 factors $M_0\subset M_1\subset M_2\subset \cdots \subset M_n\subset \cdots$. Then by [8] and [9] we get

(1.9) $\mathrm{tr}_{M_n}(x) = \mathrm{tr}_{M_0'}(x)$ for $x \in M_0'\cap M_n$.

§ 2. Factor generated by direct sums of II_1 factors.

In this section, we construct a pair of factors from two increasing sequences of finite direct sums of II_1 factors and calculate the index for the pair.

LEMMA 2.1. *Let $N\subset M$ be a pair of II_1 von Neumann algebras with finite dimensional centers acting on a Hilbert space H. Let* tr *be a faithful finite trace on M and E_N be the trace preserving conditional expectation of M onto N. Suppose a projection $e\in B(H)$ satisfies the following conditions:*

the map $N\ni x\mapsto xe\in Ne$ is a $*$-isomorphism and $exe=E_N(x)e$ for all $x\in M$.

Let L be the von Neumann algebra generated by $M\cup\{e\}$. Then:

(1) *$L=A\oplus B$, with $A\cong\langle M, e_N\rangle$ and B isomorphic to an ultraweakly closed subalgebra of M.*

(2) *Let $z\in L$ be the central projection with $A=zL$. Then z is equal to the central support of e.*

(3) *Let* Tr *be a trace on L such that* $\mathrm{Tr}|_M=\mathrm{tr}$, *then*

$$\mathrm{Tr}(e) \geqq d\cdot \mathrm{Tr}(z), \quad \textit{where } d=\min\{\varphi_i=(F_N^M)_{ii}\,;\ i=1, \cdots, n\}.$$

PROOF. (1) Let L_0 be a $*$-algebra generated by $M\cup\{e\}$, and define a $*$-homomorphism $\Phi: L_0\to\langle M, e_N\rangle$ by $\Phi(x_0+\sum_i x_iey_i)=x_0+\sum_i x_ie_Ny_i$. Suppose that $x_0+\sum_i x_iey_i=0$. For $x\in M$, we put $\bar{x}=x_0x+\sum_i x_iE_N(y_ix)$. Then we obtain that

$$\bar{x}e=(x_0+\sum_i x_iey_i)xe=0 \quad \text{and} \quad E_N(\bar{x}^*\bar{x})e=e\bar{x}^*\bar{x}e=0.$$

Hence we have $E_N(\bar{x}^*\bar{x})=0$, i.e., $\bar{x}=0$. Denote by $[x]$ the image of x under the imbedding of M into $L^2(M, \mathrm{tr})$. Then for any $x\in M$,

$$(x_0+\sum_i x_ie_Ny_i)[x]=[x_0x+\sum_i x_iE_N(y_ix)]=0.$$

Since M is dense in $L^2(M, \mathrm{tr})$, this implies $x_0+\sum_i x_ie_Ny_i=0$. Thus Φ is well-defined.

Next we prove the norm continuity of Φ. Let $x=x_0+\sum_i x_iey_i\in L_0$ and $y\in M$. Then

$$\Phi(x)[y]=[x_0y+\sum_i x_iE_N(y_iy)]=[\bar{y}]$$

and

$$\|\Phi(x)[y]\|^2=\|[\bar{y}]\|^2=\mathrm{tr}(\bar{y}^*\bar{y})=\mathrm{tr}(E_N(\bar{y}^*\bar{y})).$$

The map $N\ni x'\mapsto \mathrm{tr}_L(ex')\in \boldsymbol{C}$ is a faithful trace on N. Therefore there is a constant $\alpha>0$ such that

$$\alpha^{-1}\mathrm{tr}(x')\leqq \mathrm{tr}_L(ex')\leqq \alpha\,\mathrm{tr}(x') \quad \text{for all } x'\in N.$$

Then,

$$\begin{aligned}\|\Phi(x)[y]\|^2 &\leqq \alpha\,\mathrm{tr}_L(eE_N(\bar{y}^*\bar{y}))=\alpha\,\mathrm{tr}_L(e\bar{y}^*\bar{y}e)=\alpha\,\mathrm{tr}_L(ey^*x^*xye)\\ &\leqq \alpha\|x\|^2\mathrm{tr}_L(ey^*ye)=\alpha\,\mathrm{tr}_L(eE_N(y^*y))\\ &\leqq \alpha^2\|x\|^2\mathrm{tr}(E_N(y^*y))=\alpha^2\|x\|^2\|[y]\|^2\end{aligned}$$

so that $\|\Phi(x)[y]\|\leqq\alpha\|x\|\cdot\|[y]\|$, i.e., $\|\Phi(x)\|\leqq\alpha\|x\|$.

We prove the ultrastrong continuity of Φ. Let $\{x_\lambda\}_{\lambda\in\Lambda}\subset L_0$ be a bounded net converging to 0 in the ultrastrong topology (=us). From the previous argument, the net $\{\Phi(x_\lambda)\}_{\lambda\in\Lambda}$ is also bounded and for any $y\in M$

$$\|\Phi(x_\lambda)[y]\|^2\leqq \alpha\,\mathrm{tr}_L(ey^*x_\lambda^*x_\lambda ye)\to 0 \quad \text{as } \lambda\to\infty.$$

Therefore $\Phi(x_\lambda)\to 0$ in the strong operator topology and by norm boundedness of $\{\Phi(x_\lambda)\}_{\lambda\in\Lambda}$ it follows that $\Phi(x_\lambda)\to 0$ (us), i.e., Φ is us-continuous on $(L_0)_1$.

Since $(L_0)_1$ is us-dense in L_1, the map Φ extends to an ultrastrongly continuous homomorphism $\tilde{\Phi}$ of L_1. We denote by φ the linear extension of $\tilde{\Phi}$ to L. By the us-continuity of φ on the bounded set of L, we see that φ is ultraweakly continuous. As $\varphi|_{L_0}=\Phi$ is a $*$-homomorphism, it follows that φ is also a $*$-homomorphism. Thus we obtain a us-continuous $*$-homomorphism φ of L onto $\langle M, e_N\rangle$.

Put $B=\mathrm{Ker}(\varphi)\subset L$ then B is an ultraweakly closed two-sided ideal of L and there exists a central projection $z\in L$ such that $B=(1-z)L$. Define $A=zL$, then $\varphi: A\to\langle M, e_N\rangle$ is a $*$-isomorphism. Therefore

$$L = A\oplus B \quad \text{and} \quad A\cong\langle M, e_N\rangle.$$

(2) As $\varphi(e)=e_N$ and $z_{\langle M, e_N\rangle}(e_N)=1$, it follows that $z_L(e)\geqq z$. On the other hand, by $\|1-z\|\leqq 1$, there exists a net $\{x_\lambda\}_{\lambda\in\Lambda}\subset(L_0)_1$ such that $x_\lambda\to 1-z$ (us) and $\varphi(x_\lambda)\to\varphi(1-z)=0$ (us). For $x_\lambda=a_0+\sum_i a_i e b_i$, we put $y_\lambda=a_0+\sum_i a_i E_N(b_i)$. Then $x_\lambda e=y_\lambda e$ and

$$E_N(y_\lambda^* y_\lambda)e_N = e_N y_\lambda^* y_\lambda e_N = e_N\varphi(x_\lambda)^*\varphi(x_\lambda)e_N \to 0$$

in ultraweak topology (=uw). Hence we have that $E_N(y_\lambda^* y_\lambda)\to 0$ (uw) and

$$e x_\lambda^* x_\lambda e = e y_\lambda^* y_\lambda e = E_N(y_\lambda^* y_\lambda)e \to 0 \text{ (uw)},$$

whence $x_\lambda e\to 0$ (us). Since $x_\lambda e$ converges to $(1-z)e$ ultrastrongly, we have $(1-z)e=0$, so that $z_L(e)=z$.

(3) Let $\{p_i\,;\, i=1, \cdots, n\}$ be the minimal central projections of N and define a $*$-isomorphism $\Psi: A\to\langle M, e_N\rangle$ by $\Psi=\varphi|_A$. Then we can take the central projections $\{\tilde{p}\,;\, i=1, \cdots, n\}$ of A such that $\Psi(\tilde{p}_i)=Jp_iJ$, where J is the canonical conjugation on $L^2(M, \mathrm{tr})$. Now we define another trace Tr' on $\langle M, e\rangle$ by $\mathrm{Tr}'=\mathrm{Tr}\circ\Psi^{-1}$. Then

$$\mathrm{Tr}(e\tilde{p}_i) = \mathrm{Tr}'(e_N Jp_iJ) = \varphi_i\,\mathrm{Tr}'(Jp_iJ) = \varphi_i\,\mathrm{Tr}(\tilde{p}_i) \geqq d\cdot\mathrm{Tr}(\tilde{p}_i),$$

and therefore

$$\mathrm{Tr}(e) = \sum_i \mathrm{Tr}(e\tilde{p}_i) \geqq d\sum_i \mathrm{Tr}(\tilde{p}_i) = d\cdot\mathrm{Tr}(z). \qquad \blacksquare$$

In the rest of this section, we consider the following situation. Let $\{M_n\}_{n\in\boldsymbol{N}}$ and $\{N_n\}_{n\in\boldsymbol{N}}$ be two increasing sequences of finite direct sums of II_1 factors and assume that there exist traces tr_{M_n} and tr_{N_n} such that for each $n\in\boldsymbol{N}$,

$$\mathrm{tr}_{M_{n+1}}|_{M_n} = \mathrm{tr}_{M_n} \quad \text{and} \quad \mathrm{tr}_{M_n}|_{N_n} = \mathrm{tr}_{N_n}$$

and the following diagram

$$\begin{array}{ccc} M_n & \subset & M_{n+1} \\ \cup & & \cup \\ N_n & \subset & N_{n+1} \end{array} \tag{2.1}$$

is a commuting square, i.e., $E_{M_n}E_{N_{n+1}}=E_{N_n}$ ([2]).

Moreover we deal with the following two conditions.

CONDITION I (Periodicity). There exist $n_0 \geqq 1$ and $p \geqq 1$ such that for any $n \geqq n_0$,

(1) $T_{N_n}^{N_{n+1}}$, $T_{M_n}^{M_{n+1}}$ and $F_{N_n}^{M_n}$ are periodic modulo p.

(2) $T_{N_n}^{N_{n+p}}$ and $T_{M_n}^{M_{n+p}}$ are primitive.

CONDITION II (Lower Boundedness). There exists a constant $d>0$ such that $(F_{N_n}^{M_n})_{ii} \geqq d$ for all n and i.

It is clear that Condition II follows from Condition I.

Here we denote the inductive limit of $\{M_n\}$ (resp. $\{N_n\}$) by M_∞ (resp. N_∞) and let tr be the tracial state on M_∞ such that $\mathrm{tr}|_{M_n} = \mathrm{tr}_{M_n}$ for all $n \in \boldsymbol{N}$. Moreover let π be the GNS representation with respect to tr and put $M = \pi(M_\infty)''$ and $N = \pi(N_\infty)''$, then $N \subset M$ is a pair of finite von Neumann algebras.

Now we give a sufficient conditions for M and N to be factors and for the index $[M:N]$ to be finite.

LEMMA 2.2.

(1) *If Condition* I *holds, M and N are II_1 factors.*

(2) *If Condition* II *holds, and M and N are II_1 factors, then the index $[M:N]$ is finite.*

PROOF. (1) Let tr be a normalized trace on M and $\vec{s}_n$ the trace vector of tr for M_n. We may suppose that $n_0 = p = 1$. Then we can put $T_{M_n}^{M_{n+1}} = T$ for any $n \in \boldsymbol{N}$, and by (1.3), it follows that

$$\vec{s}_n = T^k \vec{s}_{n+k} \quad \text{for all } k \geqq 1.$$

Thus $\vec{s}_n \in \bigcap_k T^k(\boldsymbol{R}^+)^m$, where $\boldsymbol{R}^+ = \{x \in \boldsymbol{R};\ x>0\}$ and hence $\vec{s}_n$ is a Perron Frobenius eigenvector of T. Therefore the normalized trace on M is unique so that M is a II_1 factor.

(2) Let L_n be the von Neumann algebra generated by $M_n \cup \{e_N\}$ and z_n be the central support of e_N in L_n, then $z_n \to 1$ (us). Take a semifinite trace Tr on $\langle M, e_N \rangle$. Since $e_N \langle M, e_N \rangle e_N = N e_N \cong N$, we have that e_N is a finite projection and $\mathrm{Tr}(e_N) < \infty$. From Lemma 2.1(3), we get

$$\mathrm{Tr}(e_N) \geqq d \cdot \mathrm{Tr}(z_n) \quad \text{for all } n \in \boldsymbol{N},$$

and letting $n \to \infty$, we have that

$$\mathrm{Tr}(e_N) \geqq d \cdot \mathrm{Tr}(1), \quad \text{i.e.,} \quad \mathrm{Tr}(1) \leqq d^{-1}\mathrm{Tr}(e_N) < \infty .$$

Therefore $\langle M, e_N \rangle$ is finite so that the index $[M:N]$ is finite. ∎

Here we give an index formula which is one of our main results of this paper.

THEOREM 2.3. *Let M and N be defined from two increasing sequences* $\{M_n\}$ *and* $\{N_n\}$ *as above*

(1) *Assume M and N are* II_1 *factors, and* $[M:N]<\infty$. *Then*

$$[M:N]=\lim_n\langle\vec{t}_n, \vec{f}_n\rangle,$$

where $\vec{f}_n=((F_{N_n}^{M_n})_{ii}^{-1})_i$ *and* $\vec{t}_n$ *is the trace vector of* N_n *and* $\langle\cdot,\cdot\rangle$ *is the standard inner product.*

(2) *If Condition* I *holds, then for all* $n\geqq n_0$,

$$[M:N]=\langle\vec{t}_n, \vec{f}_n\rangle=[M_n:N_n].$$

PROOF. (1) Since the index $[M:N]$ is finite, there exists a normalized trace tr on $\langle M, e_N\rangle$ such that

$$\operatorname{tr}(xe_N)=[M:N]^{-1}\operatorname{tr}(x) \quad \text{for } x\in M.$$

Using Lemma 2.1, we get an ultraweakly closed subalgebra A of $L_n=(M_n\cup\{e_N\})''$, $*$-isomorphic to $K_n=(M_n\cup\{e_{N_n}\})''$. Now let $\{p_i; i=1, \cdots, m\}$ be the minimal central projections of N_n, $\{\tilde{p}_i; i=1, \cdots, m\}$ be the corresponding central projections of A. Then, by the same method as the proof of (3) in Lemma 2.1,

$$\operatorname{tr}(\tilde{p}_i)=\varphi_{n,i}^{-1}\operatorname{tr}(e_Np_i)=\varphi_{n,i}^{-1}[M:N]^{-1}\operatorname{tr}(p_i),$$

where $\varphi_{n,i}=(F_{N_n}^{M_n})_{ii}$. Denoting the trace vector of tr for N_n by $\vec{t}_n=(t_{n,1}, \cdots, t_{n,m})$, ($m$ depends on n) and the central support of e_N in L_n by z_n, we get

$$\begin{aligned}\operatorname{tr}(z_n)&=\sum_i\operatorname{tr}(\tilde{p}_i)=\sum_i\varphi_{n,i}^{-1}[M:N]^{-1}\operatorname{tr}(p_i)\\&=[M:N]^{-1}\sum_i\varphi_{n,i}^{-1}t_{n,i}=[M:N]^{-1}\langle\vec{t}_n, \vec{f}_n\rangle.\end{aligned}$$

Since $z_n\to 1$ (uw) as $n\to\infty$, it follows that

$$\lim_n\langle\vec{t}_n, \vec{f}_n\rangle=[M:N].$$

(2) If Condition I holds, then for $n\geqq n_0$ the trace vector $\vec{t}_n$ is a Perron Frobenius eigenvector of $T_{N_n}^{N_{n+1}}$ by the proof of Lemma 2.2. Since $T_{N_n}^{N_{n+1}}$ and $F_{N_n}^{M_n}$ are periodic modulo p, we see that $\vec{t}_n$ and $\vec{f}_n$ are also periodic for $n\geqq n_0$. Because $\langle\vec{t}_n, \vec{f}_n\rangle$ converges to $[M:N]$, we have for $n\geqq n_0$,

$$[M:N]=\langle\vec{t}_n, \vec{f}_n\rangle \quad \text{and} \quad z_n=1.$$

From $z_n=1$, we have $K_n \cong L_n$, so there exists a $*$-isomorphism $\Psi: K_n \to L_n$. Let tr be a Markov trace on L_n and define the trace tr$'$ on K_n by $\mathrm{tr}'=\mathrm{tr}\circ\Psi$, then tr$'$ is also a Markov trace. Denoting the trace vector for M_n by $\vec{s}_n$, we have

$$\tilde{T}_n T_n \vec{s}_n = [M:N]\vec{s}_n,$$

where $T_n=T_{N_n}^{M_n}$ and $\tilde{T}_n=\tilde{T}_{N_n}^{M_n}$. Therefore $\vec{s}_n$ is a Perron Frobenius eigenvector of $\tilde{T}_n T_n$ so that $[M:N]$ is the maximal eigenvalue of $\tilde{T}_n T_n$ and so

$$[M:N] = r(\tilde{T}_n T_n) = [M_n : N_n]. \qquad \blacksquare$$

REMARK 2.1. In case that M_n and N_n are finite direct sums of full matrix algebras, the same formula holds too. This formula is not exactly the same as Wenzl's index formula, but essentially equivalent.

Similarly as in [**13**], we get the next theorem concerned with the relative commutant.

THEOREM 2.4. *Let M, N, $\{M_n\}$ and $\{N_n\}$ be as above and $\{p_{n,i}\}_{i=1}^{m_n}$ be the minimal central projections of N_n. Suppose that $N\subset M$ is a pair of II_1 factors with finite index and there exists a constant $c>0$ such that* $\mathrm{tr}(p_{n,i})>c$ *for all i and n.*

Then for any nonzero projection $p\in N_n$, the following inequality holds:

$$\dim(N'\cap M) \leqq \dim(N_n'\cap M_n)_p .$$

§ 3. Examples.

In this section, we give examples of $\{M_n\}_{n\in N}$ and $\{N_n\}_{n\in N}$ satisfying Condition II.

Let $A_{-1}\subset A_0$ be an irreducible pair of II_1 factors with index λ. If $\lambda<4$, then there exists $k\in \boldsymbol{N}$ such that $\lambda=4\cos^2(\pi/k)$. In case $\lambda\geqq 4$ we put $k=\infty$.

By the basic construction we get a sequence of II_1 factors $A_{-1}\subset A_0\subset A_1=\langle A_0, e_1\rangle\subset A_2=\langle A_1, e_2\rangle\subset \cdots$, where $e_i=e_{A_{i-2}}$. Now we define

$$(3.1)\qquad N_0 = A_{-1},\ N_i = (A_{-1}\cup\{e_1, \cdots, e_i\})'' \ \text{ for } i\geqq 1 \text{ and } M_j = A_j \text{ for } j\geqq 0.$$

Then $N_n\cong N_0\otimes B_n$ where $B_n=\{e_1, \cdots, e_n\}''$, so we can see the structure of N_n from the structure of B_n. This fact is important in the sequel.

LEMMA 3.1. *For all n, the diagram*

$$(3.2)\qquad \begin{array}{ccc} M_n & \subset & M_{n+1} \\ \cup & & \cup \\ N_n & \subset & N_{n+1} \end{array}$$

is a commuting square.

PROOF. Let $E_{M_n}: M_{n+1}\to M_n$ be the trace preserving conditional expectation, then $E_{M_n}(e_{n+1})=\lambda^{-1}$. Denote by L_{n+1} the $*$-algebra generated by $N_n\cup\{e_{n+1}\}$. For $x=x_0+\sum_i x_i e_{n+1}y_i\in L_{n+1}(x_i,\ y_i\in N_n)$, we get

$$E_{M_n}(x)=x_0+\sum_i x_i E_{M_n}(e_{n+1})y_i=x_0+\sum_i \lambda^{-1}x_i y_i\in N_n\,.$$

By Kaplansky's density theorem, for any $x\in N_{n+1}$ there exist a sequence $\{x_i\}_{i\in N}$, $x_i\in L_{n+1}$ such that $\|x_i\|\leqq\|x\|$ and $x_i\to x$ in $L^2(M_{n+1},\mathrm{tr})$ as $i\to\infty$. Then the sequence $\{E_{M_n}(x_i)\}_i$ in N_n converges to $E_{M_n}(x)$ as $i\to\infty$ in ultraweak topology, so we have $E_{M_n}(x)\in N_n$.

Thus $E_{M_n}(N_{n+1})\subset N_n$, i.e., the diagram (3.2) is a commuting square. ■

Next we calculate the trace matrices $T_{M_n}^{M_{n+1}}$, $T_{N_n}^{N_{n+1}}$ and $T_{N_n}^{M_n}$.

It is clear that $T_{M_n}^{M_{n+1}}=(1)$, and $T_{N_n}^{N_{n+1}}$ is given in the next proposition.

PROPOSITION 3.2. *Let $\Lambda_{N_n}^{N_{n+1}}$ be the index matrix and $T_{N_n}^{N_{n+1}}$ the trace matrix of the inclusion $N_n\subset N_{n+1}$. Then,*

$$\Lambda_{N_n}^{N_n}=(d_{i,j}^{(n)})_{ij},\ d_{i,j}^{(n)}=\begin{cases}1 & j=i,\ i+1,\\ 0 & \textit{otherwise},\end{cases}$$

$$T_{N_n}^{N_{n+1}}=(c_{i,j}^{(n)})_{ij},\ c_{i,j}^{(n)}=\begin{cases}\dfrac{\alpha_{n,i}}{\alpha_{n+1,j}} & j=i,\ i+1,\\ 0 & \textit{otherwise},\end{cases}$$

where for $n\leqq k-3$,

$$i=0,1,\cdots,[(n+1)/2],\quad j=0,1,\cdots,[(n+2)/2],\quad \alpha_{n,i}=\binom{n}{i}-\binom{n}{i-2},$$

and for $n\geqq k-2$,

$$i=[(n-k+4)/2],\cdots,[(n+1)/2],\quad j=[(n-k+5)/2],\cdots,[(n+2)/2],$$

$$\alpha_{n,i}=\binom{n}{i}-\binom{n}{i-2}-\binom{n}{i+k-2}.$$

PROOF. We prove this by induction on n. Since N_0 is a factor and since $N_1=N_0e_1\oplus N_0(1-e_1)$, we see that $T_{N_0}^{N_1}$ is equal to 1×2-matrix $(1,1)$. Suppose that the statement is true for $n=m$. By Lemma 2.1, we obtain that $N_{m+2}\cong\langle N_{m+1},e_{N_m}\rangle\oplus B$, where B is an ultraweakly closed subalgebra of N_{m+1}. Thus $\Lambda_{N_{m+1}}^{N_{m+1}}=(\Lambda_{N_{m+1}}^B,\ \Lambda_{N_{m+1}}^{\langle N_{m+1},e_{N_m}\rangle})$ and $T_{N_{m+1}}^{N_{m+2}}=(T_{N_{m+1}}^B,\ T_{N_{m+1}}^{\langle N_{m+1},e_{N_m}\rangle})$. Since the central projection corresponding to B is $1-e_1\vee e_2\vee\cdots\vee e_{m+2}$, and since $N_{m+1}=N_{m+1}(1-e_1\vee\cdots\vee e_{m+1})\oplus N_{m+1,1}\oplus\cdots\oplus N_{m+1,[(m+2)/2]}$, we have $\Lambda_{N_{m+1}}^B=T_{N_{m+1}}^B=(1,0,\cdots,0)^t$. On the other hand, we have that $\Lambda_{N_{m+1}}^{\langle N_{m+1},e_{N_m}\rangle}=(\Lambda_{N_m}^{N_{m+1}})^t$ and

$T_{N_{m+1}}^{\langle N_{m+1}, e_{N_m}\rangle}=\tilde{T}_{N_m}^{N_{m+1}}F_{N_m}^{N_{m+1}}$ by (1.5) and (1.6), so the statement is true for $n=m+1$.

In case $n\geqq k-2$, the algebra $\langle N_{n+1}, e_{N_n}\rangle$ is isomorphic to N_{n+2}, so we have that $\Lambda_{N_{n+1}}^{N_{n+2}}=\Lambda_{N_{n+1}}^{\langle N_{n+1}, e_{N_n}\rangle}$ and $T_{N_{n+1}}^{N_{n+2}}=T_{N_{n+1}}^{\langle N_{n+1}, e_{N_n}\rangle}$. Therefore the assertion in this case follows by a simple calculation. ■

PROPOSITION 3.3. *Let $\Lambda_{N_n}^{M_n}$ be the index matrix and $T_{N_n}^{M_n}$ the trace matrix of the inclusion $N_n\subset M_n$. Then,*

$$T_{N_n}^{M_n}=(c_i^{(n)}) \quad with \ c_i^{(n)}=\alpha_{n,i}\lambda^{-i}P_{n+2-2i}(\lambda^{-1})$$

and

$$\Lambda_{N_n}^{M_n}=(d_i^{(n)}) \quad with \ d_i^{(n)}=\lambda^{(n+1-2i)/2}P_{n+2-2i}(\lambda^{-1}),$$

where

$$i=0, \cdots, [(n+1)/2] \ (n\leqq k-3)\,;\ i=[(n-k+4)/2], \cdots, [(n+1)/2] \ (n\geqq k-2),$$

and $\alpha_{n,i}$ is the constant in Proposition 3.2 and $P_n(t)$ is Jones polynomial defined by $P_1(t)=P_2(t)=1$ and $P_n(t)=P_{n-1}(t)-tP_{n-2}(t)$.

PROOF. Let $\{p_{n,i}\}_i$ be the minimal central projections corresponding to the factorization of N_n. Since $T_{N_n}^{M_n}=(\mathrm{tr}(p_{n,i}))_i$, it is easy to see that $c_i^{(n)}=\alpha_{n,i}\lambda^{-1}P_{n+2-2i}(\lambda^{-1})$. We prove the assertion for $\Lambda_{N_n}^{M_n}$ by induction on n. Since $d_0^{(0)}=[A_0 : A_{-1}]^{1/2}=\lambda^{1/2}$, it is clear for $n=0$. Suppose it is true for $n=m$. For $j=i$, $i+1$, we have that

$$\begin{aligned}(d_j^{(m+1)})^2 &= [(M_{m+1})_{p_{m+1,j}} : (N_{m+1})_{p_{m+1,j}}]\\ &= [(M_{m+1})_{p_{m+1,j}p_{m,i}} : (N_{m+1})_{p_{m+1,j}p_{m,i}}]\\ &= [(M_{m+1})_{p_{m+1,j}p_{m,i}} : (N_m)_{p_{m+1,j}p_{m,i}}]\\ &= \mathrm{tr}_{(M_{m+1})_{p_{m,i}}}(q)\cdot\mathrm{tr}_{(N_m)'_{p_{m,i}}}(q)\cdot[(M_{m+1})_{p_{m,i}} : (N_n)_{p_{m,i}}],\end{aligned}$$

where $q=p_{m+1,j}p_{m,1}$.

Denoting the trace on M_{m+1} by tr, we obtain that $\mathrm{tr}_{(M_{m+1})_{p_{m,i}}}(q)=\mathrm{tr}(p_{m,i})^{-1}\mathrm{tr}(q)$ and by (1.9)

$$\mathrm{tr}_{(N_m)'_{p_{m,i}}}(q)=\mathrm{tr}_{N_0'}(p_{m,i})^{-1}\mathrm{tr}_{N_0'}(q)=\mathrm{tr}(p_{m,i})^{-1}\mathrm{tr}(q).$$

Therefore

$$\begin{aligned}(d_j^{(m+1)})^2 &= \mathrm{tr}(p_{m,i})^{-2}\mathrm{tr}(q)^2[(M_{m+1})_{p_{m,i}} : (M_m)_{p_{m,i}}]\cdot[(M_m)_{p_{m,i}} : (N_m)_{p_{m,i}}]\\ &= \mathrm{tr}(p_{m,i})^{-2}\mathrm{tr}(p_{m+1,j})^2\mathrm{tr}_{(N_{m+1})_{p_{m+1,j}}}(q)\cdot(d_i^{(m)})^2\lambda.\end{aligned}$$

Using the induction hypothesis and Proposition 3.2, we obtain

$$\begin{aligned} d_j^{(m+1)} &= \alpha_{m,i}\alpha_{m+1,j}^{-1}\operatorname{tr}(p_{m,i})^{-1}\operatorname{tr}(p_{m+1,j})d_i^{(m)}\lambda^{1/2} \\ &= \lambda^{(m+2-2j)/2}P_{m+3-2i}(\lambda^{-1}). \end{aligned} \qquad \blacksquare$$

Put $M=(\bigcup_n M_n)''$ and $N=(\bigcup_n N_n)''$, then M and N are II_1 factors (cf. [1]).

THEOREM 3.4. *Let $A_{-1}\subset A_0$ be an irreducible pair of II_1 factors with index λ and construct $\{M_n\}_n$ and $\{N_n\}_n$ as in* (3.1).

(1) *$\{M_n\}_n$ and $\{N_n\}_n$ satisfy Condition* II *if and only if the index $\lambda<4$.*

(2) *The index $[M:N]$ is given by*

$$[M:N]=\begin{cases} \dfrac{k}{4\sin^2(\pi/k)} & \text{if } \lambda<4, \\ \infty & \text{if } \lambda\geqq 4, \end{cases}$$

where k is an integer such that $\lambda=4\cos^2(\pi/k)$.

PROOF. Let $\{p_{n,i}\}_i$ be the minimal central projections corresponding to the factorization of N_n. Then the trace vector $\vec{t}_n$ for N_n is equal to $(\operatorname{tr}(p_{n,i}))_i$ and the vector $\vec{f}_n=(f_{n,i})_i$ in Theorem 2.3 is given by $\vec{f}_n=(\operatorname{tr}(p_{n,i})^{-1}(d_i^{(n)})^2)_i$ with $d_i^{(n)}=[(M_n)_{p_{n,i}}:(N_n)_{p_{n,i}}]^{1/2}$. Using Proposition 3.3, we see that

$$(f_{n,i})^{-1}=\frac{\alpha_{n,i}}{\lambda^{n+1-2i}P_{n+2-2i}(\lambda^{-1})}.$$

a) Case of $\lambda<4$: Since $P_n((4\cos^2\theta_k)^{-1})=\sin n\theta_k/(2^{n-1}\cos^{n-1}\theta_k\sin\theta_k)$ with $\theta_k=\pi/k$, it follows that

$$(f_{n,i})^{-1}=\frac{\alpha_{n,i}2^{n+1-i}\sin\theta_k}{\sin(n+2-2i)\theta_k\cos^{n+1}\theta_k}\geqq\sin\theta_k.$$

Therefore we see that the Lower Boundedness Condition holds. By Theorem 2.3, we get

$$\begin{aligned} [M:N] &= \lim_n\langle\vec{t}_n,\vec{f}_n\rangle \\ &= \lim_n\sum_{i=[(n-k+4)/2]}^{[(n+1)/2]}\operatorname{tr}(p_{n,i})\operatorname{tr}(p_{n,i})^{-1}(d_i^{(n)})^2 \\ &= \lim_n\sum_{i=[(n-k+4)/2]}^{[(n+1)/2]}\frac{\sin^2(n+2-2i)\theta_k}{\sin^2\theta_k} \\ &= \frac{k}{4\sin^2(\pi/k)}. \end{aligned}$$

b) Case of $\lambda\geqq4$: By a simple calculation, it follows that

$$(f_{n,0})^{-1}=\frac{\alpha_{n,0}}{\lambda^{n+1}P_{n+2}(\lambda^{-1})}\leqq\lambda^{-n/2}\longrightarrow 0\quad(n\to\infty).$$

So Condition II does not hold. Suppose in contrary that $[M:N]<\infty$. Then by Theorem 2.3, we have that

$$
\begin{aligned}
[M:N] &= \lim_n \langle \vec{t}_n, \vec{f}_n \rangle \\
&= \lim_n \sum_{i=0}^{[(n+1)/2]} (d_i^{(n)})^2 \\
&= \lim_n \sum_{i=0}^{[(n+1)/2]} \lambda^{n+1-2j} P_{n+2-2j}^2(\lambda^{-1}) \\
&\geq \lim_n \sum_{i=0}^{[(n+1)/2]} \lambda^{-1} = \infty .
\end{aligned}
$$

This is a contradiction, so that $[M:N]=\infty$. ■

REMARK 3.1. In case that $A_{-1}\subset A_0$ is a pair of type A_n, Choda [**1**] calculated the value of index $[M:N]$ by using the Wenzl's index formula.

REMARK 3.2. By Theorem 2.4, the pair $N\subset M$ is irreducible, that is, $N'\cap M=\boldsymbol{C}$, in case $\lambda<4$.

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