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A lower bound for sectional genus of quasi-polarized manifolds

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Introduction.

Let X be a smooth projective variety over $\boldsymbol{C}$ with $\dim X=n$, and L an ample (resp. a nef and big) Cartier divisor. Then (X, L) is called a polarized (resp. a quasi-polarized) manifold.

For this (X, L), the sectional genus of L is defined to be a non negative integer valued function by the following formula ([**Fj2**]):

$$g(L)=1+\frac{1}{2}(K_X+(n-1)L)L^{n-1},$$

where K_X is the canonical divisor of X.

Then there is the following conjecture:

CONJECTURE 1 (p. 111 in [**Fj3**]). *Let (X, L) be a quasi-polarized manifold. Then $g(L)\geqq q(X)$, where $q(X)=h^1(X, \mathcal{O}_X)$ (called the irregularity of X).*

In [**Fk1**], we treat $\dim X=2$ case. But if $\dim X\geqq 3$, the problem seems difficult. So we consider the following conjecture:

CONJECTURE 2. *Let (X, L) be a quasi-polarized manifold, Y a normal projective variety with $1\leqq\dim Y<\dim X$, and $f: X\rightarrow Y$ a surjective morphism with connected fibers. Then $g(L)\geqq h^1(\mathcal{O}_{Y'})$, where Y' is a resolution of Y.*

Of course Conjecture 2 follows from Conjecture 1. The hypothesis of Conjecture 2 is natural because X has a fibration in many cases (Albanese fibration, Iitaka fibration, etc.).

In this paper, we consider Conjecture 2. In particular, we study $\dim Y=1$ or some special cases of $\dim Y\geqq 2$. Using some results with respect to Conjecture 2, we study Conjecture 1.

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§ 0. Notations and conventions.

In this paper, we shall study mainly a smooth projective variety X over $\boldsymbol{C}$.

$\mathcal{O}(D)$: invertible sheaf associated with a Cartier divisor D on X.
$\mathcal{O}_X$: the structure sheaf of X.
$\chi(\mathcal{F})$: Euler-Poincaré characteristic of a coherent sheaf $\mathcal{F}$.
$\chi(X)=\chi(\mathcal{O}_X)$
$h^i(\mathcal{F})=\dim H^i(X, \mathcal{F})$ for a coherent sheaf $\mathcal{F}$ on X.
$h^i(D)=h^i(\mathcal{O}(D))$ for a divisor D.
$D|_C$: the restriction of D to C.
$|D|$: the complete linear system associated with a divisor D.
K_X: the canonical divisor of X.
$p_g(X)$ (or p_g): the geometric genus $h^0(K_X)$ of X.
$p_m(X)$ (or p_m): the m-genus $h^0(mK_X)$ of X.
$q(X)$ (or q): the irregularity $h^1(\mathcal{O}_X)$ of a smooth projective variety X.

If X is a normal projective variety over $\boldsymbol{C}$, then we define $q(X)=h^1(\mathcal{O}_{X'})$, where X' is a resolution of X. We remark that $q(X)$ is independent of a resolution of X.

$\kappa(D)$: Iitaka dimension of a Cartier divisor D on X.
$\kappa(X)$: Kodaira dimension of X.
$\boldsymbol{P}_Y(\mathcal{E})$: the $\boldsymbol{P}^{r-1}$-bundle associated with a locally free sheaf $\mathcal{E}$ of rank r over Y.
$\mathcal{O}_{\boldsymbol{P}_Y(\mathcal{E})}(1)$: the tautological invertible sheaf of $\boldsymbol{P}_Y(\mathcal{E})$.
$\sim$ (or $=$): linear equivalence.
$\equiv$: numerical equivalence.

For $r\in\boldsymbol{R}$, we define $[r]=\max\{t\in\boldsymbol{Z}: t\leqq r\}$, $\lceil r\rceil=-[-r]$.

(f, X, Y, L) is called a polarized (resp. quasi-polarized) fiber space if X is a smooth projective variety, Y is a smooth or normal projective variety with $1\leqq\dim Y<\dim X$, $f: X\to Y$ is a surjective morphism with connected fibers, and L is an ample (resp. a nef and big) Cartier divisor on X.

We say that two quasi-polarized fiber spaces (f, X, Y, L) and (h, X, Y', L) are isomorphic if there is an isomorphism $\delta: Y\to Y'$ such that $h=\delta\circ f$. In this case we write $(f, X, Y, L)\cong(h, X, Y', L)$.

We say that (f, X, Y, L) is a scroll if Y is smooth, $f: X\to Y$ is $\boldsymbol{P}^t$-bundle, and $L|_F=\mathcal{O}(1)$ where F is a fiber of f and $t=\dim X-\dim Y$.

We say that (X, L) has a structure of scroll over Y if there exists a surjective morphism $f: X\to Y$ such that $(F, L|_F)\cong(\boldsymbol{P}^{n-m}, \mathcal{O}(1))$ for any fiber F of f, where $\dim X=n$, and $\dim Y=m$.

We say that a Cartier divisor D on a projective variety X is pseudo-effective if there is a big Cartier divisor H such that $\kappa(mD+H)\geqq 0$ for any natural number m.

A general fiber F of f for a quasi-polarized fiber space (f, X, Y, L) means a fiber of a point of the set which is intersection of at most countable many Zariski open sets.

Let D be an effective divisor on X. We call D a normal crossing divisor if D has regular components which intersect transversally.

§1. $\dim Y=1$ case.

In this section, we consider a lower bound for $g(L)$ under the following condition:

$(*)$: Let (f, X, Y, L) be a (quasi-)polarized fiber space with $\dim X=n$, where Y is a smooth projective curve.

1-1. The nefness of $K_{X/Y}+tL$.

We study the nefness of $K_{X/Y}+tL$ for $t=n$, $n-1$, $n-2$, where $K_{X/Y}=K_X-f^*K_Y$. Here Theorem A in Appendix plays an important role. (See Appendix for the statement of Theorem A and its proof.)

THEOREM 1.1.1 (cf. Theorem 1 in [**Fj2**]). *Let (f, X, Y, L) be a polarized fiber space with* $\dim X=n\geqq 2$, $\dim Y=1$.

Then $K_{X/Y}+nL$ is nef.

PROOF. If $K_{X/Y}+nL$ is not f-nef, there exists an extremal rational curve l such that $(K_{X/Y}+nL)\cdot l<0$ and $f(l)=$point. Let $\varphi: X\to Z$ be the contraction morphism of l.

Then there exists a morphism $g: Z\to Y$ such that $f=g\circ\varphi$ (Theorem 3-2-1 in [**KMM**]). In particular $\dim Z\geqq\dim Y=1$.

But by the proof of Theorem 1 in [**Fj2**], $\dim Z=0$. This contradicts $\dim Z\geqq\dim Y=1$. Hence $K_{X/Y}+nL$ is f-nef.

On the other hand, $(K_{X/Y}+nL)-K_X$ is f-ample. By the base point free theorem (Theorem 3-1-1 in [**KMM**]),

$$f^*f_*\mathcal{O}(m(K_{X/Y}+nL)) \longrightarrow \mathcal{O}(m(K_{X/Y}+nL)) \tag{1.1.1.1}$$

is surjective for any $m\gg 0$.

By Theorem A in Appendix, $f_*\mathcal{O}(m(K_{X/Y}+nL))$ is semipositive ([**Fj1**]) and by (1.1.1.1) $\mathcal{O}(m(K_{X/Y}+nL))$ is nef. Therefore $K_{X/Y}+nL$ is nef. □

THEOREM 1.1.2 (cf. Theorem 2 in [**Fj2**]). *Let (f, X, Y, L) be as in Theorem 1.1.1. Then $K_{X/Y}+(n-1)L$ is nef unless (f, X, Y, L) is a scroll.*

PROOF. If $K_{X/Y}+(n-1)L$ is not f-nef, there exists an extremal rational curve l such that $(K_X+(n-1)L)\cdot l=(K_{X/Y}+(n-1)L)\cdot l<0$ and $f(l)=$point. Let $\varphi: X\to Z$ be the contraction morphism of l.

Then there exists a morphism $g: Z\to Y$ such that $f=g\circ\varphi$. In particular $\dim Z\geqq\dim Y=1$.

By ((2.7) proof of Theorem 2 in [**Fj2**]), φ is not birational and $\dim Z=1$. Then (φ, X, Z, L) is a scroll by the proof of Theorem 2 in [**Fj2**]. On the other hand, $Z\cong Y$ because f has connected fibers. Hence (f, X, Y, L) is a scroll.

If $K_{X/Y}+(n-1)L$ is f-nef, $K_{X/Y}+(n-1)L$ is nef by the same argument as in Theorem 1.1.1. □

THEOREM 1.1.3 (cf. Theorem 3 and 3′ in [**Fj2**]). *Let (f, X, Y, L) be as in Theorem 1.1.1. Suppose that $\dim X=n\geqq 3$ and $K_{X/Y}+(n-1)L$ is nef. Then $K_{X/Y}+(n-2)L$ is nef except the following cases:*

(3-1) *There exist a smooth projective variety X', a birational morphism $\mu: X\to X'$, and a surjective morphism with connected fibers $f': X'\to Y$ such that $f=f'\circ\mu$, μ is blowing down of $E\cong\boldsymbol{P}^{n-1}$, $E|_E=\mathcal{O}(-1)$, and $L|_E=\mathcal{O}(1)$.*

(3-2) *(f, X, Y, L) is $\boldsymbol{P}^2$-bundle and $L|_F=\mathcal{O}(2)$ for any fiber F of f.*

(3-3) *F is a hyperquadric in $\boldsymbol{P}^n$ and $L|_F=\mathcal{O}(1)$, where F is a general fiber of f.*

(3-4) *(F, L_F) is a scroll over a smooth curve, where F is a general fiber of f.*

PROOF. If $K_{X/Y}+(n-2)L$ is f-nef, then $K_{X/Y}+(n-2)L$ is nef by the same argument as in Theorem 1.1.1.

If $K_{X/Y}+(n-2)L$ is not f-nef, there exists an extremal rational curve l such that $(K_{X/Y}+(n-2)L)\cdot l<0$ and $f(l)=$point. Let $\varphi: X\to Z$ be the contraction morphism of l. Then we have a morphism $g: Z\to Y$ such that $f=g\circ\varphi$.

Case (A): φ is birational.

Then by the proof of Theorem 3′ in [**Fj2**], φ is blowing down of $E\cong\boldsymbol{P}^{n-1}$, $E|_E=\mathcal{O}(-1)$ and $L|_E=\mathcal{O}(1)$. We put $\mu=\varphi$, $f'=g$, and $Z=X'$. So (3-1) is obtained.

Case (B): φ is not birational.

We remark that $\dim Z\geqq\dim Y=1$. By Theorem 3′ in [**Fj2**], we have the following three types:

(1) $\dim Z=1$, $(F_\varphi, L|_{F_\varphi})=(\boldsymbol{P}^2, \mathcal{O}(2))$ for every fiber F_φ of φ.
(2) $\dim Z=1$, F is hyperquadric and $L|_F=\mathcal{O}(1)$.
(3) $\dim Z=2$, Z is smooth, and (φ, X, Z, L) is scroll.

Case (1)
In this case, $Z\cong Y$ since every fiber of f is connected. So $(f, X, Y, L)\cong$

(φ, X, Z, L) and (3-2) is obtained.

Case (2)

By the same argument as in Case (1), $(f, X, Y, L)\cong(\varphi, X, Z, L)$. Hence (3-3) is obtained.

Case (3)

In this case, a general fiber F of f is scroll over a smooth curve. Hence (3-4) is obtained. □

1-2. $g(L)\geqq g(Y)$.

Here we shall show that the following theorem.

THEOREM 1.2.1. *Let (f, X, Y, L) be a polarized fiber space with* $\dim Y=1$. *Then $g(L)\geqq g(Y)$, where $g(Y)$ is the genus of Y.*

PROOF. First since $2(g(Y)-1)L^{n-1}F=f^*K_Y L^{n-1}$, we have

$$g(L)=g(Y)+\frac{1}{2}(K_{X/Y}+(n-1)L)L^{n-1}+(g(Y)-1)(L^{n-1}\cdot F-1), \tag{1.2.1.1}$$

where F is a general fiber of f.

Case (a): $g(Y)=0$.

$g(L)\geqq g(Y)=0$ by Corollary 1 in [**Fj2**].

Case (b): $g(Y)\geqq 1$.

In this case,

$$(g(Y)-1)(L^{n-1}\cdot F-1)\geqq 0 \tag{1.2.1.2}$$

since L is ample.

Case (b)-1: $K_{X/Y}+(n-1)L$ is nef.

By (1.2.1.1) and (1.2.1.2), we have $g(L)\geqq g(Y)$.

Case (b)-2: $K_{X/Y}+(n-1)L$ is not nef.

By Theorem 1.1.2, (f, X, Y, L) is a scroll. Let $\mathcal{E}$ be a locally free sheaf of rank n over Y such that $X=\boldsymbol{P}(\mathcal{E})$ and $L=\mathcal{O}_{\boldsymbol{P}(\mathcal{E})}(1)$. Then $K_X=f^*(K_Y+\det\mathcal{E})-\mathcal{O}_{\boldsymbol{P}(\mathcal{E})}(n)$ ((1.3) in [**Fj3**]). Hence $g(L)=1+(K_X+(n-1)L)L^{n-1}/2=1+(f^*(K_Y+\det\mathcal{E})-L)L^{n-1}/2=1+(1/2)\deg K_Y=g(Y)$.

Therefore $g(L)\geqq g(Y)$ is obtained. □

REMARK 1.2.2. There exists an example of (f, X, Y, L) with $g(L)=g(Y)$. (For example, the case (f, X, Y, L) is scroll.)

In 1-4, we shall show that (f, X, Y, L) with $g(L)=g(Y)$ has a structure of scroll over a smooth curve.

By Theorem 1.2.1, we have the following Corollary.

COROLLARY 1.2.3. *Let (X, L) be a polarized manifold. Assume that the image of the Albanese map* ([**U**]) *is a curve. Then $g(L)\geqq q(X)$.*

PROOF. Let $\alpha: X \to \mathrm{Alb}\,X$ be the Albanese map of X. By assumption, $\alpha(X)$ is a smooth curve of genus $q(X)$ and $\alpha: X \to \alpha(X)$ has connected fibers. Hence by Theorem 1.2.1, $g(L) \geqq g(\alpha(X)) = q(X)$. □

1-3. $\kappa(X) \geqq 0$.

Here we treat $\kappa(X) \geqq 0$ case.

LEMMA 1.3.1. *Let X be a projective variety with* $\dim X = n$ *and D a pseudo effective Cartier divisor on X. Then $DL^{n-1} \geqq 0$ for any nef Cartier divisor L.*

PROOF. By definition of a pseudo effective Cartier divisor (see § 0 or (11.3) in [**Mo**]), $\kappa(tD+H) \geqq 0$ for any natural number t and a big Cartier divisor H over X. Since L is nef, $mL+A$ is ample for any natural number m and an ample Cartier divisor A over X. Therefore

$$\left(D+\frac{1}{t}H\right)\left(L+\frac{1}{m}A\right)^{n-1} = \frac{1}{m^{n-1}t}(tD+H)(mL+A)^{n-1} \geqq 0.$$

Tend $t \to \infty$ and $m \to \infty$, we have $DL^{n-1} \geqq 0$. □

REMARK 1.3.2.

(1) Let X and Y be smooth projective varieties over $\boldsymbol{C}$, and $f: X \to Y$ a surjective morphism with connected fibers. Let D be a Cartier divisor on X such that $f_*\mathcal{O}(D) \neq 0$. If $f_*\mathcal{O}(D)$ is weakly positive (see Appendix), then D is pseudo effective.

(2) Let $\mathcal{E}$ be a locally free sheaf on a normal projective variety X. If $\mathcal{E}$ is semipositive ((5.1) in [**Mo**]), then $\mathcal{E}$ is weakly positive.

PROOF.

The proof of (1)

By hypothesis, the natural map

$$f^*f_*\mathcal{O}(D) \longrightarrow \mathcal{O}(D)$$

is non-trivial. If $\mathcal{O}(D-Z) = \mathrm{Im}(f^*f_*\mathcal{O}(D) \to \mathcal{O}(D))^{**}$, where Z is an effective divisor on X and $**$ is double dual, then $f^*f_*\mathcal{O}(D) \to \mathcal{O}(D-Z)$ is surjective in codimension 1. By Hironaka theory [**Hi**], there exists a birational morphism $\mu: X' \to X$ such that

$$\mu^*f^*f_*\mathcal{O}(D) \longrightarrow \mathcal{O}(\mu^*(D-Z)-E)$$

is surjective, where X' is smooth and E is an exceptional effective divisor over X'.

By hypothesis, $\mu^*f^*f_*\mathcal{O}(D)$ is weakly positive. Hence $\mathcal{O}(\mu^*(D-Z)-E)$ is weakly positive. By definition, $\mu^*(D-Z)-E$ is pseudo effective. Since Z and E are effective, μ^*D is pseudo effective. Hence D is pseudo effective.

The proof of (2)

Since $\mathcal{E}$ is semipositive, $S^{\alpha}(\mathcal{E})$ is also semipositive for any positive integer α. Let $\mathcal{H}$ be an ample invertible sheaf on X. Then $S^{\alpha}(\mathcal{E})\otimes\mathcal{H}$ is an ample locally free sheaf ([**Ha2**]). Hence $\mathcal{E}$ is weakly positive. □

THEOREM 1.3.3. *Let* (f, X, Y, L) *be a quasi-polarized fiber space with* $\dim Y=1$, $g(Y)\geqq 1$, *and* $\kappa(F)\geqq 0$, *where* F *is a general fiber of* f.

Then $g(L)\geqq g(Y)+\lceil((n-1)/2)L^{n}\rceil$.

PROOF. Since $\kappa(F)\geqq 0$, there exists a Zariski open set U of Y such that for any closed point $y\in U$,

(1) $F_{y}=f^{-1}(y)$ is smooth

(2) $h^{0}(mK_{F_{y}})$ is constant and not zero for some fixed $m\in\boldsymbol{N}$.

By Grauert's theorem (see [**Ha1**]), $f_{*}\mathcal{O}(mK_{X/Y})\neq 0$. Hence by Lemma 1.3.1, Remark 1.3.2 and the semipositivity of $f_{*}\mathcal{O}(mK_{X/Y})$ ([**Ka2**], [**V3**]), $K_{X/Y}\cdot L^{n-1}\geqq 0$.

By (1.2.1.1) in Theorem 1.2.1, we have

$$g(L)\geqq g(Y)+\frac{n-1}{2}L^{n}+(g(Y)-1)(L^{n-1}\cdot F-1).$$

Since L is nef and big, L_{F} is also nef and big. Hence $L_{F}^{n-1}\geqq 1$.

By hypothesis, $g(Y)\geqq 1$. Therefore

$$g(L)\geqq g(Y)+\left\lceil\frac{n-1}{2}L^{n}\right\rceil$$

because $g(L)$ is integer. □

THEOREM 1.3.4. *Let* (X, L) *be a quasi-polarized manifold with* $\kappa(X)=1$ *and* $L^{n}\geqq 2$. *Then* $g(L)\geqq q(X)$.

PROOF. In general, there is the following fibration (called Iitaka fibration [**Ii1**]) if $\kappa(X)\geqq 1$:

There exist a birational morphism $\mu: X'\to X$ and a surjective morphism with connected fibers $f: X'\to Y$ such that $\dim Y=\kappa(X)$ and $\kappa(F)=0$ for a general fiber F of f, where X' and Y are smooth projective varieties.

We remark that $q(X)=q(X')$ and $g(L)=g(L')$, where $L'=\mu^{*}L$.

So we may assume that there is a fibration $f: X\to Y$, where Y is a smooth projective variety.

Here $\dim Y=1$.

If $g(Y)\geqq 1$, then we apply Theorem 1.3.3 for this (f, X, Y, L). Hence $g(L)\geqq g(Y)+\lceil((n-1)/2)L^{n}\rceil$. By hypothesis, $\lceil((n-1)/2)L^{n}\rceil\geqq n-1$. Since $\kappa(F)=0$, $q(F)\leqq\dim F=n-1$ by Kawamata's theorem ([**Ka1**]). So we have $g(L)\geqq g(Y)+(n-1)\geqq g(Y)+q(F)$.

On the other hand, by Theorem B in Appendix, $q(F)+g(Y)\geqq q(X)$. Therefore $g(L)\geqq q(X)$.

If $g(Y)=0$, then $g(L)=1+(K_X+(n-1)L)L^{n-1}/2\geqq 1+n-1\geqq 1+q(F)>g(Y)+q(F)\geqq q(X)$. □

By Kawamata's theorem, we have the following theorem.

THEOREM 1.3.5. *Let (X, L) be a quasi-polarized manifold with $\kappa(X)=0$ and $L^n\geqq 2$. Then $g(L)\geqq q(X)$.*

PROOF. Since $\kappa(X)=0$, $q(X)\leqq \dim X=n$ by Kawamata's theorem.
Hence

$$\begin{aligned} g(L) &= 1+\frac{1}{2}(K_X+(n-1)L)L^{n-1} \\ &\geqq 1+\frac{n-1}{2}L^n \\ &\geqq n \\ &\geqq q(X). \end{aligned}$$ □

1-4. Classification of (f, X, Y, L) with $g(L)=g(Y)$.

Here we shall classify (f, X, Y, L) with $\dim Y=1$ and $g(L)=g(Y)$.

LEMMA 1.4.1. *If $f_*\mathcal{O}(D)$ is ample, then $DL^{n-1}>0$ for any ample line bundle L on X.*

PROOF. By hypothesis, given any coherent sheaf $\mathcal{F}$ on Y, there exists a natural number m_0 such that for every $m\geqq m_0$, $\mathcal{F}\otimes S^m(f_*(D))$ is generated by the global sections. Hence $f^*\mathcal{F}\otimes S^m(f^*\circ f_*(D))$ is generated by the global sections. We put $\mathcal{F}=\mathcal{O}(-A)$, where $\mathcal{O}(A)$ is an ample invertible sheaf on Y. Then $mD-f^*A$ is effective and $L^{n-1}(mD-f^*A)\geqq 0$. Hence $L^{n-1}D>0$. □

THEOREM 1.4.2. *Let (f, X, Y, L) be a polarized fiber space with $\dim X=n\geqq 3$ and $\dim Y=1$. Suppose that $g(L)=g(Y)$. Then (f, X, Y, L) is a scroll.*

PROOF. First we have

$$g(L) = g(Y)+\frac{1}{2}(K_{X/Y}+(n-1)L)L^{n-1}+(L^{n-1}F-1)(g(Y)-1). \tag{1.4.2.1}$$

Case (1): $g(Y)\geqq 1$

If $f_*\mathcal{O}(K_{X/Y}+(n-1)L)\neq 0$, then $f_*\mathcal{O}(K_{X/Y}+(n-1)L)$ is ample by Theorem 2.4 and Corollary 2.5 in [**E-V**], so by Lemma 1.4.1,

$$(K_{X/Y}+(n-1)L)L^{n-1}>0.$$

By (1.4.2.1), $g(L)>g(Y)$. Hence we may assume $f_*\mathcal{O}(K_{X/Y}+(n-1)L)=0$. If

$K_{X/Y}+(n-1)L$ is not nef, then (f, X, Y, L) is a scroll by Theorem 1.1.2. Hence we may assume that $K_{X/Y}+(n-1)L$ is nef.

By hypothesis, there are two possible cases:

(A) $$(K_{X/Y}+(n-1)L)L^{n-1}=0, \quad g(Y)=1$$

(B) $$(K_{X/Y}+(n-1)L)L^{n-1}=0, \quad L^{n-1}F=1$$

Case (A)

Since $g(L)=g(Y)=1$, we have

(A-1) (X, L) is a del Pezzo variety

(A-2) (X, L) is a scroll over an elliptic curve

by Fujita's classification of $g(L)=1$. ([**Fj2**])

If (X, L) is the case (A-1), then since $-K_X$ is ample, $q(X)=0$, which contradicts $q(Y)\geqq 1$. Next we consider that (X, L) is the case (A-2). Let $\pi: X\to C$ be a $\boldsymbol{P}^{n-1}$-bundle with $L_F=\mathcal{O}(1)$, where C is an elliptic curve and F is a fiber of f. Since $\boldsymbol{P}^{n-1}$ has no fibration over a curve for $n\geqq 3$, there is a morphism $\mu: C\to Y$ such that $f=\mu\circ\pi$ ((4.4) in [**EGA**] III). Since f has connected fibers, μ is an isomorphism ((7.1) in [**Mu**]). Therefore (f, X, Y, L) is a scroll.

Case (B)

In this case we can exclude $g(Y)=1$, which implies $g(Y)\geqq 2$. Since $(K_{X/Y}+(n-2)L)L^{n-1}+L^n=0$, $K_{X/Y}+(n-2)L$ is not nef. Hence we can apply Theorem 1.1.3 to this case.

Case (B-1): (f, X, Y, L) is the type (3-1) in Theorem 1.1.3.

This case cannot occur. Indeed, let $E\cong\boldsymbol{P}^{n-1}$ be as in (3-1) in Theorem 1.1.3. Either E cannot be a fiber of f, or the restriction of f to E cannot be a surjection since $\boldsymbol{P}^{n-1}$ has no fibration over a curve. If E is in a fiber of f, the fiber is not irreducible and $L^{n-1}F>1$, which is a contradiction.

Case (B-2): (f, X, Y, L) is the type (3-2) or the type (3-3) in Theorem 1.1.3.

In these cases, $L^{n-1}F>1$ which are contradictions.

Case (B-3): (f, X, Y, L) is the type (3-4) in Theorem 1.1.3.

Let $F=\boldsymbol{P}_C(\mathcal{E})$, $L_F=\mathcal{O}_{\boldsymbol{P}(\mathcal{E})}(1)$, and $\pi: \boldsymbol{P}_C(\mathcal{E})\to C$ the projection, where $\mathcal{E}$ is a locally free sheaf of rank $n-1$ over a smooth curve C.

We may assume that $\mathcal{E}$ is ample. $\det\mathcal{E}$ is also ample.

By Riemann-Roch formula on C and vanishing theorem,

$$\begin{aligned} h^0(K_C+\det\mathcal{E}) &= \chi(K_C+\det\mathcal{E}) \\ &= g(C)-1+\deg(\det\mathcal{E}). \end{aligned}$$

If $h^0(K_C+\det\mathcal{E})=0$, then we have $g(C)=0$ and $\deg(\det\mathcal{E})=1$.

Then

$$\mathcal{E}=\mathcal{O}(a_1)\oplus\mathcal{O}(a_2)\oplus\cdots\oplus\mathcal{O}(a_{n-1})$$

by Grothendieck's theorem.

Since $\mathcal{E}$ is ample, $a_i>0$ for any i. Hence

$$\deg(\det \mathcal{E}) \geqq n-1 \geqq 2$$

since $n \geqq 3$. This contradicts $\deg(\det \mathcal{E})=1$.

Therefore by the formula $K_{F/C}=\mathcal{O}_{P(\mathcal{E})}(-(n-1))\otimes\pi^* \det \mathcal{E}$,

$$\begin{aligned} h^0(K_F+(n-1)L_F) &= h^0(\pi^*(K_C+\det \mathcal{E})) \\ &= h^0(K_C+\det \mathcal{E}) > 0 . \end{aligned}$$

But by Grauert's theorem, $f_*\mathcal{O}(K_{X/Y}+(n-1)L)\neq 0$.

This contradicts the assumption.

Therefore this case cannot occur.

Case (1) is complete.

Case (2): $g(Y)=0$, i.e., $Y\cong \boldsymbol{P}^1$

In this case, $g(L)=0$. So by Fujita's classification of (X, L) with $g(L)=0$ ([**Fj2**]), (X, L) is one of the following three possible types:

(A) $(X, L)=(\boldsymbol{P}^n, \mathcal{O}(1))$.

(B) X is a hyperquadric in $\boldsymbol{P}^{n+1}$, $L=\mathcal{O}_X(1)$.

(C) (X, L) is a scroll over $\boldsymbol{P}^1$.

Note that X with $\operatorname{Pic} X\cong \boldsymbol{Z}$ has no fibration over a curve.

Case (A)

This case cannot occur since X has no fibration over a curve.

Case (B)

Since $n\geqq 3$, $\operatorname{Pic} X\cong \boldsymbol{Z}$ by Lefschetz's Theorem ((7.1) in [**Fj3**]). Hence this case cannot occur.

Case (C)

Let $h: X\to \boldsymbol{P}^1$ be the structure morphism of scroll, and F_h ($\cong \boldsymbol{P}^{n-1}$) any fiber of h, which has no fibration over a curve for $n\geqq 3$.

Then $\dim f(F_h)=0$.

Hence there is a morphism $\mu: \boldsymbol{P}^1\to Y$ such that $f=\mu\circ h$ ((4.4) in [**EGA**] III). Since f has connected fibers, μ is isomorphism ((7.1) in [**Mu**]).

Therefore (f, X, Y, L) is a scroll. □

When $\dim X=2$, we obtain the following.

PROPOSITION 1.4.3. *Let (f, X, Y, L) be a polarized fiber space, X a surface, and Y a curve. Assume that $g(L)=g(Y)$ and (f, X, Y, L) is not a scroll.*

Then $(f, X, Y, L)\cong(\pi, \boldsymbol{P}^1\times \boldsymbol{P}^1, \boldsymbol{P}^1, L)$ as a polarized fiber space, where π is one projection such that $LF_\pi\geqq 2$, where F_π is a fiber of π.

PROOF. Let F be a general fiber of f.

Case (1): $g(Y)\geqq 1$.

Case (1)-1: $g(F)\geqq 2$.

In this case, by Theorem 5.5 in [**Fk1**], $g(L)\geqq g(Y)+1$.

Hence this case is excluded.

Case (1)-2: $g(F)=1$.

In this case, $\kappa(X)\leqq\kappa(F)+\dim Y=1$ ([**Ii1**]). Let (f', X', C, L') be the relatively minimal model of (f, X, C, L) and $\mu: X\rightarrow X'$ its birational morphism, where $L'=\mu_{*}L$ in the sense of cycle theory. By the canonical bundle formula for elliptic fibrations ([**BPV**]), $K_{X}\cdot L\geqq K_{X'}\cdot L'\geqq 2g(Y)-2$. Hence taking it into account that $g(L)$ is an integer, we have $g(L)\geqq g(Y)+1$, which is a contradiction.

Case (1)-3: $g(F)=0$.

In this case, $\kappa(X)\leqq\kappa(F)+\dim Y=-\infty$. Then $g(L)\geqq q(X)$ ([**Fk1**]). Since $g(L)=g(Y)$, we have $g(L)=g(Y)=q(X)$. Thus by the classification [**L-P**] and [**Fk1**], (X, L) is one of the following two types.

(A) $(\boldsymbol{P}^{2}, \mathcal{O}(r))$, $r=1$ or 2.

(B) X is a $\boldsymbol{P}^{1}$-bundle over a smooth curve C and $L|_{F'}=\mathcal{O}(1)$, where F' is a fiber of the projection $\pi: X\rightarrow C$.

Case (A) is excluded, since $\boldsymbol{P}^{2}$ has no fibration over a curve.

Case (B)

Since π is a $\boldsymbol{P}^{1}$-bundle and $g(Y)\geqq 1$, there is a morphism $\mu: C\rightarrow Y$ such that $f=\mu\circ\pi$ ((4.4) in [**EGA**] III). Since f has connected fibers, μ is isomorphism ((7.1) in [**Mu**]).

Hence (f, X, Y, L) is a scroll.

Case (2): $g(Y)=0$.

By hypothesis, $g(L)=g(Y)=0$. By the classification [**L-P**], [**Fj2**] and [**Fj3**], (X, L) is one of (A) and (B) of the previous Case (1)-3. Hence (X, L) has a structure of scroll, since (A) never becomes a polarized fiber space as remarked previously.

Let $\pi_{1}: X\rightarrow C\cong\boldsymbol{P}^{1}$ be the $\boldsymbol{P}^{1}$-bundle such that (π_{1}, X, C, L) is a scroll. We put $X=\boldsymbol{P}_{C}(\mathcal{E})$ and $\mathcal{E}=\mathcal{O}_{C}\oplus\mathcal{O}_{C}(-e)$, where $e\geqq 0$. Let H be the $-\infty$ section of π_{1} which is a member of the complete linear system associated to the tautological invertible sheaf $\mathcal{O}_{\boldsymbol{P}(\mathcal{E})}(1)$ over X and F_{1} a fiber of π_{1}. We remark that $H^{2}=-e$ ([**Ha1**]). Let F_{f} be a fiber of f. Then we can write $F_{f}\equiv aH+bF_{1}$ for some $a, b\in\boldsymbol{Z}$. Since $F_{f}^{2}=0$, $-a^{2}e+2ab=0$. If $a=0$, $F_{f}=bF_{1}$ and $b>0$. f factors through π_{1}, which is an isomorphism since f has connected fibers. Hence we can prove $(f, X, Y, L)\cong(\pi_{1}, X, C, L)$, which is a scroll against hypothesis. Thus $a\neq 0$, $2b-ae=0$ and $F_{f}\equiv aH+(ae/2)F_{1}$. Since F_{f} is nef, we have $F_{f}\cdot F_{1}=a>0$ and $H\cdot F_{f}=-ae/2\geqq 0$. Therefore $e=0$, $X\cong\boldsymbol{P}^{1}\times\boldsymbol{P}^{1}$ and let π_{1} be one projection and π_{2} the other projection. Then H is a fiber of π_{2}. Since $F_{f}\equiv aH$

for some $a\in\boldsymbol{N}$, there exists a morphism $\theta:\boldsymbol{P}^1\to Y$ such that $f=\theta\circ\pi_2$. Since f has connected fibers, θ is an isomorphism. Hence $(f, X, Y, L)\cong(\pi_2, \boldsymbol{P}^1\times\boldsymbol{P}^1, \boldsymbol{P}^1, L)$. □

EXAMPLE 1.4.4. Let $X=\boldsymbol{P}^1\times\boldsymbol{P}^1$, $p_i:\boldsymbol{P}^1\times\boldsymbol{P}^1\to\boldsymbol{P}^1$ the i-th projection, and F_i a fiber of p_i. Then $K_X\equiv-2F_1-2F_2$. We put $L\equiv2F_1+F_2$. We remark that L is ample and $g(L)=0$.

Then $(p_1, X, \boldsymbol{P}^1, L)$ is a scroll, but $(p_2, X, \boldsymbol{P}^1, L)$ is not a scroll.

§ 2. Some special cases of $\dim Y\geqq2$.

In this section, we shall consider some special cases.

First by Lemma 1.3.1 we can prove the following lemma:

LEMMA 2.1. *Let (f, X, Y, L) be a quasi-polarized fiber space with $\dim X>\dim Y\geqq1$ and $\kappa(F)\geqq0$, where F is a general fiber of f. Then $K_{X/Y}L^{n-1}\geqq0$.*

PROOF. Since $\kappa(F)\geqq0$, we have $f_*\mathcal{O}(tK_{X/Y})\neq0$ for $t\gg0$.

By Viehweg's theorem ([**V3**]), $f_*\mathcal{O}(tK_{X/Y})$ is weakly positive. Hence by Lemma 1.3.1 and Remark 1.3.2, $K_{X/Y}L^{n-1}\geqq0$. □

THEOREM 2.2. *Let (f, X, Y, L) be a quasi-polarized fiber space with $\kappa(X)\geqq0$ and $\dim X=n\geqq3$, where Y is a normal projective variety with $\dim Y=m$ and $\kappa(Y)=0$ or 1. Then $g(L)\geqq q(Y)+\lceil((n-1)/2)L^n\rceil-m+1$. In particular, $g(L)\geqq q(Y)$ holds if $L^n\geqq2$.*

PROOF. Note that a quasi-polarized fiber space (f, X, Y, L) with Y a normal projective variety can be replaced to a quasi-polarized fiber space (f', X', Y', L') with X' and Y' smooth projective varieties and with $g(L)=g(L')$ and X' and Y' are birational to X and Y, respectively. Hence we omit the prime. Indeed, let $\mu:Y'\to Y$ be a resolution of Y. By Hironaka theory [**Hi**], there exist a birational morphism $\lambda:X'\to X$, and a surjective morphism with connected fibers $f':X'\to Y'$ such that $f\circ\lambda=\mu\circ f'$.

We remark that (f', X', Y', L') is a quasi-polarized fiber space and $g(L)=g(L')$, where $L'=(\lambda)^*L$.

Case (1): $\kappa(Y)=0$.

By Kawamata's theorem, $q(Y)\leqq\dim Y=m$.

Hence by Lemma 2.1,

$$\begin{aligned} g(L) &= 1+\frac{1}{2}K_{X/Y}(L)^{n-1}+\frac{n-1}{2}(L)^n+\frac{1}{2}f^*K_Y(L)^{n-1} \\ &\geqq 1+\frac{n-1}{2}(L)^n+\frac{1}{2}f^*K_Y(L)^{n-1}. \end{aligned}$$

Since $f^*K_Y(L)^{n-1}\geqq 0$, and $g(L)\in \boldsymbol{Z}$, we have

$$g(L) \geqq m+\left\lceil\frac{n-1}{2}L^n\right\rceil-m+1$$

$$\geqq q(Y)+\left\lceil\frac{n-1}{2}L^n\right\rceil-m+1.$$

Case (2): $\kappa(Y)=1$.

By Iitaka theory ([**Ii1**]), there exists a fiber space $g: Y\to C$ onto a curve C with a general fiber F of $\kappa(F)=0$.

By Theorem B in Appendix and Kawamata's theorem, $q(Y)\leqq g(C)+q(F)\leqq g(C)+\dim F\leqq g(C)+m-1$.

Hence if $g(C)=0$, $q(Y)\leqq m-1$.

Hence

$$g(L) \geqq 1+\left\lceil\frac{n-1}{2}L^n\right\rceil$$

$$> m-1+\left\lceil\frac{n-1}{2}L^n\right\rceil-m+1$$

$$\geqq q(Y)+\left\lceil\frac{n-1}{2}L^n\right\rceil-m+1.$$

If $g(C)\geqq 1$, applying Theorem 1.3.3 to $(g\circ f, X, C, L)$, we have $g(L)\geqq g(C)+\lceil((n-1)/2)L^n\rceil$, since $\kappa(F)+\dim C\geqq\kappa(X)\geqq 0$ ([**Ii1**]).

Hence

$$g(L) \geqq g(C)+m-1+\left\lceil\frac{n-1}{2}L^n\right\rceil-m+1$$

$$\geqq q(Y)+\left\lceil\frac{n-1}{2}L^n\right\rceil-m+1. \qquad \square$$

Next we prove that Conjecture 2 is true if $\kappa(X)\geqq 0$, $\kappa(Y)\leqq 1$, and $\dim Y=2$.

THEOREM 2.3. *Let (f, X, Y, L) be a quasi-polarized fiber space with $\kappa(X)\geqq 0$ and $\dim X=n\geqq 3$, where Y is a normal projective surface over $\boldsymbol{C}$ with $\kappa(Y)\leqq 1$. Then $g(L)\geqq q(Y)+\lceil((n-1)/2)L^n\rceil-1$.*

PROOF. As in the proof of Theorem 2.2, (f, X, Y, L) is replaced by (f', X', Y', L'). If $\kappa(Y)=0$ or 1, then, by Theorem 2.2, $g(L)\geqq q(Y)+\lceil((n-1)/2)L^n\rceil-1$ holds.

So we may assume that $\kappa(Y)=-\infty$.

If $q(Y)=0$, it is obviously proved. Since $\kappa(X)\geqq 0$ and $g(L)$ is an integer,

$$g(L) \geqq 1+\left\lceil\frac{n-1}{2}L^n\right\rceil.$$

If $q(Y)\geqq 1$, there exists an Albanese map $\pi: Y\to C$ where C is a smooth curve of genus $q(Y)$. Hence $h=\pi\circ f: X\to C$ is a fiber space. Since $\kappa(F_h)+\dim C\geqq \kappa(X)\geqq 0$ and $g(C)\geqq 1$, applying Theorem 1.3.3 to $(\pi\circ f, X, C, L)$, we have

$$g(L)\geqq g(C)+\left\lceil\frac{n-1}{2}L^n\right\rceil>q(Y)-1+\left\lceil\frac{n-1}{2}L^n\right\rceil,$$

where F_h is a general fiber of h. □

Appendix.

First we shall prove the following theorem by the same method as [**V3**].

THEOREM A. *Let X and Y be smooth quasi-projective varieties over $\mathbf{C}$, $\mathcal{L}$ a semiample invertible sheaf over X, $f: X\to Y$ a projective surjective morphism, and $\omega_{X/Y}=\omega_X\otimes f^*\omega_Y^{-1}$. Then for any positive integer k, $f_*((\omega_{X/Y}\otimes\mathcal{L})^{\otimes k})$ is weakly positive in the sense of Viehweg* [**V3**].

REMARK. If $\mathcal{L}$ is semiample over $f^{-1}(U)$ for an open set $U\subset Y$, then we can prove that for any positive integer k, $f_*((\omega_{X/Y}\otimes\mathcal{L})^{\otimes k})$ is weakly positive by the same method as the following argument.

We use the same notations as in [**V3**].

Let $\mathcal{F}$ be a torsion free coherent sheaf over Y and $\mathcal{F}^{**}$ the double dual of $\mathcal{F}$. Let $\hat{S}^\beta\mathcal{F}$ denote the double dual of the β-th symmetric power of $\mathcal{F}$.

DEFINITION. The sheaf $\mathcal{F}$ is said to be generated over an open set U by global section if the canonical map

$$\mathcal{O}_U\otimes H^0(Y, \mathcal{F})\longrightarrow\mathcal{F}_U$$

is a surjection and U is an open set dense in Y. An invertible sheaf $\mathcal{L}$ is said to be semiample over U if some tensor power of $\mathcal{L}$ is generated over U by global sections. Note that $\mathcal{F}=0$ is said to be generated over Y by global sections. $\mathcal{F}$ is said to be weakly generated over an open set U if the double dual of some symmetric power of $\mathcal{F}$ is generated over U by global sections.

Note that letting $i: Y(\mathcal{F})\subset Y$ be the biggest open set such that $\mathcal{F}$ is locally free, $\hat{S}^k(\mathcal{F})=i_*S^k(i^*\mathcal{F})$.

DEFINITION (Viehweg [**V3**]). The sheaf $\mathcal{F}$ is said to be weakly positive if there exist an ample invertible sheaf $\mathcal{H}$ over Y and an open set U such that for any positive integer α, $S^\alpha(\mathcal{F})\otimes\mathcal{H}$ is weakly generated over an open set U by global sections.

Note that $\mathcal{F}=0$ is weakly positive and that since $\mathcal{F}$ is torsion free, $\mathcal{F}$ is locally free in codimension one. Hence $H^0(Y, \hat{S}^\beta(\mathcal{F}))=H^0(Y(\mathcal{F}), S^\beta(\mathcal{F}))$. Hence to prove $f_*((\omega_{X/Y}\otimes\mathcal{L})^{\otimes k})$ is weakly positive, we may replace Y by $Y-S$ over which $f_*((\omega_{X/Y}\otimes\mathcal{L})^{\otimes k})$ is locally free with $\operatorname{codim}(Y-S)\geqq 2$.

At first we shall prove the following lemmata.

LEMMA A.1. *$f_*(\omega_{X/Y}\otimes\mathcal{L})$ is weakly positive.*

PROOF. Since $\mathcal{L}$ is semiample, for some $N\geqq 2$

$$\mathcal{L}^{\otimes N}=\mathcal{O}\Big(\sum_j \nu_j D_j\Big),$$

where D_j are non-singular prime divisors with $\nu_j=1$.

Let $\mathcal{L}^{(i)}=\mathcal{L}^{\otimes i}(-\sum_j[i\cdot\nu_j/N]D_j)$. By Lemma 5.1 in [V3], $f_*(\mathcal{L}^{(i)}\otimes\omega_{X/Y})$ is weakly positive. But since $N\geqq 2$, we have $\mathcal{L}^{(1)}=\mathcal{L}$. Therefore

$$f_*(\omega_{X/Y}\otimes\mathcal{L}^{(1)})=f_*(\omega_{X/Y}\otimes\mathcal{L})$$

is weakly positive. □

LEMMA A.2. *Let f, X, Y be as above and $\mathcal{L}$ a semiample invertible sheaf over X.*

(1) *Let $\mathcal{A}$ be an invertible sheaf over X and $\sum_j e_jE_j$ an effective divisor's irreducible decomposition such that for $N>0$, $\mathcal{A}^{\otimes N}=\mathcal{O}_X(\sum_j e_jE_j)$. Suppose that the support of $\sum_j e_jE_j$ is normally crossing over $f^{-1}(U)$ for a dense open set $U\subset Y$.*

Then, for $0\leqq i\leqq N-1$, the sheaf $f_*(\mathcal{A}^{\otimes i}(-\sum_j[i\cdot e_j/N]E_j)\otimes\omega_{X/Y}\otimes\mathcal{L})$ is weakly positive. (Therefore for $0\leqq i\leqq N-1$, the sheaf $f_*(\mathcal{A}^{\otimes i}(-\sum_j g_jE_j)\otimes\omega_{X/Y}\otimes\mathcal{L})$ is weakly positive if

$$f_*\Big(\mathcal{A}^{\otimes i}\Big(-\sum_j\Big[\frac{i\cdot e_j}{N}\Big]E_j\Big)\otimes\omega_{X/Y}\otimes\mathcal{L}\Big)\longrightarrow f_*\Big(\mathcal{A}^{\otimes i}\Big(-\sum_j g_jE_j\Big)\otimes\omega_{X/Y}\otimes\mathcal{L}\Big)$$

is an isomorphism over a dense open subset of Y.)

(2) *Let $\mathcal{N}$ be an invertible sheaf over X which is generated over $f^{-1}(U)$ by global sections for an open set $U\subset Y$. Then $\mathcal{N}=\mathcal{O}_X(B+\sum_j d_jD_j)$ as the irreducible decomposition such that B is nonsingular over $f^{-1}(U)$ and the support of $\sum_j d_jD_j$ is contained in $f^{-1}(Y-U)$.*

PROOF.

(1) We take a blowing up $\mu: T\to X$ which is an isomorphism over $f^{-1}(U)$ such that $(\mu^*\mathcal{A})^{\otimes N}=\mathcal{O}_X(\sum_{j,k}f_{j,k}F_{j,k})$ with the support of the irreducible decomposition $\sum_{j,k}F_{j,k}$ normally crossing. Note that $e_j|f_{j,k}$, and the centers of the blowing up never meet the points where $\sum_j E_j$ is normally crossing. Let d be a composite of a desingularization $Z\to\operatorname{Spec}(\bigoplus_{i=0}^{N-1}(\mu^*\mathcal{A})^{-i})$ and the structure

morphism $\mathrm{Spec}(\oplus_{i=0}^{N-1}(\mu^*\mathcal{A})^{-i})\to T$. Then by (2.3) in [**V3**], we have

$$d_*\omega_{Z/Y}=\bigoplus_{i=0}^{N-1}((\mu^*\mathcal{A})^{(i)}\otimes\omega_{T/Y}).$$

Hence

$$f_*\circ\mu_*\circ d_*(\omega_{Z/Y}\otimes d^*\circ\mu^*\mathcal{L})=\bigoplus_{i=0}^{N-1}f_*\circ\mu_*((\mu^*\mathcal{A})^{(i)}\otimes\omega_{T/Y}\otimes\mu^*\mathcal{L}).$$

By Lemma A.1,

$$f_*\circ\mu_*\circ d_*(\omega_{Z/Y}\otimes d^*\circ\mu^*\mathcal{L})$$

is weakly positive. Hence

$$f_*\circ\mu_*((\mu^*\mathcal{A})^{(i)}\otimes\omega_{T/Y}\otimes\mu^*\mathcal{L})=f_*\circ\mu_*\Big((\mu^*\mathcal{A})^{\otimes i}\Big(-\sum_{j,k}\Big[\frac{i\cdot f_{j,k}}{N}\Big]F_{j,k}\Big)\otimes\omega_{T/Y}\otimes\mu^*\mathcal{L}\Big)$$

is weakly positive. The following natural map is an isomorphism over U

$$f_*\circ\mu_*\Big((\mu^*\mathcal{A})^{\otimes i}\Big(-\sum_{j,k}\Big[\frac{i\cdot f_{j,k}}{N}\Big]F_{j,k}\Big)\otimes\omega_{T/Y}\otimes\mu^*\mathcal{L}\Big)$$

$$\to f_*\circ\mu_*\Big((\mu^*\mathcal{A})^{\otimes i}\Big(-\sum{}'\Big[\frac{i\cdot f_{j,k}}{N}\Big]F_{j,k}\Big)\otimes\omega_{T/Y}\otimes\mu^*\mathcal{L}\Big)$$

if in the last term the sum $\sum'$ tends over $F_{j,k}$'s intersecting on $(f\circ\mu)^{-1}(U)$. Hence the last term is weakly positive. On the other hand $\mathcal{O}(\sum_j[i\cdot e_j/N]\mu^*E_j)=\mathcal{O}(\sum'[i\cdot f_{j,k}/N]F_{j,k})$ over $(f\circ\mu)^{-1}(U)$.

Hence over U

$$f_*\circ\mu_*\Big((\mu^*\mathcal{A})^{\otimes i}\Big(-\sum{}'\Big[\frac{i\cdot f_{j,k}}{N}\Big]F_{j,k}\Big)\otimes\omega_{T/Y}\otimes\mu^*\mathcal{L}\Big)$$

$$=f_*\circ\mu_*\Big((\mu^*\mathcal{A})^{\otimes i}\Big(-\sum_j\Big[\frac{i\cdot e_j}{N}\Big]\mu^*E_j\Big)\otimes\omega_{T/Y}\otimes\mu^*\mathcal{L}\Big)$$

$$=f_*\Big(\mathcal{A}^{\otimes i}\Big(-\sum_j\Big[\frac{i\cdot e_j}{N}\Big]E_j\Big)\otimes\omega_{X/Y}\otimes\mathcal{L}\Big)$$

is weakly positive.

(2) Let $\mathcal{N}=\mathcal{O}_X(B+\sum_i d_iD_i)$, where $D_i\subset f^{-1}(Y-U)$ for each i. Since $\mathcal{N}$ is generated over $f^{-1}(U)$ by global sections and $\mathcal{N}|_{f^{-1}(U)}=\mathcal{O}_X(B)|_{f^{-1}(U)}$, a general section B of $\mathcal{N}|_{f^{-1}(U)}$ is nonsingular over $f^{-1}(U)$ by Bertini's theorem. □

LEMMA A.3. *Let X, Y, f, $\mathcal{L}$ be as above and $\mathcal{H}$ an ample line bundle on Y such that for given $k>0$ and some $\nu>0$ the sheaf $\hat{S}^\nu(f_*((\omega_{X/Y}\otimes\mathcal{L})^{\otimes k})\otimes\mathcal{H}^{\otimes k})$ is generated over an open set U by global sections.*

Then $f_*((\omega_{X/Y}\otimes\mathcal{L})^{\otimes k}\otimes f^*\mathcal{H}^{\otimes k-1})$ is weakly positive.

PROOF. By (1.3 iv) in [V3] we may replace Y by $Y-S$, as long as S is a closed subvariety of codimension $\geqq 2$. Hence we may assume that $f_*((\omega_{X/Y}\otimes \mathcal{L}\otimes f^*\mathcal{H})^{\otimes k})$ is locally free on Y.

We put

$$\mathcal{M}=\mathrm{Im}(f^*(f_*((\omega_{X/Y}\otimes\mathcal{L}\otimes f^*\mathcal{H})^{\otimes k}))\longrightarrow(\omega_{X/Y}\otimes\mathcal{L}\otimes f^*\mathcal{H})^{\otimes k})^{**},$$

where ** denotes the double dual.

Then $\mathcal{M}$ is a line bundle, i.e.,

$$\mathcal{M}=(\omega_{X/Y}\otimes\mathcal{L}\otimes f^*\mathcal{H})^{\otimes k}\otimes\mathcal{O}_X(-Z),$$

where Z is an effective divisor on X.

Then there exists a blowing up of X, $\rho_1: X'\to X$ such that

$$\rho_1^*\circ f^*(f_*((\omega_{X/Y}\otimes\mathcal{L}\otimes f^*\mathcal{H})^{\otimes k}))\longrightarrow\rho_1^*((\omega_{X/Y}\otimes\mathcal{L}\otimes f^*\mathcal{H})^{\otimes k})\otimes\rho_1^*\mathcal{O}(-Z)\otimes\mathcal{O}(-E)$$

is surjective, where E is an exceptional effective divisor.

In order to have the support of $\rho_2^*(\rho_1^*Z+E)=D$ in a normal crossing divisor, we take a blowing up $\rho_2: X''\to X'$. Here we put $\rho_1\circ\rho_2=\rho$ and $f\circ\rho=g$.

The pullback of the map above

$$\rho^*\circ f^*(f_*((\omega_{X/Y}\otimes\mathcal{L}\otimes f^*\mathcal{H})^{\otimes k}))\longrightarrow\rho^*((\omega_{X/Y}\otimes\mathcal{L}\otimes f^*\mathcal{H})^{\otimes k})\otimes\mathcal{O}(-D)$$

is a surjection, whose image we denote by $\mathcal{N}$. Note that $g_*\mathcal{N}\supset f_*((\omega_{X/Y}\otimes\mathcal{L}\otimes f^*\mathcal{H})^{\otimes k})=g_*((\omega_{X''/Y}\otimes\rho^*\mathcal{L})^{\otimes k})\otimes\mathcal{H}^{\otimes k}$ and that $\rho_*\omega_{X''}^{\otimes k}=\omega_X^{\otimes k}$. Then we have

$$\begin{aligned}g^*(f_*((\omega_{X/Y}\otimes\mathcal{L}\otimes f^*\mathcal{H})^{\otimes k}))&=g^*(f_*((\omega_{X/Y}\otimes\mathcal{L})^{\otimes k})\otimes\mathcal{H}^{\otimes k})\\&=g^*(g_*((\omega_{X''/Y}\otimes\rho^*\mathcal{L})^{\otimes k})\otimes\mathcal{H}^{\otimes k}).\end{aligned}$$

We remark that

$$f_*((\omega_{X/Y}\otimes\mathcal{L})^{\otimes k})\otimes\mathcal{H}^{\otimes k}\cong g_*((\omega_{X''/Y}\otimes\rho^*\mathcal{L})^{\otimes k})\otimes\mathcal{H}^{\otimes k},$$

and

$$S^\nu(f_*((\omega_{X/Y}\otimes\mathcal{L})^{\otimes k})\otimes\mathcal{H}^{\otimes k})\cong S^\nu(g_*((\omega_{X''/Y}\otimes\rho^*\mathcal{L})^{\otimes k})\otimes\mathcal{H}^{\otimes k}).$$

Since

$$g^*(g_*((\omega_{X''/Y}\otimes\rho^*\mathcal{L})^{\otimes k})\otimes\mathcal{H}^{\otimes k})\longrightarrow\rho^*((\omega_{X/Y}\otimes\mathcal{L}\otimes f^*\mathcal{H})^{\otimes k})\otimes\mathcal{O}(-D)$$

is surjective,

$$\begin{aligned}g^*S^\nu(g_*((\omega_{X''/Y}\otimes\rho^*\mathcal{L})^{\otimes k})\otimes\mathcal{H}^{\otimes k})&\longrightarrow S^\nu(\rho^*((\omega_{X/Y}\otimes\mathcal{L}\otimes f^*\mathcal{H})^{\otimes k})\otimes\mathcal{O}(-D))\\&\cong\rho^*((\omega_{X/Y}\otimes\mathcal{L}\otimes f^*\mathcal{H})^{\otimes k\nu})\otimes\mathcal{O}(-\nu D)\end{aligned}$$

is surjective.

Hence by hypothesis, $\mathcal{N}^{\otimes\nu}=\rho^*((\omega_{X/Y}\otimes\mathcal{L}\otimes f^*\mathcal{H})^{\otimes k\nu})\otimes\mathcal{O}(-\nu D)$ is generated over $g^{-1}(U)$ for an open set U of Y by global sections.

Hence we apply Lemma A.2 to $(\rho^*(\omega_{X/Y}\otimes\mathcal{L}\otimes f^*\mathcal{H}))^{\otimes k}=\mathcal{N}\otimes\mathcal{O}(D)$.

Then $g_*(\omega_{X''/Y}\otimes\rho^*\mathcal{L}\otimes(\rho^*(\omega_{X/Y}\otimes\mathcal{L}\otimes f^*\mathcal{H}))^{\otimes k-1}(-[((k-1)/k)D]))$ is weakly positive.

Since $\rho_*\omega_{X''}=\omega_X$, we have

$$(1)\qquad g_*(\omega_{X''/Y}\otimes\rho^*\mathcal{L}\otimes(\rho^*(\omega_{X/Y}\otimes\mathcal{L}\otimes f^*\mathcal{H}))^{\otimes k-1}\left(-\left[\frac{k-1}{k}D\right]\right))$$
$$\subset g_*(\omega_{X''/Y}\otimes\rho^*\mathcal{L}\otimes(\rho^*(\omega_{X/Y}\otimes\mathcal{L}\otimes f^*\mathcal{H}))^{\otimes k-1})$$
$$= f_*((\omega_{X/Y}\otimes\mathcal{L})^{\otimes k})\otimes\mathcal{H}^{\otimes k-1},$$

and since $\mathcal{O}([((k-1)/k)D])\subset\mathcal{O}(D)$ and $\rho^*\omega_X\subset\omega_{X''}$,

$$(2)\qquad \mathcal{N}\otimes g^*\mathcal{H}^{-1}\subset(\omega_{X''/Y}\otimes\rho^*\mathcal{L}\otimes\rho^*(\omega_{X/Y}\otimes\mathcal{L}\otimes f^*\mathcal{H})^{\otimes k-1})\left(-\left[\frac{k-1}{k}D\right]\right).$$

Since $g_*\mathcal{N}\supset f_*((\omega_{X/Y}\otimes\mathcal{L}\otimes f^*\mathcal{H})^{\otimes k})$, we have by (1) and (2)

$$g_*\mathcal{N}\otimes\mathcal{H}^{-1}\subset g_*(\omega_{X''/Y}\otimes\rho^*\mathcal{L}\otimes\rho^*(\omega_{X/Y}\otimes\mathcal{L}\otimes f^*\mathcal{H})^{\otimes k-1}\left(-\left[\frac{k-1}{k}D\right]\right))$$
$$\subset f_*((\omega_{X/Y}\otimes\mathcal{L})^{\otimes k})\otimes\mathcal{H}^{\otimes k-1}$$

three of which all coincide and are weakly positive. □

LEMMA A.4. *Let f, X, Y, $\mathcal{L}$ be as in Theorem* A, *Y' a smooth quasi-projective variety, $\tau: Y'\to Y$ a flat projective morphism, $S=X\times_Y Y'$, S' the normalization of S, and X' a desingularization of S'. We have the following diagram:*

$$\begin{array}{ccccccc} X' & \xrightarrow{d} & S' & \xrightarrow{\sigma} & S & \xrightarrow{\tau_2} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle h'} & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow[id]{} & Y' & \xrightarrow[id]{} & Y' & \xrightarrow[\tau]{} & Y \end{array}$$

We put $\tau_1=\tau_2\circ\sigma$ and $\tau'=\tau_1\circ d$.

Assume that S' has only rational singularities.

Then for any $k\geqq 0$ there exists a homomorphism

$$i: f'_*((\omega_{X'/Y'}\otimes(\tau')^*\mathcal{L})^{\otimes k+1})\longrightarrow\tau^*\circ f_*((\omega_{X/Y}\otimes\mathcal{L})^{\otimes k+1})$$

which is an isomorphism over an open subvariety of Y'.

PROOF. By the proof of Lemma 3.2 in [**V3**],

$$\sigma_*\circ d_*(\omega_{X'/Y'}^{\otimes k+1})\longrightarrow\tau_2^*(\omega_{X/Y}^{\otimes k+1})$$

is an isomorphism over $h^{-1}(U)$ for an open subvariety U of Y'. Then

$$\sigma_* \circ d_*((\omega_{X'/Y'} \otimes (\tau')^* \mathcal{L})^{\otimes k+1}) \cong \sigma_* \circ d_*(\omega_{X'/Y'}^{\otimes k+1}) \otimes \tau_2^* \mathcal{L}^{\otimes k+1}$$
$$\to \tau_2^*((\omega_{X/Y} \otimes \mathcal{L})^{\otimes k+1})$$

is an isomorphism over $h^{-1}(U)$.

Hence since τ is a flat morphism, by the flat base change theorem ([**Ha1**]),

$$f'_*((\omega_{X'/Y'} \otimes (\tau')^* \mathcal{L})^{\otimes k+1}) \longrightarrow h_* \circ \tau_2^*((\omega_{X/Y} \otimes \mathcal{L})^{\otimes k+1})$$
$$\cong \tau^* \circ f_*((\omega_{X/Y} \otimes \mathcal{L})^{\otimes k+1})$$

is an isomorphism over U. □

PROOF OF THEOREM A. Let $\mathcal{H}$ be any ample line bundle on Y.

Only to prove Theorem A, by (1.3 iv) in [**V3**], we may assume that $f_*((\omega_{X/Y} \otimes \mathcal{L})^{\otimes k})$ is locally free on Y.

$$r = \mathrm{Min}\{s > 0 : f_*((\omega_{X/Y} \otimes \mathcal{L})^{\otimes k}) \otimes \mathcal{H}^{\otimes sk-1} : \text{weakly positive}\}.$$

Then there exists a positive integer ν such that

$$S^{\nu}(f_*((\omega_{X/Y} \otimes \mathcal{L})^{\otimes k})) \otimes \mathcal{H}^{\otimes \nu(rk-1)} \otimes \mathcal{H}^{\otimes \nu}$$

is generated over an open set by global sections.

By Lemma A.3, $f_*((\omega_{X/Y} \otimes \mathcal{L})^{\otimes k}) \otimes \mathcal{H}^{\otimes r(k-1)}$ is weakly positive. Then by the choice of r, $(r-1)k-1 < r(k-1)$. Hence we have $r \leq k$. Hence for any surjective morphism and any $\mathcal{H}$, $f_*((\omega_{X/Y} \otimes \mathcal{L})^{\otimes k}) \otimes \mathcal{H}^{\otimes k^2-k}$ is weakly positive.

Next we take $\tau : Y' \to Y$: a finite surjective morphism such that $\tau^* \mathcal{H} = (\mathcal{H}')^{\otimes d}$ for a Cartier divisor $\mathcal{H}'$, where Y' is a smooth quasi-projective variety and d is given below. (We can take this. See [**B-G**], [**Ka1**], [**V3**].)

We use the same notations as in Lemma A.4.

We blow up X if necessary, so we may assume that the support of the ramification locus $\Delta(S'/X)$ (see [**V2**]) is a normal crossing divisor. Then the assumption of Lemma A.4 is satisfied. (See [**V1**].)

By the same argument above for $f' : X' \to Y'$ and Lemma A.4, we can prove that $\tau^* \circ f_*((\omega_{X/Y} \otimes \mathcal{L})^{\otimes k}) \otimes (\mathcal{H}')^{\otimes k^2-k}$ is weakly positive.

Let α be a positive integer, and we put $d = 2(k^2-k)\alpha + 1$.

For a sufficiently big integer β,

$$(1) \qquad S^{2\alpha\beta}(\tau^* \circ f_*((\omega_{X/Y} \otimes \mathcal{L})^{\otimes k}) \otimes (\mathcal{H}')^{\otimes k^2-k}) \otimes (\mathcal{H}')^{\otimes \beta}$$
$$\cong \tau^* S^{2\alpha\beta}(f_*((\omega_{X/Y} \otimes \mathcal{L})^{\otimes k})) \otimes (\tau^* \mathcal{H})^{\otimes \beta}$$

is generated over an open set by global sections.

Since the trace map $\tau_* \mathcal{O}_{Y'} \to \mathcal{O}_Y$ is surjective,

$$(2) \qquad \tau_* \circ \tau^*(S^{2\alpha\beta}(f_*((\omega_{X/Y} \otimes \mathcal{L})^{\otimes k})) \otimes \mathcal{H}^{\otimes \beta}) \longrightarrow S^{2\alpha\beta}(f_*((\omega_{X/Y} \otimes \mathcal{L})^{\otimes k})) \otimes \mathcal{H}^{\otimes \beta}$$

is surjective.

By (1),

$$\bigoplus \mathcal{O}_{Y'} \longrightarrow \tau^* S^{2\alpha\beta}(f_*((\omega_{X/Y}\otimes\mathcal{L})^{\otimes k}))\otimes\tau^*\mathcal{H}^{\otimes\beta}$$

is surjective over a dense open set of Y'.

Since τ is finite surjective,

$$\bigoplus \tau_*\mathcal{O}_{Y'} \longrightarrow \tau_*\circ\tau^*(S^{2\alpha\beta}(f_*((\omega_{X/Y}\otimes\mathcal{L})^{\otimes k}))\otimes\mathcal{H}^{\otimes\beta})$$

is surjective over a dense open set of Y.

Hence by (2)

$$(\bigoplus \tau_*\mathcal{O}_{Y'})\otimes\mathcal{H}^{\otimes\beta} \longrightarrow S^{2\alpha\beta}(f_*((\omega_{X/Y}\otimes\mathcal{L})^{\otimes k}))\otimes\mathcal{H}^{\otimes 2\beta}$$

is surjective over a dense open set of Y.

For a sufficiently big integer β, $\tau_*\mathcal{O}_{Y'}\otimes\mathcal{H}^{\otimes\beta}$ is generated by global sections.

Hence $S^{2\alpha\beta}(f_*((\omega_{X/Y}\otimes\mathcal{L})^{\otimes k}))\otimes\mathcal{H}^{\otimes 2\beta}$ is generated over an open set by global sections. Therefore $f_*((\omega_{X/Y}\otimes\mathcal{L})^{\otimes k})$ is weakly positive. □

We can also prove the following theorem. (This theorem was pointed out by the referee.)

THEOREM A'. *Let X and Y be smooth quasi-projective varieties over $\boldsymbol{C}$, $\mathcal{L}$ a semiample invertible sheaf over X, and $f: X\to Y$ a projective surjective morphism. Then for any positive integer k and i, $f_*(\omega_{X/Y}^{\otimes k}\otimes\mathcal{L}^{\otimes i})$ is weakly positive.*

PROOF. Let $\eta: X'\to X$ be a finite cyclic covering defined by the nonsingular divisor B such that $\mathcal{L}^{\otimes N}=\mathcal{O}(B)$. Then $\eta_*\omega_{X'/Y}=\bigoplus_{i=0}^{N-1}(\omega_{X/Y}\otimes\mathcal{L}^{\otimes i})$. Since X' is nonsingular and η is affine,

$$(\eta_*\omega_{X'/Y})^{\otimes k} = \eta_*(\omega_{X'/Y}^{\otimes k}).$$

Hence we have

$$(f\circ\eta)_*(\omega_{X'/Y}^{\otimes k}) = \bigoplus_{t=0}^{k(N-1)} f_*(\omega_{X/Y}^{\otimes k}\otimes\mathcal{L}^{\otimes t})^{\oplus\alpha(t)},$$

which is weakly positive by Viehweg [V3], where $(\sum_{i=0}^{N-1}x^i)^k=\sum_{t=0}^{k(N-1)}\alpha(t)x^t$. Thus $f_*(\omega_{X/Y}^{\otimes k}\otimes\mathcal{L}^{\otimes t})$ is also weakly positive for $0\leqq t\leqq k(N-1)$. Tend $N\to\infty$ and we complete the proof. □

THEOREM B. *Let (f, X, Y) be a fiber space with $n=\dim X>\dim Y=s$. Then $q(X)\leqq q(F)+q(Y)$, where F is a general fiber of f.*

PROOF. Note that $H^0(X, f^*\Omega_Y^1)=H^0(Y, \Omega_Y^1)$ since (f, X, Y) is a fiber space and that there exists the canonical restriction: $H^0(X, \Omega_X^1)\to H^0(F, \Omega_F^1)$, $\phi\to\phi_F$. By the following claim proved soon, we can show the inequality

$$\dim H^0(X, \Omega_X^1)/H^0(X, f^*\Omega_Y^1) \leqq \dim H^0(F, \Omega_F^1).$$

Indeed let $(\phi_i)_{1\leq i\leq q}$ be a basis of representative 1-forms of $H^0(X, \Omega_X^1)/H^0(X, f^*\Omega_Y^1)$. If there exist complex numbers $(a_i)_{1\leq i\leq q}$ such that $(\sum_{i=1}^q a_i\phi_i)_F=0$, by the claim $\sum_{i=1}^q a_i\phi_i=0 \mod H^0(X, f^*\Omega_Y^1)$, which implies the image of the basis is linearly independent in $H^0(F, \Omega_F^1)$. It is enough to show the following claim:

CLAIM. *Let φ be an element of $H^0(X, \Omega_X^1)$ such that $\varphi_F=0$ for a general fiber F of f. Then there is a $\phi\in H^0(Y, \Omega_Y^1)$ such that $\varphi=f^*\phi$, where Ω_X^1 (resp. Ω_Y^1) is the sheaf of differentials of X (resp. Y).*

Let Y_0 be a Zariski open set such that $f_0: X_0=f^{-1}(Y_0)\to Y_0$ is smooth and $\sum(f)=Y-Y_0$. Let D be irreducible components of $\sum(f)$ of codimension 1 in Y and $D=\bigcup_{i=1}^t D_i$. Then we may assume that D and $f^{-1}(D)$ are normal crossing divisors. Indeed, if $\bigcup_{i=1}^t D_i$ is not a normal crossing divisor, then by taking some blowing ups $\mu_Y: Y_1\to Y$, $(\mu_Y^*(D))_{\mathrm{red}}$ is a normal crossing divisor. Then there exist a birational morphism $\mu_1: X_1\to X$ and a surjective morphism $f_1: X_1\to Y_1$ with connected fibers such that $\mu_Y\circ f_1=f\circ\mu_1$. Let $\sum(f_1)=\mu_Y^{-1}(\sum(f))$ and $Y_{1,0}=Y_1-\sum(f_1)$. Then $Y_{1,0}$ is a Zariski open set such that $f_1: f_1^{-1}(Y_{1,0})=X_{1,0}\to Y_{1,0}$ is smooth. Let A be the union of irreducible components of $\sum(f_1)$ of codimension 1 in Y_1. Then A is a normal crossing divisor. If $(f_1^{-1}(A))_{\mathrm{red}}$ is not a normal crossing divisor, then we take some blowing ups $\mu_2: X_2\to X_1$ such that $((f_1\circ\mu_2)^{-1}(A))_{\mathrm{red}}$ is a normal crossing divisor. We remark that $f_2=f_1\circ\mu_2: X_2\to Y_1$ is a fiber space, $q(X)=q(X_2)$, $q(Y)=q(Y_1)$, and $q(F)=q(F_2)$, where F (resp. F_2) is a general fiber of f (resp. f_2). If we can prove $q(X_2)\leqq q(F_2)+q(Y_1)$, then $q(X)\leqq q(F)+q(Y)$ is proved.

(Step 1)

We remark that there is an exact sequence

$$0\longrightarrow f_0^*\Omega_{Y_0}^1\longrightarrow\Omega_{X_0}^1\longrightarrow\Omega_{X_0/Y_0}^1\longrightarrow 0,$$

where Ω_{X_0/Y_0}^1 is the sheaf of relative differentials of X_0 over Y_0.

Hence

$$0\longrightarrow H^0(X_0, f_0^*\Omega_{Y_0}^1)\xrightarrow{\alpha} H^0(X_0, \Omega_{X_0}^1)\xrightarrow{\beta} H^0(X_0, \Omega_{X_0/Y_0}^1)$$

is exact.

Let $\varphi\in H^0(X, \Omega_X^1)$. We assume that $\varphi_{F_y}=0$ for some $y\in Y_0$, where F_y is the fiber of f over y.

Note that

$$H^0(X_0, \Omega_{X_0/Y_0})=H^0(Y_0, f_*\Omega_{X_0/Y_0})\cong \mathrm{Hom}(\mathcal{O}_{Y_0}, f_*\Omega_{X_0/Y_0}).$$

Hence there corresponds $\Phi: \mathcal{O}_{Y_0}\to f_*\Omega_{X_0/Y_0}$ to the given $\beta(\varphi_{X_0})$.

By Hodge theory, $\dim H^0(F_y, \Omega_{F_y}^1)$ is constant for any $y\in Y_0$. Thus $f_*\Omega_{X_0/Y_0}\otimes\mathcal{O}_y/m_y=H^0(F_y, \Omega_{F_y}^1)$ for any $y\in Y_0$. Hence $\varphi_{F_y}=0$ for some $y\in Y_0$

implies the following composite map is zero; $\mathcal{O}_{Y_0}\to f_*\Omega_{X_0/Y_0}\otimes\mathcal{O}_y/m_y$. By NAK lemma, the map $\mathcal{O}_{Y_0}\to f_*\Omega_{X_0/Y_0}\otimes\mathcal{O}_y$ is zero and $\Phi:\mathcal{O}_{Y_0}\to f_*\Omega_{X_0/Y_0}$ is zero. Hence $\beta(\varphi_{X_0})=0$.

Therefore by the above exact sequence there exists $\psi_0\in H^0(X_0, f_0^*\Omega_{Y_0}^1)\cong H^0(Y_0, \Omega_{Y_0}^1)$ such that $f_0^*\psi_0=\varphi$ on X_0.

(Step 2)

Let $A=Y-(D\cup Y_0)$ and $Y_1=A\cup Y_0$. Then A is an analytic subspace of Y_1 and $\operatorname{codim}(A)\geqq 2$ in Y_1. Hence by Hartog's theorem, there exists $\psi_1\in H^0(X_1, f^*\Omega_X^1)$ such that $f^*\psi_1=\varphi$ on $X_1=f^{-1}(Y_1)$.

(Step 3)

The following argument is the same as in the proof of Proposition 6.7 of [**F-R**] p. 975.

Let $D=\bigcup_{i=1}^t D_i$, $f^{-1}(D)=W=\bigcup_j W_j$ and for each D_i we take an irreducible component W_i of $f^{-1}(D_i)$ such that $f(W_i)=D_i$.

Let $M_i=\{x\in W_i \mid f_{W_i}: W_i\to D_i$ is of maximal rank at $x\in W\setminus\bigcup_{j\neq i}W_j$ and $f(x)\notin D_j$ for $j\neq i\}$, and $N_i=\{y\in D_i \mid y=f(x),\ x\in M_i\}$. We remark that D_i and W_i are smooth by assumption. Let $x\in M_i$. Then we take a coordinate system $(x_1, x_2, \cdots, x_n)$ on X around $x\in M_i$ and a coordinate system $(y_1, y_2, \cdots, y_s)$ on Y around $y=f(x)$ such that $W_i=\{x_1=0\}$, $D_i=\{y_1=0\}$, and f is defined by $(x_1, x_2, \cdots, x_n)\to(x_1^\mu, x_2, \cdots, x_s)=(y_1, y_2, \cdots, y_s)$ around x, where $\mu\in\boldsymbol{N}$. Let $T_i(x)$ be the germ of manifold defined by $x_{s+1}=\cdots=x_n=0$ around x. We will identify $T_i(x)$ with a representing neighbourhood of x. Then $U_i(y)=f(T_i(x))$ is a neighbourhood of y in Y. Let G be the group generated by $g\in\operatorname{Aut}(T_i(x))$, where $g:(x_1, x_2, \cdots, x_s)\to(\rho x_1, x_2, \cdots, x_s)$ with $\rho=\exp(2\pi i/\mu)$. Then $f(T_i(x))$ is the quotient of $T_i(x)$ by G. By (Step 2), we have $\psi_{2,i}^y\in H^0(U_i(y)-D_i, \Omega_Y^1)$ such that $\varphi=f^*\psi_{2,i}^y$ on $f^{-1}(U_i(y))-f^{-1}(D_i)$. Hence $\varphi_{T_i(x)}=g^*\varphi_{T_i(x)}$ off W_i, where $\varphi_{T_i(x)}$ is the restriction of φ to $T_i(x)$. This implies that $\varphi_{T_i(x)}$ is G-invariant as a holomorphic 1-form. Hence $\varphi_{T_i(x)}$ is a pullpack of a holomorphic 1-form $(\psi_{2,i}^y)'$ on $U_i(y)=f(T_i(x))=T_i(x)/G$. We remark that $(\psi_{2,i}^y)'$ is an extension of $\psi_{2,i}^y$. Therefore $\varphi=f^*((\psi_{2,i}^y)')$ on $f^{-1}(U_i(y))-f^{-1}(D_i)$. Since φ and $(\psi_{2,i}^y)'$ are holomorphic, $\varphi=f^*((\psi_{2,i}^y)')$ on $f^{-1}(U_i(y))$.

(Step 4)

Let $Y_2=Y_1\cup\bigcup_{i=1}^t(\bigcup_{y\in N_i}U_i(y))$. Since ψ_1 and $(\psi_{2,i}^y)'$ are holomorphic, there exists $\psi_2\in H^0(Y_2, \Omega_Y^1)$ such that $\varphi=f^*\psi_2$ on $f^{-1}(Y_2)$ by the above argument. Because $Y-Y_2$ is contained in an analytic subset B of Y with $\operatorname{codim}(B)\geqq 2$ in Y, by Hartog's theorem, there exists $\psi\in H^0(Y, \Omega_Y^1)$ such that $\varphi=f^*\psi$ on $f^{-1}(Y_2)$. Since φ and ψ are holomorphic, $\varphi=f^*\psi$ on $X=f^{-1}(Y)$. □

References

[B-G] S. Bloch and D. Gieseker, The positivity of the Chern classes of an ample vector bundle, Invent. Math., **12** (1971), 112–117.

[BPV] W. Barth, C. Peters and A. Van de Ven, Compact complex surfaces, 1984, Springer-Verlag.

[EGA] J. Dieudonné and A. Grothendieck, Eléments de Géométrie Algébrique, Publ. Math. Inst. HES., **4, 8, 11, 17, 20, 24, 28, 32**.

[E-V] H. Esnault and E. Viehweg, Effective bounds for semipositive sheaves and for the height of points on curves over complex function fields, Compositio Math., **76** (1990), 69–85.

[F-R] T. Fujita and J. Roberts, Varieties with small secant varieties: extremal case, Amer. J. Math., **103** (1981), 953–976.

[Fj1] T. Fujita, On Kähler fiber spaces over curves, J. Math. Soc. Japan, **30** (1978), 779–794.

[Fj2] T. Fujita, On polarized manifolds whose adjoint bundles are not semipositive, in Algebraic Geometry, Sendai 1985, Adv. Stud. Pure Math., **10** (1987), 167–178.

[Fj3] T. Fujita, Classification Theories of Polarized Varieties, London Math. Soc. Lecture Note Ser., **155**, 1990, Cambridge.

[Fk1] Y. Fukuma, A lower bound for the sectional genus of quasi-polarized surfaces, to appear in Geom. Dedicata.

[Fk2] Y. Fukuma, Master Thesis (in Japanese), (1993).

[H-N] G. Harder and M. S. Narashimhan, On the cohomology groups of moduli spaces of vector bundles on curves, Math. Ann., **212** (1975), 215–248.

[Ha1] R, Hartshorne, Algebraic Geometry, Graduate Texts in Math., **52**, Springer-Verlag, New York, Heidelberg, Berlin, 1977.

[Ha2] R. Hartshorne, Ample subvarieties of Algebraic Varieties, Lecture Notes in Math., **156**, 1986, Springer-Verlag.

[Hi] H. Hironaka, Resolution of singularities of an algebraic variety over a field of characteristic zero I, II, Ann. of Math., **79** (1964), 109–326.

[Ii1] S. Iitaka, On D-dimension of algebraic varieties, J. Math. Soc. Japan, **23** (1971), 356–373.

[Ii2] S. Iitaka, Algebraic Geometry, Graduate Texts in Math., **76**, Springer-Verlag, New York, Heidelberg, Berlin, 1982.

[Io] P. Ionescu, Generalized adjunction and applications, Math. Proc. Cambridge Philos. Soc., **99** (1986), 457–472.

[KMM] Y. Kawamata, K. Matsuda and K. Matsuki, Introduction to the minimal model problem, in Algebraic Geometry, Sendai 1985, Adv. Stud. Pure Math., **10** (1987), 283–360.

[Ka1] Y. Kawamata, Characterization of Abelian varieties, Compositio Math., **43** (1981), 253–276.

[Ka2] Y. Kawamata, Kodaira dimension of algebraic fiber spaces over curves, Invent. Math., **66** (1982), 59–71.

[L-P] A. Lanteri and M. Palleschi, About the adjunction process for polarized algebraic surfaces, J. Reine Angew. Math., **352** (1984), 15–23.

[Mh] K. Maehara, The weak 1-positivity of direct image sheaves, J. Reine Angew. Math., **364** (1986), 112–129.

[Mo] S. Mori, Classification of higher-dimensional varieties, in Algebraic Geometry, Bowdoin 1985, Proc. Sympos. Pure Math., **46** (1987), 269–331.

[Mu] D. Mumford, Algebraic Geometry I Complex projective varieties, 1970, Springer-Verlag.

[O] K. Ohno, Some inequalities for minimal fibrations of surfaces of general types over curves, J. Math. Soc. Japan, **44** (1992), 643–666.

[U] K. Ueno, Classification Theory of Algebraic Varieties, Lecture Notes in Math., **439**, 1975, Springer-Verlag.

[V1] E. Viehweg, Rational singularity of higher dimensional schemes, Proc. Amer. Math. Soc., **63** (1977), 6–8.

[V2] E. Viehweg, Klassifikationstheorie algebraischer varietäten der dimension drei, Compositio Math., **41** (1980), 361–400.

[V3] E. Viehweg, Weak positivity and the additivity of the Kodaira dimension for certain fiber spaces, Adv. Stud. Pure Math., **1** (1983), 329–353.

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