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Some results on Igusa local zeta functions associated with simple prehomogeneous vector spaces

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1. Introduction.

The aim of this paper is to show that the Γ-factor is completely determined by the b-function of a simple prehomogeneous vector space satisfying some conditions, defined over a p-adic number field.

Let K be a p-adic number field. Let (G, ρ, V) be a regular prehomogeneous vector space defined over K and Y the Zariski-open $\rho(G)$-orbit in V. It is well-known that the dual (G, ρ^*, V^*) is also a regular prehomogeneous vector space, where ρ^* is the contragredient representation of ρ on the dual vector space V^* of V, [[**S-K**], §4 Proposition 10, Remark 11]. In this paper, we assume the following assumption:

ASSUMPTOIN (A): G is K-split and $Y_K=Y(K)$ is a single $\rho(G)_K$-orbit.

Let $Z(s)$, $Z^*(s)$ $(s\in \boldsymbol{C}^N)$ be the zeta distribution associated with (G, ρ, V), (G, ρ^*, V^*), respectively. The fundamental theorem of the theory of prehomogeneous vector spaces is roughly speaking that the Fourier transform $\widehat{Z(s)}$ of $Z(s)$ coincides with $Z^*(s^*)$ for a certain s^* up to a constant multiple $\Gamma_K(s)$ depending only meromorphically on s:

$$\widehat{Z(s)}=\Gamma_K(s)\cdot Z^*(s^*).$$

When (G, ρ, V) is a reduced regular irreducible prehomogeneous vector space, J.-I. Igusa proved that the Γ-factor $\Gamma_K(s)$ is completely determined by the b-function $b(s)$ of (G, ρ, V):

$$\Gamma_K(s)=\prod_{\lambda}\mathrm{Tate}\,\gamma(s-\kappa+\lambda), \quad b(s)=\prod_{\lambda}(s+\lambda).$$

[[**Igusa-3**]]. We shall show an analogy of Igusa's result in the case of simple prehomogeneous vector spaces. We remark that, in order to settle explicit forms of Γ-factors, we shall calculate explicitly Igusa local zeta functions.

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Finally, as an application, we might mention that we can calculate explicitly the Fourier transform of relative invariants over the real number field $\boldsymbol{R}$ from the explicit form of these Igusa local zeta functions, since the generalized Iwasawa-Tate theory holds for all regular simple prehomogeneous vector spaces with the assumption (A) [[**Kimura-1**], [**K-K**]].

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NOTATIONS. Throughout this paper, we denote by K a p-adic number field, namely a finite extension of the p-adic number field $\boldsymbol{Q}_p$. Let O_K be the ring of integers in K. We fix a prime element π once and for all, then πO_K is the unique maximal ideal of O_K. The cardinality of the residue field $O_K/\pi O_K$ is denoted by q. We denote by $|\cdot|_K$ the absolute value on K normalized by $|\pi|_K=q^{-1}$. We denote by $O_K[x_1, \cdots, x_n]$ the polynomial ring of n variables over O_K.

For a commutative ring R, $M(m, n; R)$ stands for the set of m by n matrices with entries in R. If $m=n$, then we write simply $M(m; R)$ instead of $M(m, m; R)$. For any $x\in M(m, n; R)$, ${}^t x(\in M(n, m; R))$ is the transpose of x. The determinant of $x\in M(m; R)$ is denoted by $\det(x)$. $\mathrm{Alt}(m; R)$ stands for the set of m by m alternating matrices with entries in R; $\mathrm{Alt}(m; R)=\{x\in M(m; R) \mid {}^t x=-x\}$. The pfaffian of $x\in\mathrm{Alt}(2n, R)$ is denoted by $Pf(x)$.

As usual, we denote by $\boldsymbol{C}$ and $\boldsymbol{Z}$, respectively, the complex number field and the ring of rational integers.

2. Preliminaries.

In this section, let (G, ρ, V) be a reductive regular prehomogeneous vector space defined over K, satisfying the assumption (A) [see § 1].

Take K-irreducible polynomials $P_1, \cdots, P_N$ defining the K-irreducible hypersurfaces contained in the singular set of (G, ρ, V), then they are relative invariants of (G, ρ, V). Let χ_j be the K-rational character of G corresponding to P_j:

$$P_j(\rho(g)x) = \chi_j(g)P_j(x) \quad (g\in G)$$

for $j=1, \cdots, N$. The polynomials $P_1, \cdots, P_N$ are called the *K-basic relative invariants* of (G, ρ, V).

By the regularity of (G, ρ, V), there exists an element $\kappa=(\kappa_1, \cdots, \kappa_N)\in(1/2\cdot\boldsymbol{Z})^N$ satisfying

$$\det(\rho(g))^2=\prod_{j=1}^{N}\chi_j(g)^{2\kappa_j}\quad(g\in G).$$

Since we assume that (G, ρ, V) is reductive, we have K-relatively invariant irreducible polynomials $P_1^*, \cdots, P_N^*$ corresponding to the characters $\chi_1^{-1}, \cdots, \chi_N^{-1}$, respectively [[**S-K**], §4 Proposition 24]. They are the K-basic relative invariants of (G, ρ^*, V^*), which is the dual of (G, ρ, V). Moreover we have

$$\det(\rho^*(g))^2=\prod_{j=1}^{N}\chi_j(g)^{-2\kappa_j}\quad(g\in G).$$

Let $S(V_K)$ be the space of Schwartz-Bruhat functions on V_K. For $s=(s_1, \cdots, s_N)\in\boldsymbol{C}^N$, we consider the integral:

$$Z(s)(\Phi)=\int_{Y_K}\prod_{j=1}^{N}|P_j(x)|_K^{s_j}\Phi(x)\mu(x)\quad(\Phi\in S(V_K)),$$

where $\mu(x)=\prod_{j=1}^{N}|P_j(x)|_K^{-\kappa_j}dx$ is a G_K-invariant measure on Y_K, while $dx=dx_1\cdots dx_n$ denotes the Haar measure on $V_K=K^n$ ($n=\dim V$) normalized by

$$\mathrm{vol}(O_K^n)=\int_{O_K^n}dx=1.$$

This integral is absolutely convergent for $\mathrm{Re}(s_j)>\kappa_j$ ($j=1, \cdots, N$). This integral can be analytically continued to a meromorphic function of $s\in\boldsymbol{C}^N$ and we define a tempered distribution

$$Z(s):\Phi\longmapsto Z(s)(\Phi)$$

on V_K depending on s meromorphically, which we call the *zeta distribution* associated with (G, ρ, V) over K. Starting from (G, ρ^*, V^*), we can similarly define the zeta distribution $Z^*(s)$ on V_K^*.

We fix an additive character ψ of K such that ψ is non-trivial on $\pi^{-1}O_K$ and trivial on O_K. We define the Fourier transform $\widehat{\Phi^*}$ of $\Phi^*\in S(V_K^*)$ by

$$\widehat{\Phi^*}(x)=\int_{V_K^*}\Phi^*(y)\psi(\langle x, y\rangle)dy.$$

We define the Fourier transform $\widehat{Z(s)}$ of $Z(s)$ by

$$\widehat{Z(s)}(\Phi^*)=Z(s)(\widehat{\Phi^*})\quad(\Phi^*\in S(V_K^*)).$$

In the above notations, we obtain the functional equation:

$$\hat{Z}(s)=\Gamma_K(s)\cdot Z^*(\kappa-s),\tag{2.1}$$

where $\Gamma_K(s)$ is a rational function of $q^{s_1}, \cdots, q^{s_N}$. The functional equation (2.1) is nothing but the p-adic fundamental theorem in the theory of prehomogeneous vector spaces. The p-adic fundamental theorem is proved by J.-I. Igusa and F. Sato with the *finite orbits condition* [[**Igusa-4**] and [**F. Sato**]]. Moreover, A. Gyoja succeeded in proving it without this condition in J. A. M. I. 1992-1993. We call $\Gamma_K(s)$ the *Γ-factor* of (G, ρ, V). Now we shall give the definitions of b-functions and Igusa local zeta functions. We put

$$P=\prod_{j=1}^{N} P_j, \quad P^*=\prod_{j=1}^{N} P_j^*,$$

then there exists a polynomial $b(s)$ in $s_1, \cdots, s_N$ of degree $\deg(P)$ satisfying

$$P^*(\partial/\partial x)P^{s+1}(x)=b(s)P^s(x),$$

where we put $P^s=\prod_{j=1}^{N} P_j^{s_j}$, $P^{s+1}=\prod_{j=1}^{N} P_j^{s_j+1}$ for $s=(s_1, \cdots, s_N)$. The polynomial $b(s)$ is called the *b-function* of (G, ρ, V). We define the *Igusa local zeta function* of (G, ρ, V) by the analytic continuation of the integral:

$$Z_K(s)=\int_{O_K^n} \prod_{j=1}^{N} |P_j(x)|_K^{s_j} dx$$

for $s=(s_1, \cdots, s_N) \in C^N$, $\mathrm{Re}(s_j)>0$ $(j=1, \cdots, N)$.

3. Igusa's Result.

In this section, we shall present J.-I. Igusa's result on Γ-factors $\Gamma_K(s)$ of reduced regular irreducible prehomogeneous vector spaces.

For every reduced regular irreducible prehomogeneous vector space (G, ρ, V), the number N of basic relative invariants is 1. Moreover κ is given as follows:

$$\kappa=\dim V/\deg(P) \quad (P\text{: the basic relative invariant})$$

In [[**Igusa-3**]], J.-I. Igusa restricts (G, ρ, V) by the additional assumption:

ASSUMPTION (A)$'$:
G is K-split and all roots of the b-function $b(s)$ are integers.

There exist 8 reduced regular irreducible prehomogeneous vector spaces satisfying the assumption (A)$'$. We remark that the assumption (A)$'$ is equivalent to the assumption (A) [see § 1] in the case of reduced regular irreducible prehomogeneous vector spaces.

DEFINITION 3.1. *We define the Tate local factor* Tate $\gamma(s)$ *by*

$$\text{Tate}\,\gamma(s)=(1-q^{-(1-s)})/(1-q^{-s}).$$

The Tate local factor Tate $\gamma(s)$ is nothing but the Γ-factor of the prehomogeneous vector space $(GL(1), \Lambda_1, \bar{K})$ [[**Tate**]].

PROPOSITION 3.1 ([**Igusa-3**], § 6). *Let (G, ρ, V) be the reduced regular irreducible prehomogeneous vector space satisfying the Assumption* (A)′. *Let $\Gamma_K(s)$ be its Γ-factor and $b(s)$ its b-function, then we have (after some normalization),*

$$\Gamma_K(s) = \prod_{\lambda} \text{Tate}\,\gamma(s-\kappa+\lambda), \quad b(s) = \prod_{\lambda} (s+\lambda).$$

The proof of this proposition is based on the computation of Igusa local zeta functions of such prehomogeneous vector spaces ([**Igusa-3**], § 6 Lemma 5).

4. A classification of simple prehomogeneous vector spaces.

We shall give the definition of simple prehomogeneous vector spaces.

Let G_s be a simple algebraic group and we put

$G = GL(1)^l \times G_s \quad (l \geqq 2)$,

$V = V_1 \oplus \cdots \oplus V_l$ (a direct sum of l finite-dimensional vector spaces).

We define a finite-dimensional representation ρ of G on V by

$$\rho(\alpha_1, \cdots, \alpha_l\,; g)x = (\alpha_1 \cdot \rho_1(g)x_1, \cdots, \alpha_l \cdot \rho_l(g)x_l)$$

for $x=(x_1, \cdots, x_l) \in V$ and $(\alpha_1, \cdots, \alpha_l : g) \in G$, where each $\rho_k : G \to GL(V_k)$ is an irreducible representation of G_s on V_k $(k=1, \cdots, l)$. We shall simply write $\rho = \rho_1 \oplus \cdots \oplus \rho_l$. If this triplet $(G, \rho, V) = (GL(1)^l \times G_s, \rho_1 \oplus \cdots \oplus \rho_l, V_1 \oplus \cdots \oplus V_l)$ is a prehomogeneous vector space, namely V has a Zariski-open $\rho(G)$-orbit, then we call it a *simple prehomogeneous vector space.*

We shall consider the Γ-factors $\Gamma_K(s)$ of the regular simple prehomogeneous vector spaces satisfying the assumption (A) [see § 1]. By [[**K-K-H**], § 2 Theorem 2.19 and Corollary 2.20], there exist 14 such prehomogeneous vector spaces. Moreover, their basic relative invariants are given in [[**Kimura-3**], § 3] and we can easily obtain $\kappa = (\kappa_1, \cdots, \kappa_N) \in (1/2 \cdot \boldsymbol{Z})^N$ for them.

PROPOSITION 4.1 ([**Kimura-3**], [**K-K-H**]). *All regular simple prehomogeneous vector spaces satisfying*

ASSUMPTION (A): *G is K-split and Y_K is a single $\rho(G)_K$-orbit.*

are given as follows:

(1) $(GL(1)^2 \times SL(n), \Lambda_1 \oplus \Lambda_1^*, V(n) \oplus V(n)^*)$

(2) $(GL(1)^n \times SL(n), \overbrace{\Lambda_1 \oplus \cdots \oplus \Lambda_1}^{n}, \overbrace{V(n) \oplus \cdots \oplus V(n)}^{n}) \quad (n \geqq 2)$

(3) $(GL(1)^2\times SL(2m+1),\ \Lambda_2\oplus\Lambda_1,\ V(m(2m+1))\oplus V(2m+1))$

(4) $(GL(1)^2\times Sp(n),\ \Lambda_1\oplus\Lambda_1,\ V(2n)\oplus V(2n))$

(5) $(GL(1)^2\times Spin(10),\ \Lambda_e\oplus\Lambda_e,\ V(16)\oplus V(16))$
Λ_e; *the even half-spin representation*

(6) $(GL(1)^3\times SL(2m),\ \Lambda_2\oplus\Lambda_1\oplus\Lambda_1,\ V(m(2m-1))\oplus V(2m)\oplus V(2m))$

(7) $(GL(1)^3\times SL(2m),\ \Lambda_2\oplus\Lambda_1\oplus\Lambda_1^*,\ V(m(2m-1))\oplus V(2m)\oplus V(2m)^*)$

(8) $(GL(1)^3\times SL(2m),\ \Lambda_2\oplus\Lambda_1^*\oplus\Lambda_1^*,\ V(m(2m-1))\oplus V(2m)^*\oplus V(2m)^*)$

(9) $(GL(1)^2\times Spin(8)$, *the vector rep*$\oplus$*a half-spin rep*, $V(8)\oplus V(8))$

(10) $(GL(1)^2\times Spin(10)$, *the vector rep*$\oplus$*a half-spin rep*, $V(10)\oplus V(16))$

(11) $(GL(1)^4\times SL(2m+1),\ \Lambda_2\oplus\Lambda_1\oplus\Lambda_1\oplus\Lambda_1,$
$V(m(2m+1))\oplus V(2m+1)\oplus V(2m+1)\oplus V(2m+1))$

(12) $(GL(1)^4\times SL(2m+1),\ \Lambda_2\oplus\Lambda_1\oplus\Lambda_1^*\oplus\Lambda_1^*,$
$V(m(2m+1))\oplus V(2m+1)\oplus V(2m+1)^*\oplus V(2m+1)^*)$

(13) $(GL(1)^{n+1}\times SL(n),\ \Lambda_1\overbrace{\oplus\cdots\oplus}^{n+1}\Lambda_1,\ V(n)\overbrace{\oplus\cdots\oplus}^{n+1}V(n))$

(14) $(GL(1)^{n+1}\times SL(n),\ \Lambda_1\overbrace{\oplus\cdots\oplus}^{n}\Lambda_1\oplus\Lambda_1^*,\ V(n)\overbrace{\oplus\cdots\oplus}^{n}V(n)\oplus V(n)^*)$.

Their basic relative invariants and $\kappa\in(1/2\cdot\boldsymbol{Z}^N)$ *are given as follows*:

(1) $N=1$
$P(X)=\langle x, y\rangle\ (X=(x, y)\in V(n)\oplus V(n)^*)$.
$\kappa=n$.

(2) $N=1$
$P(X)=\det(X)\ (X\in M(n\,;\bar{K})=V(n)\overbrace{\oplus\cdots\oplus}^{n}V(n))$.
$\kappa=n$.

(3) $N=1$
$P(X)=Pf(X)\ (X\in Alt(2m+2\,;\bar{K})=V(m(2m+1))\oplus V(2m+1))$.
$\kappa=2m+1$.

(4) $N=1$
$P(X)=Pf({}^tXJ_nX)$, *where* $J_n=\begin{pmatrix}0 & 1_n\\ -1_n & 0\end{pmatrix}$ $(X\in M(2n, 2\,;\bar{K})$
$=V(2n)\oplus V(2n))$.
$\kappa=2n$.

(5) $N=1$
We take as the basic relative invariant $P(X)$ *the polynomial given in* [[**Kimura-4**], §4 Proposition 4-2].
$\kappa=8$.

For type (6), (7) *and* (8), *we identified the representation space with*

$$\{X=(x, y, z)\mid x\in Alt(2m;\bar{K}),\ y, z\in\bar{K}^{2m}\}.$$

(6) $N=2$
$P_1(X)=Pf(x)$, $P_2(X)={}^t y\Delta(x)z$,
in which, for any $x\in Alt(2m;\bar{K})$, *we define the alternating matrix* $\Delta(x)$ *by*

$$\Delta(x)\cdot x=x\cdot\Delta(x)=Pf(x)\cdot 1_{2m}.$$

$\kappa=(1, 2m)$.

(7) $N=2$
$P_1(X)=Pf(x)$, $P_2(X)=\langle y, z\rangle$.
$\kappa=(2m-1, 2m)$.

(8) $N=2$
$P_1(X)=Pf(x)$, $P_2(X)={}^t yxz$.
$\kappa=(2m-3, 2m)$.

(9) $N=2$
$P_1(X)=q_1(x)$, $P_2(X)=q_2(y)$ *quadratic forms* $(X=(x, y)\in V(8)\oplus V(8))$.
$\kappa=(4, 4)$.

(10) $N=2$
$P_1(X)=q(x)$ *quadratic form on* $V(16)$,
$P_2(X)=\langle x, Q(y)\rangle$, $Q(y)={}^t(Q_1(y), \cdots, Q_{10}(y))$.
For an element y *of* $V(16)$ *is of the form*

$$y=y_0+\sum_{1\le i<j\le 5}y_{ij}e_ie_j+\sum_{k=1}^{5}y_k^*e_k^*$$

where $e_ke_k^*=e_1\cdots e_5$ $(k=1, \cdots, 5)$, *let* $Y=(y_{ij})$ *be the alternating matrix obtained from* y, *and* Y_i *the* 4×4 *alternating matrix obtained from* $(-1)^iY$ *by crossing out its i-th row and column* $(i=1, \cdots, 5)$. *We define ten quadratic forms* $Q_i(y)$ *on* $V(16)$ *by*

$$Q_i(y)=\sum_{j=1}^{5}y_{ij}y_j^*$$

and

$$Q_{i+5}=y_0y_i^*+Pf(Y_i)$$

for $i=1, \cdots, 5$ [[**S-K**], p. 119 and [**Kimura-4**], p. 21].
$\kappa=(1, 8)$.

(11) $N=4$
$P_k(X)=Pf\begin{pmatrix}x & {}^ty_k\\ -y_k & 0\end{pmatrix}$ *for* $k=1, 2, 3$,

$P_4(X)=Pf\begin{pmatrix} x & {}^t y \\ -y & 0 \end{pmatrix}$ $(y=(y_1, y_2, y_3))$

$(X=(x, y_1, y_2, y_3)\in Alt(2m+1)\oplus \bar{K}^{2m+1}\oplus \bar{K}^{2m+1}\oplus \bar{K}^{2m+1})$.

(12) $N=4$

$P_1(X)=Pf\begin{pmatrix} x & {}^t y \\ -y & 0 \end{pmatrix}$, $P_2(X)=\langle y, z\rangle$, $P_3(X)=\langle y, w\rangle$, $P_4(X)={}^t zxw$

$(X=(x, y, z, w)\in Alt(2m+1)\oplus \bar{K}^{2m+1}\oplus \bar{K}^{2m+1}\oplus \bar{K}^{2m+1})$.

(13) $N=n+1$

$P_k(X)=\det(x_1, \cdots, \check{x_k}, \cdots, x_{n+1})$ *for* $k=1, \cdots, n+1$, *where* "$\vee$" *means crossing out* $(X=(x_1, \cdots, x_{n+1})\in M(n, n+1; \bar{K})=\overbrace{V(n)\oplus \cdots \oplus V(n)}^{n+1})$.

$\kappa=(\overbrace{1, \cdots, 1}^{n+1})$.

(14) $N=n+1$

$P_k(X)=\langle x_k, y\rangle$ *for* $k=1, \cdots, n$, $P_{n+1}(X)=\det(x_1, \cdots, x_n)$

$(X=(x_1, \cdots, x_n; y)\in M(n; \bar{K})=\overbrace{V(n)\oplus \cdots \oplus V(n)}^{n}\oplus V(n)^*)$.

$\kappa=(\overbrace{1, \cdots, 1}^{n}, n-1)$.

S. Kasai determines the b-functions of the prehomogeneous vector spaces in Proposition 4.1, except for type (11) and (12) [[**Kasai**]].

Proposition 4.2 ([**Kasai**]). *Let* (G, ρ, V) *be one of the simple prehomogeneous vector spaces given in Proposition* 4.1, *except for type* (11) *and* (12). *The b-function* $b(s)$ *of* (G, ρ, V) *has the following expression in the terms of the gamma function* $\Gamma(s)$:

$$b(s) = c\gamma(s+1)/\gamma(s) \tag{4.1}$$

where c *is a non-zero constant and* $\gamma(s)$ *is given by*

$$\gamma(s) = \prod_{\lambda} \Gamma(a_\lambda s+b_\lambda) \quad (\textit{finite product}),$$

$$a_\lambda s+b_\lambda = \sum_{i=1}^{N} a_\lambda^i s_i+b_\lambda, \quad a_\lambda^i=1 \text{ or } 0,\ b_\lambda\in \boldsymbol{Z},\ \text{and } b_\lambda>0.$$

We remark that $s+1$ *in the right hand side of* (2.1) *means*

$$s+1 = (s_1+1, \cdots, s_N+1)$$

for $s=(s_1, \cdots, s_N)\in \boldsymbol{C}^N$. *Moreover the set* $\{a_\lambda s+b_\lambda\}$ *is given as follows*:

(1) $\{s+1, s+n\}$ $(s\in \boldsymbol{C})$

(2) $\{s+j \mid j=1, \cdots, n\}$ $(s\in \boldsymbol{C})$

(3) $\{s+2j-1 \mid j=1, \cdots, m+1\}$ $(s\in \boldsymbol{C})$

(4) $\{s+1,\ s+2n\}$ $(s\in \boldsymbol{C})$

(5) $\{s+1,\ s+4,\ s+5,\ s+8\}$ $(s\in \boldsymbol{C})$

(6) $\{s_1+1,\ s_1+s_2+2j-1\ (2\leqq j\leqq m),\ s_2+1,\ s_2+2m\}$ $(s=(s_1,\ s_2)\in \boldsymbol{C}^2)$

(7) $\{s_1+2j-1\ (1\leqq j\leqq m),\ s_2+1,\ s_2+2m\}$ $(s=(s_1,\ s_2)\in \boldsymbol{C}^2)$

(8) $\{s_1+2j-1\ (1\leqq j\leqq m-1),\ s_1+s_2+2m-1,\ s_2+1,\ s_2+2m\}$ $(s=(s_1,\ s_2)\in \boldsymbol{C}^2)$

(9) $\{s_1+1,\ s_1+4,\ s_2+1,\ s_2+4\}$ $(s=(s_1,\ s_2)\in \boldsymbol{C}^2)$

(10) $\{s_1+1,\ s_1+s_2+5,\ s_2+1,\ s_2+8\}$ $(s=(s_1,\ s_2)\in \boldsymbol{C}^2)$

(11) *unknown case*

(12) *unknown case*

(13) $\{s_i+1,\ (1\leqq i\leqq n+1),\ s_1+\cdots+s_{n+1}+j,\ (2\leqq j\leqq n)\}$
$(s=(s_1,\ \cdots,\ s_{n+1})\in \boldsymbol{C}^{n+1})$

(14) $\{s_i+1,\ (1\leqq i\leqq n),\ s_{n+1}+j,\ (1\leqq j\leqq n-1),\ s_1+\cdots+s_{n+1}+n\}$
$(s=(s_1,\ \cdots,\ s_{n+1})\in \boldsymbol{C}^{n+1})$.

5. The Main Theorem and its Proof.

The main theorem of this paper is as follows:

THEOREM 5.1. *Let (G, ρ, V) be one of simple prehomogeneous vector spaces given in Proposition* 4.1, *except for type* (11) *and* (12). *Let $Z_K(s)$ be the Igusa local zeta function and $\Gamma_K(s)$ the Γ-factor of (G, ρ, V), then we have*

$$Z_K(s)=\prod_{\lambda}(1-q^{-b_\lambda})/(1-q^{-(a_\lambda s+b_\lambda)}), \tag{5.1}$$

and

$$\Gamma_K(s)=\prod_{\lambda} \mathit{Tate}\,\gamma(a_\lambda(s-\kappa)+b_\lambda), \tag{5.2}$$

where the set $\{a_\lambda s+b_\lambda\}$ is given in Proposition 4.2.

If Φ_0 denotes the characteristic function of O_K^n, then we have

$$Z(s)(\Phi_0)=Z_K(s-\kappa),$$

in which $Z(s)$ is the zeta distribution associated with (G, ρ, V) [see § 2]. Since the Fourier transform $\hat{\Phi}_0$ of Φ_0 equals Φ_0 and the zeta distribution $Z^*(s)$ associated with (G, ρ^*, V^*), coincides with $Z(s)$ in our case, we have

$$\hat{Z}(s)(\Phi_0)=Z(s)(\hat{\Phi}_0)=Z_K(s-\kappa),$$

and

$$Z^*(\kappa-s)(\Phi_0)=Z(\kappa-s)(\Phi_0)=Z_K(-s).$$

By evaluating both sides of the functional equation (2.1) at Φ_0, we have

$$\Gamma_K(s)=Z_K(s-\kappa)/Z_K(-s).$$

Therefore, if (5.1) is proved, we have

$$\Gamma_K(s)=\prod_{\lambda}(1-q^{a_\lambda s-b_\lambda})/(1-q^{-a_\lambda(s-\kappa)-b_\lambda}).$$

From Proposition 4.1 and 4.2, we have

$$\{a_\lambda s-b_\lambda\}=\{a_\lambda(s-\kappa)+b_\lambda-1\}.$$

Hence we have (5.2). Therefore our task is to prove (5.1).

For type (1), (2), (3), (4) and (5) (i.e., the case of $N=1$), (5.1) has been already proved, because we can find out the same Igusa local zeta function which can be computed [[**Igusa-2**]]. For example, the Igusa local zeta function of type (2) coincides with the Igusa local zeta function of the reduced regular irreducible prehomogeneous vector space:

$$(G\times GL(m),\ \rho\otimes\Lambda_1,\ V(m)\otimes V(m)),$$

where $\rho: G\to GL(V(m))$ is an m-dimensional irreducible representation of a connected semi-simple algebraic group G. We can immediately obtain (5.1) for type (7) and (9) from the above results. For type (6), (8), (10), (13) and (14), we shall prove (5.1) by case by case computation.

We shall collect some propositions.

Proposition 5.1 ([**S-K**], §2 Proposition 9). *Let G be a linear algebraic group and let $\rho: G\to GL(V(m))$ be a faithful irreducible representation of G on the m-dimensional vector space $V(m)$. Let n be a positive integer with $m>n\geqq 1$. Then a triplet $(G\times SL(n), \rho\otimes\Lambda_1, V(m)\otimes V(n))$ is a prehomogeneous vector space if and only if $(G\times SL(m-n), \rho^*\otimes\Lambda_1, V(m)^*\otimes V(m-n))$ is a prehomogeneous vector space.*

We say that two triplets $(G\times SL(n), \rho\otimes\Lambda_1, V(m)\otimes V(n))$ and $(G\times SL(m-n), \rho^*\otimes\Lambda_1, V(m)^*\otimes V(m-n))$ are *castling transforms* of each other. We assume that they are prehomogeneous vector spaces.

Proposition 5.2 ([**S-K**], §4 Proposition 18). *There is a one-to-one correspondence between the relative invariants $P(x)$ of $(G\times SL(n), \rho\otimes\Lambda_1, V(m)\otimes V(n))$ $(m>n\geqq 1)$ and the relative invariants $\tilde{P}(\tilde{x})$ of its castling transform $(G\times SL(m-n), \rho^*\otimes\Lambda_1, V(m)^*\otimes V(m-n))$. If $P(x)$ is irreducible, then $\tilde{P}(\tilde{x})$ is also irreducible.*

By Proposition 5.2, the number of basic relative invariants of $(G\times SL(n), \rho\otimes\Lambda_1, V(m)\otimes V(n))$ is equal to the number, say N, of basic relative invariants of its castling transform $(G\times SL(m-n), \rho^*\otimes\Lambda_1, V(m)^*\otimes V(m-n))$. Let $P_1(x), \cdots, P_N(x)$ be the basic relative invariants of $(G\times SL(n), \rho\otimes\Lambda_1, V(m)\otimes V(n))$, and $\tilde{P}_1(\tilde{x}), \cdots, \tilde{P}_N(\tilde{x})$ the basic relative invariants of $(G\times SL(m-n), \rho^*\otimes\Lambda_1,$

$V(m)^*\otimes V(m-n))$. Let $Z_K(s)$, $\tilde{Z}_K(s)$ be the Igusa local zeta function of $(G\times SL(n), \rho\otimes\Lambda_1, V(m)\otimes V(n))$, $(G\times SL(m-n), \rho^*\otimes\Lambda_1, V(m)^*\otimes V(m-n))$ respectively:

$$Z_K(s)=\int_{O_K^{mn}}\prod_{j=1}^{N}|P_j(x)|_K^{s_j}dx\,,\quad \tilde{Z}_K(s)=\int_{O_K^{m(m-n)}}\prod_{j=1}^{N}|\tilde{P}_j(\tilde{x})|_K^{s_j}d\tilde{x}\,.$$

PROPOSITION 5.3 ([**Igusa-1**], § 8). *In the above notations, if $m-n>n$, we have*

$$\tilde{Z}_K(s)/Z_K(s)=\prod_{n<j\leq m-n}(1-q^{-j})/(1-q^{-(ds+j)})\,, \tag{5.3}$$

where we put $\deg(P_j)=d_j\cdot n$, $\deg(\tilde{P}_j)=d_j\cdot(m-n)$ *for* $j=1, \cdots, N$, *and* $ds=d_1s_1+\cdots+d_Ns_N$.

J.-I. Igusa proves this proposition in the case of $N=1$. However, his proof holds in the case of $N\geqq 2$.

We put

$$I(s)=\int_{O_K^n\times O_K^m}\prod_{j=1}^{N}|F_j(x, y)|_K^{s_j}dxdy$$

for $s=(s_1, \cdots, s_N)$, $Re(s_j)>0$ $(j=1, \cdots, N)$, where we take N polynomials $F_j(x, y)$ $(j=1, \cdots, N)$ from $O_K[x, y]=O_K[x_1, \cdots, x_n, y_1, \cdots, y_m]$. We define subsets D_i $(i=1, \cdots, n)$ of O_K^n by

$$D_i=\{(x_1^{(i)}, \cdots, x_{i-1}^{(i)}, \mu_i, x_{i+1}^{(i)}, \cdots, x_n^{(i)})\in O_K^n\,|\,(x_1^{(i)}, \cdots, x_{i-1}^{(i)})\in\pi O_K^{i-1}\}\,.$$

PROPOSITION 5.4 ([**Kimura-2**], § 1 Theorem 1.2). *In the above notations, we have*

$$I(s)=\sum_{i=1}^{n}\int_{D_i\times O_K^m}\prod_{j=1}^{N}|F_j^{(i)}(x, y)|_K^{s_j}dx^{(i)}dy\,, \tag{5.4}$$

where we put

$$F_j^{(i)}(x, y)=F_j(\mu_i x_1^{(i)}, \cdots, \mu_i x_{i-1}^{(i)}, \mu_i, \mu_i x_{i+1}^{(i)}, \cdots, \mu_i x_n^{(i)}, y)$$

and

$$dx^{(i)}=|\mu_i|_K^{n-1}d\mu_i dx_1^{(i)}\cdots dx_{i-1}^{(i)}dx_{i+1}^{(i)}\cdots dx_n^{(i)},$$

for $i=1, \cdots, n$, $j=1, \cdots, N$.

This proposition is proved by T. Kimura, hence we call the formula (5.4) *Kimura's integral formula.*

We define some new notations.

For $j\geqq 1$, we put

$$(j)=1-q^{-j},\quad (j)_+=1+q^{-j}.$$

We put

$$U_n = O_K^n - \pi O_K^n$$

for any $n \geqq 1$. Then we can easily obtain the following proposition.

PROPOSITION 5.5. (1) $vol(U_n)=(n)$.
(2) U_n *is the* $GL(n\,;O_K)$*-orbit of* $e_1={}^t(\overbrace{1, 0, \cdots, 0}^{n})$: $U_n=GL(n\,;O_K)\cdot e_1$.
(3) $O_K^n=\bigcup_{k\geq 0}\pi^k U_n$ *(disjoint union).*

Type (6): $(GL(1)^3\times SL(2m),\ \Lambda_2\oplus\Lambda_1\oplus\Lambda_1,\ V(m(2m-1))\oplus V(2m)\oplus V(2m))$
We shall consider the computation of the Igusa local zeta function:

$$Z_K(s)=\int_{x\in Alt(2m,\,O_K),\ y\in O_K^{2m},\ z\in O_K^{2m}}|Pf(x)|_K^{s_1}|{}^t y\Delta(x)z|_K^{s_2}dx\,dy\,dz.$$

Since the polynomial ${}^t y\Delta(x)z$ is homogeneous of degree 1 with respect to y, we have

$$Z_K(s)=(2m)/(1-q^{-(s_2+2m)})\cdot\int_{x\in Alt(2m,\,O_K),\ z\in O_K^{2m}}|Pf(x)|_K^{s_1}|{}^t e_1\Delta(x)z|_K^{s_2}dx\,dz.$$

In fact, by Proposition 5.5, we have

$$\begin{aligned}Z_K(s)&=\sum_{k\geq 0}\int_{y\in\pi^k U_{2m}}\left\{\int_{x\in Alt(2m;\,O_K),\ z\in O_K^{2m}}|Pf(x)|_K^{s_1}|{}^t y\Delta(x)z|_K^{s_2}dx\,dz\right\}dy\\&=\left\{\sum_{k\geq 0}q^{-(s_2+2m)k}\right\}\cdot\int_{y\in U_{2m}}\left\{\int_{x\in Alt(2m;\,O_K),\ z\in O_K^{2m}}|Pf(x)|_K^{s_1}|{}^t y\Delta(x)z|_K^{s_2}dx\,dz\right\}dy\\&=\left\{\sum_{k\geq 0}q^{-(s_2+2m)k}\right\}\cdot vol(U_{2m})\cdot\int_{x\in Alt(2m;\,O_K),\ z\in O_K^{2m}}|Pf(x)|_K^{s_1}|{}^t e_1\Delta(x)z|_K^{s_2}dx\,dz\\&=(2m)/(1-q^{(s_2+2m)})\cdot\int_{x\in Alt(2m;\,O_K),\ z\in O_K^{2m}}|Pf(x)|_K^{s_1}|{}^t e_1\Delta(x)z|_K^{s_2}dx\,dz.\end{aligned}$$

If we write $z\in O_K^{2m}$ as $z=\binom{z_1}{z^*}$, with $z_1\in O_K$, $z^*\in O_K^{2m-1}$, then the polynomial ${}^t e_1\Delta(x)z={}^t e_1\Delta(x)\binom{z_1}{z^*}$ is homogeneous of degree 1 with respect to z^*, hence we have

$$\begin{aligned}&\int_{x\in Alt(2m;\,O_K),\ z\in O_K^{2m}}|Pf(x)|_K^{s_1}|{}^t e_1\Delta(x)z|_K^{s_2}dx\,dz\\&=(2m-1)/(1-q^{(s_2+2m-1)})\cdot\int_{x\in Alt(2m;\,O_K),\ z\in O_K}|Pf(x)|_K^{s_1}\left|{}^t e_1\Delta(x)\binom{z_1}{e_1^*}\right|_K^{s_2}dx\,dz,\end{aligned}$$

where we put $e_1^*={}^t(\overbrace{1, 0, \cdots, 0}^{2m-1})$. By a suitable variable exchange, we have

$$Z_K(s)=(2m)(2m-1)/(1-q^{(s_2+2m)})(1-q^{(s_2+2m-1)})\cdot I_m(s), \tag{5.5}$$

where we define $I_m(s)$ by

$$I_m(s)=\int_{x\in Alt(2m;\,O_K)}|Pf(x)|_K^{s_1}|Pf(x)_{2m-1,2m}|_K^{s_2}dx\,.$$

Notation. For $1\leqq i<j\leqq 2m$, we denote by $Pf(x)_{i,j}$ the pfaffian of the submatrix of x which is obtained from x by crossing out its i-th and j-th rows and columns, then $\Delta(x)$ is given by

$$\Delta(x)=((-1)^{i+j}Pf(x)_{i,j})_{1\leq i<j\leq 2m}\,.$$

It is sufficient to compute the integral $I_m(s)$. We shall compute it by the induction on m.

When $m=1$, we have

$$I_1(s)=(1)/(1-q^{-(s_1+1)})\,. \tag{5.6}$$

When $m\geqq 2$, the polynomials $Pf(x)$ and $Pf(x)_{2m-1,2m}$ are homogeneous of degree 1 with respect to the $2m-1$ variables x_{12}, x_{13}, $\cdots$, x_{12m-1}, x_{12m}, and the 2 variables x_{12m-1}, x_{12m} disappear in the polynomial $Pf(x)_{2m-1,2m}$. We apply Kimura's integral formula to $I_m(s)$ with respect to the $2m-1$ variables x_{12}, x_{13}, $\cdots$, x_{12m-1}, x_{12m}, in this order, then we have

$$I_m(s)=(1)/(1-q^{-(s_1+s_2+2m-1)})\cdot\left\{\sum_{k=2}^{2m}q^{-(k-2)}\cdot J_k(s)\right\}, \tag{5.7}$$

For $k=2,\cdots,2m$, $J_k(s)$ is defined as follows:

$$J_k(s)=\int|Pf(x')|_K^{s_1}|Pf(x')_{2m-1,2m}|_K^{s_2}dx'.$$

with

$$x'=\left(\begin{array}{c|ccc} 0 & \pi{}^t x_1 & 1 & {}^t x_2 \\ \hline -\pi x_1 & & & \\ -1 & & x'' & \\ -x_2 & & & \end{array}\right),$$

where the domain of integral is defined by

$$x_1\in O_K^{k-2},\ x_2\in O_K^{2m-k},\ x''\in Alt(2m-1;\,O_K)\,.$$

We have

$$J_k(s)=\begin{cases} I_{m-1}(s) & (k=2,\cdots,2m-2)\\ q^{-s_2}\cdot(2m-3)/(1-q^{-(s_2+2m-3)})\cdot I_{m-1}(s) & (k=2m-1,\,2m)\end{cases} \tag{5.8}$$

In fact, for $k=2,\cdots,2m-2$, we have

$$J_k(s)=\int_{x'\in Alt(2m-2;\,O_K)}|Pf\begin{pmatrix}J_1 & 0\\ 0 & x'\end{pmatrix}|_K^{s_1}|Pf\begin{pmatrix}J_1 & 0\\ 0 & x'\end{pmatrix}_{2m-1,2m}|_K^{s_2}dx'$$
$$=I_{m-1}(s)$$

by a suitable variable exchange. For $k=2m-1$, $2m$, by a suitable variable exchange, we have

$$J_{2m-1}(s)=J_{2m}(s)$$
$$=q^{-s_2}\int|Pf\begin{pmatrix}x' & z^*\\ -{}^tz^* & 0\end{pmatrix}|_K^{s_1}|Pf\begin{pmatrix}0 & {}^tx^*\\ -x^* & x'\end{pmatrix}|_K^{s_2}dx'dx^*dz^*,$$

where the domain of integral is defined by

$$x'\in Alt(2m-3\,;\,O_K),\ x^*,\ z^*\in O_K^{2m-3}.$$

Since the polynomial $Pf\begin{pmatrix}0 & {}^tx^*\\ -x^* & x'\end{pmatrix}$ is homogeneous of degree 1 with respect to x^*, we have

$$J_{2m-1}(s)=J_{2m}(s)$$
$$=q^{-s_2}\cdot(2m-3)/(1-q^{-(s_2+2m-3)})\cdot\int_{x\in Alt(2m-2;\,O_K)}|Pf(x)|_K^{s_1}|Pf(x)_{1,2}|_K^{s_2}dx$$
$$=q^{-s_2}\cdot(2m-3)/(1-q^{-(s_2+2m-3)})\cdot I_{m-1}(s).$$

By (5.7) and (5.8), we have the induction formula

(5.9) $$I_m(s)=I_{m-1}(s)\cdot c_m(s)\quad(m\geqq 2),$$

where we put $c_m(s)=(2m-3)/(1-q^{-(s_2+2m-3)})\cdot(1-q^{-(s_2+2m-1)})/(1-q^{-(s_1+s_2+2m-1)})$.

By (5.6) and (5.9), we have

$$I_1(s)=(1)/(1-q^{-(s_1+1)}),$$

and

$$I_m(s)=(1)/(1-q^{-(s_1+1)})\cdot(1)/(1-q^{-(s_2+1)})$$
$$\times\prod_{j=2}^{m-1}(2j-1)/(1-q^{-(s_1+s_2+2j-1)})\cdot(1-q^{-(s_2+2m-1)})/(1-q^{-(s_1+s_2+2m-1)}),$$

for $m\geqq 2$. Hence, by (5.5), we have our result:

$$Z_K(s)=(1)/(1-q^{-(s_1+1)})\cdot\prod_{j=2}^{m}(2j-1)/(1-q^{-(s_1+s_2+2j-1)})$$
$$\times(1)(2m)/(1-q^{-(s_2+1)})(1-q^{-(s_2+2m)}).$$

Type (8): $(GL(1)^3\times SL(2m),\ \Lambda_2\oplus\Lambda_1^*\oplus\Lambda_1^*,\ V(m(2m-1))\oplus V(2m)^*\oplus V(2m)^*)$

We shall consider the computation of the Igusa local zeta function:

$$Z_K(s)=\int_{x\in Alt(2m;\,O_K),\,y\in O_K^{2m},\,z\in O_K^{2m}}|Pf(x)|_K^{s_1}|{}^t yxz|_K^{s_2}dxdydz.$$

We can compute $Z_K(s)$ by the same way with type (6).

Since the polynomial ${}^t yxz$ is homogeneous of degree 1 with respect to y, we have

$$Z_K(s)=(2m)/(1-q^{-(s_2+2m)})\cdot\int_{x\in Alt(2m;\,O_K),\,z\in O_K^{2m}}|Pf(x)|_K^{s_1}|{}^t e_1xz|_K^{s_2}dxdz.$$

If we write $x\in Alt(2m;O_K)$ as $x=\begin{pmatrix}0 & {}^t x_1\\ -x_1 & x'\end{pmatrix}$, with $x_1\in O_K^{2m-1}$, $x'\in Alt(2m-1;O_K)$, then the polynomials $Pf(x)$ and ${}^t e_1xz$ are homogeneous of degree 1 with respect to x_1. Hence we have

$$\int_{x\in Alt(2m;\,O_K),\,z\in O_K^{2m}}|Pf(x)|_K^{s_1}|{}^t e_1xz|_K^{s_2}dxdz$$

$$=(2m-1)/(1-q^{-(s_1+s_2+2m-1)})\cdot\int_{x''\in Alt(2m-2;\,O_K),\,z_1\in O_K}|Pf(x'')|_K^{s_1}|z_1|_K^{s_2}dx''dz_1$$

$$=(2m-1)/(1-q^{-(s_1+s_2+2m-1)})\cdot\prod_{j=1}^{m-1}(2j-1)/(1-q^{-(s_1+2j-1)})\cdot(1)/(1-q^{-(s_2+1)}).$$

Therefore we have our result:

$$Z_K(s)=\prod_{j=1}^{m-1}(2j-1)/(1-q^{-(s_1+2j-1)})\cdot(1)(2m)/(1-q^{-(s_2+1)})(1-q^{-(s_2+2m)})$$
$$\times(2m-1)/(1-q^{-(s_1+s_2+2m-1)}).$$

Type (10): ($GL(1)^2\times Spin(10)$, *the vector rep$\oplus$a half-spin rep,* $V(10)\oplus V(16)$)
We shall consider the computation of the Igusa local zeta function:

$$Z_K(s)=\int_{x\in O_K^{10},\,y\in O_K^{16}}|q(x)|_K^{s_1}|\langle x,\,Q(y)\rangle|_K^{s_2}dxdy.$$

Since the polynomial $\langle x,Q(y)\rangle$ is homogeneous of degree 2 with respect to y, we have

$$Z_K(s)=1/(1-q^{-(2s_2+16)})\cdot I(s), \tag{5.10}$$

where we define

$$I(s)=\int_{x\in O_K^{10},\,y\in U_{16}}|q(x)|_K^{s_1}|\langle x,\,Q(y)\rangle|_K^{s_2}dxdy.$$

We define

$$V_k=\{y\in U_{16}\,|\,Q(y)\in\pi^kU_{10}\}\quad(k\geqq 0),$$

then we have

$$U_{16}=\Big(\bigcup_{k\geq 0}V_k\Big)\cup\{y\in U_{16}\,|\,Q(y)=0\}\ \text{(disjoint union)}.$$

Since the measure of the subset $\{y\in U_{16}\,|\,Q(y)=0\}$ of U_{16} is 0, we have

$$I(s)=\sum_{k\geq 0}I_k(s)\,, \tag{5.11}$$

where we put

$$I_k(s)=\int_{x\in O_K^{10},\,y\in V_k}|q(x)|_K^{s_1}|\langle x,\,Q(y)\rangle|_K^{s_2}dxdy\,.$$

LEMMA 5.1 ([**Igusa-3**], §7). *In the above notations, we have*

(1) V_k *is the* $Spin_{10}(O_K)$*-orbit of* $1+\pi^k e_5^*$.

(2) $vol(V_k)=\begin{cases}(5)(8) & (k=0)\\ (3)_+(5)(8)q^{-5k} & (k\geq 1).\end{cases}$

(3) $Q(1+ce_5^*)={}^t(\overbrace{0,\,\cdots,\,0,\,c}^{10})$.

By Lemma 5.1, we have

$$\begin{aligned}I_k(s)&=\int_{x\in O_K^{10},\,y\in V_k}|q(x)|_K^{s_1}|\langle x,\,Q(1+\pi^k e_5^*)\rangle|_K^{s_2}dxdy\\&=vol(V_k)\cdot\int_{x\in O_K^{10}}|q(x)|_K^{s_1}|\pi^k x_{10}|_K^{s_2}dx\\&=\begin{cases}(5)(8)\cdot J(s) & (k=0)\\ (3)_+(5)(8)q^{-ks_2}\cdot J(s) & (k\geq 1),\end{cases}\end{aligned}$$

where we define

$$J(s)=\int_{O_K^{10}}|x_1x_6+\cdots+x_5x_{10}|_K^{s_1}|x_{10}|_K^{s_2}dx\,.$$

By (5.11), we have

$$\begin{aligned}I(s)&=I_0(s)+\sum_{k\geq 1}I_k(s)\\&=(5)(8)\cdot\Big\{1+(3)_+\sum_{k\geq 1}q^{-(s_2+5)k}\Big\}\cdot J(s)\\&=(5)(8)\cdot(1+q^{-(s_2+8)})/(1-q^{-(s_2+5)})\cdot J(s)\,.\end{aligned}$$

Hence we have, by (5.10),

$$Z_K(s)=(5)(8)/(1-q^{-(s_2+5)})(1-q^{-(s_2+8)})\cdot J(s)\,. \tag{5.12}$$

It is sufficient to consider $J(s)$. The polynomials $x_1x_6+\cdots+x_5x_{10}$ and x_{10} are homogeneous of degree 1 with respect to the 5 variables $x_6,\cdots,x_{10}$. We apply Kimura's integral formula to $J(s)$ with respect to the 5 variables x_{10}, x_9, $\cdots$, x_6 in this order, then we have

$$J(s)=(1)/(1-q^{-(s_1+s_2+5)})\cdot\left\{\sum_{k=1}^{5}q^{-(k-1)}\cdot J_k(s)\right\},$$

where we define

$$J_1(s)=\int_{O_K^{10}}|x_1x_6+\cdots+x_5|_K^{s_1}dx.$$

and

$$J_k(s)=\int_{O_K^{10}}|x_1x_6+\cdots+x_{5-(k-1)}+\cdots+\pi x_5x_{10}|_K^{s_1}|\pi x_{10}|_K^{s_2}dx \quad (k=2,\cdots,5).$$

We can easily obtain

$$J_k(s)=\begin{cases}(1)/(1-q^{-(s_1+1)}) & (k=1)\\ q^{-s_2}\cdot(1)/(1-q^{-(s_1+1)})\cdot(1)/(1-q^{-(s_2+1)}) & (k=2,\cdots,5).\end{cases}$$

Hence we have

$$(5.13)\quad J(s)=(1)/(1-q^{-(s_1+1)})\cdot(1)/(1-q^{-(s_2+1)})\cdot(1-q^{-(s_2+5)})/(1-q^{-(s_1+s_2+5)}).$$

By (5.12) and (5.13), we have our result:

$$Z_K(s)=(1)/(1-q^{-(s_1+1)})\cdot(1)(8)/(1-q^{-(s_2+1)})(1-q^{-(s_2+8)})\cdot(5)/(1-q^{-(s_1+s_2+5)}).$$

Type (13): $(GL(1)^{n+1}\times SL(n),\ \overbrace{\Lambda_1\oplus\cdots\oplus\Lambda_1}^{n+1},\ \overbrace{V(n)\oplus\cdots\oplus V(n)}^{n+1})$

We consider the prehomogeneous vector space:

$$(GL(1)^{n+1},\ \rho,\ V(n+1)),$$

where we define the representation ρ of $GL(1)^{n+1}$ on $V(n+1)=\overline{K}^{n+1}$ by

$$\rho(\alpha_1,\cdots,\alpha_{n+1})x={}^t(\alpha_1\cdot x_1,\cdots,\alpha_{n+1}\cdot x_{n+1})\quad (x={}^t(x_1,\cdots,x_{n+1})\in\overline{K}^{n+1}).$$

This is the castling transform of our prehomogeneous vector space. The Igusa local zeta function of $(GL(1)^{n+1},\ \rho,\ V(n+1))$ is as follows

$$Z_K(s)=\int_{O_K^{n+1}}\prod_{i=1}^{n+1}|x_i|_K^{s_i}dx=\prod_{i=1}^{n+1}(1)/(1-q^{-(s_i+1)}).$$

Let $\tilde{Z}_K(s)$ be the Igusa local zeta function of our prehomogeneous vector space:

$$\tilde{Z}_K(s)=\int_{O_K^{n(n+1)}}\prod_{i=1}^{n+1}|P_i(x)|_K^{s_i}dx,$$

where the basic relative invariants $P_i(x)$ $(i=1,\cdots,n+1)$ are given in Proposition 4.1.

By Proposition 5.3, we have our result:

$$\tilde{Z}_K(s)=\prod_{i=1}^{n+1}(1)/(1-q^{-(s_i+1)})\cdot\prod_{j=2}^{n}(j)/(1-q^{-(s_1+\cdots+s_{n+1}+j)}).$$

Type (14): $(GL(1)^{n+1}\times SL(n),\ \overbrace{\Lambda_1\oplus\cdots\oplus\Lambda_1}^{n}\oplus\Lambda_1^*,\ \overbrace{V(n)\oplus\cdots\oplus V(n)}^{n}\oplus V(n)^*)$
We shall consider the computation of the Igusa local zeta function:

$$Z_K(s)=\int_{x=(x_1,\cdots,x_n)\in M(n;O_K),\,y\in O_K^n}\prod_{i=1}^{n}|\langle x_i,y\rangle|_K^{s_i}|\det(x)|_K^{s_{n+1}}dxdy,$$

The polynomials $\langle x_i, y\rangle$ $(i=1,\cdots,n)$ are homogeneous of degree 1 with respect to y. We apply Kimura's integral formula to $Z_K(s)$ with respect to y, then we have

$$Z_K(s)=(1)/(1-q^{-(s_1+\cdots+s_n+n)})\cdot\left\{\sum_{j=1}^{n}q^{-(j-1)}\cdot I_j(s)\right\},\tag{5.14}$$

where we define

$$I_j(s)=\int_{x\in M(n;O_K),\,y\in O_K^n}\prod_{i=1}^{n}|\langle x_i,{}^t(\pi y_1,\cdots,\overset{i}{1},\cdots,y_n)\rangle|_K^{s_i}|\det(x)|_K^{s_n}dxdy.$$

By a suitable variable exchange, we have

$$I_j(s)=\int_{{}^t(x_1,\cdots,x_n)\in O_K^n,\,x'\in M(n-1,n;O_K)}\prod_{i=1}^{n}|x_i|_K^{s_i}|\det\begin{pmatrix}x_1\cdots x_n\\ x'\end{pmatrix}|_K^{s_2}dx_1\cdots dx_n dx',$$

hence $I_j(s)$ is independent of the index j. We denote by $I(s)$ this integral. We have, by (5.14),

$$Z_K(s)=(n)/(1-q^{-(s_1+\cdots+s_n+n)})\cdot I(s).\tag{5.15}$$

It is sufficient to consider $I(s)$. The polynomials $x_1,\cdots,x_n$ and $\det(x_1\cdots x_n/x')$ are homogeneous of degree 1 with respect to the n variables $x_1,\cdots,x_n$. We apply Kimura's integral formula to $I(s)$ with respect to these n variables x_1, $\cdots$, x_n, then we have

$$I(s)=(1)/(1-q^{-(s_1+\cdots+s_{n+1}+n)})\cdot\left\{\sum_{k=1}^{n}q^{-(k-1)}\cdot J_k(s)\right\},\tag{5.16}$$

where we define

$$J_k(s)=q^{-(s_1+\cdots+s_{k-1})}\cdot\int\prod_{1\le i\le n,\,i\ne k}|x_i|_K^{s_i}|\det(x'')|_K^{s_{n+1}}dx_1\cdots d\check{x}_k\cdots dx_n dx'',$$

where the domain of integral is defined by

$$ {}^t(x_1, \cdots, \check{x_k}, \cdots, x_n) \in O_K^{n-1},\ x'' \in M(n-1\,; O_K) $$

We can obtain

$$ (5.17) \qquad J_k(s) = q^{-(s_1+\cdots+s_{k-1})} \cdot \prod_{1 \leq i \leq n,\ i \neq k} (1)/(1-q^{-(s_i+1)}) \cdot \prod_{j=1}^{n-1} (j)/(1-q^{-(s_{n+1}+j)}) . $$

By (5.15), (5.16) and (5.17), we obtain our result:

$$ Z_K(s) = \prod_{i=1}^{n} (1)/(1-q^{-(s_i+1)}) \cdot \prod_{j=1}^{n-1} (j)/(1-q^{-(s_{n+1}+j)}) \cdot (n)/(1-q^{-(s_1+\cdots+s_{n+1}+n)}) . $$

6. On type (11) and (12).

In this section, we consider the prehomogeneous vector spaces of type (11) and (12):

(11) $(GL(1)^4 \times SL(2m+1),\ \Lambda_2 \oplus \Lambda_1 \oplus \Lambda_1 \oplus \Lambda_1,$
$V(m(2m+1)) \oplus V(2m+1) \oplus V(2m+1) \oplus V(2m+1)),$

(12) $(GL(1)^4 \times SL(2m+1),\ \Lambda_2 \oplus \Lambda_1 \oplus \Lambda_1^* \oplus \Lambda_1^*,$
$V(m(2m+1)) \oplus V(2m+1) \oplus V(2m+1)^* \otimes V(2m+1)^*),$

which are excepted from our main theorem—Theorem 5.1.

The b-functions of them are still unsettled, hence we can not conclude that Theorem 5.1 holds for them. However we can settle the Igusa local zeta function of type (12).

PROPOSITION 6.1. *Let $Z_K(s)$ be the Igusa local zeta function of the prehomogeneous vector space of type* (12):

$$ Z_K(s) = \int |Pf\begin{pmatrix} x & {}^t y \\ -y & 0 \end{pmatrix}|_K^{s_1} |\langle y, z\rangle|_K^{s_2} |\langle y, w\rangle|_K^{s_3} |{}^t z x w|_K^{s_4} dx\, dy\, dz\, dw , $$

where the domain of integral is defined by

$$ x \in Alt(2m+1\,; O_K),\ y,\ z,\ w \in O_K^{2m+1}, $$

then we have

$$ Z_K(s) = \prod_{i=1}^{m} (2i-1)/(1-q^{-(s_1+2i-1)}) \cdot (1)/(1-q^{-(s_2+1)}) \cdot (1)/(1-q^{-(s_3+1)}) $$
$$ \times (1)(2m)/(1-q^{-(s_4+1)})(1-q^{-(s_4+2m)}) \cdot (2m+1)/(1-q^{-(s_1+s_2+s_3+s_4+2m+1)}) . $$

PROOF. Since the polynomials $Pf\begin{pmatrix} x & {}^t y \\ -y & 0 \end{pmatrix}$, $\langle y, z\rangle$ and $\langle y, w\rangle$, are homogeneous of degree 1 with respect to y, then we have

$$ (6.1) \qquad Z_K(s) = (2m+1)/(1-q^{-(s_1+s_2+s_3+2m+1)}) \cdot I_m(s) . $$

where we define

$$I_m(s)=\int |Pf(x'))|_K^{s_1}|z_0|_K^{s_2}|w_0|_K^{s_3}|(z_0,\ {}^t z_1)\begin{pmatrix} 0 & {}^t x_1 \\ -x_1 & x' \end{pmatrix}\begin{pmatrix} w_0 \\ w_1 \end{pmatrix}|_K^{s_4} dx'dx_1dz_0dz_1dw_0dw_1,$$

where the domain of integral is given by

$$x'\in Alt(2m\,;O_K),\ z_0,\ w_0\in O_K,\ x_1,\ z_1,\ w_1\in O_K^{2m}.$$

The polynomials z_0 and $(z_0,\ {}^t z_1)\begin{pmatrix} 0 & {}^t x_1 \\ -x_1 & x' \end{pmatrix}\begin{pmatrix} w_0 \\ w_1 \end{pmatrix}$ are homogeneous of degree 1 with respect to z_0 and z_1. We apply Kimura's integral formula to $I_m(s)$ with respect to z_0 and z_1, then we have

$$\text{(6.2)}\qquad I_m(s)=(1)/(1-q^{-(s_2+s_4+2m+1)})\cdot\left\{\sum_{k=0}^{2m} q^{-k}\cdot J_k(s)\right\},$$

where we define

$$J_0(s)=\int |Pf(x')|_K^{s_1}|w_0|_K^{s_3}|{}^t e_1\begin{pmatrix} 0 & {}^t x_1 \\ -x_1 & x' \end{pmatrix}\begin{pmatrix} w_0 \\ w_1 \end{pmatrix}|_K^{s_4} dx'dx_1dw_0dw_1,$$

$$x'\in Alt(2m\,;O_K),\ w_0\in O_K,\ x_1,\ w_1\in O_K^{2m}\quad (e_1={}^t(\overbrace{1,\ 0,\ \cdots,\ 0}^{2m+1})),$$

and

$$J_k(s)=q^{-s_2}\cdot\int |Pf(x')|_K^{s_1}|z_0|_K^{s_2}|w_0|_K^{s_3}$$
$$|(\pi z_0,\ {}^t e_1^*)\begin{pmatrix} 0 & {}^t x_1 \\ -x_1 & x' \end{pmatrix}\begin{pmatrix} w_0 \\ w_1 \end{pmatrix}|_K^{s_4} dx'dx_1dz_0dw_0dw_1,$$

$$x'\in Alt(2m\,;O_K),\ z_0,\ w_0\in O_K,\ x_1,\ w_1\in O_K^{2m}\quad (e_1^*={}^t(\overbrace{1,\ 0,\ \cdots,\ 0}^{2m})).$$

for $k=1,\ \cdots,\ 2m$.

Since

$${}^t e_1\begin{pmatrix} 0 & {}^t x_1 \\ -x_1 & x' \end{pmatrix}\begin{pmatrix} w_0 \\ w_1 \end{pmatrix}={}^t x_1 w_1=\langle x_1,\ w_1\rangle,$$

we have

$$J_0(s)=\int_{x'\in Alt(2m;O_K)}|Pf(x')|_K^{s_1}dx'\cdot\int_{w_0\in O_K}|w_0|_K^{s_3}dw_0\cdot\int_{x_1,\ w_1\in O_K^{2m}}|\langle x_1,\ w_1\rangle|_K^{s_4}dx_1dw_1$$
$$=\prod_{i=1}^{m}(2i-1)/(1-q^{-(s_1+2i-1)})\cdot(1)/(1-q^{-(s_3+1)})$$
$$\times(1)(2m)/(1-q^{-(s_4+1)})(1-q^{-(s_4+2m)}).$$

For $k=1,\ \cdots,\ 2m$, the integral $J_k(s)$ is independent of the index k, we denote by $J(s)$ this integral. If we write $x'\in Alt(2m\,;O_K)$ as $x'=\begin{pmatrix} 0 & {}^t x_1' \\ -x_1' & x'' \end{pmatrix}$

with $x_1' \in O_K^{2m-1}$, $x'' \in Alt(2m-1; O_K)$, and $w_1 \in O_K^{2m}$ as $w_1 = \begin{pmatrix} w_1' \\ w_1'' \end{pmatrix}$ with $w_1' \in O_K$, $w_1'' \in O_K^{2m}$, then we have

$$(\pi z_0, {}^t e_1^*)\begin{pmatrix} 0 & {}^t x_1 \\ -x_1 & x' \end{pmatrix}\begin{pmatrix} w_0 \\ w_1 \end{pmatrix} = \pi z_0 \langle x_1, w_1 \rangle - w_0 \langle x_1, e_1^* \rangle + \langle x_1', w_1 \rangle .$$

The polynomials $Pf(x')$, z_0, w_0 and $\pi z_0\langle x_1, w_1\rangle - w_0\langle x_1, e_1^*\rangle + \langle x_1', w_1\rangle$ are homogeneous of degree 1 with respect to the $2m+1$ variables w_0, ${}^t x_1' = (x_{12}, \cdots, x_{1 2m})$, z_0. We apply Kimura's integral formula to $J(s)$ with respect to these variables in the above order, then we have

$$\begin{aligned} J(s) = {} & q^{-s_2} \cdot (1)/(1-q^{-(s_1+s_2+s_3+s_4+2m+1)}) \\ & \times \{J'_{w_0}(s) + q^{-1} \cdot J'_{x_{12}}(s) + \cdots + q^{-(2m-1)} \cdot J'_{x_{12m}}(s) + q^{-2m} \cdot J'_{z_0}(s)\} . \end{aligned}$$

By suitable variables exchanges, we can compute $J'_{w_0}(s)$, $J'_{x_{12}}(s)$, $\cdots$, $J'_{x_{12m}}(s)$, $J'_{z_0}(s)$ as follows:

$$\begin{aligned} J'_{w_0}(s) &= \int_{x' \in Alt(2m; O_K),\, x_0,\, z_0 \in O_K} |Pf(x')|_K^{s_1} |z_0|_K^{s_2} |x_0|_K^{s_4} dx' dx_0 dz_0 , \\ &= \prod_{i=1}^{m} (2i-1)/(1-q^{-(s_1+2i-1)}) \cdot (1)/(1-q^{-(s_2+1)}) \cdot (1)/(1-q^{-(s_4+1)}) , \end{aligned}$$

$$\begin{aligned} J'_{x_{1j}}(s) &= q^{-s_3} \cdot \int_{x'' \in Alt(2m-2; O_K),\, z_0,\, w_0,\, w_0' \in O_K} |Pf(x'')|_K^{s_1} |z_0|_K^{s_2} |w_0|_K^{s_3} |w_0'|_K^{s_4} dx'' dz_0 dw_0 dw_0' \\ &= q^{-s_3} \cdot \prod_{i=1}^{m-1} (2i-1)/(1-q^{-(s_1+2i-1)}) \\ &\quad \times (1)/(1-q^{-(s_2+1)}) \cdot (1)/(1-q^{-(s_3+1)}) \cdot (1)/(1-q^{-(s_4+1)}), \quad (j=2, \cdots, 2m) \end{aligned}$$

$$\begin{aligned} J'_{z_0}(s) &= q^{-(s_1+s_3+s_4)} \\ &\quad \times \int_{x' \in Alt(2m; O_K),\, w_0 \in O_K,\, x_1,\, w_1 \in O_K^{2m}} |Pf(x')|_K^{s_1} |w_0|_K^{s_3} |\langle x_1, w_1 \rangle|_K^{s_4} dx' dw_0 dx_1 dw_1 \\ &= q^{-(s_1+s_3+s_4)} \prod_{i=1}^{m} (2i-1)/(1-q^{-(s_1+2i-1)}) \\ &\quad \times (1)/(1-q^{-(s_3+1)}) \cdot (1)(2m)/(1-q^{-(s_4+1)})(1-q^{-(s_4+2m)}) . \end{aligned}$$

Therefore we have,

$$\begin{aligned} J(s) = {} & q^{-s_2} \cdot \prod_{i=1}^{m} (2i-1)/(1-q^{-(s_1+2i-1)}) \\ & \times (1)/(1-q^{-(s_2+1)}) \cdot (1)/(1-q^{-(s_3+1)}) \cdot (1)/(1-q^{-(s_4+1)}) \\ & \times N/(1-q^{-(s_4+2m)})(1-q^{-(s_1+s_2+s_3+s_4+2m+1)}) , \end{aligned}$$

where we put

$$N = (1-q^{-(s_1+s_3+2m)})(1-q^{-(s_4+2m)}) + q^{-(s_1+s_3+s_4+2m)} \cdot (2m) \cdot (1-q^{-(s_2+1)}) .$$

By (6.1) and (6.2) and the above result, we have our result. Q. E. D.

Let $Z_K(s)$ be the Igusa local zeta function of type (11):

$$Z_K(s)=\int\prod_{j=1}^{3}\left|Pf\begin{pmatrix} x & {}^t y_k \\ -y_k & 0\end{pmatrix}\right|_K^{s_j}\left|Pf\begin{pmatrix} x & {}^t y \\ -y & 0\end{pmatrix}\right|_K^{s_4}dx\,dy,$$

where the domain of integral is defined by

$$x\in Alt(2m+1\,;O_K),\ y=(y_1,\ y_2,\ y_3)\in M(2m+1,\ 3\,;O_K),$$

then we have

$$(6.3)\qquad Z_K(s)=\prod_{i=1}^{3}(1)/(1-q^{-(s_i+1)})\cdot(1)(2m)/(1-q^{-(s_4+1)})(1-q^{-(s_4+2m)})$$
$$\times\prod_{j=2}^{m+1}(2j-1)/(1-q^{-(s_1+s_2+s_3+s_4+2j-1)}),$$

for $m=1, 2$. We can prove it by *Igusa's Key lemma* [[**Igusa-4**], §7] and Kimura's integral formula. However we can not prove it for $m\geqq 3$.

At the end of this paper, we shall give some conjectures:

(C-1) (6.3) holds for every $m\geqq 3$.

(C-2) The b-functions of prehomogeneous vector spaces of type (11) and (12) are of the form:

$$b(s)=c\gamma(s+1)/\gamma(s),\quad \gamma(s)=\prod_{\lambda}\Gamma(a_\lambda s+b_\lambda).$$

Moreover the sets $\{a_\lambda s+b_\lambda\}$ are given by

(11) $\{s_1+2j-1\ (1\leqq i\leqq m), s_2+1,\ s_3+1,\ s_4+1,\ s_4+2m,\ s_1+s_2+s_3+s_4+2m+1\}$,

(12) $\{s_1+1,\ s_2+1,\ s_3+1,\ s_4+1,\ s_4+2m,\ s_1+s_2+s_3+s_4+2j-1\ (2\leqq j\leqq m+1)\}$.

If the above conjectures (C-1) and (C-2) hold, then we can conclude that our main theorem—Theorem 5.1 holds for every simple prehomogeneous vector space satisfying the assumption (A). We shall give it as our last conjecture:

(C-3) For every simple prehomogeneous vector space $(G,\ \rho,\ V)$ defined over a p-adic number field K, satisfying that G is K-split and Y_K is a single $\rho(G)_K$-orbit, the Γ-factor is completely determined by the b-function.

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