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Del Pezzo surfaces as hyperplane sections

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Introduction.

Let L be a very ample line bundle on an n-dimensional complex projective manifold X. In this article we classify pairs (X, L) as above with some smooth $A\in|L|$ being del Pezzo, i.e., with a smooth $A\in|L|$ such that $-K_A=(n-2)H$ for some ample line bundle H on A. We assume that $n\geqq 3$ since otherwise we are in the completely understood case when A is an elliptic curve.

If $H=L_A$, then the problem reduces to the classification of del Pezzo manifolds, which has been done by Fujita [**Fu**] in the more general setting of ample divisors. However there are several examples (e.g., [**LPS**], [**LPS1**]) showing that $H\neq L_A$ can occur. This suggests the development of a detailed structure theory in which both Fujita's theory (in the very ample setting) and all known examples fit. This is exactly what we do in this paper.

If $n\geqq 4$, then the structure of pairs (X, L) with $A\in|L|$ del Pezzo is simple: we work it out in the appendix. In particular this shows that, apart from few obvious exceptions, the situation $H\neq L_A$ can occur only when $n=3$, which we assume from here on in this introduction.

In section 0 we summarize background material. We also prove some very ampleness results (Theorems (0.3) and (0.5)) in order to show that a number of pairs coming up in the classification do really occur.

In section 2, by using adjunction theory, we prove a structure theorem (Theorem (2.4)) giving a breakup of the possible pairs (X, L) we are dealing with into 9 classes. Of these the most complicated are quadric fibrations over $\boldsymbol{P}^1$, Veronese bundles over $\boldsymbol{P}^1$ and scrolls over surfaces.

We study quadric fibrations over $\boldsymbol{P}^1$ in sections 1 and 4. To do this we embed X in $\boldsymbol{P}(\pi_*L)$ where $\pi: X\rightarrow\boldsymbol{P}^1$ is the quadric fibration map. The smoothness of X imposes very strong restrictions on which vector bundles π_*L are possible and on the homology class of X in $\boldsymbol{P}(\pi_*L)$.

In section 3 we classify the Veronese bundles over $\boldsymbol{P}^1$ that arise in our structure theory.

In section 5 we give a precise description of the scrolls over surfaces appearing in the breakup result assuming $K_A^2 \geqq 2$. Actually, if $K_A^2 \geqq 2$, we have that $-(K_X+L)$ is nef, hence X is a Fano bundle, this allowing us to apply classification results for such special manifolds, e.g. [**SzW**].

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0. Background material.

We work over the complex number field. A projective k-fold is an irreducible smooth projective scheme of dimension k. Vector bundles are holomorphic vector bundles. We use standard notation from algebraic geometry. We also adopt some current abuses. Everywhere we do not distinguish between line bundles and invertible sheaves. We freely shift from the multiplicative to the additive notation for line bundles; multiplicative notation with "$\cdot$" omitted is reserved for the intersection product in the Chow rings.

Let V be a projective k-fold and let $\mathcal{L}$ be a line bundle on V. We let $\mathcal{L}^r = c_1(\mathcal{L})^r$; $\mathcal{L}_W$ will denote the restriction of $\mathcal{L}$ to a subvariety W of V; K_V will stand for the canonical bundle of V.

If p_X, p_Y are the projections of a product $X \times Y$ onto the factors, we set $\mathcal{O}_{X\times Y}(m, n) = p_X{}^*\mathcal{O}_X(m) + p_Y{}^*\mathcal{O}_Y(n)$.

A line bundle $\mathcal{L}$ on V is said to be numerically effective (nef, for short) if $\mathcal{L}C \geqq 0$ for all curves $C \subset V$. In addition $\mathcal{L}$ is said to be big if $\mathcal{L}^k > 0$. We say that $\mathcal{L}$ is spanned if it is spanned at all points by $\Gamma(V, \mathcal{L})$.

For an ample line bundle $\mathcal{L}$ on V, the sectional genus $g(V, \mathcal{L})$ of $(V, \mathcal{L})$ is defined by

$$2g(V, \mathcal{L}) - 2 = (K_V + (k-1)\mathcal{L})\mathcal{L}^{k-1}.$$

If $\mathcal{L}$ is also spanned, then $g(V, \mathcal{L})$ is simply the geometric genus of the smooth curve obtained by intersecting $k-1$ general elements of the complete linear system $|\mathcal{L}|$. We also set $d(V, \mathcal{L}) = \mathcal{L}^k$.

(0.1) Special Varieties.

We denote by $\boldsymbol{Q}^k$ a smooth quadric hypersurface of $\boldsymbol{P}^{k+1}$. Let V be a projective k-fold and let $\mathcal{L}$ be an ample line bundle on V. We say that $(V, \mathcal{L})$ is a Del Pezzo k-fold if $-K_V = (k-1)\mathcal{L}$. We say that $(V, \mathcal{L})$ is a scroll (respectively a quadric bundle) over a normal variety W of dimension h, if there exists a surjective morphism with connected fibers $p: V \to W$ and an ample line bundle H on W such that $K_V+(k-h+1)\mathcal{L}=p^*H$ (respectively $K_V+(k-h)\mathcal{L}=p^*H$). In particular, if $(V, \mathcal{L})$ is a scroll over either a curve or a surface W with $k-h>0$, then W is smooth and V is a $\boldsymbol{P}^{k-h}$-bundle over W and $\mathcal{L}_f=\mathcal{O}_{P^{k-h}}(1)$ for every fibre f of p [**S2**, (3.3)]. A projective 3-fold V is said to be Fano if $-K_V$ is ample.

(0.2) Reductions [**S2**, (0.5)].

Let $\mathcal{L}$ be an ample and spanned line bundle on a projective k-fold V. We say that a pair $(V', \mathcal{L}')$, consisting of a projective k-fold V' and an ample line bundle $\mathcal{L}'$, is a reduction of $(V, \mathcal{L})$ if

(0.2.1) there exists a morphism $\rho: V\to V'$ expressing V as V' blown-up at a finite set B,

(0.2.2) $\mathcal{L}=\rho^*\mathcal{L}'-[\rho^{-1}(B)]$ (equivalently $K_V+(k-1)\mathcal{L}=\rho^*(K_{V'}+(k-1)\mathcal{L}')$).

Recall that if $K_V+(k-1)\mathcal{L}$ is nef and big, then there exists a reduction $(V', \mathcal{L}')$ of $(V, \mathcal{L})$ and $K_{V'}+(k-1)\mathcal{L}'$ is ample [**S2**, (4.5)]. Note that in this case such a reduction is unique up to isomorphisms and that the positive dimensional fibres of ρ are precisely the linear $\boldsymbol{P}^{k-1}\subset V$ with normal bundle $\mathcal{O}_{P^{k-1}}(-1)$. Furthermore ρ induces a bijection between the smooth elements of $|\mathcal{L}|$ and the smooth divisors of $|\mathcal{L}'|$, passing through B.

In particular, in the special case of threefolds, we need to recall the following fact (e.g. see [**SV**, (0.3.3)]).

(0.2.3) Let $(V', \mathcal{L}')$ be the reduction of $(V, \mathcal{L})$, let $\rho: V\to V'$ be the reduction morphism. Let S be any smooth element of $|\mathcal{L}|$ and let $S'=\rho(S)$. Then $(S', \mathcal{L}'_{S'})$ is the reduction of $(S, \mathcal{L}_S)$. In particular, if ρ contracts t (-1)-planes of $(V, \mathcal{L})$, then

$$K_{S'}^2=K_S^2+t\geqq K_S^2.$$

For all the results of adjunction theory we will need for pairs $(V, \mathcal{L})$ with $\mathcal{L}$ very ample, we refer to [**SV**], [**S3**] and especially [**BS**].

Now we prove a very ampleness result which we need in sec. 3.

(0.3) THEOREM. *Let $P_1, \cdots, P_r$ be r totally disjoint linear subspaces of $\boldsymbol{P}^n$ (i.e. the linear space they generate has dimension $a_1+\cdots+a_r+r-1$, where $a_i=\dim P_i$). Let $\pi: P\to \boldsymbol{P}^n$ be the blowing-up of the union of the P_i's and let $E_i=\pi^{-1}(P_i)$. If $t\geqq r\geqq 3$, then the line bundle $L=\pi^*\mathcal{O}_{P^n}(t)-\Sigma_i E_i$ is very ample.*

PROOF. Any two points or a point and a direction in $\boldsymbol{P}^n$ generate a line, say l.

CLAIM. *At least one of the projections from P_i maps l to a line.*

PROOF. Assume otherwise. Then l must meet every P_i and thus l is not contained in any of them, since they are totally disjoint. So there exists a point $x \in l$ not on any P_i. Project to $\boldsymbol{P}^{n-1}$ from x. Note that every P_i maps isomorphically onto its image, say Q_i since x is in none of them. On the other hand in $\boldsymbol{P}^{n-1}$ all the subspaces Q_i meet since l is a line through x meeting all the P_i's. This implies that the Q_i's span a linear space of dimension $\leqq a_1 + \cdots + a_r$. Thus coming back to $\boldsymbol{P}^n$ we conclude that the P_i's span a linear space of dimension $\leqq a_1 + \cdots + a_r + 1$, which in view of our assumption implies $r \leqq 2$, a contradiction. This proves the claim.

Since $\pi^*\mathcal{O}_{\boldsymbol{P}^n}(t-r)$ is spanned, it suffices to assume $t=r$. To prove the very ampleness of L, note that $L = \Sigma(\pi^*\mathcal{O}_{\boldsymbol{P}^n}(1) - E_i)$; so if two points or directions are separated by $|\pi^*\mathcal{O}_{\boldsymbol{P}^n}(1) - E_i|$ for some i, then they are also separated by $|L|$. Moreover note that the morphism associated with the i-th summand is induced by the projection of $\boldsymbol{P}^n$ from the linear space P_i.

Now consider two points x, y on P. If they are not on the same E_i with the same image in P_i, then they are separated by $|\pi^*\mathcal{O}_{\boldsymbol{P}^n}(1) - E_j|$ for some j, by the above claim. Therefore assume that x, y are in the same E_i with the same image, say z, in P_i. This means that x, y correspond to two different normal directions to P_i at z. But then, since the projection from P_i separates normal directions we see that $|\pi^*\mathcal{O}_{\boldsymbol{P}^n}(1) - E_i|$ separates x, y.

Similarly, if y is a tangent direction at x, by using the claim we reduce to the case when x is in E_i with image, say z, in P_i and y goes to zero at z. Then $|\pi^*\mathcal{O}_{\boldsymbol{P}^n}(1) - E_i|$ maps the fibre $\pi^{-1}(z)$ biholomorphically onto its image.

The argument proving the above theorem does not work for $r=2$. In this case however we have the following weaker result.

(0.4) THEOREM ([**LPS**, (0.4)]). *Let P_1, P_2 be two disjoint linear subspaces of $\boldsymbol{P}^n$. Let $\pi: P \to \boldsymbol{P}^n$ be the blow-up at P_1 and P_2 and let $L = \pi^*\mathcal{O}_{\boldsymbol{P}^n}(2) - \pi^{-1}(P_1) - \pi^{-1}(P_2)$. Then L is very ample outside the proper transform of $\langle P_1, P_2 \rangle$, the linear span of P_1 and P_2.*

By using this fact we now prove a very ampleness result we need in sec. 2.

(0.5) THEOREM. *Let $\boldsymbol{Q}^3 \subset \boldsymbol{P}^4$ be a smooth hyperquadric and let $x_1, x_2, x_3 \in \boldsymbol{Q}^3$ be three points no two on a line contained in $\boldsymbol{Q}^3$. Let $\rho: X \to \boldsymbol{Q}^3$ be the blowing up at x_1, x_2, x_3 and let $E_i = \rho^{-1}(x_i)$. Then the line bundle $L := \rho^*\mathcal{O}_{\boldsymbol{Q}^3}(2) - E_1 - E_2 - E_3$ is very ample on X.*

PROOF. Let $\theta_i: M_i \to \boldsymbol{P}^4$ be the blowing up of $\boldsymbol{P}^4$ along x_i $(i=1, 2, 3)$ and along the line $\langle x_j, x_k \rangle$ $(i \neq j, k)$ and let $\mathcal{E}_i$ and $\mathcal{E}_{jk}$ denote the exceptional divisors respectively. Look at the commutative diagram

$$\begin{array}{ccccc} \boldsymbol{P}^3 & \overset{p}{\leftarrow} & M_i & \overset{q}{\rightarrow} & \boldsymbol{P}^2 \\ p'\uparrow & \sigma''\swarrow & \downarrow\theta_i & \searrow\tau'' & \uparrow q' \\ P_i & \underset{\sigma'}{\rightarrow} & \boldsymbol{P}^4 & \underset{\tau'}{\leftarrow} & P_{jk} \end{array}$$

where p' and q' are the morphisms obtained by resolving the indeterminacies of the projections of $\boldsymbol{P}^4$ from x_i and from the line $\langle x_j, x_k\rangle$ respectively and σ'' (τ'') denotes the blowing-up of P_i (P_{jk}) along the proper transform of $\langle x_j, x_k\rangle$ via σ' (of x_i via τ'). We have

$$p'^*\mathcal{O}_{\boldsymbol{P}^3}(1)=\sigma'^*\mathcal{O}_{\boldsymbol{P}^4}(1)-\sigma'^{-1}(x_i) \quad\text{and}\quad q'^*\mathcal{O}_{\boldsymbol{P}^2}(1)=\tau'^*\mathcal{O}_{\boldsymbol{P}^4}(1)-\tau'^{-1}(\langle x_j, x_k\rangle);$$

so that

$$p^*\mathcal{O}_{\boldsymbol{P}^3}(1)+\mathcal{E}_i=\theta_i^*\mathcal{O}_{\boldsymbol{P}^4}(1)=q^*\mathcal{O}_{\boldsymbol{P}^2}(1)+\mathcal{E}_{jk}.$$

So, letting $g_i=(p, q): M_i\rightarrow\boldsymbol{P}^3\times\boldsymbol{P}^2$, we get

$$\theta_i^*\mathcal{O}_{\boldsymbol{P}^4}(2)-\mathcal{E}_i-\mathcal{E}_{jk}=p^*\mathcal{O}_{\boldsymbol{P}^3}(1)+q^*\mathcal{O}_{\boldsymbol{P}^2}(1)=g_i^*\mathcal{O}_{\boldsymbol{P}^3\times\boldsymbol{P}^2}(1, 1). \tag{0.5.1}$$

Now come to our quadric. In view of the general position assumption, the plane $\langle x_1, x_2, x_3\rangle$ cuts $\boldsymbol{Q}^3$ along a smooth conic; call C its proper transform on X. In particular since every line $\langle x_j, x_k\rangle$ is transverse to $\boldsymbol{Q}^3$ we get for every $i=1, 2, 3$ a commutative diagram

$$\begin{array}{ccc} M_i & \overset{\theta_i}{\longrightarrow} & \boldsymbol{P}^4 \\ \cup & & \cup \\ X & \underset{\rho}{\longrightarrow} & \boldsymbol{Q}^3 \end{array}$$

and $[\mathcal{E}_i]_X=E_i$, $[\mathcal{E}_{jk}]_X=E_j+E_k$. Thus restricting (0.5.1) to X we see that

$$L=(g_i^*\mathcal{O}_{\boldsymbol{P}^3\times\boldsymbol{P}^2}(1, 1))_X \quad\text{for } i=1, 2, 3.$$

So, due to the very ampleness of $\mathcal{O}_{\boldsymbol{P}^3\times\boldsymbol{P}^2}(1, 1)$, to show that sections of L separate two points x, y of X (possibly y being a tangent direction at x) it is enough to show that g_i separates them for some i. Note that, according to the definition of g_i,

(0.5.2) the p component of g_i separates any couple of points (or point, direction) whose images in $\boldsymbol{P}^4$ are not collinear with x_i, while

(0.5.3) the q component of g_i separates any couple of points (or point, direction) not on the same fibre of E_{jk}, i.e. whose images in $\boldsymbol{P}^4$ are not coplanar with x_j, x_k.

Note also that the line bundle $\theta_i^*\mathcal{O}_{P^4}(2)-\mathcal{E}_i-\mathcal{E}_{jk}$ is very ample on M_i outside the proper transform of $\langle x_1, x_2, x_3\rangle$, by (0.4); hence L is very ample on $X\backslash C$. Property (0.5.2) will be enough to show that some g_i separates x and y in the remaining cases. Actually, let $x\in X\backslash C$ and $y\in C$; then necessarily $y\neq x_i$ for some i, so that g_i separates x and y, by (0.5.2). Now suppose that $x\in C\backslash(E_1\cup E_2\cup E_3)$; then x and any point $y\in C$ (or tangent direction) are separated by some g_i by (0.5.2), since they cannot be collinear with all x_i's. Now let $x\in E_i$; if y is a tangent direction or another point on the same E_i, then projecting from x_i we get different images, hence g_i separates x and y, by (0.5.2). Finally assume that $x\in E_1$ and $y\in E_2$; then g_3 separates them, by (0.5.2), since x_3 and the images of x, y in $\boldsymbol{P}^4$ are not collinear. This concludes the proof.

In sec. 4 we will have to decide whether a line bundle on a projective bundle has a smooth element in its linear system. The remainder of this section deals with preparatory material to this end. A key condition translating smoothness is the following

(0.6) PROPOSITION. *Let $\mathcal{L}$ be a line bundle on a compact complex manifold X. Let Z be a compact complex submanifold of X. Let $|\mathcal{L}-Z|$ be the linear system of the zero sets of sections of $\mathcal{L}$ that vanish on Z. Let N_Z^* be the conormal bundle of Z in X and let*

$$d_Z\colon \Gamma(\mathcal{L}) \longrightarrow \Gamma(N_Z^*\otimes\mathcal{L}) \tag{0.6.1}$$

be the homomorphism locally given by the differential along Z. Then the linear system $|\mathcal{L}-Z|$ contains an element smooth except possibly on the set $Bs|\mathcal{L}-Z|-Z$ if and only if there exists a section $s\in\Gamma(\mathcal{L}-Z)$ such that d_Zs is nowhere zero on Z.

PROOF. Let $s\in\Gamma(\mathcal{L}-Z)$. Since s vanishes on Z, the zero locus of s is smooth along Z if and only if locally the differential of s does not vanish on Z. Since this is an open condition, if satisfied, then the general element of $|\mathcal{L}-Z|$ is smooth along Z; on the other hand by the Bertini theorem the general element of $|\mathcal{L}-Z|$ is smooth outside $Bs|\mathcal{L}-Z|$. This gives the existence of an element smooth outside $Bs|\mathcal{L}-Z|-Z$. The converse is obvious.

We can explicitly describe N_Z^* when X is a $\boldsymbol{P}$-bundle and Z is a $\boldsymbol{P}$-subbundle of it. First we recall the following general result.

(0.7) LEMMA. *Let $\mathcal{E}\to\mathcal{B}\to 0$ be a surjection of vector bundles over a smooth connected manifold Y. Then there is an inclusion $\boldsymbol{P}(\mathcal{B})\subseteq\boldsymbol{P}(\mathcal{E})$ and the restriction of the tautological bundle $\xi_{\mathcal{E}}$ of $\mathcal{E}$ to $\boldsymbol{P}(\mathcal{B})$ is equal to the tautological bundle $\xi_{\mathcal{B}}$ of $\mathcal{B}$.*

PROOF. Let $\pi_{\mathcal{E}}: \mathbf{P}(\mathcal{E}) \to Y$ and $\pi_{\mathcal{B}}: \mathbf{P}(\mathcal{B}) \to Y$ denote the bundle projections. Since the inclusion $\mathbf{P}(\mathcal{B}) \subseteqq \mathbf{P}(\mathcal{E})$ corresponds to an injection $0 \to \mathcal{B}^* \to \mathcal{E}^*$ between the dual bundles, it follows that $\pi_{\mathcal{E}|\mathbf{P}(\mathcal{B})} = \pi_{\mathcal{B}}$. Note that $(\xi_{\mathcal{E}})^*$ is the subbundle of $\pi_{\mathcal{E}}{}^*\mathcal{E}^*$ with each fibre over a point, e, of $\mathbf{P}(\mathcal{E})$ equal to the line through the origin of the fibre $\mathcal{E}^*{}_{\pi(e)}$ corresponding to e. Similarly $(\xi_{\mathcal{B}})^*$ is the subbundle of $\pi_{\mathcal{B}}{}^*\mathcal{B}^*$ with each fibre over a point, b, of $\mathbf{P}(\mathcal{B})$ equal to the line through the origin of the fibre $\mathcal{B}^*{}_{\pi(b)}$ corresponding to b. Under the inclusion $\pi_{\mathcal{B}}^*\mathcal{B}^* \subseteqq \pi_{\mathcal{E}}^*\mathcal{E}^*$ induced by $\mathcal{B}^* \subseteqq \mathcal{E}^*$ we have that $(\xi_{\mathcal{B}})^* = ((\xi_{\mathcal{E}})^*)_{\mathbf{P}(\mathcal{B})}$.

In particular this implies

(0.7.1) $\quad \xi_{\mathcal{E}}{}^{(\mathrm{rk}\,\mathcal{B}-1+\dim Y)}\mathbf{P}(\mathcal{B}) = \xi_{\mathcal{B}}{}^{(\mathrm{rk}\,\mathcal{B}-1+\dim Y)}$, and if $\dim Y = 1$, this last number is $\deg \mathcal{B}$.

Keeping notation as in (0.7), we have

(0.8) LEMMA. *Let $0 \to \mathcal{A} \to \mathcal{E} \to \mathcal{B} \to 0$ be an exact sequence of vector bundles over a connected manifold Y. Then the conormal bundle of $\mathbf{P}(\mathcal{B})$ inside $\mathbf{P}(\mathcal{E})$ is isomorphic to*

$$\pi_{\mathcal{B}}{}^*\mathcal{A} \otimes \xi_{\mathcal{B}}{}^*.$$

PROOF. To see this let $V_{\mathcal{E}}$, $V_{\mathcal{B}}$ denote the vertical tangent bundles of $\mathbf{P}(\mathcal{E})$ and $\mathbf{P}(\mathcal{B})$, i.e., the subbundles with vectors going to zero under the differential $d\pi$. Let Q denote the quotient of $(V_{\mathcal{E}})_{\mathbf{P}(\mathcal{B})}$ by $V_{\mathcal{B}}$ under the natural inclusion. Considering the following diagram it follows that it suffices to show that $Q \cong \pi_{\mathcal{B}}{}^*\mathcal{A}^* \otimes \xi_{\mathcal{B}}$.

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \uparrow & & \uparrow & & & & \\
0 & \longrightarrow & Q & \longrightarrow & N_{\mathbf{P}(\mathcal{B})|\mathbf{P}(\mathcal{E})} & \longrightarrow & 0 & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & (V_{\mathcal{E}})_{\mathbf{P}(\mathcal{B})} & \longrightarrow & (T_{\mathbf{P}(\mathcal{E})})_{\mathbf{P}(\mathcal{B})} & \longrightarrow & (\pi_{\mathcal{E}}{}^*T_Y)_{\mathbf{P}(\mathcal{B})} & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & V_{\mathcal{B}} & \longrightarrow & T_{\mathbf{P}(\mathcal{B})} & \longrightarrow & \pi_{\mathcal{B}}{}^*T_Y & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & 0 & & 0 & & 0. & &
\end{array}$$

Recall that the bundle $V_{\mathcal{E}}$ fits into the Euler sequence

$$0 \longrightarrow \mathcal{O}_{\mathbf{P}(\mathcal{E})} \longrightarrow \pi_{\mathcal{E}}{}^*\mathcal{E}^* \otimes \xi_{\mathcal{E}} \longrightarrow V_{\mathcal{E}} \longrightarrow 0.$$

Note that the first term of this sequence tensored with $\xi_{\mathcal{E}}^*$ gives the tautological inclusion of $\xi_{\mathcal{E}}^*$ in $\pi_{\mathcal{E}}^*\mathcal{E}^*$. Of course $V_{\mathcal{B}}$ fits into a similar sequence. Putting them together we get the following diagram

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \uparrow & & \uparrow & & \\
 & & 0 & \longrightarrow & \pi_{\mathcal{B}}^*\mathcal{A}^*\otimes\xi_{\mathcal{B}} & \cong & N_{P(\mathcal{B})|P(\mathcal{E})} & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & (\mathcal{O}_{P(\mathcal{E})})_{P(\mathcal{B})} & \longrightarrow & (\pi_{\mathcal{E}}^*\mathcal{E}^*\otimes\xi_{\mathcal{E}})_{P(\mathcal{B})} & \longrightarrow & (V_{\mathcal{E}})_{P(\mathcal{B})} & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \mathcal{O}_{P(\mathcal{B})} & \longrightarrow & \pi_{\mathcal{B}}^*\mathcal{B}^*\otimes\xi_{\mathcal{B}} & \longrightarrow & V_{\mathcal{B}} & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & 0 & & 0 & & 0. & &
\end{array}$$

An inspection of it, noting that $(\pi_{\mathcal{E}}^*\mathcal{E}^*\otimes\xi_{\mathcal{E}})_{P(\mathcal{B})}\cong\pi_{\mathcal{B}}^*\mathcal{E}^*\otimes\xi_{\mathcal{B}}$ by (0.7), concludes the proof.

1. Quadric fibrations over P^1: general properties.

In this section we discuss some general properties of quadric fibrations over P^1, which we will need in sec. 4. For the sake of completeness we start considering polarized n-folds (X, L), $n\geqq 3$, where the line bundle L is assumed to be very ample. Let $p: X\to P^1$ be the morphism expressing X as a quadric fibration. Then $K_X+(n-1)L=p^*H$, for some ample line bundle H on P^1. First of all note that all fibres of p are irreducible. To see this note that all fibres of p are embedded by $|L|$ as quadric hypersurfaces of P^n. Assume that there is a reducible fibre $Q=A+B$ of p. Since

$$\mathcal{O}_A=[Q]_A=[A]_A+[B]_A$$

and $[B]_A=\mathcal{O}_{P^{n-1}}(1)$ we would get $[A]_A=\mathcal{O}_{P^{n-1}}(-1)$, so that A could be contracted, hence also the hyperplane $B\cap A$ would be contractible, a contradiction. Moreover all fibres of p are reduced. Otherwise, by cutting out $(n-2)$ general elements of $|L|$ we would get a smooth surface fibered in conics over P^1 having a double line as a fibre, a contradiction.

Let $\mathcal{E}=p_*L$. For every fibre Q of p we have $h^0(L_Q)=n+1$, since $|L|$ embeds Q as a quadric of P^n. This implies that $\mathcal{E}$ is a rank-$(n+1)$ vector bundle on P^1. Moreover $\mathcal{E}$ is spanned. To see this let $t\in P^1$, consider the fibre $Q_t=p^{-1}(t)$ and look at the following diagram

$$\begin{array}{ccc} \Gamma(L) & \longrightarrow & \Gamma(L_t) \\ \downarrow & & \downarrow \\ \Gamma(\mathcal{E}) & \longrightarrow & \Gamma(\mathcal{E}_t) \end{array}$$

where the vertical arrows are isomorphisms. Since L is very ample and $|L|$ embeds Q_t as a quadric of $\boldsymbol{P}^n$, the restriction homomorphism $\Gamma(L)\rightarrow\Gamma(L_t)$ is surjective and then so is also the homomorphism $\Gamma(\mathcal{E})\rightarrow\Gamma(\mathcal{E}_t)$. So we have

(1.0.1) $\qquad \mathcal{E}=\bigoplus_{i=0,\cdots,n}\mathcal{O}_{\boldsymbol{P}^1}(a_i)$, with $a_n\geqq a_{n-1}\geqq\cdots\geqq a_1\geqq a_0\geqq 0$.

We let $\delta=\deg\mathcal{E}=\sum a_i$. Consider the projective bundle $P=\boldsymbol{P}(\mathcal{E})$, let $\pi:P\rightarrow\boldsymbol{P}^1$ be the projection and let ξ be the tautological line bundle of $\mathcal{E}$ on P. Then, from the relation $\xi^{n+1}-\xi^n\pi^*c_1(\mathcal{E})=0$ we get

(1.0.2) $$\xi^{n+1}=\delta.$$

Moreover since $\mathcal{E}=p_*L$ and X embeds fibrewise inside P, we have that

(1.0.3) $$\xi_X=L \quad\text{and}\quad X\in|2\xi-bF|,$$

where F stands for a fibre of π. We denote by z_j the homogeneous coordinate on the general fibre of P corresponding to the summand $\mathcal{O}_{\boldsymbol{P}^1}(a_j)$; so the quadric Q cut out by X on F is represented by a second degree homogeneous equation in the z_j's. By (1.0.3) and in view of the isomorphism

$$\Gamma(\boldsymbol{P}(\mathcal{E}),2\xi-bF)\cong\Gamma(\boldsymbol{P}^1,\mathcal{E}^{(2)}\otimes\mathcal{O}_{\boldsymbol{P}^1}(-b))=\bigoplus_{i\leq j}\Gamma(\mathcal{O}_{\boldsymbol{P}^1}(a_i+a_j-b))$$

we have that

(1.0.4) every summand z_iz_j appearing in the equation of Q corresponds to a section of $\mathcal{O}_{\boldsymbol{P}^1}(a_i+a_j-b)$.

A sort of converse to the above setting is the following situation

(1.0.5) Let $\mathcal{E}$ be as in (1.0.1), let $\pi:P=\boldsymbol{P}(\mathcal{E})\rightarrow\boldsymbol{P}^1$ be the corresponding projective bundle and set $\delta=\sum a_i$. Let ξ and F denote the tautological bundle of $\mathcal{E}$ and a fibre of π respectively.

(1.1) LEMMA. *Let things be as in* (1.0.5) *and assume that* $|2\xi-bF|$ *contains a smooth element* Y *and that* ξ_Y *is very ample. Then either* $b=0$ *or* -1, *or* $\mathcal{E}$ *is very ample.*

PROOF. We have that $\pi_*(-\xi+bF)=\pi_*(-\xi)\otimes\mathcal{O}_{\boldsymbol{P}^1}(b)=0$, since $\pi_*(-\xi)=0$. Since also $R^1\pi_*(-\xi+bF)=0$ we get $h^i(-\xi+bF)=0$ for $i=0,1$. This implies that $h^i(\xi-Y)=0$ for $i=0,1$. So, looking at the exact cohomology sequence of

$$0\longrightarrow[\xi-Y]\longrightarrow\xi\longrightarrow\xi_Y\longrightarrow 0,$$

we see that $H^0(P, \xi)=H^0(Y, \xi_Y)$. In view of the assumption we thus conclude that the map $\varphi_\xi: P \to \boldsymbol{P}(H^0(P, \xi))$ embeds Y. Now assume that $\mathcal{E}$ is not ample, so $a_0=0$. Letting σ denote the section of π corresponding to the surjection $\mathcal{E} \to \mathcal{O}_{P^1}(a_0)=\mathcal{O}_{P^1}$, we have that $\varphi_\xi(\sigma)$ is a point, as ξ_σ is trivial. This implies that $Y\sigma$ consists of one point at most. On the other hand

$$Y\sigma = (2\xi - bF)\sigma = -b.$$

This shows that $\mathcal{E}$ is ample, hence very ample, unless $b=0, -1$.

In sec. 4 we will make explicit the numerical conditions assuring the existence of a smooth element in $|2\xi - bF|$, for $n=3$. Coming back to our pair (X, L), (1.1) gives the following fact:

(1.2) Since L is very ample, then either $b=0, -1$, or $\delta \geqq n+1$.

Let S be the smooth surface obtained by intersecting $(n-2)$ general elements of $|L|$ and set $k=K_S^2$.

(1.3) LEMMA. *Let (X, L) be as at the beginning of this section. Then the integers δ, b, n and k are related as follows:*

i) $2\delta = 3b+8-k$ *(in particular $b-k$ is even)*;

ii) $b \geqq k-2$;

iii) $2\delta \geqq (n+1)b$, *with equality if and only if X is a bundle (i.e. there are no singular fibres)*;

iv) $(n-2)b \leqq 8-k$;

v) $k \leqq 2+6/(n-1)$.

PROOF. By the canonical bundle formula we know that $K_P = -(n+1)\xi + \pi^*\mathcal{O}_{P^1}(\delta-2)$ and then, by adjunction,

(1.3.1) $\quad K_X = (-(n-1)\xi + \pi^*\mathcal{O}_{P^1}(\delta-b-2))_X$ and $K_S = (-\xi + (\delta-b-2)F)_S$.

Now since $k=K_S^2$ we get by (1.0.2)

$$\begin{aligned} k = K_S^2 &= (-\xi+(\delta-b-2)F)_S^2 = (-\xi+(\delta-b-2)F)^2 X \xi^{(n-2)} \\ &= (-\xi+(\delta-b-2)F)^2(2\xi-bF)\xi^{(n-2)} = -2\delta+3b+8. \end{aligned}$$

This proves i). To prove ii) note that by (1.3.1) we have $p^*H = K_X + (n-1)L = p^*\mathcal{O}_{P^1}(\delta-b-2)$. Thus the ampleness of H implies that $\delta-b-2=\deg H \geqq 1$, hence $\delta \geqq b+3$. Then the assertion follows by combining i) with this inequality. To prove iii) note that the singular fibres of X correspond to the zeroes of the section representing the determinant of the matrix of the terms $z_i z_j$, which in view of (1.0.4) is an element of $\mathcal{O}_{P^1}(2\delta-(n+1)b)$. Hence its degree must be $\geqq 0$ equality meaning that X is a bundle. iv) simply follows by combining i) and iii). Finally ii) and iv) give v).

We can also compute the numerical invariants of (X, L) in terms of b and k.

(1.4) REMARK. $d=d(X, L)=2b+8-k$ and $g=g(X, L)=3+(b-k)/2$.

PROOF. We have, recalling (1.0.2),

$$d = L^n = \xi^n X = \xi^n(2\xi-bF) = 2\xi^{(n+1)}-b\xi^n F = 2\delta-b,$$

hence (1.3; i) gives d. Genus formula, taking into account (1.3; i), gives g.

(1.5) THEOREM. *Let (X, L) be a quadric fibration over $\boldsymbol{P}^1$ as at the beginning of this section. Then X embeds fibrewise in a projective bundle $\boldsymbol{P}(\mathcal{E})$, where $\mathcal{E}=\bigoplus_{i=0,\dots,n}\mathcal{O}_{\boldsymbol{P}^1}(a_i)$, with $a_n\geqq a_{n-1}\geqq\cdots\geqq a_1\geqq a_0\geqq 0$, and*

$$X\in|2\xi-bF|,\quad L=\xi_X,$$

where ξ is the tautological bundle of $\mathcal{E}$ and F is a fibre of $\boldsymbol{P}(\mathcal{E})$. Moreover, if $k\geqq 1$, then the possible values of the invariants k, b, $\delta=\sum a_i$, d, g, n ($\geqq 3$) are those listed in the following table

k	b	δ	d	g	n
5	3	6	9	2	3
4	2	5	8	2	3, 4
	4	8	12	3	3
3	1	4	7	2	3
	3	7	11	3	3
	5	10	15	4	3
2	0	3	6	2	3
	2	6	10	3	3, 4, 5
	4	9	14	4	3
	6	12	18	5	3
1	−1	2	5	2	3
	1	5	9	3	3, 4
	3	8	13	4	3, 4
	5	11	17	5	3
	7	14	21	6	3.

PROOF. From (1.3; v) we have $k\leqq 5$, equality implying that $n=3$. Fix $k=1, \cdots, 5$ and note that (1.3; ii) provides a lower bound for b. By comparing it with the upper bound given by (1.3; iv) and recalling that $b-k$ is even we get an upper bound of n for any admissible value of b, unless $b=-1$ or 0, in which cases (1.4) shows that $g=2$. However in these cases it must be $n=3$, as proven by Fujita [**Fu1**, (3.25), (3.26)]. Since L is very ample, in the remaining cases (1.2) applies giving $n \leqq \delta - 1$, where δ is determined by (1.3; i). By comparing these two upper bounds for n we get the last column in the table. The values of d and g stem from (1.4).

(1.6) REMARKS. i) As to the effectiveness of the list provided by (1.5) in case $g=2$ note that all cases do really occur and the explicit description of the vector bundle $\mathcal{E}$ is known [**Fu1**, (3.30), (3.31)] (where L is simply assumed to be ample but is in fact very ample). We recall it in the following table for the convenience of the reader.

k	b	δ	$(a_0, \cdots, a_n)$	d	n
5	3	6	(1, 1, 2, 2)	9	3
4	2	5	(1, 1, 1, 2)	8	3
3	1	4	(1, 1, 1, 1)	7	3
2	0	3	(0, 1, 1, 1)	6	3
1	-1	2	(0, 0, 1, 1)	5	3
4	2	5	(1, 1, 1, 1, 1)	8	4.

For an alternative description of the pair (X, L) see also [**Io**, Thm. 3.4].

ii) Note that in all the above cases the surface S is $\boldsymbol{P}^2$ blown-up at $13-d$ points in general position; this means that S is a Del Pezzo surface with $K_S{}^2 = d-4=k$.

2. The general result.

We first recall our set-up

(2.0) Let A be a Del Pezzo surface contained as a smooth very ample divisor in a projective 3-fold X and set $L=[A]$.

In this section we prove a general result concerning our pairs (X, L), while the next sections are devoted to special subcases. We set $k=K_A{}^2$. Recall that $1 \leqq k \leqq 9$ and that A is $\boldsymbol{P}^2$ blown-up at $(9-k)$ points for $1 \leqq k \leqq 7$, A is either $\boldsymbol{P}^1 \times \boldsymbol{P}^1$ or $\boldsymbol{F}_1$ for $k=8$, while $A=\boldsymbol{P}^2$ for $k=9$.

By adjunction we know that there exists a line bundle $\mathcal{H} \in \mathrm{Pic}(X)$ such that $\mathcal{H}_A = -K_A$; of course

$$K_X + L = -\mathcal{H}.$$

(2.1) PROPOSITION. *Let $k \geqq 2$. If K_X+2L is nef and $(K_X+2L)^2 L > 0$, then $\mathcal{H}$ is nef. In particular X is Fano.*

PROOF. Let $E = K_X + 2L$ and look at the exact sequence

$$0 \longrightarrow -E \longrightarrow \mathcal{H} \longrightarrow \mathcal{H}_A \longrightarrow 0.$$

Since E is nef and $E^2L=(K_X+2L)^2L>0$, [**LPS**, (0.7)] and Serre duality give $h^1(-E)=0$. As a consequence the restriction homomorphism

$$H^0(X, \mathcal{H}) \longrightarrow H^0(A, \mathcal{H}_A)$$

induced in cohomology by the above sequence is surjective. Assume that $\mathcal{H}$ is not nef. Then there exists a curve Z in X such that $\mathcal{H}Z<0$; such a curve Z, which is contained in the base locus of $|\mathcal{H}|$, has a nonempty intersection with A, which is ample, this producing base points for the trace on A of $|\mathcal{H}|$, which is $|\mathcal{H}_A|=|-K_A|$. Since $-K_A$ is spanned for $k\geqq 2$, this implies $k=1$, contradiction. The last assertion follows from the fact that $-K_X=L+\mathcal{H}$.

(2.2) REMARK. In case $k=1$, under the same assumptions as in (2.1), the same argument shows that $|\mathcal{H}|$ is a pencil whose base locus is a line Z of (X, L) with $\mathcal{H}Z=-1$. In this case $-K_X$ is nef and big, and ample off Z.

Before stating the main result of this section it is convenient to recall the following fact, which is an immediate consequence of the Nakai-Moishezon ampleness criterion.

(2.3) REMARK. Let S be a Del Pezzo surface. Every smooth surface S' dominated by S via a birational morphism is a Del Pezzo surface too.

(2.4) THEOREM. *Let things be as in* (2.0). *Then the possible pairs* (X, L) *and the corresponding values of* k *are the following*:

(2.4.a) $(\boldsymbol{P}^3, \mathcal{O}_{\boldsymbol{P}^3}(1))$, $k=9$;

(2.4.b) $(\boldsymbol{Q}^3, \mathcal{O}_{\boldsymbol{Q}^3}(1))$, $k=8$ *and* $A=\boldsymbol{P}^1\times\boldsymbol{P}^1$;

(2.4.c) (X, L) *is a scroll over* $\boldsymbol{P}^1$, $k=8$ (*for more information see* (2.8)).

(2.4.d) (X, L) *is a Del Pezzo threefold of degree* $k\geqq 3$ (*see* (2.9)).

(2.4.e) (X, L) *is a quadric fibration over* $\boldsymbol{P}^1$, $k\leqq 8$ (*for more information see sec.* 4),

(2.4.f) (X, L) *is a scroll over a surface* (*for a precise description when* $k\geqq 2$ *see sec.* 5),

(2.4.g) (X, L) *admits* $(\boldsymbol{P}^3, \mathcal{O}_{\boldsymbol{P}^3}(3))$ *as a reduction, the reduction morphism* $X\to\boldsymbol{P}^3$ *being the blow-up at* $0\leqq 3-k\leqq 2$ *points,*

(2.4.h) (X, L) *admits* $(\boldsymbol{Q}^3, \mathcal{O}_{\boldsymbol{Q}^3}(2))$ *as a reduction, the reduction morphism* $X\to\boldsymbol{Q}^3$ *being the blow-up at* $0\leqq 4-k\leqq 3$ *points no two of them lying on a line of* $\boldsymbol{Q}^3$,

(2.4.i) (X, L) *admits as reduction* (X', L') *a Veronese bundle over* $\boldsymbol{P}^1$, *i.e.* X' *is a* $\boldsymbol{P}^2$*-bundle over* $\boldsymbol{P}^1$ *and* $2K_{X'}+3L'=\phi^*H$, *where* $\phi: X'\to\boldsymbol{P}^1$ *is the bundle projection and* H *is an ample line bundle on* $\boldsymbol{P}^1$ (*for a precise description see sec.* 3).

The proof of (2.4) takes the remainder of this section. The first step is the following

(2.5) LEMMA. *Let* (X, L) *be as in* (2.0). *Then* K_X+2L *is nef and big unless in cases* (2.4.a-f).

PROOF. As a first thing assume that K_X+2L is not nef. Then by [**SV**], (X, L) is either $(\boldsymbol{P}^3, \mathcal{O}_{\boldsymbol{P}^3}(1))$, $(\boldsymbol{Q}^3, \mathcal{O}_{\boldsymbol{Q}^3}(1))$, or a scroll over a smooth curve C. In the last case A, which is a smooth element of $|L|$, is a $\boldsymbol{P}^1$-bundle over C and then, since A is rational, it follows that $C=\boldsymbol{P}^1$. Note that in all the above cases we have $K_A{}^2=9$ or 8. So, apart from cases (2.4.a-c), K_X+2L is nef. Assume that it is not big. Then, according to adjunction theory [**S3**, (0.3)], either (X, L) is a quadric bundle over a smooth curve C and $C=\boldsymbol{P}^1$, $k\leqq 8$, since A, which is rational, has to be a conic bundle over C (case (2.4.e)), or (X, L) is as in (2.4.f), or $K_X=-2L$. In the last case (X, L) is a Del Pezzo 3-fold; moreover, since by adjunction $-K_A=L_A$, which is very ample, we have $k=d(A, L_A)\geqq 3$. This give (2.4.d).

In view of (2.5) we can proceed assuming that K_X+2L is nef and big. Let (X', L') be the reduction of (X, L) and let $\rho: X\to X'$ be the corresponding reduction morphism.

(2.6) REMARK. $K_{X'}+L'$ is not nef.

PROOF. Let $S'=\rho(A)$. Then S' is a smooth element of $|L'|$ and by adjunction $(K_{X'}+L')_{S'}=K_{S'}$. If $K_{X'}+L'$ were nef, then so would $K_{S'}$ be. On the other hand $-K_{S'}$ has to be ample in view of (2.3), a contradiction.

(2.7) By adjunction theory [**S3**, (0.4)] and [**BS**, (1.2)] it follows from (2.6) that (X', L') is one of the following pairs:

(2.7.1) $(\boldsymbol{P}^3, \mathcal{O}_{\boldsymbol{P}^3}(3))$,

(2.7.2) $(\boldsymbol{Q}^3, \mathcal{O}_{\boldsymbol{Q}^3}(2))$,

(2.7.3) a Veronese bundle, i.e. X' is a $\boldsymbol{P}^2$-bundle over a smooth curve C and $2K_{X'}+3L'=\phi^*H$, where $\phi: X'\to C$ is the bundle projection and H is an ample line bundle on C.

We show that these pairs lead respectively to cases g), h) and i) in (2.4). Let $S'=\rho(A)$ as before; then according to (0.2.3), ρ is the blowing-up of X' at t points, where

$$t = K_{S'}{}^2 - K_A{}^2 = K_{S'}{}^2 - k.$$

Note that if A is $\boldsymbol{P}^2$ or $\boldsymbol{P}^1\times\boldsymbol{P}^1$ it must be $t=0$, i.e. $(X, L)=(X', L')$; on the other hand for none of the above pairs it can be $L'=[A]$ (e.g. [**Ba1**]). So from now on we can assume $k\leqq 8$ and $A\neq \boldsymbol{P}^1\times\boldsymbol{P}^1$.

As to case (2.7.3) note that $C=\boldsymbol{P}^1$, since A, which is a rational surface, has to fibre over C; this gives (2.4.i). A complete description of pairs occurring in this case is given in (3.1). Here we study in detail the first two cases.

In case (2.7.1) we have $K_{S'}{}^2=3$, whence $k\leqq 3$ and so $\rho: X\to\boldsymbol{P}^3$ is the blowing-up at $0\leqq 3-k\leqq 2$ points and $L=\rho^*\mathcal{O}_{\boldsymbol{P}^3}(3)-\rho^{-1}((3-k)$ points).

Note that all these cases really occur. This is obvious for $k=3$, while if $k=1, 2$, for (X', L') and ρ as above, the line bundle $\rho^*\mathcal{O}_{\boldsymbol{P}^3}(3)-\rho^{-1}((3-k)$ points) is in fact very ample, by [**LPS**, (0.4) and (0.5.1)]. This gives (2.4.g).

In case (2.7.2) we have $K_{S'}{}^2=4$, hence $k\leqq 4$ and $\rho: X\to\boldsymbol{Q}^3$ is the blowing-up at $0\leqq 4-k\leqq 3$ points and $L=\rho^*\mathcal{O}_{\boldsymbol{Q}^3}(2)-\rho^{-1}((4-k)$ points). Note that no two of these points can lie on a line $\mathfrak{l}\subset\boldsymbol{Q}^3$; otherwise we would get $L\rho^{-1}(\mathfrak{l})=0$, contradicting the ampleness of L. On the other hand assume that the $4-k$ points satisfy the above condition, then the very ampleness of L follows from [**LPS**, (0.5; 1) and (0.6)] in cases $k=3$ and 2 respectively and from (0.5) in case $k=1$.

Note that in both cases the general element of $|L|$ is a Del Pezzo surface with $K_A{}^2=k$; in fact it is either a cubic surface blown-up at $(3-k)$ general points or a complete intersection of two quadrics blown-up at $(4-k)$ general points.

(2.8) Let (X, L) be as in (2.4.c). Then according to [**Ba1, 2**] we have that $X=\boldsymbol{P}(\mathcal{E})$, where $\mathcal{E}=\bigoplus_{i=1,\dots,3}\mathcal{O}_{\boldsymbol{P}^1}(a_i)$ with $a_i>0$ for all i and $c_1(\mathcal{E})=a_1+a_2+a_3$ is even if $A=\boldsymbol{P}^1\times\boldsymbol{P}^1$, odd if $A=\boldsymbol{F}_1$.

(2.9) The list of Del Pezzo threefolds occurring in (2.4.d), is the following [**Fu**, I and II]:

k	description of (X, L); in all cases $-K_X=2L$
3	$(V_3, \mathcal{O}_{\boldsymbol{P}}(1)_V)$ a smooth cubic hypersurface;
4	$(V_{2,2}, \mathcal{O}_{\boldsymbol{P}}(1)_V)$ a general complete intersection of type (2, 2);
5	$(V, \mathcal{O}_{\boldsymbol{P}}(1)_V)$ the section of the grassmannian $G(1, 4)$ embedded in $\boldsymbol{P}^9$ via the Plücker embedding by three general hyperplanes;
6	$(\boldsymbol{P}^1\times\boldsymbol{P}^1\times\boldsymbol{P}^1, \mathcal{O}_{\boldsymbol{P}^1\times\boldsymbol{P}^1\times\boldsymbol{P}^1}(1, 1, 1))$ or $(\boldsymbol{P}(T_{\boldsymbol{P}^2}), \xi)$, where $T_{\boldsymbol{P}^2}$ is the tangent bundle to $\boldsymbol{P}^2$ and ξ its tautological bundle;
7	$(B_p(\boldsymbol{P}^3), \sigma^*\mathcal{O}_{\boldsymbol{P}}(2)-\sigma^{-1}(p))$, where $\sigma: B_p(\boldsymbol{P}^3)\to\boldsymbol{P}^3$ is the blowing-up at a point p;
8	$(\boldsymbol{P}^3, \mathcal{O}_{\boldsymbol{P}}(2))$.

3. More on Veronese Bundles.

Here we look more closely at case (2.4.i). Recall that (X, L) has a reduction (X', L') where

(3.0) X' is a $\boldsymbol{P}^2$-bundle over $\boldsymbol{P}^1$ and $2K_{X'}+3L'=\phi^*H$, where $\phi: X'\to\boldsymbol{P}^1$ is the bundle projection and H is an ample line bundle on $\boldsymbol{P}^1$.

The reduction morphism $\rho: X\to X'$ is the blowing-up of X' at t points, and as observed in sec. 2, if $S'=\rho(A)$ then

(3.0.0) $$t=K_{S'}{}^2-K_A{}^2=K_{S'}{}^2-k\,.$$

The general properties of (X', L') have been worked out in [**LPS**, sec. 3]. Here we recall the situation for the convenience of the reader. Let $X'=\boldsymbol{P}(\mathcal{E})$ and let F be a fibre of $\phi: X'\to\boldsymbol{P}^1$. Since $L'_F=\mathcal{O}_{\boldsymbol{P}^2}(2)$ according to (3.0), we have $(K_{X'}+2L')_F=\mathcal{O}_{\boldsymbol{P}^2}(1)$ and so we can assume that $\mathcal{E}=\phi_*(K_{X'}+2L')$. For shortness let ξ be the tautological bundle of $\mathcal{E}$; then

(3.0.1) $$\xi=K_{X'}+2L'.$$

Note that

$$2\xi=2(K_{X'}+2L')=2K_{X'}+3L'+L'=\phi^*H+L'$$

is the sum of a nef and an ample line bundle, hence ξ is ample and so is $\mathcal{E}$. Therefore

(3.0.2) $$\mathcal{E}=\bigoplus_{i=1,\dots,3}\mathcal{O}_{\boldsymbol{P}^1}(a_i),\quad \text{where } a_i>0\ (i=1,2,3).$$

By the canonical bundle formula

$$K_{X'}=-3\xi+(\alpha-2)F,\quad \text{where } \alpha=c_1(\mathcal{E})=a_1+a_2+a_3\,.$$

From (3.0.1) and the relation above we see that

(3.0.3) $$L'=2\xi+(1-(\alpha/2))F.$$

In particular, recalling also (3.0.2), we have that

(3.0.4) $$\alpha \text{ is even and } \geqq 4\,.$$

On the other hand the basic relation for the tautological bundle ξ gives $\xi^3=\alpha$ and thus, by adjunction,

$$K_{S'}{}^2=(K_{X'}+L')^2L'=(-\xi+((\alpha/2)-1)F)^2(2\xi+(1-(\alpha/2))F)=5-(\alpha/2)\,.$$

So recalling (3.0.0) we get

(3.0.5) $$t+k=5-(\alpha/2)\,.$$

As $t \geqq 0$ and $k \geqq 1$, by combining (3.0.4) with (3.0.5) we get only the following possibilities:

$$\alpha = 8, \quad \text{with } (k, t) = (1, 0);$$

$$\alpha = 6, \quad \text{with } (k, t) = (1, 1) \text{ or } (2, 0);$$

$$\alpha = 4, \quad \text{with } (k, t) = (1, 2), (2, 1), \text{ or } (3, 0).$$

Cases with $k=2$ have already been studied in [**LPS**, sec. 3]. The case with $k=3$ simply corresponds to the reduction of the pair found in case $\alpha=4$, $k=2$ in the same study and the corresponding line bundle L is in fact very ample in this case. Moreover note that for $\alpha=6$, 4, in case $k=1$ we get a pair (X, L) the contraction of a (-1)-plane of which gives rise to the corresponding pair found for $k=2$. In these cases it only remains to decide about the very ampleness of L. We now show that the line bundles L occurring for these pairs are in fact very ample.

In case $\alpha=6$, $k=1$, (X, L) has the pair $(X', L')=(\boldsymbol{P}^1 \times \boldsymbol{P}^2, \mathcal{O}_{\boldsymbol{P}^1 \times \boldsymbol{P}^2}(2, 2))$ as simple reduction [**LPS**, (3.2.5)]. Let $x \in X'$ be the point blown-up by ρ. To show that the line bundle $L=\rho^*\mathcal{O}_{\boldsymbol{P}^1 \times \boldsymbol{P}^2}(2, 2)-\rho^{-1}(x)$ is very ample on X, consider the Segre embedding $s: X' \to \boldsymbol{P}^5$ and note that $\mathcal{O}_{\boldsymbol{P}^1 \times \boldsymbol{P}^2}(1, 1)=s^*\mathcal{O}_{\boldsymbol{P}^5}(1)$. Let θ: $P \to \boldsymbol{P}^5$ be the blow-up of $\boldsymbol{P}^5$ at the point $s(x)$ and let E_x be the corresponding exceptional divisor. Then, looking at the inclusion of X in P, we get $L=\rho^*\mathcal{O}_{\boldsymbol{P}^1 \times \boldsymbol{P}^2}(2, 2)-\rho^{-1}(x)=(\theta^*\mathcal{O}_{\boldsymbol{P}^5}(2)-E_x)_X$. Thus L is very ample since the line bundle $\theta^*\mathcal{O}_{\boldsymbol{P}^5}(2)-E_x$ is very ample on P, as shown in [**LPS**, (0.5; 1)].

In case $\alpha=4$, with $k=1$, the reduction (X', L') of (X, L) is the following pair [**LPS**, (3.2.6)]: X' is $\boldsymbol{P}^3$ blown-up along a line $\mathfrak{l}$ via σ and $L'=\sigma^*\mathcal{O}_{\boldsymbol{P}^3}(2)+F$. Let x', $y' \in X'$ be the points blown-up by ρ. We already know from [**LPS**] that neither $x=\sigma(x')$ nor $y=\sigma(y')$ can lie on $\mathfrak{l}$. Look at the composite morphism $\beta=\rho \circ \sigma: X \to \boldsymbol{P}^3$, which exhibits X as $\boldsymbol{P}^3$ blown-up along a line $\mathfrak{l}$ and two points x, $y \notin \mathfrak{l}$. Since $F=\sigma^*\mathcal{O}_{\boldsymbol{P}^3}(1)-\sigma^{-1}(\mathfrak{l})$, we have

$$L = \beta^*\mathcal{O}_{\boldsymbol{P}^3}(3)-\beta^{-1}(x)-\beta^{-1}(y)-\beta^{-1}(\mathfrak{l}).$$

So the lines $\langle x, y\rangle$ and $\mathfrak{l}$ are skew; otherwise it would be $L\beta^{-1}(\langle x, y\rangle)=0$, contradicting the ampleness. Thus x, y and $\mathfrak{l}$ are three totally disjoint linear subspaces of $\boldsymbol{P}^3$ and (0.3) applies. Hence L is very ample.

So it only remains to look at the new case $\alpha=8$. In this case $(X, L)=(X', L')$ since $t=0$, and $L=2\xi-3F=2(\xi-F)-F$, by (3.0.2). So from the ampleness of $L+F$ it follows that $\mathcal{E}\otimes\mathcal{O}_{\boldsymbol{P}^1}(-1)$ is ample. Therefore, recalling (3.0.2) we get $a_i-1>0$ $(i=1, 2, 3)$, so that, up to exchanging the summands, we get only the following possibilities for $\mathcal{E}$:

i) $\mathcal{E}=\mathcal{O}_{P^1}(2)\oplus\mathcal{O}_{P^1}(2)\oplus\mathcal{O}_{P^1}(4)$,

ii) $\mathcal{E}=\mathcal{O}_{P^1}(2)\oplus\mathcal{O}_{P^1}(3)\oplus\mathcal{O}_{P^1}(3)$,

Note that $\mathcal{E}=\mathcal{E}'\otimes\mathcal{O}_{P^1}(2)$, where $\mathcal{E}'$ is either $\mathcal{O}_{P^1}{}^{\oplus 2}\oplus\mathcal{O}_{P^1}(2)$ or $\mathcal{O}_{P^1}\oplus\mathcal{O}_{P^1}(1)^{\oplus 2}$ according to cases i) and ii). Let ξ' be the tautological bundle of $\mathcal{E}'$; then $\xi'=\xi-2F$ and so we get $L=2\xi'+F=\xi'+(\xi'+F)$. Note that ξ' is spanned, since $\mathcal{E}'$ is so; moreover $(\xi'+F)$ is very ample since it is the tautological bundle of $\mathcal{E}'\otimes\mathcal{O}_{P^1}(1)$, which is a direct sum of very ample line bundles. This shows that L is very ample in both cases i), ii). Notice that, according to [**Ha**, sec. 3], X' is the desingularization of a quadric cone of $\boldsymbol{P}^4$ having as vertex a line in case i), a point in case ii).

All the above proves the following

(3.1) THEOREM. *Let (X, L) be as in* (2.4.i), *let (X', L') be its reduction and let t be the number of blowing-ups the reduction morphism $\rho: X\to X'$ factors through. Then:*

(3.1.1) *X' is a minimal desingularization $\mu: X'\to Q$ of a quadric cone of $\boldsymbol{P}^4$ of rank* 3 *or* 4, *$L'=\mu^*\mathcal{O}_Q(2)+F$, F being the proper transform of a plane of Q, and $(k, t)=(1, 0)$.*

(3.1.2) $(X', L')=(\boldsymbol{P}^2\times\boldsymbol{P}^1, \mathcal{O}_{P^2\times P^1}(2, 2))$ *and* $(k, t)=(2, 0)$ *or* $(1, 1)$;

(3.1.3) *X' is the blow-up $\sigma: X'\to\boldsymbol{P}^3$ along a line $\mathfrak{l}$, $L'=\sigma^*\mathcal{O}_{P^3}(2)+F$, F being the proper transform of a plane through $\mathfrak{l}$, and $(k, t)=(1, 2)$, $(2, 1)$ or $(3, 0)$; for $t\geqq 1$ the points blown-up by ρ do not lie on $\sigma^{-1}(\mathfrak{l})$ and if $t=2$ their images in $\boldsymbol{P}^3$ generate a line skew with $\mathfrak{l}$.*

4. More on quadric fibrations.

In order to give a better description of pairs as in (2.4.e), in this section we specialize the quadric fibrations (X, L) considered in sec. 1 to the case of 3-folds. So let (X, L) be a quadric fibration over $\boldsymbol{P}^1$ with $n=3$ and let $\pi: P=\boldsymbol{P}(\mathcal{E})\to\boldsymbol{P}^1$ be the $\boldsymbol{P}^3$-bundle in which X embeds fibrewise as in (1.5).

Let us start with the following useful

(4.1) REMARK. Let things be as in (1.0.1) with $n=3$ and let (X, L) be a quadric fibration embedded fibrewise in $\boldsymbol{P}(\mathcal{E})$. If $2a_1<b$ then $a_0+a_3\geqq b$ and $a_1+a_2\geqq b$.

PROOF. By assumption, in view of the isomorphism

(4.1.1) $\Gamma(\boldsymbol{P}(\mathcal{E}), 2\xi-bF)\cong\Gamma(\boldsymbol{P}^1, \mathcal{E}^{(2)}\otimes\mathcal{O}_{P^1}(-b))=\oplus_{0\leqq i\leqq j\leqq 3}\Gamma(\mathcal{O}_{P^1}(a_i+a_j-b))$

every fibre of X has an equation containing neither the term z_1^2 nor, a fortiori, the terms z_0z_1 and z_0^2. Assume that $a_0+a_3<b$ then the equation would also contain neither z_0z_3 nor z_0z_2. Similarly, assuming $a_1+a_2<b$ then the equation would not contain the term z_1z_2. In both cases it turns out that every fibre of X would be a singular quadric, a contradiction.

(4.2) THEOREM. *Let* $\mathcal{E}$ *be as in* (1.0.1) *with* $n=3$ *and, keeping the same notation as in* (1.0.5), *set* $\mathcal{L}=[2\xi-bF]$.

A) *Assume that* $2a_1<b$. *Then there is a smooth element* X *in* $|\mathcal{L}|$ *if and only if* $a_0+a_3=a_1+a_2=b$.

B) *Assume that* $2a_1\geqq b$; *then*

B_1) *if* $a_0+a_2\geqq b$ *then there is a smooth element* X *in* $|\mathcal{L}|$;

B_2) *if* $a_0+a_2<b$ *then there is a smooth element* X *in* $|\mathcal{L}|$ *if and only if* $a_0+a_3=b$.

PROOF. Case A). Let $a_0+a_3=a_1+a_2=b$; then there are two sections generating $\Gamma(\mathcal{O}_{P^1}(a_0+a_3-b))$ and $\Gamma(\mathcal{O}_{P^1}(a_1+a_2-b))$ respectively. Hence in view of the isomorphism (4.1.1) we can choose an element X of $|\mathcal{L}|$ whose restriction to every fibre of $\boldsymbol{P}(\mathcal{E})$ has $z_0z_3-z_1z_2=0$ as equation. Of course such an X is smooth. To prove the converse assume that there is a smooth $X\in|\mathcal{L}|$. Consider $Y:=\boldsymbol{P}(\mathcal{O}(a_0)\oplus\mathcal{O}(a_1))$. Note that $Y\subset P$; moreover $Y\subset X$. Indeed let C be the section of $Y\to\boldsymbol{P}^1$, corresponding to the surjection

$$\mathcal{O}_{P^1}(a_0)\oplus\mathcal{O}_{P^1}(a_1)\longrightarrow\mathcal{O}_{P^1}(a_1).$$

Then

$$C^2=a_1-a_0\geqq 0 \quad\text{and}\quad \xi C=a_1,$$

by (0.7.1). Since $CX=C(2\xi-bF)=2a_1-b<0$ we see that C and all deformations of C are in X. On the other hand, since $C^2\geqq 0$, C has deformations in Y covering an open set of Y. It thus follows that $Y\subset X$. Now consider the exact sequence of the normal bundles

$$0\longrightarrow N_{Y|X}\longrightarrow N_{Y|P}\longrightarrow(N_{X|P})_Y\longrightarrow 0.$$

Since $N_{X|P}=\mathcal{L}$, tensoring the above sequence by $\mathcal{L}^{-1}$ we see that $c_2(N_{Y|P}\otimes\mathcal{L}^{-1})=0$. On the other hand we know by (0.8) that

$$N_{Y|P}=\pi_Y^*(\mathcal{O}(-a_2)\oplus\mathcal{O}(-a_3))\otimes\xi_Y.$$

We thus get $0=(-\xi+(b-a_2)F)(-\xi+(b-a_3)F)Y=a_2+a_3-2b+\xi^2Y=\delta-2b$, by (0.7.1). Hence $a_0+a_1+a_2+a_3=2b$. Recalling (4.1), this gives $a_0+a_3=a_1+a_2=b$.

Case B). It is immediate to check by using [**BS1**, p. 74] that the line bundle $\mathcal{L}=[2\xi-bF]$ is spanned if and only if $2a_0\geqq b$. So we can assume that $2a_0<b$. First consider subcase B1). Under the condition $a_0+a_2\geqq b$, the isomorphism (4.1.1)

shows that given any fibre F of π, there certainly are global sections of $\mathcal{L}$ which restrict to non-zero multiples of the monomials $z_0 z_j$ for $j=2, 3$ and $z_i z_j$ for $1 \leqq i \leqq j \leqq 3$, but not of the monomial z_0^2. The base locus of the surviving monomials is the point $(1:0:0:0)$. Thus, as a set, the base locus of $\mathcal{L}$ is the section $C := \boldsymbol{P}(\mathcal{O}(a_0))$ corresponding to the surjection $\mathcal{E} \to \mathcal{O}_{P^1}(a_0)$. According to (0.6) a general section $s \in \Gamma(\mathcal{L})$ will have a smooth zero set if the differential $d_C s \in \Gamma(N_C^* \otimes \mathcal{L})$ is nowhere zero on C. Note that, by (0.8), under the identification given by $\pi_C{}^*$ we get

$$(4.2.1) \qquad N_C^* \otimes \mathcal{L} \cong (\mathcal{O}_{P^1}(a_1) \oplus \mathcal{O}_{P^1}(a_2) \oplus \mathcal{O}_{P^1}(a_3)) \otimes (-\xi_C) \otimes (2\xi - bF)_C$$
$$= \mathcal{O}_{P^1}(a_0 + a_1 - b) \oplus \mathcal{O}_{P^1}(a_0 + a_2 - b) \oplus \mathcal{O}_{P^1}(a_0 + a_3 - b).$$

So the differentials of the global sections span a 2-dimensional vector subbundle of $N_C^* \otimes \mathcal{L}$. Actually, in local coordinates, the section corresponding to the monomial $z_0 z_j$ goes to dz_j. Thus we can choose a global section $s \in \Gamma(\mathcal{L})$ whose differential vanishes nowhere on C.

Finally consider subcase B2). Since $a_0 + a_2 < b$, the isomorphism (4.1.1) shows that given any fibre F of π, there are global sections of $\mathcal{L}$ which restrict to non-zero multiples of the monomials $z_0 z_3$ and $z_i z_j$ for $1 \leqq i \leqq j \leqq 3$, and no other monomials are restrictions of global sections of $\mathcal{L}$. As before we thus see that the base locus of $|\mathcal{L}|$ is still C as a set. Now, looking at (4.2.1) we see that the differentials on C of the global sections of $\mathcal{L}$ span only the subbundle of $N_C^* \otimes \mathcal{L}$ given by $\mathcal{O}_{P^1}(a_0 + a_3 - b)$, which corresponds to the differential of the monomial $z_0 z_3$. So we will have a general section, whose differential vanishes nowhere on C, if and only if the subbundle $\mathcal{O}_{P^1}(a_0 + a_3 - b)$ is the trivial bundle, i.e. if and only if $a_0 + a_3 = b$. This concludes the proof in view of (0.6).

From now on assume that (X, L) is as in (2.4.e), i.e.

(4.3) a smooth element $A \in |L|$ is a Del Pezzo surface.

(4.4) PROPOSITION. *Let things be as above and assume that* (4.3) *holds. If* $2a_1 < b$ *then* $a_3 = a_2 = a_1 + 1 = a_0 + 1$ *and* $a_0 + a_3 = b$, *implying in particular that* b *is odd and* $\delta = 2b$.

PROOF. Let $Y := \boldsymbol{P}(\mathcal{O}(a_0) \oplus \mathcal{O}(a_1))$. As shown in the proof of (4.2), case A), we have that $Y \subset X$. Since $A \in |\xi_X|$, it meets Y along a curve C and then $-K_A C > 0$ since A is Del Pezzo. By adjunction recalling (1.3.1) this gives

$$(4.4.1) \qquad (\xi - (\delta - 2 - b)F)\xi Y \geqq 1.$$

By using (0.7.1) we get $\xi^2 Y = a_0 + a_1$, so that (4.4.1) yields

$$b + 1 \geqq a_2 + a_3.$$

On the other hand by (4.1) we have the inequalities $a_1+a_2 \geqq b$ and $a_0+a_3 \geqq b$. Putting together all these inequalities we get

$$b+1 \geqq a_2+a_3 \geqq b-a_1+a_3 \geqq 2a_1+1-a_1+a_3 = a_1+a_3+1 \geqq a_1+a_2+1 \geqq b+1,$$

and similarly $b+1 \geqq a_2+a_3 \geqq a_1+a_3+1 \geqq a_0+a_3+1 \geqq b+1$, so that all the above inequalities are equalities. This proves the assertion.

(4.5) PROPOSITION. *Let things be as above and assume that* (4.3) *holds. Then* $b+3 \geqq 2a_3$. *In particular, if* $2a_1 \geqq b$, *then* $a_3-a_1 \leqq 1$.

PROOF. Let $Z := \boldsymbol{P}(\mathcal{O}(a_0) \oplus \mathcal{O}(a_1) \oplus \mathcal{O}(a_2))$ and note that Z meets X along a surface. This surface intersects A along a curve and by ampleness $-K_A$ has to restrict positively to it. Since $A \in |\xi_X|$, recalling (1.3.1) this gives

$$(2\xi-bF)\xi(\xi-(\delta-2-b)F)Z \geqq 1. \tag{4.5.1}$$

As $\xi^3 Z = a_0+a_1+a_2$ by (0.7.1) and $\xi^2 FZ = 1$, we get from (4.5.1) that

$$2(a_0+a_1+a_2)-b-2(\delta-2-b) \geqq 1,$$

which proves the assertion.

The above discussion allows us to list the possible invariants occurring for pairs as in (2.4.e). Note that the first 5 pairs corresponding to the invariants listed in (1.6) do really fit into case (2.4.e), by (1.6, ii), and by [**Fu1**] there are no more pairs with $g=2$. So the following statement takes care of the remaining case $g \geqq 3$. It follows by combining the list in (1.5) with the smoothness conditions given by (4.2) and finally checking the conditions of (4.4), (4.5).

(4.6) THEOREM. *Let* (X, L) *be a quadric fibration as in* (2.4.e), *let* $\boldsymbol{P}(\mathcal{E})$ *be the* $\boldsymbol{P}^3$*-bundle over* $\boldsymbol{P}^1$ *in which* X *embeds fibrewise and keep notation as in* (1.5). *If* $g \geqq 3$, *then only the following invariants can occur* (*where* * *stands for* X *being a bundle*):

k	b	δ	a_0	a_1	a_2	a_3	d	g	
4	4	8	1	2	2	3	12	3	*
			2	2	2	2			*
3	3	7	1	2	2	2	11	3	
	5	10	2	2	3	3	15	4	*
2	2	6	1	1	2	2	10	3	
	4	9	1	2	3	3	14	4	
			2	2	2	3			
	6	12	2	3	3	4	18	5	*
			3	3	3	3			*
1	1	5	1	1	1	2	9	3	

Continued.

k	b	δ	a_0	a_1	a_2	a_3	d	g	
1	3	8	1	2	2	3	13	4	
			2	2	2	2			
	5	11	1	3	3	4	17	5	
			2	3	3	3			
	7	14	3	3	4	4	21	6	*.

Note that in cases $k=2$ and 3 the above result improves the description given in [**LPS**, (4.4)] and in [**LPS1**, (3.3.1)].

5. More on scrolls.

This section is devoted to the study of pairs (X, L) as in case (2.4.f) satisfying $k \geqq 2$. Let $\pi: X \to S$ be the morphism expressing X as a $\boldsymbol{P}^1$-bundle over a smooth surface S; we have $K_X+2L=\pi^* H$ for an ample $H \in \mathrm{Pic}(S)$. Note also that H is spanned, since so is K_X+2L [**SV**, (0.1)]. We have $X=\boldsymbol{P}(\mathcal{E})$, where $\mathcal{E}=\pi_* L$ is a very ample rank-2 vector bundle on S, since so is its tautological line bundle L on X. Moreover $(S, \det \mathcal{E})$ is simply the reduction of (A, L_A); so, as to the invariants $d=d(X, L)$ and $g=g(X, L)$ we have

$$d = d(A,\ L_A) = d(S,\ \det\mathcal{E}) - c_2(\mathcal{E}), \quad g = g(A,\ L_A) = g(S,\ \det\mathcal{E}).$$

Recall that $K_X+L=-\mathcal{H} \in \mathrm{Pic}(X)$, where according to (2.1) $\mathcal{H}$ is nef since we assume that $k=K_A{}^2 \geqq 2$. In this case $-K_X$ is ample, so that X is a Fano bundle. This makes our analysis relatively easy, due to [**SzW**].

(5.1) THEOREM. *Let (X, L) be as in* (2.4.f) *and assume that X is Fano. Then the data S, $\mathcal{E}$ and the invariants d, g are those listed in the following table, according to the corresponding values of k.*

k	S	$\mathcal{E}$	d	g
5	$\boldsymbol{P}^2$	$\mathcal{O}_{\boldsymbol{P}^2}(2)^{\oplus 2}$	12	3
4	$\boldsymbol{P}^2$	given by $0 \to \mathcal{O}_{\boldsymbol{P}^2}(2) \to \mathcal{E} \to \mathcal{J}_x(2) \to 0$	11	3
3	$\boldsymbol{P}^2$	given by $0 \to \mathcal{O}_{\boldsymbol{P}^2}{}^{\oplus 2} \to \mathcal{O}_{\boldsymbol{P}^2}(1)^{\oplus 4} \to \mathcal{E} \to 0$	10	3
	$\boldsymbol{P}^1 \times \boldsymbol{P}^1$	$\mathcal{O}_{\boldsymbol{P}^1 \times \boldsymbol{P}^1}(1,\ 2) \oplus \mathcal{O}_{\boldsymbol{P}^1 \times \boldsymbol{P}^1}(2,\ 1)$	13	4
2	$\boldsymbol{P}^2$	$T_{\boldsymbol{P}^2}(1)$	18	6
	$\boldsymbol{P}^2$	given by $0 \to \mathcal{O}_{\boldsymbol{P}^2}(-1) \to \mathcal{O}_{\boldsymbol{P}^2}(1)^{\oplus 3} \to \mathcal{E} \to 0$	9	3
	$\boldsymbol{P}^1 \times \boldsymbol{P}^1$	given by $0 \to \mathcal{O}_{\boldsymbol{P}^1 \times \boldsymbol{P}^1} \to \mathcal{O}_{\boldsymbol{P}^1 \times \boldsymbol{P}^1}(1,\ 1)^{\oplus 3} \to \mathcal{E} \to 0$	12	4
	$\boldsymbol{F}_1$	$\beta^* T_{\boldsymbol{P}^2} \otimes [2s+3f]$ ($\beta: \boldsymbol{F}_1 \to \boldsymbol{P}^2$ being the blowing-up)	15	5.

In the above table $\mathcal{J}_x$ stands for the ideal sheaf of a point x, while s, f denote the section of minimal self-intersection and a fibre of $\boldsymbol{F}_1$ respectively.

PROOF. Since X is Fano we have to check the list in [**SzW**]. We call $\mathcal{F}$ the vector bundle in [**SzW**], so that $\mathcal{E}=\mathcal{F}\otimes\mathcal{L}$ for a suitable line bundle $\mathcal{L}$ on S. Note that in all cases S is a Del Pezzo surface by (2.3), since it is dominated by the Del Pezzo surface $A\in|L|$ via $\pi_{|A}$. First let us prove the following

(5.1.1) LEMMA. *If $\mathcal{E}=\mathcal{L}\oplus\mathcal{L}$, then $(S, \mathcal{L})=(\boldsymbol{P}^2, \mathcal{O}_{\boldsymbol{P}^2}(2))$ (which gives rise to the first case in* (5.1)).

PROOF. We have that $X=\boldsymbol{P}(\mathcal{E})=S\times\boldsymbol{P}^1$. Let $q:X\to\boldsymbol{P}^1$ be the second projection and call F any fibre of it. So $(K_X)_F=K_F$. Let D be the effective divisor cut out on F by A. Then $K_AD<0$ since $-K_A$ is ample and then by adjunction we get

$$0>(K_X+L)D=(K_X+L)_FD=(K_F+L_F)D,$$

which shows that K_F+L_F is not nef. Therefore (F, L_F) is either $(\boldsymbol{P}^2, \mathcal{O}_{\boldsymbol{P}^2}(2))$, $(\boldsymbol{P}^2, \mathcal{O}_{\boldsymbol{P}^2}(1))$, $(\boldsymbol{P}^1\times\boldsymbol{P}^1, \mathcal{O}_{\boldsymbol{P}^1\times\boldsymbol{P}^1}(1, 1))$ or a scroll. Note that $(F, L_F)=(S, \mathcal{L})$ under the isomorphism induced by π, as we see by restricting to F the canonical bundle formula $K_X+2L=\pi^*(K_S+c_1(\mathcal{E}))=\pi^*(K_S+2\mathcal{L})$. So $(S, \mathcal{L})$ is one of the four pairs above. In order to prove that the last three cases cannot happen, it is enough to show that $K_S+2\mathcal{L}$ is ample. But this follows from the fact that $K_F+2L_F=(K_X+2L)_F=(\pi^*H)_F$ is ample, so being H on S.

Continuing the proof of (5.1). In view of (5.1.1) all cases corresponding to $\mathcal{F}=\mathcal{O}_S\oplus\mathcal{O}_S$ listed in [**SzW**] are ruled out apart from case 3) there which gives rise to the first case in (5.1). As to the remaining cases in [**SzW**] check the condition $c_2(\mathcal{E})=c_1(\mathcal{E})^2-L_A{}^2=K_S{}^2-K_A{}^2$ for the Fano bundle $\pi:X=\boldsymbol{P}(\mathcal{E})\to S$, with $L=\mathcal{O}_{\boldsymbol{P}(\mathcal{E})}(1)$ in order to determine the normalizing bundle $\mathcal{L}$. This check leads to the cases listed in (5.1) and the following ones:

(5.1.2)

k	S	$\mathcal{E}$
6	$\boldsymbol{P}^2$	$\mathcal{O}_{\boldsymbol{P}^2}(3)\oplus\mathcal{O}_{\boldsymbol{P}^2}(1)$
4	$\boldsymbol{P}^1\times\boldsymbol{P}^1$	$\mathcal{O}_{\boldsymbol{P}^1\times\boldsymbol{P}^1}(2, 2)\oplus\mathcal{O}_{\boldsymbol{P}^1\times\boldsymbol{P}^1}(1, 1)$
3	$\boldsymbol{P}^2$	$\mathcal{O}_{\boldsymbol{P}^2}(3)\oplus\mathcal{O}_{\boldsymbol{P}^2}(2)$
3	$\boldsymbol{F}_1$	$[2s+3f]\oplus[s+2f]$
2	$\boldsymbol{P}^1\times\boldsymbol{P}^1$	$\mathcal{O}_{\boldsymbol{P}^1\times\boldsymbol{P}^1}(2, 2)\oplus\mathcal{O}_{\boldsymbol{P}^1\times\boldsymbol{P}^1}(2, 1)$
1	$\boldsymbol{P}^2$	$\mathcal{O}_{\boldsymbol{P}^2}(4)\oplus\mathcal{O}_{\boldsymbol{P}^2}(2)$.

Note that in the first 5 cases $\mathcal{E}=[-K_S]\oplus\mathcal{M}$. They are thus ruled out in view of the following

(5.1.3) LEMMA. *Let things be as above. If* $\mathcal{E}=[-K_S]\oplus\mathcal{M}$, *then* A *cannot be a Del Pezzo surface.*

PROOF. We have $K_X=-2L+\pi^*(K_S+\det\mathcal{E})=-2L+\pi^*\mathcal{M}$. So $-(K_X+L)=L-\pi^*\mathcal{M}$ is the tautological bundle for the projectivization of $\mathcal{E}'=[-K_S-\mathcal{M}]\oplus\mathcal{O}_S$ and then it is trivial when restricted to the section Σ of π corresponding to the surjection $\mathcal{E}'\to\mathcal{O}_S$. Then by adjunction we see that $-K_A$ cannot be ample.

Concluding the proof of (5.1). The same argument for proving (5.1.3) rules out also case $k=1$ in the table above; actually $-(K_X+L)$ is the tautological bundle of $\mathcal{E}'=\mathcal{O}_{P^2}(1)\oplus\mathcal{O}_{P^2}(-1)$; if Σ stands for the section corresponding to the surjection $\mathcal{E}'\to\mathcal{O}_{P^2}(-1)$, we thus see that $(-K_A)_{\Sigma\cap A}$ is negative, a contradiction.

So it only remains to show that all cases listed in the statement of (5.1) do really occur. As to the 4 cases corresponding to $k=2$ this was shown in [**LPS**, sec. 5]. We prove the same in the remaining cases.

Let $k=5$. In this case, $-(K_X+L)$ is the tautological bundle of $\mathcal{E}'=\mathcal{O}_{P^2}(1)^{\oplus 2}$, hence it is very ample, so being $\mathcal{E}'$. This shows that every smooth element A of $|L|$ is a Del Pezzo surface.

Let $k=4$. We prove that $\mathcal{H}=-(K_X+L)$ is ample. By adjunction this implies that for every smooth element A of $|L|$, $-K_A=\mathcal{H}_A$ is ample, hence A is a Del Pezzo surface. By contradiction assume that X contains a curve C on which $\mathcal{H}$ fails to be ample. Recalling that $\mathcal{H}$ is nef since $4=k>1$, we thus get

$$\mathcal{H}C=0. \tag{5.1.4}$$

Note that $\mathcal{H}$ is the tautological bundle of $\mathcal{E}'=\mathcal{E}(-1)$. Since C is not a fibre of π we have that $D=\pi(C)$ is a curve and $\mathcal{E}'_D$ is not ample. Restrict to D the exact sequence defining $\mathcal{E}'$: the first term is $\mathcal{O}_{P^2}(1)_D$, which is ample, while the third one is $\mathcal{J}_x(1)_D$, which is ample unless D is a line through x, in which case it is simply $\mathcal{O}_D$. So D is a line through x and this sequence reads as follows:

$$0\longrightarrow\mathcal{O}_{P^1}(1)\longrightarrow\mathcal{O}_{P^1}(1)\oplus\mathcal{O}_{P^1}\longrightarrow\mathcal{O}_{P^1}\longrightarrow 0.$$

Hence $\pi^{-1}(D)$ is the Segre-Hirzebruch surface $\boldsymbol{F}_1$ and C is the fundamental section on it, corresponding to the surjection $\mathcal{O}_{P^1}(1)\oplus\mathcal{O}_{P^1}\to\mathcal{O}_{P^1}$. We have $K_X=-2L+\pi^*(K_S+\det\mathcal{E})$; moreover $\det\mathcal{E}=\mathcal{O}_{P^2}(4)$ as we see from the exact sequence defining $\mathcal{E}$, since x, which is 0-dimensional, does not affect the computation of c_1. Therefore $\mathcal{H}=-(K_X+L)=L-\pi^*\mathcal{O}_{P^2}(1)$ and so (5.1.4) implies that $0=(L-\pi^*\mathcal{O}_{P^2}(1))C=LC-\mathcal{O}_{P^2}(1)D=LC-1$, i.e.

$$LC=1. \tag{5.1.5}$$

Now look at the normal bundle N of C in X. We have the exact sequence

$$0 \longrightarrow \mathcal{O}_C(-1) \longrightarrow N \longrightarrow \mathcal{O}_C(1) \longrightarrow 0,$$

$\mathcal{O}_C(-1)$ being the normal bundle of C inside $\pi^{-1}(D)=\boldsymbol{F}_1$ and $\mathcal{O}_C(1)$ being the pull-back of the normal bundle of the line D in $\boldsymbol{P}^2$. We thus get by adjunction

$$-2 = K_X C + \deg(\det N) = K_X C .$$

This, recalling (5.1.5), gives $(K_X+L)C=-1$, which contradicts (5.1.4).

Let $k=3$. There are two cases. In the former case note that $\boldsymbol{P}(\mathcal{O}_{\boldsymbol{P}^2}(1)^{\oplus 4})=\boldsymbol{P}^2\times\boldsymbol{P}^3$ Segre embedded in $\boldsymbol{P}^{11}$ by means of the tautological bundle. From the exact sequence defining $\mathcal{E}$ we thus see that X is the intersection of it with two general hyperplanes of $\boldsymbol{P}^{11}$. Thus every smooth element A of $|L|$ is a Del Pezzo surface.

Now come to the second case. Let $\mathcal{H}=-(K_X+L)$ again and note that $\mathcal{H}$ is the tautological bundle of $\mathcal{E}'=\mathcal{O}_{\boldsymbol{P}^1\times\boldsymbol{P}^1}(0, 1)\oplus\mathcal{O}_{\boldsymbol{P}^1\times\boldsymbol{P}^1}(1, 0)$; so $\mathcal{H}$ is spanned. Note that $\mathcal{H}$ restricts as $\mathcal{O}_{\boldsymbol{P}^1}(1)$ to the fibres of π. So, if X contains an irreducible curve C such that $\mathcal{H}_C$ is trivial, then $D=\pi(C)$ is a curve and $\mathcal{E}'_D$ is not ample. As $\mathcal{E}'$ is the sum of the pull-backs of two ample line bundles via the projections of $\boldsymbol{P}^1\times\boldsymbol{P}^1$ on the factors, we see that D must be either a horizontal or a vertical factor of $\boldsymbol{P}^1\times\boldsymbol{P}^1$. Without loss of generality we can assume $D=\boldsymbol{P}^1$ to be vertical. Thus $\mathcal{E}'_D=\mathcal{O}_{\boldsymbol{P}^1}(1)\oplus\mathcal{O}_{\boldsymbol{P}^1}$. Note that the tautological bundle of $\mathcal{E}'_D$ is $\mathcal{H}_{\pi^{-1}(D)}$; also C corresponds to the surjection $\mathcal{O}_{\boldsymbol{P}^1}(1)\oplus\mathcal{O}_{\boldsymbol{P}^1}\rightarrow\mathcal{O}_{\boldsymbol{P}^1}$. So $\pi^{-1}(D)=\boldsymbol{F}_1$ and C is the fundamental section on it. Now let S_1 be the surface in X swept out by those curves C, as D varies among the vertical fibres of $\boldsymbol{P}^1\times\boldsymbol{P}^1$. Note that $C^2=0$ in S_1 and in fact $S_1=\boldsymbol{P}^1\times\boldsymbol{P}^1$, with C as vertical fibre. Similarly there is a surface S_2 in X generated in the same way as D varies among the horizontal fibres of $\boldsymbol{P}^1\times\boldsymbol{P}^1$. The above argument shows that the only curves to which $\mathcal{H}$ restricts trivially are the vertical fibres of S_1 and the horizontal fibres of S_2. As L is very ample, the general element $A\in|L|$ cuts S_1, S_2 along smooth curves and so does not contain curves like C. Thus, by adjunction, $-K_A=\mathcal{H}_A$ is ample. So the general element of $|L|$ is Del Pezzo.

Appendix. Del Pezzo manifolds as ample divisors.

Del Pezzo manifolds have been classified by Fujita [**Fu**]. Let A be a Del Pezzo n-fold; then $-K_A=(n-1)h$, where h is an ample element of Pic(A). Let $d(A)=d(A, h)$. Here we assume that A is contained as an ample divisor in a smooth projective $(n+1)$-fold X and we classify pairs (X, L), where $L=[A]$, under the assumption that $n=\dim A\geqq 3$. The results are summarized in the following table, where in the last column an indication for the argument proving the result is given. Here $\boldsymbol{V}_n$ stands for the cone over the Veronese manifold $(\boldsymbol{P}^{n-1}, \mathcal{O}_{\boldsymbol{P}}(2))$.

$d(A)$	A	$\dim A$	X	L	where
1	$\pi: A \to \mathbf{V}_n$ double cover	any	$\Pi: X \to \mathbf{V}_{n+1}$ double cover	$\Pi^*\mathcal{O}_{\mathbf{V}}(1)$	(A.1, i)
2	$\pi: A \to \mathbf{P}^n$ double cover	any	$\Pi: X \to \mathbf{P}^{n+1}$ double cover	$\Pi^*\mathcal{O}_{\mathbf{P}}(1)$	(A.1, i)
			$\mathbf{Q}^{n+1}$	$\mathcal{O}_{\mathbf{Q}}(1)$	(A.2)
			$\mathbf{P}^{n+1}$	$\mathcal{O}_{\mathbf{P}}(2)$	(A.2)
3	cubic hypersurface of $\mathbf{P}^{n+1}$	any	cubic hypersurface of $\mathbf{P}^{n+2}$	$\mathcal{O}_X(1)$	(A.1, i)
			$\mathbf{P}^{n+1}$	$\mathcal{O}_{\mathbf{P}}(3)$	(A.1, iii)
4	complete intersection of two quadrics in $\mathbf{P}^{n+2}$	any	complete intersection of two quadrics in $\mathbf{P}^{n+3}$	$\mathcal{O}_X(1)$	(A.1, i)
			$\mathbf{Q}^{n+1}$	$\mathcal{O}_{\mathbf{Q}}(2)$	(A.1, ii)
5	section $G \cap H^i$ with i hyperplanes of $G=G(1, 4) \subset \mathbf{P}^9$ Plücker embedded ($i=0, 1, 2, 3$)	3, 4, 5, 6	no if $i=0$	—	(A.1, i)
			$G \cap H^{i-1}$ if $i>0$	$\mathcal{O}_X(1)$	
6	$\mathbf{P}^1 \times \mathbf{P}^1 \times \mathbf{P}^1$	3	no	—	(A.3)
	$\mathbf{P}(T_{\mathbf{P}^2})$	3	$\mathbf{P}^2 \times \mathbf{P}^2$	$\mathcal{O}_{\mathbf{P}^2 \times \mathbf{P}^2}(1, 1)$	(A.4)
	$\mathbf{P}^2 \times \mathbf{P}^2$	4	no	—	(A.3)
7	$B_p(\mathbf{P}^3)$	3	no	—	(A.5)
8	$\mathbf{P}^3$	3	$\mathbf{P}^4$	$\mathcal{O}_{\mathbf{P}}(1)$	(A.1, iv)

Note that L is very ample except in cases $d(A)=1$ and $d(A)=2$ when the double cover $\Pi: X \to \mathbf{P}^{n+1}$ has a branch locus of degree $2b$, $b \geqq 2$. Obviously, when $b=1$ the pair $(X, \Pi^*\mathcal{O}_{\mathbf{P}}(1))$ coincides with $(\mathbf{Q}^{n+1}, \mathcal{O}_{\mathbf{Q}}(1))$.

(A.1) PROPOSITION. *Let A be a Del Pezzo n-fold contained as an ample divisor in a smooth projective $(n+1)$-fold X and let $L=[A]$. Assume that* $\mathrm{Pic}(A) = \mathbf{Z}$. *Then (X, L) is either*

i) *a Del Pezzo $(n+1)$-fold of the same degree,*

ii) $(\mathbf{Q}^{n+1}, \mathcal{O}_{\mathbf{Q}}(2))$,

iii) $(\mathbf{P}^{n+1}, \mathcal{O}_{\mathbf{P}}(3))$, *or*

iv) $(\mathbf{P}^4, \mathcal{O}_{\mathbf{P}}(1))$.

PROOF. Let $-K_A=(n-1)h$, where h is an ample element of $\mathrm{Pic}(A)$. As $n \geqq 3$ we have $\mathrm{Pic}(X) \cong \mathrm{Pic}(A) = \mathbf{Z}$, by the Lefschetz theorem; let $\mathscr{H} \in \mathrm{Pic}(X)$ be the element such that $\mathscr{H}_A = h$. Note that $\mathscr{H}$ is ample. Assume that h generates $\mathrm{Pic}(A)$; then $\mathscr{H}$ generates $\mathrm{Pic}(X)$ and so we can write $L=a\mathscr{H}$ and $K_X=r\mathscr{H}$ for some integers r and $a>0$. By adjunction we have

$$(*)\qquad (n-1)h=-K_A=-(K_X+L)_A=-(r+a)\mathcal{H}_A=-(r+a)h\,,$$

hence $-K_X=(n-1+a)\mathcal{H}=(\dim X-(2-a))\mathcal{H}$, which implies $a\leqq 3$ by well known properties of Fano manifolds. Since $a\geqq 1$, we get the following possibilities:

i) $a=1$, in which case by definition X is a Del Pezzo manifold and $d(X, \mathcal{H})=\mathcal{H}^{n+1}=\mathcal{H}^n L=\mathcal{H}_A{}^n=d(A)$;
ii) $a=2$, in which case $(X, \mathcal{H})=(\boldsymbol{Q}^{n+1}, \mathcal{O}_{\boldsymbol{Q}}(1))$ by the Kobayashi-Ochiai theorem [**KO**] and $2=d(X, \mathcal{H})=\mathcal{H}^{n+1}=\mathcal{H}^n L/2=\mathcal{H}_A{}^n/2=d(A)/2$, hence $d(A)=4$;
iii) $a=3$, in which case $(X, \mathcal{H})=(\boldsymbol{P}^{n+1}, \mathcal{O}_{\boldsymbol{P}}(1))$ by the Kobayashi-Ochiai theorem [**KO**] and $1=d(X, \mathcal{H})=\mathcal{H}^{n+1}=\mathcal{H}^n L/3=\mathcal{H}_A{}^n/3=d(A)/3$, hence $d(A)=3$.

Thus in all the above cases the assertion follows from Fujita's classification [**Fu**, I, II and III]. Recall that in case $d(A)=1$, we have $\mathrm{Pic}(A)=\boldsymbol{Z}$ [**Is**, (6.11)]. Now assume that h does not generate $\mathrm{Pic}(A)=\boldsymbol{Z}$; note that this happens just when $d(A)=8$, in which case $h\in 2\,\mathrm{Pic}(A)$. Let $\mathcal{G}$ denote the positive generator of $\mathrm{Pic}(X)$ and set $L=\alpha\mathcal{G}$ and $K_X=\rho\mathcal{G}$ for some integers ρ and $\alpha>0$ as before. Then adjunction gives $-K_X=(4+\alpha)\mathcal{H}$, hence $\alpha=1$ and then $X=\boldsymbol{P}^4$ by [**KO**], with $L=\mathcal{G}=\mathcal{O}_{\boldsymbol{P}}(1)$. This gives case iv).

Note that (A.1) covers cases $d(A)=1$, 3, 4, 5, 8 and partially case $d(A)=2$.

(A.2) In case $d(A)=2$, when A is a quadric, the above argument still works with a small modification. In fact in this case we have $-K_A=nh$. Modifying (*) accordingly this gives $-K_X=(n+a)\mathcal{H}$, which implies $a\leqq 2$. Now use again [**KO**]; then case $a=1$ gives $(X, L)=(\boldsymbol{Q}^{n+1}, \mathcal{O}_{\boldsymbol{Q}}(1))$, while in case $a=2$ we get $(X, L)=(\boldsymbol{P}^{n+1}, \mathcal{O}_{\boldsymbol{P}}(2))$.

In case $d(A)=6$, if A is a product the assertion is an immediate consequence of the following fact:

(A.3) Proposition ([**S1**, Prop. IV]). *Let A be a projective n-fold, which is a product, contained as an ample divisor inside a projective $(n+1)$-fold. Then A has exactly two factors, one of which has dimension* 1.

(A.4) For the case $A=\boldsymbol{P}(T_{\boldsymbol{P}^2})$ we first recall the following fact

(A.4.1) Proposition ([**FSaSo**, (2.0)]. *Let $p: A\to\boldsymbol{P}^2$ be a $\boldsymbol{P}^1$-bundle contained as an ample divisor in a projective 4-fold X. If $A\neq\boldsymbol{P}^1\times\boldsymbol{P}^2$ then p extends to a morphism $P: X\to\boldsymbol{P}^2$ giving X the structure of a $\boldsymbol{P}^2$-bundle.*

Recall also that one can realize A as the obvious incidence correspondence in $\boldsymbol{P}^2\times\boldsymbol{P}^{2*}$, where $\boldsymbol{P}^{2*}$ stands for the dual projective plane; this endows A with two distinct $\boldsymbol{P}^1$-bundle structures on $\boldsymbol{P}^2$, from which X inherits two distinct $\boldsymbol{P}^2$-bundle structures, due to (A.4.1). This implies that $X=\boldsymbol{P}^2\times\boldsymbol{P}^2$ in view of [**Sa**]. Moreover as $P_{|A}=p$, we conclude that $A\in|\mathcal{O}_{\boldsymbol{P}^2\times\boldsymbol{P}^2}(1, 1)|$. Note that in fact $X=\boldsymbol{P}(\mathcal{E})$, where $\mathcal{E}$ is the extension of $T_{\boldsymbol{P}^2}$ defined by the Euler sequence.

(A.5) We finally come to case $d(A)=7$. In this case $A=B_p(\boldsymbol{P}^3)$ is the blow-up of $\boldsymbol{P}^3$ at a point p. Let X be a smooth 4-fold containing A as an ample divisor; then by [**Fu2**, (7.15) and (7.16)] there exists a projective 4-fold Y containing $\boldsymbol{P}^3$ as an ample divisor such that X is the blow-up of Y at p. It thus follows that $Y=\boldsymbol{P}^4$ (e.g. see [**S1**, p. 67]) and so X has a structure of a $\boldsymbol{P}^1$-bundle over $\boldsymbol{P}^3$. On the other hand A has a structure of $\boldsymbol{P}^1$-bundle over $\boldsymbol{P}^2$; in fact $A=\boldsymbol{P}(\mathcal{O}_{\boldsymbol{P}^2}\oplus\mathcal{O}_{\boldsymbol{P}^2}(-1))$. Therefore by (A.4.1) X would also be a $\boldsymbol{P}^2$-bundle over $\boldsymbol{P}^2$, which gives a contradiction since the two $\boldsymbol{P}$-bundle structures of X are topologically not compatible.

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