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Strong approximation theorem for division algebras over $\boldsymbol{R}(\boldsymbol{X})$

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Introduction.

Let R be a Dedekind domain with the quotient field K, D be a central simple K-algebra. We call that (D, R) has the strong approximation property iff the commutator subgroup $[D^{\times}, D^{\times}]$ of $D^{\times}$ is dense in its adelization (for the precise meaning, see § 3). In this paper, when K is the rational function field $\boldsymbol{R}(X)$ of one variable over the reals, we shall prove:

SAT: (D, R) has the strong approximation property if and only if $D\otimes_{K}K_{v}$ is not a division algebra for some non-prime place v (i.e. the place v which does not come from any prime ideal of R).

If K is a global field (i.e. $[K: \boldsymbol{Q}]<\infty$ or $[K: \boldsymbol{F}_{q}(X)]<\infty$), then $[D^{\times}, D^{\times}]$ coincides with the norm 1 group $D^{(1)}$, and the result SAT of the above type is well known as SAT (Strong Approximation Theorem) of Eichler.

Swan [**11**] systematically applied SAT of Eichler to the theory of lattices over orders. Recently Hijikata [**6**], extending the scope of Swan's approach to arbitrary Dedekind domains and remarking that non-division D always has the strong approximation property, pointed out the importance of establishing SAT for a central division D over the quotient field of an arbitrary Dedekind domain.

Our result gives a first example of non-trivial SAT other than the global field. In § 1, 2, we describe the structure of the Brauer group $Br(K)$ for the algebraic function field $K=\boldsymbol{R}(X, y)$ of one variable. Although the structure of $Br(K)$ is known as "abstract groups", even for K's with much more general constant fields ([**2**], [**4**], [**5**]), we need to know some explicit isomorphism reflecting the ramification of D's. A remarkable fact is that Hasse's principle holds for $K=\boldsymbol{R}(X, y)$ (i.e. $Br(K)\rightarrow\prod Br(K_{v})$ is injective). In § 3, we formulate the strong approximation property. In § 4, we prove SAT for $R=\boldsymbol{R}[X]$. In § 5, we prove the only if part of SAT for $K=\boldsymbol{R}(X, y)$. We prove the if part of SAT for any R in $K=\boldsymbol{R}(X)$.

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1. Brauer groups of $\boldsymbol{R}((X))$ and $\boldsymbol{R}(X)$.

Let $\boldsymbol{R}((X))$ be the field of formal power series over $\boldsymbol{R}$. It is a complete valuation field with the residue field $\boldsymbol{R}$. By J. P. Serre "Corps locaux" Chap. 12, we have

$$Br(\boldsymbol{R}((X))) \cong Gal(\boldsymbol{C}/\boldsymbol{R}) \times Br(\boldsymbol{R}) \cong (\boldsymbol{Z}/2\boldsymbol{Z})^2 .$$

($Br(K)$ denotes the Brauer group of K). We shall determine it more concretely.

Let D be a central division algebra over $\boldsymbol{R}((X))$. Since $Br(\boldsymbol{C}((X)))$ is trivial, D splits over $\boldsymbol{C}((X))$, so that D contains a maximal subfield isomorphic to $\boldsymbol{C}((X))$. Thus we have

$$D = K+Ki+Kj+Kij, \qquad K = \boldsymbol{R}((X)),$$

$$i^2 = -1, \quad j^2 = f \in K^\times, \quad ji = -ij .$$

We shall denote this D by $\{-1, f\}$.

Since $\{-1, f\} \cong \{-1, f'\} \Leftrightarrow ff'^{-1} \in N_{K(\sqrt{-1})/K}(K(\sqrt{-1})^\times) = (K^2+K^2) \cap K^\times$, we have $Br(K) \cong K^\times/(K^2+K^2) \cap K^\times$, whose complete representative system is given by $\{1, -1, X, -X\}$ so that

$$Br(\boldsymbol{R}((X))) = \{\boldsymbol{R}((X)), \boldsymbol{H}((X)), \{-1, X\}, \{-1, -X\}\} .$$

Note that $\boldsymbol{R}((X)) = \{-1, 1\}$ and $\boldsymbol{H}((X)) = \{-1, -1\}$ where $\boldsymbol{H}$ is the usual quaternion algebra over $\boldsymbol{R}$. $\boldsymbol{H}((X))$ is unramified over $\boldsymbol{R}((X))$, while $\{-1, X\}$ and $\{-1, -X\}$ are ramified.

Next, we shall determine the Brauer group of $\boldsymbol{R}(X)$.

THEOREM 1. (1) *Every central division algebra over* $\boldsymbol{R}(X)$ *has the index* $\leqq 2$, *hence if it is not trivial, it is a quaternion algebra over* $\boldsymbol{R}(X)$,

(2) $Br(\boldsymbol{R}(X)) \cong \boldsymbol{Z}/2\boldsymbol{Z} \times (\boldsymbol{Z}/2\boldsymbol{Z})_0^{\boldsymbol{R}} \cong (\boldsymbol{Z}/2\boldsymbol{Z})_0^{\boldsymbol{R} \amalg \{sgn\}}$, *where* $(\boldsymbol{Z}/2\boldsymbol{Z})_0^{\boldsymbol{R}}$ *denotes the continuous direct sum of* $\boldsymbol{Z}/2\boldsymbol{Z}$, *namely the aggregation of all finite subsets of* $\boldsymbol{R}$ *with the group operation*: $A \cdot B =$ *the symmetric difference of* A *and* B.

PROOF. Let D be a central division algebra over $\boldsymbol{R}(X)$. Then by the same reason as before, $\boldsymbol{C}(X)$ is a splitting field of D. This proves (1), and some maximal subfield of D is isomorphic to $\boldsymbol{C}(X)$. Thus D is in the form of $D = \{-1, f\}$ for some $f \in K^\times$, $K = \boldsymbol{R}(X)$, and we have $Br(\boldsymbol{R}(X)) \cong K^\times/(K^2+K^2) \cap K^\times$.

If $f \in K^2+K^2$, then $f(a) \geqq 0$ for $^\forall a \in \boldsymbol{R}$. Conversely, if $f(a) \geqq 0$ for $^\forall a \in \boldsymbol{R}$, then f is decomposed into the product $f = \prod_i (X-a_i)^2 \prod_j (X-\alpha_j)(X-\bar{\alpha}_j)$, $a_i \in \boldsymbol{R}$, $\alpha_j \in \boldsymbol{C} \backslash \boldsymbol{R}$. Since $(X-\alpha_j)(X-\bar{\alpha}_j) = N_{K(\sqrt{-1})/K}(X-\alpha_j)$, we have $\in K^2+K^2$. Therefore, as a complete representative system of $K^\times/(K^2+K^2) \cap K^\times$, we get $\{\pm(X-a_1)\cdots(X-a_n) \mid a_i \in \boldsymbol{R}$, mutually distinct$\}$.

A "place" means a valuation on K which is trivial on $\boldsymbol{R}^\times$. The residue field of a place v is $\boldsymbol{R}$ or $\boldsymbol{C}$, according to which we call v "real" or "imaginary". (Note that this terminology differs from the ones used for algebraic number fields). For an imaginary place v, D_v is trivial over K_v. For a real place v, D_v is one of four algebras over K_v.

For $f=\pm(X-a_1)\cdots(X-a_n)$, $D=\{-1, f\}$ is trivial at $a\in\boldsymbol{R}$ such that $f(a)>0$. It is ramified at a_i and at ∞ (the place corresponding to X^{-1}) if the degree of f is odd. Since $\{-1, f_1\}\otimes_{\boldsymbol{R}(X)}\{-1, f_2\}\sim\{-1, f_1f_2\}$, the multiplication in $Br(\boldsymbol{R}(X))$ corresponds to the symmetric difference of the sets of ramified places. Thus we have obtained the desired result (2). Note that the isomorphism prescribes the ramification of each division algebra.

REMARK. The set of all real places will be denoted by $RP(K)$. For $K=\boldsymbol{R}(X)$, we have $RP(K)=\boldsymbol{R}\amalg\{\infty\}$.

Then, we have $Br(\boldsymbol{R}(X))\cong(\boldsymbol{Z}/2\boldsymbol{Z})_0^{RP(\boldsymbol{R}(X))}$. The isomorphism is given as follows. Suppose that a central division algebra D over $\boldsymbol{R}(X)$ corresponds to a finite subset A of $RP(\boldsymbol{R}(X))=\boldsymbol{R}\amalg\{\infty\}$. D is ramified at every $a\in A\backslash\{\infty\}$, and at ∞ if $|A\backslash\{\infty\}|$ is odd. There are two Ds which are ramified at no place. They are attributed to $\boldsymbol{Z}/2\boldsymbol{Z}$ at ∞.

COROLLARY. *D is trivial if and only if D_v is trivial for any place v. In this sense, Hasse's principle holds for $\boldsymbol{R}(X)$.*

$\{-1, -1\}$ is unramified but non-trivial at every place. All other non-trivial $\{-1, f\}$ are ramified at some places.

2. Brauer group of $\boldsymbol{R}(X, y)$.

Let K be a finite extension of $\boldsymbol{R}(X)$, namely an algebraic function field of one variable over $\boldsymbol{R}$. In other words, $K=\boldsymbol{R}(X, y)$, y is algebraic over $\boldsymbol{R}(X)$.

If $\sqrt{-1}\in K$, then K is an algebraic function field of one variable over $\boldsymbol{C}$, so that $Br(K)$ is trivial. (Theorem of Tsen, c.f. [**10**], Part III).

Hereafter we shall assume that $\sqrt{-1}\notin K$. Since $Br(K(\sqrt{-1}))$ is trivial, a central division algebra D over K splits over $K(\sqrt{-1})$. This implies that D is a quaternion algebra and $D=\{-1, f\}$ for some $f\in K^\times$. From this we see that $Br(K)$ has the exponent 2, and $Br(K)\cong K^\times/(K^2+K^2)\cap K^\times$.

Let $RP(K)$ be the set of all real places. Since the places of $K(\sqrt{-1})=\boldsymbol{C}(X, y)$ are in one-to-one correspondence with points of a compact Riemann surface $\mathfrak{R}$, and since a real place v of K does not decompose in $K(\sqrt{-1})$, $RP(K)$ is identified with a subset of $\mathfrak{R}$.

For a real place v of K, we have $^\exists\varphi\in K$, $\mathrm{ord}_v(\varphi)=1$. Then, $\varphi(z)$ is a local coordinate in a neighbourhood of the corresponding $z_v\in\mathfrak{R}$. Since $z\in RP(K)$ is equivalent to $\varphi(z)\in\boldsymbol{R}$ in this neighbourhood, $RP(K)$ is a one-dimensional real manifold. Since $\mathfrak{R}$ is compact, $RP(K)$ consists of ν closed curves, where ν is the number of connected components of $RP(K)$.

THEOREM 2. *We have* $Br(K)\cong(\boldsymbol{Z}/2\boldsymbol{Z})_0^{RP(K)}$.

The isomorphism is given as follows: Fix a point $z_i(1\leqq i\leqq\nu)$ *from each connected component of* $RP(K)$. *Suppose that* $Br(K)\ni D$ *corresponds to a finite subset* A *of* $RP(K)$. *Then,* D *is ramified at* $A\backslash\{z_1,\cdots,z_\nu\}$ *and possibly at* z_i. *The ramification at* z_i *is determined by the rule that* D *is ramified at even number of places on each connected component of* $RP(K)$.

There are 2^ν *different division algebras which are ramified at no real place. They are attributed to* $(\boldsymbol{Z}/2\boldsymbol{Z})^{\{z_1,\cdots,z_\nu\}}$.

PROOF. Let $Br_1(K)$ be the group of all division algebras which are ramified at no real place. Then, $D=\{-1,f\}\in Br_1(K)$ is equivalent to that $\mathrm{ord}_z(f)$ is even for every $z\in RP(K)$, namely that $f(z)$ has definite sign on each connected component of $RP(K)$.

As shown later, Hasse's principle holds for $K=\boldsymbol{R}(X,y)$. Therefore, $D=\{-1,f\}$ is trivial if and only if f is non-negative on $RP(K)$, so that we have $|Br_1(K)|\leqq 2^\nu$. The equality holds if for any connected component C of $RP(K)$, there exists $f\in K^\times$ such that $f(z)\leqq 0$ on C but $f(z)\geqq 0$ on $RP(K)\backslash C$. Since $RP(K)$ is mapped homeomorphically into $\boldsymbol{R}^4$ by $z\mapsto(T_i(z))_{1\leq i\leq 4}$, $T_1(z)=X(z)/X(z)^2+1$, $T_2(z)=1/X(z)^2+1$, $T_3(z)=y(z)/y(z)^2+1$, $T_4(z)=1/y(z)^2+1$, and since the function F defined by $F(z)=-1$ on C and $F(z)=1$ on $RP(K)\backslash C$ is continuous on $RP(K)$, the polynomial approximation theorem of Weierstrass assures that there exists a polynomial $P(T_i)$ such that $P(T_i(z))<0$ on C but $P(T_i(z))>0$ on $RP(K)\backslash C$. This completes the proof of $Br_1(K)\cong(\boldsymbol{Z}/2\boldsymbol{Z})^{\{z_1,\cdots,z_\nu\}}$.

Take any $f\in K^\times$. If $\mathrm{ord}_{z_0}(f)$ is odd for $z_0\in RP(K)$, then $f(z)$ changes its sign when z crosses z_0. Since a connected component C of $RP(K)$ is a closed curve, $f(z)$ must change its sign even times on C, therefore $D=\{-1,f\}$ is ramified at even number of places on C.

Now, we shall show that for any two points ζ and ζ' on C, there exists $f\in K^\times$ such that $D=\{-1,f\}$ is ramified at ζ and ζ', but not ramified at other real places. Again we shall map $RP(K)$ into $\boldsymbol{R}^4$ by $z\mapsto(T_i(z))_{1\leq i\leq 4}$. Since C is a closed analytic curve, there are $\zeta=\zeta_0,\zeta_1,\cdots\zeta_n=\zeta'(\zeta_i\in C)$ and small spheres S_j: $\sum_{i=1}^4(T_i-a_{ij})^2=r_j^2$ such that $S_j\cap RP(K)=\{\zeta_{j-1},\zeta_j\}$.

Then $f=\prod_{j=1}^n\{\sum_{i=1}^4(T_i(z)-a_{ij})^2-r_j^2\}$ satisfies $\mathrm{ord}_\zeta(f)=\mathrm{ord}_{\zeta'}(f)=1$, $\mathrm{ord}_{\zeta_i}(f)=2(1\leqq i\leqq n-1)$, and $\mathrm{ord}_z(f)=0$ for $z\in RP(K)\backslash\{\zeta_i\}$. This f is the desired element of $K^\times$.

Thus we have proved $Br(K)/Br_1(K)\cong(\boldsymbol{Z}/2\boldsymbol{Z})_0^{RP(K)\backslash\{z_1,\cdots,z_\nu\}}$, so combining with the result for $Br_1(K)$, we get $Br(K)\cong(\boldsymbol{Z}/2\boldsymbol{Z})_0^{RP(K)}$.

REMARK. K satisfies Hasse's principle as a result of the following lemma.

Let $\square K$ be the set of all sums of squares, $\square K=\{\sum x_i^2 \mid x_i \in K\}$.

LEMMA. *Let $K=\boldsymbol{R}(X, y)$ be an algebraic function field over $\boldsymbol{R}$.*

(1) *For $f \in K^{\times}$, $f \in \square K$ if and only if $f(z) \geqq 0$ for ${}^{\forall}z \in RP(K)$. Especially, if $RP(K)=\phi$ then $\square K=K$.*

(2) *Every element of $\square K$ can be written as a sum of two squares.*

We shall omit the proof here, and refer to [**9**], Th. 3.2, Chap. 3 and Th. 2.1, Chap. 4.

COROLLARY. *$K=\boldsymbol{R}(X, y)$ satisfies Hasse's principle.*

PROOF. $D=\{-1, f\}$ is locally trivial if and only if $f(z) \geqq 0$ for ${}^{\forall}z \in RP(K)$, which is equivalent to $f \in K^2+K^2=N_{K(\sqrt{-1})/K}(K(\sqrt{-1})^{\times})$, hence $D=\{-1, f\}$ is trivial.

3. Approximation in idele groups.

Let R be a Dedekind domain, and K be its quotient field. Every prime ideal p of R defines the p-adic valuation on K. We call this a prime valuation. Besides p-adic valuations, we often consider some others, which we call non-prime valuations. For instance, if K is an algebraic function field over the constant field k, it seems to be inevitable to consider all valuations trivial on $k^{\times}$.

We define the adele ring R_A of R by $R_A=\prod_p R_p$, where p runs over all prime valuations and R_p denotes the completion of R at the place p. Also we define the adele ring K_A of K by $K_A=K \otimes_R R_A \cong \bigcup_S(\prod_{p \in S} K_p \times \prod_{p \notin S} R_p)$ where S runs over all finite set of prime valuations. The idele group $K_A^{\times}$ is defined as the group of inversible elements of K_A. It is written in the form of $K_A^{\times}=\bigcup_S(\prod_{p \in S} K_p^{\times} \times \prod_{p \notin S} R_p^{\times})$.

The fundamental system of neighbourhoods of 0 in K_A is given by $\{V(S, n)\}$, where

$$V(S, n)=\prod_{p \in S} p^n R_p \times \prod_{p \notin S} R_p .$$

Similarly, the fundamental system of neighbourhoods of 1 in $K_A^{\times}$ is given by $\{U(S, n)\}$, where

$$U(S, n)=\prod_{p \in S}(1+p^n R_p) \times \prod_{p \notin S} R_p^{\times} .$$

Let D be a central division algebra over K. A finitely generated R-submodule of D is called an R-lattice, and if it spans D as a K-vector space, it is called a full R-lattice. An R-lattice is called an R-order, if it is a subring including 1 (=the unit element of D).

The adele ring D_A of D is defined by $D_A=D \otimes_K K_A$. It is written in the form of $D_A=\bigcup_S(\prod_{p \in S} D_p \times \prod_{p \notin S} \Gamma_p)$, where Γ is a full R-order of D and $\Gamma_p=$

$\Gamma\otimes_R R_p$. The idele group $D_A^\times$ is defined similarly. The fundamental system of neighbourhoods of 1 in $D_A^\times$ is given by

$$U(S, n) = \prod_{p\in S}(1+p^n\Gamma_p)\times\prod_{p\notin S}\Gamma_p^\times .$$

D is diagonally imbedded into D_A, and $D^\times$ is diagonally imbedded into $D_A^\times$. D is dense in D_A (by the Chinese remainder theorem), but $D^\times$ is not dense in $D_A^\times$. But $D^\times$ may be dense in some subgroup of $D_A^\times$.

Let $\mathfrak{N}_{D/K}$ be the reduced norm $D\to K$. $\mathfrak{N}_{D/K}$ maps $D^\times$ homomorphically into $K^\times$. We shall denote its kernel by $D^{(1)}$. $\mathfrak{N}_{D/K}$ is uniquely extended as a K_A-valued polynomial function on D_A. This extension is denoted by the same symbol $\mathfrak{N}_{D/K}$, and its kernel in $D_A^\times$ is denoted by $D_A^{(1)}$.

Eichler's theorem ascertains that for global fields, $D^{(1)}$ is dense in $D_A^{(1)}$ (in the topology of $D_A^\times$) if and only if D_v is not a division algebra for some non-prime v.

For global fields, we have also $D^{(1)}=[D^\times, D^\times]$, the commutator subgroup of $D^\times$. But for a general K, this relation does not hold (Platonov [8]).

For a general K, in the connection with the cancellation problem of Γ, it seems natural to consider $[D^\times, D^\times]$ rather than $D^{(1)}$. Thus we define the strong approximation property as follows: A central division algebra D is said to have strong approximation property if $[D^\times, D^\times]$ is dense in $[D_A^\times, D_A^\times]$. To find a necessary and sufficient condition for strong approximation property is a generalization of Eichler's theorem to a general case.

In the connection with the cancellation problem of Γ, we consider a little weaker approximation property. We say that D has $D^\times$-approximation property, if the closure of $D^\times$ (in the topology of $D_A^\times$) contains $[D_A^\times, D_A^\times]$. We say that D has $R_A^\times D^\times$-approximation property, if the closure of $R_A^\times D^\times$ contains $[D_A^\times, D_A^\times]$. (Both of $D^\times$ and $R_A^\times$ are contained in $D_A^\times$, so $R_A^\times D^\times\subset D_A^\times$.) The last and weakest approximation property is necessary and sufficient for the cancellation of every full R-order Γ of D (namely $\Gamma\oplus\Gamma\cong L\oplus\Gamma$ implies $\Gamma\cong L$, the isomorphism being as Γ-lattices).

4. Eichler's theorem for $\boldsymbol{R}(X)$.

In § 1 we have seen that for $R=\boldsymbol{R}[X]$ and $K=\boldsymbol{R}(X)$, $D=\{-1, f\}$ is trivial at the non-prime place ∞ if and only if f is monic of even degree.

THEOREM 3. *If D_∞ is not trivial, then $D^\times$ is discrete in $D_A^\times$ and $R_A^\times D^\times$ is closed in $D_A^\times$.*

COROLLARY. *If D_∞ is not trivial, then $R_A^\times D^\times$-approximation property does not hold.*

PROOF OF COROLLARY. It suffices to show $[D_A^\times, D_A^\times]\not\subset R_A^\times D^\times$. For a real place a, we shall identify $D_a^\times$ with the subgroup $D_a^\times\times\prod_{p\neq a}(1)_p$ of $D_A^\times$. It is clear that $[D_A^\times, D_A^\times]\cap D_a^\times=[D_a^\times, D_a^\times]$. Since D_a is a quaternion (or a matrix) algebra over K_a, we have $[D_a^\times, D_a^\times]=D_a^{(1)}$, so that $[D_a^\times, D_a^\times]\not\subset K_a^\times$.

On the other hand, if $x=(x_p)\in R_A^\times D^\times\cap D_a^\times$, then we have ${}^\exists d\in D^\times$, ${}^\forall p$(prime place), ${}^\exists r_p\in R_p^\times$, $x_p=r_pd$. For $p\neq a$, we have $x_p=1$ so that $d=r_p^{-1}\in R_p^\times\subset K_p^\times$, so that $d\in D^\times\cap K_p^\times=K^\times$, hence $x_a=r_ad\in R_a^\times K^\times\subset K_a^\times$. This assures $R_A^\times D^\times\cap D_a^\times\subset K_a^\times$ so that $[D_A^\times, D_A^\times]\not\subset R_A^\times D^\times$.

PROOF OF THEOREM 3. $D=\{-1, f\}$ means that

$$D=K+Ki+Kj+Kij$$

$$i^2=-1,\ j^2=f,\ ji=-ij.$$

Then $\Gamma=R+Ri+Rj+Rij$ is a full R-order of $D\,(K=\boldsymbol{R}(X),\ R=\boldsymbol{R}[X])$.

A fundamental neighbourhood of 1 in $D_A^\times$ is $U(g)=\prod_p(1+g\Gamma_p)\cap\prod_p\Gamma_p^\times$ for $g\in R$ and we have $U(g)\cap D^\times=(1+g\Gamma)\cap\Gamma^\times$, so the first half of Theorem 3 is $(1+g\Gamma)\cap\Gamma^\times=(1)$.

Suppose that $d=\varphi_1+\varphi_2i+\varphi_3j+\varphi_4ij\in(1+g\Gamma)\cap\Gamma^\times$, $\varphi_i\in R$. This means that $\varphi_1\equiv1 \bmod g$, $\varphi_i\equiv0 \bmod g$ for $i\geqq2$, and $\mathfrak{N}_{D/K}(d)=\varphi_1^2+\varphi_2^2-f(\varphi_3^2+\varphi_4^2)\in R^\times=\boldsymbol{R}^\times$. If $g\in R\backslash R^\times$, substituting a zero of g, we see that $\mathfrak{N}_{D/K}(d)=1$.

Since each φ_i^2 has, if not zero, a positive coefficient of the highest degree term, such terms of φ_1^2 and φ_2^2 (resp. φ_3^2 and φ_4^2) do not cancel.

From $\varphi_1^2+\varphi_2^2-1=f(\varphi_3^2+\varphi_4^2)$, if f is of odd degree, both hand sides should be zero. This implies that $\varphi_3=\varphi_4=0$ and $\varphi_1, \varphi_2\in\boldsymbol{R}$, which implies $\varphi_2=0$ and $\varphi_1=1$ because φ_2 is a multiple of g.

If f is of even degree with a negative coefficient of the highest degree term, then the highest degree terms of $\varphi_1^2+\varphi_2^2-f(\varphi_3^2+\varphi_4^2)$ do not cancel, so that we have ${}^\forall i$, $\varphi_i\in\boldsymbol{R}$. This again implies $\varphi_i=0$ for $i\geqq2$, and so $\varphi_1=1$.

Thus the first half of Theorem 3 has been proved. Similar discussions show that $(R+g\Gamma)^\times=R^\times$, if $g\in R\backslash R^\times$.

$\Gamma_g=R+g\Gamma$ is a full R-order of D, and $(\Gamma_g)_A^\times=\prod_p(R_p+g\Gamma_p)^\times$ is an open subgroup of $D_A^\times$, so $(\Gamma_g)_A^\times D^\times$ is open and closed, hence $\bigcap_g(\Gamma_g)_A^\times D^\times$ is a closed subgroup of $D_A^\times$, containing $R_A^\times D^\times$.

We shall show the inverse inclusion. Take any $x\in\bigcap_g(\Gamma_g)_A^\times D^\times$, then ${}^\forall g$, ${}^\exists\gamma_g\in(\Gamma_g)_A^\times$, ${}^\exists d_g\in D^\times$, $x=\gamma_gd_g$. Since $(\Gamma_g)_A^\times\cap D^\times=(R+g\Gamma)^\times=R^\times$, d_g is determined modulo $R^\times$, so if g_1 is a multiple of g, then d_{g_1} differs from d_g only modulo $R^\times$. This implies that we can choose d_g independently of g, thus ${}^\exists d\in D^\times$, $xd^{-1}\in\bigcap_g(\Gamma_g)_A^\times$.

But we have $R_A^\times=\bigcap_g(\Gamma_g)_A^\times$, because ${}^\forall p$, $\bigcap_g(R_p+g\Gamma_p)^\times=R_p^\times$. Thus the proof of the second half of Theorem 3 is completed.

THEOREM 4. *If D_∞ is trivial, hence if f is monic of even degree, then* $[D^\times, D^\times]=D^{(1)}$ *is dense in* $[D_A^\times, D_A^\times]$.

This theorem is divided into the following two parts.

THEOREM 4.1. *If f is monic of even degree, then for $g, h\in R$ such that* $(h, gf)=1$, *we have*

$$(1+g\Gamma)\cap(i+h\Gamma)\cap\Gamma^\times\neq\phi.$$

THEOREM 4.2. *The conclusion part of Theorem* 4.1 *is equivalent to strong approximation property.*

PROOF OF THEOREM 4.1. It suffices to show the existence of $\varphi_i\in R$, $1\leqq i\leqq 4$ such that

$$\varphi_1\equiv 1(\mathrm{mod}\, g),\qquad \varphi_i\equiv 0(\mathrm{mod}\, g),\qquad 2\leqq i\leqq 4$$
$$\varphi_2\equiv 1(\mathrm{mod}\, h),\qquad \varphi_i\equiv 0(\mathrm{mod}\, h),\qquad i=1,3,4$$

and

$$\varphi_1^2+\varphi_2^2-f(\varphi_3^2+\varphi_4^2)=1. \tag{1}$$

Put $\varphi_1=1+g^2fu_1$, $\varphi_2=g^2fu_2$, $\varphi_i=gu_i(i=3,4)$, then the required congruence modulo g is automatically satisfied. Substituting them into (1) and dividing both sides by g^2f, we get

$$2u_1+g^2fu_1^2+g^2fu_2^2-(u_3^2+u_4^2)=0. \tag{2}$$

Since $(h^2, g^2f)=1$, g^2f is inversible in R/h^2R, so there exist $\psi, \psi'\in R$ such that

$$g^2f\psi=1+h^2\psi'.$$

Put $u_1=-\psi+h^2v_1$, $u_2=\psi+h^2v_2$, $u_i=hv_i(i=3,4)$, then the required congruence modulo h is automatically satisfied. Substituting them into (2), we get

$$-2\psi+2h^2v_1+g^2f\{2\psi^2+2h^2\psi(v_2-v_1)+h^4(v_1^2+v_2^2)\}=h^2(v_3^2+v_4^2).$$

Since $-2\psi+2g^2f\psi^2=-2\psi(1-g^2f\psi)=2h^2\psi\psi'$, we have

$$2\psi\psi'+2(1-g^2f\psi)v_1+2g^2f\psi v_2+g^2fh^2(v_1^2+v_2^2)=v_3^2+v_4^2. \tag{3}$$

Put $v_1=(1-g^2f\psi)w$ and $v_2=g^2f\psi w$, then we get

$$2\psi\psi'+\{(1-g^2f\psi)^2+(g^2f\psi)^2\}(2w+g^2fh^2w^2)=v_3^2+v_4^2. \tag{4}$$

A polynomial $P\in R=\boldsymbol{R}[X]$ belongs to R^2+R^2, if and only if $P(a)\geqq 0$ for $\forall a\in\boldsymbol{R}$, as shown in the proof of Theorem 1. So it suffices to show that the left hand side of (4) is everywhere non-negative for some $w\in R$.

Put $2\psi\psi'=F$ and $g^2fh^2=G$, then $(1-g^2f\psi)^2+(g^2f\psi)^2=1-2g^2f\psi(1-g^2f\psi)$ $=1+2g^2f\psi h^2\psi'=1+FG$, so we have

$$F+(1+FG)(2w+Gw^2) \geqq 0. \tag{5}$$

The above calculation also shows $1+FG \geqq (1/2)$, namely $FG \geqq -(1/2)$. Since f is monic of even degree, we have $\lim_{t\to\pm\infty} G(t)=\infty$ so that $^\exists M>0$, $^\forall t\in \boldsymbol{R}$, $G(t)\geqq -M$. Since $\{t \mid G(t)\leqq 0\}$ is compact, F is bounded there, so $^\exists N>0$, $|F(t)|\leqq N$ for $G(t)\leqq 0$.

The left hand side of (5) is zero for

$$w=\frac{1}{G}\{-1\pm(1+FG)^{-1/2}\}.$$

Since $(1+t)^{-1/2}\leqq 1-(t/2)+(3/\sqrt{2})t^2$ for $t\geqq -(1/2)$, if we set $w=-(F/2)+(3/\sqrt{2})F^2G$, then (5) is satisfied for $G\geqq 0$. Let P be an everywhere positive polynomial of two variables s and t, then $w=-(F/2)+(3/\sqrt{2})F^2G+P(G, FG)$ satisfies (5) for $G\geqq 0$.

The condition (5) is satisfied also for $G<0$, if

$$-1\leqq -\frac{t}{2}+\frac{3}{\sqrt{2}}t^2+sP(s, t)\leqq -1+(1+t)^{-1/2} \tag{6}$$

on $\Delta=\{(s, t) \mid -M\leqq s\leqq 0,\ t\geqq -(1/2),\ |t|\leqq N|s|\}$. The condition (6) is satisfied if

$$\varepsilon \geqq \frac{1}{s}\left\{1-(1+t)^{-1/2}-\frac{t}{2}+\frac{3}{\sqrt{2}}t^2\right\}+P(s, t)\geqq 0$$

on Δ, where $\varepsilon\leqq(1+NM)^{-1/2}/M$. Since $\alpha(s, t)=(1/s)\{1-(1+t)^{-1/2}-(t/2)+(3/\sqrt{2})t^2\}$ is nonpositive and continuous on Δ (it is continuous at $(0, 0)$ because of $|t|\leqq N|s|$), such a polynomial $P(s, t)$ exists by virtue of polynomial approximation theorem of Weierstrass. $P(s, t)$ can be assumed everywhere positive, because we can put $P=Q^2+(\varepsilon/2)$, Q being an approximating polynomial of $\sqrt{|\alpha(s, t)|}$. Thus Theorem 4.1 has been proved.

PROOF OF THEOREM 4.2. Let H be the closure of $[D^\times, D^\times]=D^{(1)}$ in $D_A^\times$. Let p_0 be a prime place where D is unramified, and let $i_{p_0}=(1, \cdots, 1, i, 1, \cdots)\in D_A^\times$ be the element of $D_A^\times$ whose p_0-coordinate is i, while other coordinates are 1.

The proof is completed by the following steps, which are slight modifications of ones given in [3] §51.

Step 1. The conclusion part of Theorem 4.1 is equivalent to that $^\forall p_0$(where D_{p_0} is unramified), $i_{p_0}\in H$(note that $i_{p_0}\in D_{p_0}^{(1)}=[D_{p_0}^\times, D_{p_0}^\times]\subset[D_A^\times, D_A^\times]$).

Step 2. Identify $D_{p_0}^{(1)}$ with a subgroup $D_{p_0}^{(1)}\times\prod_{p\neq p_0}(1)_p$ of $D_A^\times$, then $H\cap D_{p_0}^{(1)}$ is a closed normal subgroup of $D_{p_0}^{(1)}$.

Step 3. If D is unramified at p_0, then $i_{p_0}\in H$ implies $D_{p_0}^{(1)}\subset H$.

If D_{p_0} is a matrix algebra, the assertion is a result of simplicity of $PSL(2, K_{p_0})$. If D_{p_0} is an unramified quaternion algebra, since $x=a+bi+cj+dij\in D_{p_0}^{(1)}$ satisfies $x^2-2ax+1=0$, the condition $x\in H$ depends only on a. (Here we identify $x\in D_{p_0}^{(1)}$ with $x_{p_0}=(1, \cdots, 1, x, 1, \cdots)\in D_A^\times$).

Take any $x=a+bi+cj+dij\in D_{p_0}^{(1)}$. Since $b^2+c^2+d^2$ has a root in K_{p_0}, we have $^{\exists}e\in K_{p_0}$, $b^2+c^2+d^2=e^2$. If $i\in H$, then $-ai+ej\in H$, therefore $i(-ai+ej)=a+eij\in H$, hence $x\in H$. This means $D_{p_0}^{(1)}\subset H$.

Step 4. Assume the conclusion part of Theorem 4.1. For a finite set S of prime places, we have $\prod_{p\in S}D_p^{(1)}\times\prod_{p\notin S}(1)_p\subset H$.

If D is unramified on S, the assertion is a consequence of Step 3.

Let S_0 be the set of all prime places where D is ramified. The assertion for $S=S_0$ follows from the fact that $D^{(1)}$ is dense in $\prod_{p\in S_0}D_p^{(1)}$ in the product topology of $D_p^\times$.

Step 5. $\bigcup_S(\prod_{p\in S}D_p^{(1)}\times\prod_{p\notin S}(1)_p)$ is dense in $[D_A^\times, D_A^\times]$.

Combining the five assertions above, we complete the proof of Theorem 4.2.

5. Eichler's theorem for $\boldsymbol{R}(X, y)$.

For an algebraic function field $K=\boldsymbol{R}(X, y)$, we shall fix a set P of valuations (which are trivial on $\boldsymbol{R}^\times$). We call valuation $v\in P$ a prime place and $v\notin P$ a non-prime place. We assume that there exists a non-prime place. Then, $R_P=\{x\in K\mid {}^{\forall}v\in P,\ v(x)\leqq 1\}$ is a Dedekind domain and K is its quotient field. A prime ideal of R_P is given by $p_v=\{x\in R_P\mid v(x)<1\}$ for $v\in P$.

The adele ring and the idele group are constructed using prime places only. Fixing the set P, we shall write R instead of R_P.

We consider the following Property (E):

(E) A central division algebra D over K has strong approximation property, if D is trivial at some non-prime place.

The converse of the Property (E) holds always as shown below.

THEOREM 5. *If a central division algebra D is non-trivial at every non-prime place, then D does not have $R_A^\times D^\times$-approximation property.*

REMARK. Before proving this theorem, we shall mention about the product formula. The formula is expressed as follows using ord_v; $v(x)=\theta^{\mathrm{ord}_v(x)}(0<\theta<1)$.

$$^{\forall}x \in K^\times, \qquad \sum_{v:\mathrm{real}}\mathrm{ord}_v(x)+2\sum_{v:\mathrm{imag}}\mathrm{ord}_v(x)=0,$$

where the sum is taken over all places, prime or not.

PROOF. Similar discussions as the proof of Theorem 3 show that it suffices to prove that

$$(R+g\Gamma)^\times = R^\times \qquad \text{for} \quad g\in R\backslash R^\times.$$

Let $D=\{-1, f\}$, $f\in R$. The assumption of Theorem 5 means that all non-prime places are real and that for every non-prime place v, $\mathrm{ord}_v(f)$ is odd or $\mathrm{ord}_v(f)$ is even with a negative coefficient of the lowest degree term with respect to the prime element π_v.

Suppose that $\varphi_1+\varphi_2 i+\varphi_3 j+\varphi_4 ij \in (R+g\Gamma)_\times$, then $\varphi_1 \in R$, $\varphi_i \in gR (2\leqq i\leqq 4)$, and $\varphi_1^2+\varphi_2^2-f(\varphi_3^2+\varphi_4^2) \in R^\times$. Put $\varphi = \varphi_1^2+\varphi_2^2-f(\varphi_3^2+\varphi_4^2)$, then $\varphi \in R^\times$ implies $\mathrm{ord}_v(\varphi)=0$ for every prime place v. As for a non-prime place v, the assumption on f implies that the lowest degree terms do not cancel, so that $\mathrm{ord}_v(\varphi)=\mathrm{Min}(2\mathrm{ord}_v(\varphi_1), 2\mathrm{ord}_v(\varphi_2), \mathrm{ord}_v(f)+2\mathrm{ord}_v(\varphi_3), \mathrm{ord}_v(f)+2\mathrm{ord}_v(\varphi_4))$, if $\varphi_i\neq 0$.

Combining this with the product formula, we have

$$\sum_{\text{non-prime}} \mathrm{ord}_v(\varphi_i) \geqq 0 \qquad (i=1, 2),$$

$$\sum_{\text{non-prime}} \mathrm{ord}_v(\varphi_i) \geqq \frac{1}{2}\sum_{\text{prime}} \alpha_v \mathrm{ord}_v(f) \qquad (i=3, 4)$$

where $\alpha_v=1$ for a real v and $\alpha_v=2$ for an imaginary v. Since $\varphi_i \in R$ and $f\in R$, we have $\mathrm{ord}_v(\varphi_i)\geqq 0$ and $\mathrm{ord}_v(f)\geqq 0$ for a prime place v, hence again from the product formula, we must have $\mathrm{ord}_v(\varphi_i)=0$ for every prime place v. This means $\varphi_i\in R^\times$. For $i\geqq 2$, this contradicts with $\varphi_i\in gR$, so we must have $\varphi_i=0$, which in turn implies $\varphi_1\in R^\times$. This completes the proof of $(R+g\Gamma)^\times=R^\times$.

REMARK. Property (E) depends not only on K, but also on R, or equivalently on the choice of non-prime places. However:

THEOREM 6. (1) *Suppose that Property* (*E*) *holds whenever R has only one non-prime place, then it holds for any R.*

(2) *For the rational function field $K=\boldsymbol{R}(X)$, Property* (*E*) *holds for any R.*

PROOF OF (1). Let $P(R)$ be the set of all prime places for the Dedekind domain R. Then $P(R')\subset P(R)$ implies $R\subset R'$. We shall denote the idele group of D with respect to R by $D_A^\times(R)$. Then $P(R)=P(R')\amalg P(R_1)$ implies that $D_A^\times(R)$ is the product topological group of $D_A^\times(R')$ and $D_A^\times(R_1)$, because of $D_A^\times(R) =\bigcup_S(\prod_{v\in S}D_v^\times\times\prod_{v\in P(R)\setminus S}\Gamma_v^\times)$ where S runs over all finite subsets of $P(R)$.

$D^\times$ is imbedded diagonally in $D_A^\times$, and strong approximation property means precisely that the image $i_R(D^{(1)})$ is dense in $[D_A^\times(R), D_A^\times(R)]$.

If $P(R')\subset P(R)$, then the projection $D_A^\times(R)\to D_A^\times(R')$ maps $i_R(D^{(1)})$ onto $i_{R'}(D^{(1)})$ and $[D_A^\times(R), D_A^\times(R)]$ onto $[D_A^\times(R'), D_A^\times(R')]$. Therefore, if $i_R(D^{(1)})$ is dense in $[D_A^\times(R), D_A^\times(R)]$, then $i_{R'}(D^{(1)})$ is dense in $[D_A^\times(R'), D_A^\times(R')]$.

Now suppose that D is trivial at some non-prime place v of a given R. Let P_0 be the set of all places other than v, and suppose that Property (E) holds for R_0 corresponding to P_0, then $i_{R_0}(D^{(1)})$ is dense in $[D_A^\times(R_0), D_A^\times(R_0)]$, hence $i_R(D^{(1)})$ is dense in $[D_A^\times(R), D_A^\times(R)]$, so Property ($E$) holds for R.

REMARK. The proof of Theorem 4.2 does work for a general algebraic function field $K=\boldsymbol{R}(X, y)$ and its Dedekind domain R. So, strong approximation property holds for $D=\{-1, f\}$, if $(1+g\Gamma)\cap(i+h\Gamma)\cap\Gamma^\times\neq\phi$ for $^\forall g, h\in R$

such that $(gf, h)=1$.

Also the proof of Theorem 4.1 works partially. For ψ, $\psi'\in R$ such that $g^2f\psi=1+h^2\psi'$, put $F=2\psi\psi'$ and $G=g^2fh^2$. Then, we have $(1+g\Gamma)\cap(i+h\Gamma)\cap\Gamma^{\times}\neq\phi$ if $^{\exists}w\in R$, $F+(1+FG)(2w+Gw^2)\in R^2+R^2$.

Suppose that R has only one non-prime place v, then $f\in R$ means that f does not have a pole other than v. If v is real and D_v is trivial, then $\mathrm{ord}_v(f)$ is even and $f(z)$ is positive near v. Since $RP(K)$ is compact, this implies that f, hence G, is bounded from below on $RP(K)$, and that F is bounded on $\{z\in RP(K)\,|\,G(z)\leqq 0\}$. If v is imaginary, then both F and G are bounded on $RP(K)$. So, similar discussions as the proof of Theorem 4.1 show that $^{\exists}w\in R$, $F+(1+FG)(2w+Gw^2)\geqq 0$ on $RP(K)$.

The proof for general K fails only because the condition "$\varphi\in R$ and $\varphi\geqq 0$ on $RP(K)$" does not imply $\varphi\in R^2+R^2$. Since Hasse's principle is satisfied, $\varphi\in K^2+K^2$ is assured, but $\varphi\in R^2+R^2$ is not concluded. We shall give a counter example for an elliptic function field $K=\boldsymbol{R}(X, y)$, $y^2=(X-a)(X-b)(X-c)$. If $\alpha\in\boldsymbol{R}$ is smaller than $\mathrm{Min}(a, b, c)$, then we have $X-\alpha>0$ on $RP(K)$. $X-\alpha$ has a double pole at the non-prime place v, while an element of $R^2+R^2=N_{K(\sqrt{-1})/K}(R+\sqrt{-1}R)$ should have $\mathrm{ord}_v\leqq -4$.

PROOF OF THEOREM 6 (2).

Let $K=\boldsymbol{R}(X)$ and suppose that R has only one non-prime place v.

If $R\neq\boldsymbol{R}[X]$, then v corresponds to an irreducible polynomial p, and $\varphi\in R$ is equivalent to $\varphi=g/p^{\nu}$, $g\in\boldsymbol{R}[X]$ and $\deg g\leqq\nu\deg p$. Here we can assume that ν is even. Then $\varphi\geqq 0$ on $RP(K)$ implies $g\geqq 0$ on $RP(K)$, so g is of even degree and can be written as $g=g_1^2+g_2^2$, $g_i\in\boldsymbol{R}[X]$, $\deg g_i\leqq(1/2)\deg g$. Therefore $\varphi=(g_1/p^{\nu/2})^2+(g_2/p^{\nu/2})^2$ and $\deg g_i\leqq(\nu/2)\deg p$, so that $\varphi\in R^2+R^2$.

From the remark above, this completes the proof of Theorem 6 (2).

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