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Asymptotic behavior of least energy solutions to a semilinear Dirichlet problem near the critical exponent

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1. Introduction

Let Ω be a smooth bounded domain in R^n with $n \geq 3$ and $p = (n+2)/(n-2)$ (the Sobolev exponent). Consider the problem

$$(1.1) \qquad \begin{cases} -\Delta u = u^{p-\varepsilon} \text{ in } \Omega, \\ u > 0 \text{ in } \Omega, \\ u\,|_{\partial\Omega} = 0, \end{cases}$$

where $\varepsilon > 0$. It is well-known that when $\varepsilon > 0$, problem (1.1) has at least one solution. On the other hand, when $\varepsilon = 0$, problem (1.1) becomes delicate. Pohozaev [**12**] derived the so-called "Pohozaev identity" for (1.1) and showed the nonexistence of solutions to (1.1) when Ω is star-shaped. In other cases, Bahri and Coron [**2**] showed that there exists a solution for equation (1.1) when Ω has a nontrivial topology, while Ding [**D**] constructed a solution to (1.1) when Ω is contractible. Here arises an interesting question: what happens to the solutions of (1.1) as $\varepsilon \to 0$? The first result was due to Atkinson and Peletier in [**1**]. They studied the radial case and characterized the asymptotic behavior of radial solutions. Later, Brezis and Peletier [**3**] used PDE methods to give another proof of the same result in spherical domains. Finally, Z. Han [**9**] (independently by O. Rey [**13**]) proved the same result in the general case, namely:

Theorem A. *Let u_ε be a solution of problem* (1.1) *and assume*

$$\frac{\int_\Omega |\nabla u_\varepsilon|^2}{\|u_\varepsilon\|^2_{L^{p+1-\varepsilon}(\Omega)}} = S + o(1) \quad as\ \varepsilon \to 0,$$

where S is the best Sobolev constant in $R^n : S = \pi n(n-2)(\Gamma(n/2)^{n/2}/\Gamma(n))$. Suppose u_ε assumes its maximum at x_ε. Then we have (after passing to a subsequence):

1. *There exists $x_0 \in \Omega$ such that as $\varepsilon \to 0$, $x_\varepsilon \to x_0$, $u_\varepsilon \to 0$ in $C^1_{\mathrm{loc}}(\bar{\Omega}\backslash\{x_0\})$ and $|\nabla u_\varepsilon|^2 \to (n(n-2))^{-(n-2)/4}\delta_{x_0}$ in the sense of distribution, where δ_{x_0} is the Dirac function at point x_0.*

2. *The* x_0 *above is a critical point of* φ, *i.e.* $\nabla\varphi(x_0) = 0$, *where* $\varphi(x) = g(x,x)$, $x \in \Omega$ *and* $g(x,y)$ *is the regular part of the Green's function* $G(x,y)$, *i.e.*

$$g(x,y) = G(x,y) - \frac{1}{(n-2)\sigma_n|x-y|^{n-2}},$$

where σ_n *is the area of the unit sphere in* R^n.

3. $\lim_{\varepsilon\to 0} \varepsilon\|u_\varepsilon\|^2_{L^\infty(\Omega)} = 2\sigma_n^2(n(n-2))^{n-1}S^{-n/2}|\varphi(x_0)|$, *where* x_0 *is the same as in* (1).

In this paper, we return to problem (1.1). We are concerned with a particular family of solutions to problem (1.1), namely, the least energy solution u_ε to problem (1.1). The purpose of this paper is to further locate the blow up point x_0 and to give a precise asymptotic expansion of the least energy solutions.

Before we state our result, we first give some definitions.

Define

$$J_\varepsilon = \inf\left\{\frac{\int_\Omega |\nabla u|^2}{\|u\|_{L^{p+1-\varepsilon}(\Omega)}} : u \in W_0^{1,2}(\Omega), u \not\equiv 0\right\}. \tag{1.2}$$

It is well known that J_ε is attained by a solution u_ε to problem (1.1). Furthermore, $J_\varepsilon = S + o(1)$ (Throughout this paper, $A = o(a)$ means $A/a \to 0$ as $\varepsilon \to 0$ and $A = O(a)$ means that $|A/a| \le C$). Let u_ε assume its maximum at some point x_ε. If some sequence $\{x_{\varepsilon_j}\}$ converges to some point x_0, then by Theorem A, $u_\varepsilon(x)$ blows up and concentrates at x_0. Moreover, x_0 is a critical point of $\varphi(x)$. Intuitively, one would conjecture that x_0 should be a global maximum point of $\varphi(x)$. In this paper, we shall confirm this conjecture. More precisely, we shall prove the following:

THEOREM 1.1. *Suppose* $n \ge 3$. *Let* u_ε *and* x_ε *be defined as above. Then* $\varphi(x_\varepsilon) \to \max_{x\in\Omega}\varphi(x)$ *as* $\varepsilon \to 0$.

To prove Theorem 1.1, we adopt the method developed by Ni and Takagi [**11**] and Wang [**16**]. In particular in [**16**], he proved that the maximum points of least energy solutions to the problem

$$\Delta u - k(x)u + u^{p-\varepsilon} = 0, \quad u > 0, \quad x \in R^N \tag{1.3}$$

approach a global minimum point of $k(x)$ as $\varepsilon \to 0$.

The basic idea in proving Theorem 1.1 is to get an asymptotic formula for J_ε as $\varepsilon \to 0$ (Propositions 2.1 and 3.4). In order to have this asymptotic expansion, we first rescale u_ε. Define μ_ε by $\mu_\varepsilon^{-2/(p-1-\varepsilon)} = \|u_\varepsilon\|_{L^\infty(\Omega)}$. Let $v_\varepsilon(y) = \mu_\varepsilon^{2/(p-1-\varepsilon)}u_\varepsilon(\mu_\varepsilon y + x_\varepsilon)$. Then $0 < v_\varepsilon \le 1$, $v_\varepsilon(0) = 1$ and

$$\begin{cases} \Delta v_\varepsilon(y) + v_\varepsilon^{p-\varepsilon}(y) = 0 & \text{in } \Omega_{\mu_\varepsilon}, \\ v_\varepsilon \mid_{\partial\Omega_{\mu_\varepsilon}} = 0, \end{cases} \tag{1.4}$$

where $\Omega_{\mu_\varepsilon} = \{y \mid \mu_\varepsilon y + x_\varepsilon \in \Omega\}$.

Then, by the elliptic interior estimates and the uniqueness result of [4] or [5], we have

$$v_\varepsilon \to U \quad \text{in } C^2_{\text{loc}}(R^n), \tag{1.5}$$

where $U(y) = 1/(1 + (|y|^2/n(n-2)))^{(n-2)/2}$ is the unique positive solution of

$$\Delta u + u^p = 0, \quad y \in R^n, \quad u(0) = 1. \tag{1.6}$$

By using a nice test function, we get an upper bound for J_ε. To get a lower bound, we expand v_ε in μ_ε. More precisely, the following asymptotic expansion of u_ε up to the second order is established.

THEOREM 1.2. *Suppose $n \geq 3$. Let u_ε and x_ε be defined as above. Then, as $\varepsilon \to 0$,*

$$v_\varepsilon(y) = u_\varepsilon^{2/(p-1-\varepsilon)} u_\varepsilon(x_\varepsilon + \mu_\varepsilon y) = U(y) + \mu_\varepsilon^{n-2}(H(x_\varepsilon, x_\varepsilon + \mu_\varepsilon y) + w(y) + o(1)) \tag{1.7}$$

where $H(x,y) = -(n-2)[n(n-2)]^{(n-2)/2}\sigma_n g(x,y)$, and w is the unique solution of (3.3)*; moreover, the term $o(1)$ is uniform in the ball $|y| < K/\mu_\varepsilon$ with K depending only on Ω.*

To prove Theorem 1.2, we note that the first approximation of v_ε should be U. However, since $v_\varepsilon \in W_0^{1,2}(\Omega_{\mu_\varepsilon})$, we write $v_\varepsilon = \mu_\varepsilon^{(n-2)/2} P_\Omega U + \mu_\varepsilon^{n-2}\phi_\varepsilon$, where $P_\Omega U$ is the projection of U from $W^{1,2}(\Omega)$ to $W_0^{1,2}(\Omega)$ (see (2.3)). We shall show that $\phi_\varepsilon \to w$ in $L^\infty(B_{(K/\mu_\varepsilon)}(x_0))$ where $B_{3K}(x_0) \subset \Omega$ and w is the unique solution of some elliptic equation involving the operator $L = \Delta + pU^{p-1}$. To this end, we need some regularity estimates and some properties of the operator L established in Wang [16].

This paper is organized as follows: in Section 2, we obtain an upper bound for J_ε (Proposition 2.1). In Section 3, we use Proposition 3.3 to get a lower bound for J_ε (Proposition 3.4) and show that Theorem 1.1 follows immediately from Propositions 2.1 and 3.4. Finally in Section 4, we prove Proposition 3.3. Theorem 1.2 follows easily.

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2. An upper bound for J_ε

The goal of this section is to choose a good test function to get an upper bound for J_ε. That is:

PROPOSITION 2.1. *If $n \geq 3$ and $x_\varepsilon \to x_0$ as $\varepsilon \to 0$, then for any point $x_1 \in \Omega$, we have:*

$$\begin{aligned}\mu_\varepsilon^{(n-2)\varepsilon/(p+1-\varepsilon)} J_\varepsilon \leq S + \mu_\varepsilon^{n-2}\Big\{ & H(x_1,x_1) S^{(2-n)/2}\int_{R^n} U^p(y)\,dy \\ & + \frac{n-2}{n} C(n,x_0) S^{(2-n)/2}\int_{R^n} U^{p+1}(y)\log U(y)\,dy \\ & - \frac{n}{(p+1)^2} C(n,x_0) S \log S\Big\} + o(\mu_\varepsilon^{n-2}),\end{aligned} \tag{2.1}$$

where $H(x,y) = -(n-2)(n(n-2))^{(n-2)/2}\sigma_n g(x,y)$ *and* $C(n,x_0) = 2\sigma_n^2(n(n-2))^{n-1}\cdot S^{-n/2}|\varphi(x_0)|$.

Before we prove it, we need some preparations.

We first recall that $\mu_\varepsilon^{-2/(p-1-\varepsilon)} = \|\mu_\varepsilon\|_{L^\infty}$. By Theorem A, we have

$$\varepsilon = C(n,x_0)\mu_\varepsilon^{n-2} + o(\mu_\varepsilon^{n-2}). \tag{2.2}$$

Let $U_{a,\lambda}(x) = \lambda^{(n-2)/2}/(1+(\lambda^2|x-a|^2/n(n-2)))^{(n-2)/2}$ and $a \in \Omega$. We define $P_\Omega U_{a,\lambda}$ to be the unique solution of

$$\begin{cases} \Delta w + U_{a,\lambda}^p = 0, \\ w > 0, \quad x \in \Omega, \\ w = 0 \quad \text{on } \partial\Omega. \end{cases} \tag{2.3}$$

We recall the following important lemma in Rey [**14**].

LEMMA 2.2. *Let* $\lambda = \mu_\varepsilon^{-1}$, *then*

$$U_{a,\mu_\varepsilon^{-1}}(x) = P_\Omega U_{a,\mu_\varepsilon^{-1}}(x) + \mu_\varepsilon^{(n-2)/2}H(a,x) + f_{\mu_\varepsilon}$$

where $f_{\mu_\varepsilon}(x) = O(\mu_\varepsilon^{(n+2)/2})$ *and* $\partial f_{\mu_\varepsilon}/\partial x_i = O(\mu_\varepsilon^{(n+2)/2}/d(x,\partial\Omega))$.

We are ready to prove Proposition 2.1.

PROOF OF PROPOSITION 2.1: Let $x_1 \in \Omega$ and $u(x) = P_\Omega U_{x_1,\mu_\varepsilon^{-1}}(x)$. Set $\Omega_\varepsilon = \{y \mid x_1 + \mu_\varepsilon y \in \Omega\}$. Then

$$\begin{aligned}
\int_\Omega |\nabla u|^2\,dx &= \int_\Omega |\nabla P_\Omega U_{x_1,\mu_\varepsilon^{-1}}(x)|^2\,dx \\
&= \int_\Omega U_{x_1,\mu_\varepsilon^{-1}}^p(x) P_\Omega U_{x_1,\mu_\varepsilon^{-1}}(x)\,dx \\
&= \int_\Omega U_{x_1,\mu_\varepsilon^{-1}}^{p+1}(x)\,dx - \mu_\varepsilon^{(n-2)/2}\int_\Omega U_{x_1,\mu_\varepsilon^{-1}}^p(x)H(x_1,x)\,dx - \int_\Omega U_{x_1,\mu_\varepsilon^{-1}}^p(x) f_{\mu_\varepsilon}(x)\,dx \\
&= \int_{\Omega_\varepsilon} U^{p+1}(y)\,dy - \mu_\varepsilon^{n-2}\int_{\Omega_\varepsilon} U^p(y)H(x_1,x_1+\mu_\varepsilon y)\,dy \\
&\quad + O(\mu_\varepsilon^{(n+2)/2})\int_{\Omega_\varepsilon} U_{x_1,\mu_\varepsilon^{-1}}^p(x)\,dx \\
&= \int_{R^n} U^{p+1}(y)\,dy - \mu_\varepsilon^{n-2}\int_{\Omega_\varepsilon} U^p(y)H(x_1,x_1+\mu_\varepsilon y)\,dy + O(\mu_\varepsilon^n) \\
&= S^{n/2} - \mu_\varepsilon^{n-2}\int_{\Omega_\varepsilon} U^p(y)H(x_1,x_1+\mu_\varepsilon y)\,dy + O(\mu_\varepsilon^n).
\end{aligned}$$

But,

$$\begin{aligned}
&\int_{\Omega_\varepsilon} U^p(y)H(x_1,x_1+\mu_\varepsilon y)\,dy \\
&= \int_{\Omega_\varepsilon} U^p(y)[H(x_1,x_1+\mu_\varepsilon y) - H(x_1,x_1)]\,dy + H(x_1,x_1)\int_{\Omega_\varepsilon} U^p(y)\,dy \\
&= H(x_1,x_1)\int_{R^n} U^p(y)\,dy + O(\mu_\varepsilon).
\end{aligned}$$

For, $\int_{R^n} U^p(y)|y|\,dy < \infty$.

Thus we have:

$$\int_\Omega |\nabla u|^2\,dx = S^{n/2} - \mu_\varepsilon^{n-2} H(x_1, x_2)\int_{R^n} U^p(y)\,dy + o(\mu_\varepsilon^{n-2}).$$

On the other hand, we have:

$$\begin{aligned}
&\mu_\varepsilon^{(2-n)\varepsilon/(p+1-\varepsilon)}\left[\int_\Omega (P_\Omega U_{x_1,\mu_\varepsilon^{-1}})^{p+1-\varepsilon}\,dx\right]^{2/(p+1-\varepsilon)}\\
&= \left[\int_{\Omega_\varepsilon} (U - \mu_\varepsilon^{n-2} H(x_1, x_1+\mu_\varepsilon y) - \mu_\varepsilon^{(n-2)/2} f_{\mu_\varepsilon})^{p+1-\varepsilon}\,dy\right]^{2/(p+1-\varepsilon)}\\
&= \left[\int_{R^n} U^{p+1-\varepsilon}\,dy - (p+1)\mu_\varepsilon^{n-2}\int_{R^n} U^p H(x_1,x_1)\,dy + o(\mu_\varepsilon^{n-2})\right]^{2/(p+1-\varepsilon)}\\
&= \left[\int_{R^n} U^{p+1}\,dy - C(n,x_0)\mu_\varepsilon^{n-2}\int_{R^n} U^{p+1}\log U\,dy\right.\\
&\quad\left. - (p+1)H(x_1,x_1)\mu_\varepsilon^{n-2}\int_{R^n} U^p(y)\,dy + o(\mu_\varepsilon^{n-2})\right]^{2/(p+1-\varepsilon)}\\
&= \left[\int_{R^n} U^{p+1}\,dy - C(n,x_0)\mu_\varepsilon^{n-2}\int_{R^n} U^{p+1}\log U\,dy\right.\\
&\quad\left. - (p+1)H(x_1,x_1)\mu_\varepsilon^{n-2}\int_{R^n} U^p(y)\,dy + o(\mu_\varepsilon^{n-2})\right]^{2/(p+1)}\\
&\quad + C(n,x_0)\frac{2}{(p+1)^2}\left(\int_{R^n} U^{p+1}\,dy\right)^{2/(p+1)}\log\left(\int_{R^n} U^{p+1}\,dy\right)\mu_\varepsilon^{n-2} + o(\mu_\varepsilon^{n-2})\\
&= \left(\int_{R^n} U^{p+1}\,dy\right)^{2/(p+1)} - \frac{2}{p+1}\left(\int_{R^n} U^{p+1}\,dy\right)^{2/(p+1)-1}\left[C(n,x_0)\mu_\varepsilon^{n-2}\int_{R^n} U^{p+1}\log U\,dy\right.\\
&\quad\left. + (p+1)H(x_1,x_1)\mu_\varepsilon^{n-2}\int_{R^n} U^p(y)\,dy\right]\\
&\quad + C(n,x_0)\frac{2}{(p+1)^2}\left(\int_{R^n} U^{p+1}\,dy\right)^{2/(p+1)}\log\left(\int_{R^n} U^{p+1}\,dy\right)\mu_\varepsilon^{n-2} + o(\mu_\varepsilon^{n-2}).
\end{aligned}$$

Hence:

$$\begin{aligned}
&\mu_\varepsilon^{(2-n)\varepsilon/(p+1-\varepsilon)}\left[\int_\Omega (P_\Omega U_{x_1,\mu_\varepsilon^{-1}})^{p+1-\varepsilon}\,dx\right]^{2/(p+1-\varepsilon)}\\
&= (S^{n/2})^{2/(p+1)} - \mu_\varepsilon^{n-2}C(n,x_0)\frac{2}{p+1}(S^{n/2})^{2/(p+1)-1}\int_{R^n} U^{p+1}\log U\,dy\\
&\quad + \mu_\varepsilon^{n-2}C(n,x_0)\frac{2}{(p+1)^2}(S^{n/2})^{2/(p+1)}\log\left(\int_{R^n} U^{p+1}\,dy\right)\\
&\quad + (p+1)\mu_\varepsilon^{n-2}H(x_1,x_1)\int_{R^n} U^p\,dy\,\frac{2}{p+1}(S^{n/2})^{2/(p+1)-1} + o(\mu_\varepsilon^{n-2}).
\end{aligned}$$

Straightforward computations show that,

$$\begin{aligned}
\mu_\varepsilon^{(n-2)\varepsilon/(p+1-\varepsilon)} J_\varepsilon &\le \mu_\varepsilon^{(n-2)\varepsilon/(p+1-\varepsilon)} \frac{\int_\Omega |\nabla u|^2\,dx}{\left[\int_\Omega u^{p+1-\varepsilon}\,dx\right]^{2/(p+1-\varepsilon)}} \\
&= \frac{S^{n/2}}{(S^{n/2})^{2/(p+1)}} + \mu_\varepsilon^{n-2}\Bigg[H(x_1,x_1)\frac{\int_{R^n} U^p(y)\,dy}{(S^{n/2})^{2/(p+1)}} \\
&\quad + C(n,x_0)\frac{2}{p+1}(S^{n/2})^{-2/(p+1)}\int_{R^n} U^{p+1}\log U\,dy \\
&\quad - C(n,x_0)\frac{2}{(p+1)^2}(S^{n/2})^{1-2/(p+1)}\log\left(\int_{R^n} U^{p+1}\,dy\right)\Bigg] + o(\mu_\varepsilon^{n-2}) \\
&= S + \mu_\varepsilon^{n-2}\Bigg\{ H(x_1,x_1)S^{(2-n)/2}\int_{R^n} U^p(y)\,dy \\
&\quad + \frac{n-2}{n}C(n,x_0)S^{(2-n)/2}\int_{R^n} U^{p+1}(y)\log U(y)\,dy \\
&\quad - \frac{n}{(p+1)^2}C(n,x_0)S\log S\Bigg\} + o(\mu_\varepsilon^{n-2}). \qquad \square
\end{aligned}$$

3. Proofs of Theorems 1.1 and 1.2

In this section, we shall prove Theorem 1.1. To this end, we recall that $v_\varepsilon(y) = \mu_\varepsilon^{2/(p-1-\varepsilon)} u_\varepsilon(\mu_\varepsilon y + x_\varepsilon)$. Then v_ε satisfies:

$$(3.1)\qquad \begin{cases} \Delta v_\varepsilon(y) + v_\varepsilon^{p-\varepsilon}(y) = 0 & \text{in } \Omega_{\mu_\varepsilon}, \\ v_\varepsilon > 0 & \text{in } \Omega_{\mu_\varepsilon}, \\ v_\varepsilon\,|_{\partial\Omega_{\mu_\varepsilon}} = 0, \end{cases}$$

where $\Omega_{\mu_\varepsilon} = \{y \mid \mu_\varepsilon y + x_\varepsilon \in \Omega\}$.

We are in need of the following lemma in Han [9]:

LEMMA 3.1. *$u_\varepsilon(x) \le \Lambda U_{x_\varepsilon,\mu_\varepsilon^{-1}}$, that is: $v_\varepsilon(y) \le \Lambda U(y)$, for some Λ.*

Let $v_\varepsilon(y) = \mu_\varepsilon^{(n-2)/2} P_\Omega U_{x_\varepsilon,\mu_\varepsilon^{-1}}(x) + \mu_\varepsilon^{n-2}\phi_\varepsilon(y)$, where $x = \mu_\varepsilon y + x_\varepsilon$. Then ϕ_ε satisfies:

$$(3.2)\qquad \begin{cases} \Delta\phi_\varepsilon(y) + pU^{p-1}\phi_\varepsilon + F(\phi_\varepsilon) = 0, & \text{in } \Omega_{\mu_\varepsilon}, \\ \phi_\varepsilon(y)\,|_{\partial\Omega_{\mu_\varepsilon}} = 0, \end{cases}$$

where $F(\phi_\varepsilon(y)) = [v_\varepsilon^{p-\varepsilon} - U^p - \mu_\varepsilon^{n-2} pU^{p-1}\phi_\varepsilon]/(\mu_\varepsilon^{n-2})$.

We now give the following estimate for $F(\phi_\varepsilon(y))$.

LEMMA 3.2. *If $n > 5$, then we have*

$$|F(\phi_\varepsilon)| \le C(U^{p-1-\varepsilon}(|\log U| + 1) + |\phi_\varepsilon|\,|v_\varepsilon - U|^{p-1-\varepsilon} + \mu_\varepsilon^4 + |U^{p-1-\varepsilon}|).$$

If $n \le 5$, then we have

$$|F(\phi_\varepsilon)| \le C(U^{p-1-\varepsilon}(|\log U| + 1) + U^{p-2-\varepsilon}|\phi_\varepsilon|\,|v_\varepsilon - U| + \mu_\varepsilon^{n-2} + |U^{p-1-\varepsilon}|).$$

PROOF: First, by Lemma 2.1, we have

$$\begin{aligned}\mu_\varepsilon^{(n-2)/2} P_\Omega U_{x_\varepsilon,\mu_\varepsilon^{-1}}(x) &= U(y) - \mu_\varepsilon^{n-2} H(x_\varepsilon, x_\varepsilon + \mu_\varepsilon y) + o(\mu_\varepsilon^{n-2}) \\ &= U(y) - \mu_\varepsilon^{n-2} H(x_0, x_0 + \mu_\varepsilon y) + o(\mu_\varepsilon^{n-2}).\end{aligned}$$

Using this, we obtain

$$\begin{aligned}|\mu_\varepsilon^{n-2} F(\phi_\varepsilon)| &= |v_\varepsilon^{p-\varepsilon} - U^p - \mu_\varepsilon^{n-2} p U^{p-1} \phi_\varepsilon| \\ &\le |v_\varepsilon^{p-\varepsilon} - \mu_\varepsilon^{n-2}(p-\varepsilon) U^{p-1-\varepsilon}(\phi_\varepsilon - H(x_0, x_0 + \mu_\varepsilon y) + o(\mu_\varepsilon^{n-2})) - U^{p-\varepsilon}| \\ &\quad + |U^{p-\varepsilon} - U^p| \\ &\quad + \mu_\varepsilon^{n-2} |p U^{p-1} \phi_\varepsilon - (p-\varepsilon) U^{p-1-\varepsilon}(\phi_\varepsilon - H(x_0, x_0 + \mu_\varepsilon y) + o(\mu_\varepsilon^{n-2}))| \\ &= I_1 + I_2 + I_3,\end{aligned}$$

where I_1, I_2, I_3 are defined by the last equality.

We estimate I_2, I_3 as follows:

$$\begin{aligned}|I_2| &\le C\mu_\varepsilon^{n-2} U^{p-\varepsilon} |\log U|, \\ |I_3| &\le C\mu_\varepsilon^{n-2}(\mu_\varepsilon^{n-2} |\phi_\varepsilon| U^{p-1-\varepsilon}(|\log U| + 1) + U^{p-1-\varepsilon}).\end{aligned}$$

For I_1, by using the following inequality (that is where we need to treat the two cases $n > 5$ and $n \le 5$ separately):

$$|(1+\xi)_+^t - 1 - t\xi| \le C|\xi|^t$$

for $1 \le t \le 2$ and

$$|(1+\xi)_+^t - 1 - t\xi| \le C|\xi|^2$$

for $t \ge 2$, we have

$$\begin{aligned}|I_1| &\le C\mu_\varepsilon^{n+2} |\phi_\varepsilon - H(x_0, x_0 + \mu_\varepsilon y) + o(\mu_\varepsilon^{n-2})|^{p-\varepsilon} \\ &\le C\mu_\varepsilon^{n+2}(|\phi_\varepsilon|^{p-\varepsilon} + |H|^{p-\varepsilon}) \\ &\le C[\mu_\varepsilon^{n+2} |\phi_\varepsilon| (\mu_\varepsilon^4 + |v_\varepsilon - U|^{p-1-\varepsilon}) + \mu_\varepsilon^{n+2}]\end{aligned}$$

for $n > 5$ and

$$\begin{aligned}|I_1| &\le C U^{p-\varepsilon-2} \mu_\varepsilon^{2(n-2)} |\phi_\varepsilon - H(x_0, x_0 + \mu_\varepsilon y) + o(\mu_\varepsilon^{n-2})|^2 \\ &\le C U^{p-\varepsilon-2} \mu_\varepsilon^{2(n-2)} (|\phi_\varepsilon|^2 + |H|^2) \\ &= C U^{p-\varepsilon-2} [\mu_\varepsilon^{n-2} |\phi_\varepsilon| \, |v_\varepsilon - U| + \mu_\varepsilon^{2(n-2)}]\end{aligned}$$

for $n \le 5$. □

PROPOSITION 3.3. *Assume that* $n \ge 3$ *and that* $B_{3K}(x_0) \subset \Omega$ *for some positive constant* $K > 0$. *Then* $\phi_\varepsilon \to w$ *in* $L^\infty(B_{K/\mu_\varepsilon}(0))$ *as* $\varepsilon \to 0$, *where* w *is a bounded* *solution of*

$$\Delta w + pU^{p-1}w - C(n,x_0)U^p \log U - pH(x_0,x_0)U^{p-1} = 0, \quad \text{in } R^n \tag{3.3}$$

and $w \in W^{2,s}(R^n)$ *for* $s > n/3$.

Assuming Proposition 3.3 now, we show that

PROPOSITION 3.4. *Let* J_ε *be defined by* (1.2), *then*

$$\mu_\varepsilon^{(n-2)\varepsilon/(p+1-\varepsilon)} J_\varepsilon = S + \mu_\varepsilon^{n-2}\Big\{H(x_0,x_0)S^{(2-n)/2}\int_{R^n} U^p(y)\,dy + \frac{n-2}{n}C(n,x_0)S^{(2-n)/2}\int_{R^n} U^{p+1}(y)\log U(y)\,dy - \frac{n}{(p+1)^2}C(n,x_0)S\log S\Big\} + o(\mu_\varepsilon^{n-2}).$$

PROOF: We begin with:

$$\begin{aligned}
\mu_\varepsilon^{(n-2)\varepsilon/(p+1-\varepsilon)} J_\varepsilon &= \mu_\varepsilon^{(n-2)\varepsilon/(p+1-\varepsilon)} J_\varepsilon(u_\varepsilon)\\
&= \mu_\varepsilon^{(n-2)\varepsilon/(p+1-\varepsilon)} \frac{\int_\Omega |\nabla u_\varepsilon|^2\,dx}{\left[\int_\Omega u_\varepsilon^{p+1-\varepsilon}\,dx\right]^{2/(p+1-\varepsilon)}}\\
&= \mu_\varepsilon^{(n-2)\varepsilon/(p+1-\varepsilon)}\left[\int_\Omega u_\varepsilon^{p+1-\varepsilon}\,dx\right]^{1-2/(p+1-\varepsilon)}\\
&= \left(\int_{\Omega_{\mu_\varepsilon}} v_\varepsilon^{p+1-\varepsilon}\,dy\right)^{1-2/(p+1-\varepsilon)}\\
&= \left(\int_{B_{K/\mu_\varepsilon}} v_\varepsilon^{p+1-\varepsilon}\,dy + O(\mu_\varepsilon^n)\right)^{1-2/(p+1-\varepsilon)}\\
&= \Big\{\int_{B_{K/\mu_\varepsilon}} (U + \mu_\varepsilon^{n-2}(\phi_\varepsilon - H(x_0,x_0+\mu_\varepsilon y))\\
&\quad + o(\mu_\varepsilon^{n-2}))^{p+1-\varepsilon}\,dy\Big\}^{1-2/(p+1-\varepsilon)} + o(\mu_\varepsilon^{n-2})\\
&= \Big\{\int_{R^n} U^{p+1-\varepsilon}\,dy + (p+1)\int_{B_{K/\mu_\varepsilon}} \mu_\varepsilon^{n-2}U^p(\phi_\varepsilon - H(x_0,x_0+\mu_\varepsilon y))\,dy\\
&\quad + o(\mu_\varepsilon^{n-2})\Big\}^{(p-1-\varepsilon)/(p+1-\varepsilon)} + o(\mu_\varepsilon^{n-2})\\
&= \Big\{\int_{R^n} U^{p+1-\varepsilon}\,dy - C(n,x_0)\mu_\varepsilon^{n-2}\int_{R^n} U^{p+1}\log U\,dy\\
&\quad + (p+1)\int_{B_{K/\mu_\varepsilon}} \mu_\varepsilon^{n-2}U^p(\phi_\varepsilon - H(x_0,x_0+\mu_\varepsilon y))\,dy\\
&\quad + o(\mu_\varepsilon^{n-2})\Big\}^{(p-1-\varepsilon)/(p+1-\varepsilon)} + o(\mu_\varepsilon^{n-2})
\end{aligned}$$

$$= \left\{ \int_{R^n} U^{p+1}\,dy - C(n,x_0)\mu_\varepsilon^{n-2} \int_{R^n} U^{p+1} \log U\,dy + (p+1) \int_{R^n} \mu_\varepsilon^{n-2} U^p w\,dy \right.$$
$$\left. - (p+1) \int_{R^n} \mu_\varepsilon^{n-2} U^p H(x_0,x_0)\,dy + o(\mu_\varepsilon^{n-2}) \right\}^{(p-1-\varepsilon)/(p+1-\varepsilon)} + o(\mu_\varepsilon^{n-2}).$$

But, by equation (3.3), we have

$$\int_{R^n} U^p w\,dy = \frac{1}{p-1}\left(C(n,x_0) \int_{R^n} U^{p+1} \log U\,dy + pH(x_0,x_0) \int_{R^n} U^p\,dy \right).$$

Hence,

$$\mu_\varepsilon^{(n-2)\varepsilon/(p+1-\varepsilon)} J_\varepsilon = \left(\int_{R^n} U^{p+1}\,dy + C(n,x_0)\mu_\varepsilon^{n-2} \frac{2}{p-1} \int_{R^n} U^{p+1} \log U\,dy \right.$$
$$\left. + \frac{p+1}{p-1} \int_{R^n} \mu_\varepsilon^{n-2} U^p H(x_0,x_0)\,dy \right)^{(p-1)/(p+1)}$$
$$- \frac{2}{(p+1)^2} C(n,x_0)\mu_\varepsilon^{n-2} \int_{R^n} U^{p+1}\,dy \log\left(\int_{R^n} U^{p+1}\,dy \right) + o(\mu_\varepsilon^{n-2})$$
$$= \text{Right hand side of Proposition 3.4.}$$

We now in a position to prove Theorems 1.1 and 1.2. In fact, Theorem 1.2 follows easily from Proposition 3.3. By Propositions 2.1 and 3.4, we immediately get $H(x_0,x_0) \le H(x_1,x_1)$ for any $x_1 \in \Omega$. Hence $\varphi(x_0) = \max_{x\in\Omega}\varphi(x)$, which proves Theorem 1.1 □

4. Proof of Proposition 3.3

The purpose of this section is to prove Proposition 3.3. To simplify our proof, we assume that $n \ge 5$. By making minor modifications, one can see that the same proof works for the case $n \le 5$.

There are some prelimilaries to be done before we go into the proof. First of all, we recall an important property of the linearized operator $L = \Delta + pU^{p-1}$.

Lemma 4.1 (Lemma 2.3 in Wang [**16**]). *If the domain of L is* $W^{2,r}(R^n)$, *where* $n/(n-2) < r < \infty$, *then* $Ker(L) = X = span\{e_1,\dots,e_n,e_{n+1}\}$ *where* $e_i = (\partial U/\partial x_i)$, $i = 1,\dots,n$ *and* $e_{n+1} = x\cdot\nabla U + ((n-2)/2)U$.

The following lemma plays a crucial role.

Lemma 4.2 (1). *Let u be the solution of*

$$\begin{cases} -\Delta u(y) = f(y), & \text{in } \Omega_{\mu_\varepsilon}, \\ u\,|_{\partial\Omega_{\mu_\varepsilon}} = 0. \end{cases} \tag{4.1}$$

Then:

$$\|u\|_{W^{2,r}(\Omega_{\mu_\varepsilon})} \le C(\|f\|_{L^q(\Omega_{\mu_\varepsilon})} + \|f\|_{L^r(\Omega_{\mu_\varepsilon})}), \tag{4.2}$$

where C is a constant independent of μ_ε *and u*, $1/q = 1/r + 2/n$ *and* $r > 2$.

(2). *Let* $k(x) \in C^2(\bar{\Omega}_{\mu_\varepsilon})$. *Then we can extend it to a function* $K(x) \in C_0^2(R^n)$ *in such a way that*

$$\|K(x)\|_{W^{2,p}(R^n)} \leq C\|k(x)\|_{W^{2,p}(\Omega_{\mu_\varepsilon})},$$

where C *is independent of* k *and* μ_ε, $p > 1$.

PROOF: Without loss of generality, in the following proof, we may assume $x_\varepsilon = 0$ (since we can always make a translation which does not change the inequality).

(1). First of all, by the well-known regularity theorem (see, e.g. Corollary 9.10 of [**8**]) and a simple scaling argument, we have

$$\|D^2 u\|_{L^r(\Omega_{\mu_\varepsilon})} \leq C\|f\|_{L^r(\Omega_{\mu_\varepsilon})}, \tag{4.3}$$

where C is independent of μ_ε and u.

Secondly, by integration by parts, we can prove that

$$\|\nabla u\|_{L^r(\Omega_{\mu_\varepsilon})} \leq C(n)\|u\|^{1/2}_{L^r(\Omega_{\mu_\varepsilon})}\|D^2 u\|^{1/2}_{L^r(\Omega_{\mu_\varepsilon})}.$$

Thus, we have

$$\|\nabla u\|_{L^r(\Omega_{\mu_\varepsilon})} \leq C(n)(\|u\|_{L^r(\Omega_{\mu_\varepsilon})} + \|D^2 u\|_{L^r(\Omega_{\mu_\varepsilon})}). \tag{4.4}$$

Combining (4.3) and (4.4), we get

$$\|u\|_{W^{2,r}(\Omega_{\mu_\varepsilon})} \leq C(\|u\|_{L^r(\Omega_{\mu_\varepsilon})} + \|f\|_{L^r(\Omega_{\mu_\varepsilon})}) \tag{4.5}$$

where C is a constant independent of μ_ε and u.

Finally, we extend f equal to 0 outside Ω_{μ_ε} and denote it by f_1. Let $u_1 = \int_{R^n} \Gamma(x-y)|f_1(y)|\,dy$, where Γ is the fundamental solution of $-\Delta$, then by Maximum Principle, we have: $|u| \leq u_1$ on Ω_{μ_ε}.

Therefore, $\|u\|_{L^r(\Omega_{\mu_\varepsilon})} \leq \|u_1\|_{L^r(\Omega_{\mu_\varepsilon})} \leq \|u_1\|_{L^r(R^n)}$.

By virtue of the Hardy-Littlewood-Sobolev inequality ([**15**]), we have

$$\|u_1\|_{L^r(R^n)} \leq C(n,q)\|f_1\|_{L^r(R^n)} \leq C(n,q)\|f\|_{L^q(\Omega_{\mu_\varepsilon})}. \tag{4.6}$$

Thus, we obtain

$$\|u\|_{L^r(\Omega_{\mu_\varepsilon})} \leq C(n,q)\|f\|_{L^q(\Omega_{\mu_\varepsilon})}. \tag{4.7}$$

Now from (4.5) and (4.7), we have (4.1).

(2). For each point $P \in \partial\Omega$, we can find a homeomorphism Ψ_p and a neighborhood $U_p \subset \bar{\Omega}$ such that $P \in U_p$ and $\Psi_p : U_p \to B^+_{r_p}$. From $\{U_p \,|\, p \in \partial\Omega\}$, we can select a finite cover of $\partial\Omega$ and denote it by $\{U_1, \ldots, U_N\}$, we denote the corresponding homomeophism as $\{\Psi_1, \ldots, \Psi_N\}$. Let $U_0 = \Omega$, then $\{U_0, U_1, \ldots, U_N\}$ forms a finite cover of $\bar{\Omega}$. Let $\{\chi_0, \ldots, \chi_N\}$ be a partition of unity subordinate to the open cover $\{U_0, U_1, \ldots, U_N\}$. Hence we have $\sum_{i=0}^{i=N} \chi_i = 1$, for $x \in \bar{\Omega}$. It is easy to see that $\{\mu_\varepsilon^{-1} U_0, \mu_\varepsilon^{-1} U_1, \ldots, \mu_\varepsilon^{-1} U_N\}$ forms a finite cover of $\overline{\Omega_{\mu_\varepsilon}}$ and $\{\chi_i(y/\mu_\varepsilon), i = 1, \ldots, N\}$ is a partion of unity subordinate to this open cover. Then, (4.2) follows from the proof of Lemma 5.2 of Friedman [7]. □

Now we explain the plan of the proof of Proposition 3.3. Following the strategy of [11] and [16], we first prove that $\|\phi_\varepsilon\|_{L^r(\Omega_{\mu_\varepsilon})}$ is bounded for $r > n$. Then let $w_\varepsilon = \chi(\mu_\varepsilon y)w(y)$ where $\chi(x) = 1$ when $x \in B_K(x_0) \subset \Omega$ and $\chi(x) = 0$ when $x \notin B_{3K}(x_0) \subset \Omega$, we show that $\|\phi_\varepsilon - w_\varepsilon\|_{W^{2,r}(\Omega_{\mu_\varepsilon})} = o(1)$, which, by Sobolev Imbedding Theorem, proves Proposition 3.3.

Lemma 4.3 *Let $n < r < \infty$. Then $\|\phi_\varepsilon\|_{L^r(\Omega_{\mu_\varepsilon})} \le C(r)$.*

Proof: Suppose on the contrary, there exists a sequence of $\varepsilon_j \to 0$ such that $\|\phi_{\varepsilon_j}\|_{L^r(\Omega_{\mu_{\varepsilon_j}})} \to \infty$.

Let $M_j = \|\phi_{\varepsilon_j}\|_{L^r(\Omega_{\mu_{\varepsilon_j}})}$, $\Psi_j = \phi_{\varepsilon_j}/M_j$. We denote $\Omega_{\mu_{\varepsilon_j}}$ as Ω_j, μ_{ε_j} as μ_j and ϕ_{ε_j} as ϕ_j, etc. Then Ψ_j satisfies

$$\text{(4.8)} \qquad \begin{cases} \Delta\Psi_j(y) + pU^{p-1}\Psi_j + F(\phi_j)/M_j = 0, & \text{in } \Omega_j, \\ \Psi_j(y)\,|_{\partial\Omega_j} = 0. \end{cases}$$

We divide our proof into the following steps:

Step 1: we show that $\|\Psi_j\|_{W^{2,r}(\Omega_j)}$ is bounded.

Step 2: we extend Ψ_j to R^n and prove that $\Psi_j \to 0$ weakly in $W^{2,r}(R^n)$.

Step 3: we prove that $\|\Psi_j\|_{W^{2,r}(R^n)} = o(1)$, which gives a contradiction (because $\|\Psi\|_{L^r(\Omega_i)} = 1$).

Now we begin to prove step 1. In fact, by (4.5), we just need to estimate $\|F(\phi_j)/M_j\|_{L^r(\Omega_j)}$.

But by Lemma 3.2,

$$\begin{aligned} \|F(\phi_j)\|_{L^r(\Omega_j)} &\le C(\|U^{p-1-\varepsilon}(|\log U| + 1)\|_{L^r(\Omega_j)} \\ &\quad + \|\,|\phi_{\varepsilon_j}\|v_{\varepsilon_j} - U|^{p-1-\varepsilon_j}\|_{L^r(\Omega_j)} + \|\mu_{\varepsilon_j}^4\|_{L^r(\Omega_j)} + \|U^{p-1-\varepsilon_j}\|_{L^r(\Omega_j)}) \\ &\le C(1 + \|\phi_j\|_{L^r(\Omega_j)}). \end{aligned}$$

This gives rise to

$$\|\Psi_j\|_{W^{2,r}(\Omega_j)} \le C.$$

Next, from Lemma 4.2, we can extend Ψ_j to R^n in such a way that

$$\text{(4.9)} \qquad \|\Psi_j\|_{W^{2,r}(R^n)} \le C\|\Psi_j\|_{W^{2,r}(\Omega_j)}.$$

By Sobolev Imbedding Theorem, we have,

$$\text{(4.10)} \qquad \|\Psi_j\|_{L^\infty(R^n)} \le C.$$

Thus there exists a function $z \in W^{2,r}(R^n)$ such that $\Psi_j \to z$ weakly in $W^{2,r}(R^n)$ and $\Psi_j \to z$ in $C^1_{\mathrm{loc}}(R^n)$ along some subsequences.

To finish the second step, we just need to show that $z = 0$. To this end, we estimate $\|F(\phi_\varepsilon)/M_j\|_{L^\infty(\Omega_j)}$. By (4.10) and Lemma 3.2, it is easy to see that $\|F(\phi_\varepsilon)/M_j\|_{L^\infty(\Omega_j)} \to 0$ (because $\|v_j - U\|_{L^\infty(\Omega_j)} \to 0$). Hence z is a weak (thus classical) solution of the following equation:

$$\begin{cases} \Delta\Psi(y) + pU^{p-1}\Psi(y) = 0, & \text{in } R^n \\ \Psi \in W^{2,r}(R^n),\ n < r. \end{cases} \tag{4.11}$$

By Lemma 4.1, $z \in X$. That is

$$z = \sum_{i=1}^{n+1} a_i e_i$$

for some constants $a_i, i = 1, 2, \dots, n+1$.

But note that by definition, $\Psi_j(0) = H(x_0, x_0)/M_j + o(1)$, $\nabla\Psi_j(0) = o(1)$ (since $v_j(0) = 1 = \max_{y \in \Omega_j} v_j$). Thus, we have, $\Psi(0) = 0$, $\nabla\Psi(0) = 0$. Therefore,

$$\sum_{i=1}^{n+1} a_i e_i(0) = 0,$$
$$\sum_{i=1}^{n+1} a_i \nabla e_i(0) = 0.$$

Observe that $e_i(0) = \partial U/\partial x_i(0) = 0, i = 1, 2, \dots, n$, $e_{n+1}(0) = (n-2)/2$, $\nabla e_{n+1}(0) = 0$ and that $\nabla e_1(0), \dots, \nabla e_n(0)$ are linearly independent. Therefore, we get $a_i = 0$, $i = 1, 2, \dots, n+1$.

Hence $z = 0$ and $\Psi_j \to 0$ weakly in $W^{2,r}(R^n)$, which completes step 2.

We now show that $\|\Psi_j\|_{W^{2,r}(\Omega_j)} = o(1)$. By Lemma 4.2, we just need to estimate $\|pU^{p-1}\Psi_j\|_{L^q(\Omega_j)}, \|pU^{p-1}\Psi_j\|_{L^r(\Omega_j)}, \|F(\phi_j)\|_{L^q(\Omega_j)}$ and $\|F(\phi_j)\|_{L^r(\Omega_j)}$. We begin with

$$\begin{aligned} \|F(\phi_j)\|_{L^q(\Omega_j)} &\le C(\|U^{p-1-\varepsilon_j}(|\log U| + 1)\|_{L^q(\Omega_j)} \\ &\quad + \|\phi_j|v_j - U|^{p-1-\varepsilon_j}\|_{L^q(\Omega_j)} + \mu_{\varepsilon_j}^{4-n/q} + \|U^{p-1-\varepsilon_j}\|_{L^q(\Omega_j)}) \\ &\le C(1 + \|\phi_j\|_{L^r(\Omega_j)}\| \, |v_j - U|^{p-1-\varepsilon_j}\|_{L^{n/2}(\Omega_j)}) \\ &\le C(1 + o(1)\|\phi_j\|_{L^r(\Omega_j)}). \end{aligned}$$

For, $\| \, |v_j - U|^{p-1-\varepsilon_j}\|_{L^{n/2}(\Omega_j)} = o(1)$, by Lebesgue's Dominated Convergence Theorem and $q > n/3$.

$$\begin{aligned} \|F(\phi_j)\|_{L^r(\Omega_j)} &\le C(\|U^{p-1-\varepsilon_j}(|\log U| + 1)\|_{L^r(\Omega_j)} \\ &\quad + \| \, |\phi_j| \, |v_j - U|^{p-1-\varepsilon_j}\|_{L^r(\Omega_j)} + \mu_{\varepsilon_j}^{4-n/r} + \|U^{p-1-\varepsilon_j}\|_{L^r(\Omega_j)}) \\ &\le C(1 + \|\phi_j\|_{L^\infty(\Omega_j)}\| \, |v_j - U|^{p-1-\varepsilon_j}\|_{L^r(\Omega_j)}) \\ &\le C(1 + o(1)\|\phi_j\|_{L^\infty(\Omega_j)}). \end{aligned}$$

Hence

$$\|F(\phi_j)/M_j\|_{L^q(\Omega_j)} + \|F(\phi_j)/M_j\|_{L^r(\Omega_j)} = o(1). \tag{4.12}$$

Now let $f_j = pU^{p-1}\Psi_j$. Let R be a fixed number. Then

$$\begin{aligned} \|f_j\|_{L^q(\Omega_j)} &\le \left(\int_{\Omega_j \cap B_R(0)} U^{(p-1)q}|\Psi_j|^q\right)^{1/q} + \left(\int_{\Omega_j \cap B_R^c(0)} U^{(p-1)q}|\Psi_j|^q\right)^{1/q} \\ &\le C\|\Psi_j\|_{L^q(B_R(0))} + \left(\int_{|y|\ge R} U^{p+1}\right)^{2/n} (\|\Psi_j\|_{L^r(\Omega_j)}). \end{aligned} \tag{4.13}$$

Similarly, we have

$$\|f_j\|_{L^r(\Omega_j)} \le C\|\Psi_j\|_{L^r(B_R(0))} + R^{-4}(\|\Psi_j\|_{L^r(\Omega_j)}). \tag{4.14}$$

Note that by Step 2 $\Psi_j \to 0$ weakly in $W^{2,r}(R^n)$ and strongly in $W^{1,r}(B_R(0))$ for any fixed R. By Lemma 4.3, (4.12), (4.13) and (4.14), letting $j \to \infty$ and $R \to \infty$, we get $\|\Psi_j\|_{W^{2,r}(R^n)} = o(1)$. □

From Lemma 4.4, we see that $\|\phi_{\varepsilon_j}\|_{L^r(\Omega_{\mu_\varepsilon})} \le C(r)$ for $n < r < \infty$. Hence by (4.5), we have $\|\phi_\varepsilon\|_{W^{2,r}(\Omega_{\mu_\varepsilon})} \le C(r)$. By Lemma 4.2, we can extend ϕ_ε to R^n, still denote it by ϕ_ε, such that $\|\phi_\varepsilon\|_{W^{2,r}(R^n)} \le C(r)$ for $n < r < \infty$. Now we fix $r > n$. For any subsequence ε_j, we can take a further sequence, still denoted by ε_j, such that $\phi_{\varepsilon_j} \to w$ weakly in $W^{2,r}(R^n)$ and $\phi_{\varepsilon_j} \to w$ in $C^1_{\mathrm{loc}}(R^n)$. As before, from now on, we denote ϕ_{ε_j} by $\phi_j, \ldots$.

We first show that w is a bounded solution of equation (3.3). To this end, we need to show that $F(\phi_j) \to -C(n,x_0)U^p \log U - pH(x_0,x_0)U^{p-1}$ in $L^\infty(R^n)$. In fact,

$$\begin{aligned} &|\mu_j^{n-2}(F(\phi_j) + C(n,x_0)U^p \log U + pH(x_0,x_0)U^{p-1})| \\ &\quad \le C(|v_j^{p-\varepsilon_j} - \mu_j^{n-2}(p-\varepsilon_j)U^{p-1-\varepsilon_j}(\phi_j - H(x_0,x_0+\mu_j y)) - U^{p-\varepsilon_j}| \\ &\qquad + |U^{p-\varepsilon_j} - U^p - C(n,x_0)\mu_j^{n-2}U^p \log U| \\ &\qquad + \mu_j^{n-2}|pU^{p-1}\phi_j - (p-\varepsilon_j)U^{p-1-\varepsilon_j}(\phi_j - H(x_0,x_0+\mu_j y)) - pH(x_0,x_0)U^{p-1}| \\ &\quad = II_1 + II_2 + II_3, \end{aligned}$$

where II_1, II_2 and II_3 are defined at the last equality.

By using the fact that $|\phi_j|_{L^\infty(R^n)} \le C$ and $|H(x_0,x_0+\mu_j y) - H(x_0,x_0)| \le C\mu_j|y|$, we have that

$$\begin{aligned} |II_1| &\le C\mu_j^{n+2}|\phi_j - H| \le C\mu_j^{n+2} \\ |II_2| &\le o(1)\mu_j^{n-2}U^{p-\varepsilon}|\log U|^2 \\ |II_3| &\le C\mu_j^{n-1}|y|U^{p-1} \le \mu_j^{n-1}. \end{aligned}$$

So, we have that $|II_l|_{L^\infty(\Omega_j)} \to 0$, $l = 1,2,3$.

Hence w is a weak (thus classical) solution of equation (3.3).

Let $w_j = \chi(\mu_j y)w(y)$ where $\chi(x) = 1$ when $x \in B_K(x_0) \subset \Omega$ and $\chi(x) = 0$ when $x \notin B_{3K}(x_0) \subset \Omega$. One can see that $\phi_j - w_j$ satisfies the following equation

$$\begin{aligned}\Delta(\phi_j - w_j) + pU^{p-1}(\phi_j - w_j) &= -F(\phi_j) - C(n,x_0)U^p \log U - pH(x_0,x_0)U^{p-1} \\ &\quad + (1-\chi)C(n,x_0)U^p \log U + p(1-\chi)H(x_0,x_0)U^{p-1} \\ &\quad - 2\nabla_y\chi\nabla w - \Delta_y\chi w \\ &= J_1 + J_2 + J_3 \qquad (4.15)\end{aligned}$$

where J_1, J_2, J_3 are defined by the last equality.

To finish the proof of Proposition 3.3, we just need to show that $\|\phi_j - w_j\|_{W^{2,r}(\Omega_{\mu_j})} = o(1)$.

Let us first estimate J_2, J_3. Observe that w satisfies equation (3.3). Let $G(y) = pU^{p-1}w + C(n,x_0)U^p \log U + pH(x_0,x_0)U^{p-1}$. It is easy to see that $|G(y)| \le C|y|^{-4}$ for $|y| \ge 1$. By Lemma 2.3 in Li and Ni [**10**], we have that $|w| \le C|y|^{-2}$ for $|y| \ge 1$. Similarly, we see that $|\nabla w| \le C|y|^{-3}$ for $|y| \ge 1$.

Hence we have that

$$\|J_2\|_{L^q(\Omega_j)} \le C\mu_\varepsilon^{4-n/q} = o(1), \qquad (4.16)$$

$$\|J_2\|_{L^r(\Omega_j)} \le C\mu_\varepsilon^{4-n/r} = o(1) \qquad (4.17)$$

and that

$$\|J_3\|_{L^r(\Omega_j)} \le C\mu_\varepsilon\|w\|_{W^{2,r}(R^n)} = o(1), \qquad (4.18)$$

$$\|J_3\|_{L^q(\Omega_j)} \le C\mu_\varepsilon^{4-n/q} = o(1). \qquad (4.19)$$

For J_1, we estimate as we did in Lemma 4.3 and we will get $\|J_1\|_{L^q(\Omega_j)} + \|J_1\|_{L^r(\Omega_j)} = o(1)$.

Similar to Lemma 4.3, we have

$$\begin{aligned}&\|pU^{p-1}(\phi_j - w_j)\|_{L^q(\Omega_j)} \\ &\quad\le C\|\phi_j - w_j\|_{L^q(B_R(0))} + \left(\int_{|y|\ge R} U^{p+1}\right)^{2/n} (\|\phi_j - w_j\|_{L^r(\Omega_j)}),\end{aligned} \qquad (4.20)$$

$$\|pU^{p-1}(\phi_j - w_j)\|_{L^r(\Omega_j)} \le C\|\phi_j - w_j\|_{L^r(B_R(0))} + R^{-4}(\|\phi_j - w_j\|_{L^r(\Omega_j)}). \qquad (4.21)$$

Now (4.16)–(4.21) imply that $\|\phi_j - w_j\|_{W^{2,r}(\Omega_j)} = o(1)$. By Sobolev Imbedding Theorem, we have $\|\phi_j - w_j\|_{L^\infty(B_{K/\mu_\varepsilon})} = o(1)$.

Finally, if there are two sequences ε_j and ε_j', such that $\phi_{\varepsilon_j} \to w$ and $\phi_{\varepsilon_j'} \to w'$, we claim that $w = w'$. In fact, both w and w' satisfy equation (3.3) and have the properties that $w(0) = w'(0) = H(x_0,x_0)$, and $\nabla w(0) = \nabla w'(0) = 0$. Now let $z' = w - w'$, then $z' \in X$. By the same argument as we did in Lemma 4.4, we have $z' = 0$. We conclude that $\phi_\varepsilon \to w$ as $\varepsilon \to 0$. Hence $\phi_\varepsilon \to w$ in $L^\infty(B_{K/\mu_\varepsilon}(x_0))$ as $\varepsilon \to 0$. □

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