J. Math. Soc. Japan
Vol. 50, No. 3, 1998

Ahlfors functions on non-planar Riemann surfaces whose double are hyperelliptic

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(Received Mar. 29, 1996)
(Revised Sept. 27, 1996)

1. Introduction.

Let R be a finite bordered Riemann surface which is a regular subregion of a Riemann surface. The genus of R is finite and the boundary ∂R of R consists of a finite number of contours. Let $\mathfrak{F}$ be the class of holomorphic functions h on R satisfying $|h| < 1$ on R. Let P be a point on R. If $f \in \mathfrak{F}$ satisfies

$$|(f \circ \varphi^{-1})'(\varphi(P))| = \sup\{|(h \circ \varphi^{-1})'(\varphi(P))| : h \in \mathfrak{F}\}$$

for a fixed local parameter φ, then we call f the Ahlfors function at P on R. It is known, by Ahlfors [1], that the Ahlfors function exists at each point of R and is uniquely determined up to a constant multiple of absolute value 1. We note that the Ahlfors function does not depend on a choice of a local parameter φ.

We summarize some basic properties of Ahlfors functions. We define the subclass $\mathfrak{F}_1$ of $\mathfrak{F}$ as $\mathfrak{F}_1 = \{h \in \mathfrak{F} : |h| = 1 \text{ on } \partial R\}$. Then each Ahlfors function is an element of $\mathfrak{F}_1$, and gives a complete covering on the unit disk, that is, it covers each point of the unit disk the same number of times, provided that the branch points are counted as many times as their multiplicity indicates. The number will be called the degree of the Ahlfors function. Each Ahlfors function f is prime in $\mathfrak{F}_1$, that is, if $f = ab (a, b \in \mathfrak{F}_1)$, then either a or b is a constant of absolute value 1. The Ahlfors function at P vanishes to order 1 at P.

Let p be the genus of R and let q be the number of contours of R. Let N be the degree of an Ahlfors function on R. Then Ahlfors [1] showed that $q \le N \le 2p + q$. If $p = 0$, that is, if R is a planar region, then $N = q$. Let $N(R)$ be the set of degree of Ahlfors functions on R. Then $N(R) = \{q\}$, if $p = 0$. In this case, Ahlfors functions can be expressed by theta functions (see Fay [3] Proposition 6.17). On the other hand, if $p > 0$, then it is not well-known what is $N(R)$. There is only one example constructed by Yamada [4] (section 4. Example). Let Y be the example of a finite Riemann surface: the genus of Y is one and its boundary consists of two components. Yamada [4] (section 4. Example.) showed $\{2, 4\} \subset N(Y)$ in the example.

In this paper, we deal with the Ahlfors functions on non-planar Riemann surfaces whose double are hyperelliptic. In Section 2, we shall show that the double $\hat{R}$ of such Riemann surface R can be expressed as

$$y^2 = \prod_{j=1}^{g+1} (x - \alpha_j)(1 - \overline{\alpha_j} x), \tag{1}$$

where $g = 2p + q - 1$ denotes the genus of $\hat{R}$, $|\alpha_j| < 1$ $(j = 1, \dots, g+1)$ and $\alpha_j \neq \alpha_k$ $(j \neq k)$, and $R = \{P \in \hat{R} : |x(P)| < 1\}$ (Lemma 2 and its followings). By using the equation (1), we shall show that an Ahlfors function on R is of degree 2 or $g+1$ (Proposition 3), that is, $N(R) \subset \{2, g+1\}$. Yamada [**4**] (Theorem 2) showed that $\{g+1\} \subset N(R)$. In Section 3, we consider the relation between Ahlfors functions and conformal automorphisms of Riemann surfaces. As the consequence of this section, we may not move points on fixed Riemann surface, but we move the branch points α_j $(j = 1, \dots, g+1)$. It will give the same results obtained by moving points. Thus we fix the point O with $x(O) = 0$ and we consider the Ahlfors function at O on the Riemann surface defined by the equation (1). In Section 4, we consider necessary and sufficient conditions that the Ahlfors function at $O \in R$ with $x(O) = 0$ is of degree 2. We define the parameter space $\mathfrak{S} \subset \boldsymbol{C}^{g+1}$. Each point in $\mathfrak{S}$ corresponds to the branch points α_j $(j = 1, \dots g+1)$ in (1). We say that a point of $\mathfrak{S}$ is of degree 2(or $g+1$), if the degree of the Ahlfors function at O on the Riemann surface corresponding to the point of $\mathfrak{S}$. Let $\mathfrak{S}_2$ be the set of all points of degree 2 and let $\mathfrak{S}_{g+1}$ be the set of all points of degree $g+1$. We shall show that $\mathfrak{S}_2$ is closed in $\mathfrak{S}$ and $\mathfrak{S}_{g+1}$ is open in $\mathfrak{S}$ (Theorem 8). In the proof of Theorem 8, we shall construct a Riemann surface R with $N(R) = \{2, g+1\}$.

2. The degree of Ahlfors functions.

LEMMA 1. *Let R be a finite Riemann surface and let $\hat{R}$ be the double of R. Suppose that $\hat{R}$ is a hyperelliptic Riemann surface of genus $g \geq 2$. Then the hyperelliptic involution J of $\hat{R}$ and the anti-conformal involution ϕ of $\hat{R}$ satisfy $\phi \circ J \circ \phi = J$.*

PROOF. By Farkas and Kra [**2**] (Proposition III.7.9 and its corollaries), a conformal involution of $\hat{R}$ which fixes $2g+2$ points is the hyperelliptic involution J. For a meromorphic function f of degree 2 on $\hat{R}$, by Theorem III.7.3 of [**2**] we have $f \circ \phi \circ J = f \circ \phi$. Hence $f \circ \phi \circ J \circ \phi = f \circ \phi \circ \phi = f$. In particular, for any Weierstrass point $W \in \hat{R}$, $f \circ \phi \circ J \circ \phi(W) = f(W)$. Thus the conformal involution $\phi \circ J \circ \phi$ fixes $2g+2$ Weierstrass points and it is the hyperelliptic involution J. □

LEMMA 2. *Let R be a finite bordered Riemann surface of genus ≥ 1 with contours and let $\hat{R}$ be the double of R. Then $\hat{R}$ is hyperelliptic if and only if R can be expressed as a two-sheeted unlimited covering surface of the unit disk $\{z \in \boldsymbol{C} : |z| < 1\}$.*

PROOF. If R is a two-sheeted unlimited covering surface of the unit disk, then, by reflection principle, R can be extended to a two-sheeted unlimited covering surface of the Riemann sphere $\hat{\boldsymbol{C}}$. Thus $\hat{R}$ is hyperelliptic.

Suppose $\hat{R}$ is hyperelliptic. We shall show that $J(\partial R) = \partial R$ and $J(R) = R$. Let $P \in \partial R$ and let ϕ be an anti-conformal involution which fix the boundary ∂R. Since $\phi \circ J \circ \phi = J$, $J(P) = \phi \circ J \circ \phi(P) = \phi \circ J(P)$. Hence $J(P) \in \partial R$. If $P \in \partial R$, then $P = J \circ J(P) \in J(\partial R)$, and so $J(\partial R) = \partial R$. Let $P \in R$ and let $\gamma \subset R$ be a curve joining from P to a Weierstrass point in R. We note that Weierstrass points do not exist on ∂R (see Yamada [**4**] Theorem 2). Assume that $J(P) \in \phi(R)$. Since J fixes Weierstrass points, we must have $J(\gamma) \cap \partial R \neq \varnothing$. This contradicts the above assertion that $\partial R = J(\partial R)$.

Hence $J(P) \in R$. On the other hand, we obtain $P = J \circ J(P) \in J(R)$ for $P \in R$. Consequently $J(R) = R$.

For a meromorphic function f of degree 2 on $\hat{R}$, we know that $f = f \circ J$ and f is a locally conformal mapping around ∂R. These imply that f maps R onto a connected domain in $\hat{C}$ and $f(\partial R)$ consists of a finite number of analytic curves. Similarly f maps $\phi(R)$ onto the complement of $f(R \cup \partial R)$ in $\hat{C}$ and $f(\phi(R))$ is connected. Hence $f(R)$ must be simply connected and the boundary of $f(R)$ is equal to $f(\partial R)$. By the Riemann mapping theorem, there exists a mapping h which maps $f(R)$ onto the unit disk. Hence R can be expressed as a two-sheeted unlimited covering surface of the unit disk. □

The functions f and h appeared in the proof of Lemma 2 satisfy $|h \circ f| = 1$ on ∂R. We put $x = h \circ f$. We can extend x to a meromorphic function on $\hat{R}$ of degree 2. There is a meromorphic function y on $\hat{R}$ such that $\hat{R}$ can be expressed by

$$y^2 = \prod_{j=1}^{g+1} (x - \alpha_j)(1 - \overline{\alpha_j} x),$$

where $|\alpha_j| < 1$, $\alpha_j \neq \alpha_k$ $(j \neq k)$ and g the genus of $\hat{R}$. We note that $R = \{P \in \hat{R} : |x(P)| < 1\}$. We also note that $2g+2$ Weierstrass points $\{P_1, \dots, P_{g+1}\} \subset R$ and $\{\phi(P_1), \dots, \phi(P_{g+1})\} \subset \phi(R)$ satisfy $x(\{P_1, \dots, P_{g+1}\}) = \{\alpha_1, \dots, \alpha_{g+1}\}$ and $x(\{\phi(P_1), \dots, \phi(P_{g+1})\}) = \{1/\overline{\alpha_1}, \dots, 1/\overline{\alpha_{g+1}}\}$.

PROPOSITION 3. *Let R be a finite bordered Riemann surface with contours such that its double $\hat{R}$ is hyperelliptic. Then every Ahlfors function on R is of degree* 2 *or* $g+1$.

PROOF. Ahlfors [**1**] proved that the degree N of an Ahlfors function on a finite Riemann surface R satisfies $q \leq N \leq 2p + q = g + 1$, where p is the genus of R, q is the number of boundary components of R and g is the genus of $\hat{R}$. If $g = 0$ or 1, then $p = 0$ and $N = q = g + 1$. If $g \geq 2$, then $N \neq 1$, and so

$$2 \leq N \leq g + 1.$$

It is known that the degree of the Ahlfors function at a Weierstrass point on R is equal to $g+1$ (see Yamada [**4**] Theorem 2). Assume that Q_1 is not a Weierstrass point and take x as a local parameter around Q_1. Since each Ahlfors function on R can be extended to $\hat{R}$ meromorphically, in what follows we regard it as a function defined on $\hat{R}$. By Farkas and Kra [**2**] (Proposition III.7.10), on a hyperelliptic Riemann surface of genus g any function of degree $\leq g$ must be even degree and a rational function of a meromorphic function x of degree 2. Assume that the degree of the Ahlfors function f at Q_1 is less than $g+1$. Let $2n$ be the degree of f. The Ahlfors function f has zeros Q_j $(j = 1, \dots, 2n)$ and has poles $\phi(Q_j)$ $(j = 1, \dots, 2n)$, because $|f| = 1$ on ∂R. Since f is a rational function of x, we put $\{\gamma_1, \dots, \gamma_n\} = \{x(Q_1), \dots, x(Q_{2n})\}$, especially $\gamma_1 = x(Q_1)$. Clearly we have $|\gamma_j| < 1$ $(j = 1, \dots, n)$. Thus f can be expressed as

$$e^{i\theta} \prod_{j=1}^{n} \frac{x - \gamma_j}{1 - \overline{\gamma_j} x} \quad \text{for some } \theta \in \mathbf{R}.$$

The absolute value of the derivative of f at Q_1 (i.e. $|f'(Q_1)|$) is equal to

$$\frac{1}{1-|\gamma_1|^2}\prod_{j=2}^{n}\frac{\gamma_1-\gamma_j}{1-\overline{\gamma_j}\gamma_1},$$

where we take the function x as a local parameter. On the other hand, the absolute value of the derivative of

$$\frac{x-\gamma_1}{1-\overline{\gamma_1}x}$$

at Q_1 is equal to

$$\frac{1}{1-|\gamma_1|^2}.$$

This value is greater than that of f, because clearly

$$\left|\frac{\gamma_j-\gamma_1}{1-\overline{\gamma_1}\gamma_j}\right|<1 \qquad (j=2,\dots,n).$$

This shows that $n=1$. Hence we have $N=2$ or $g+1$. □

We see that if $g \geq 2$ and if the degree of the Ahlfors function at $P \in R$ is equal to 2, then the Ahlfors function is of the form

$$e^{i\theta}\,\frac{x-x(P)}{1-\overline{x(P)}x} \qquad (\theta \in \boldsymbol{R}).$$

According to Yamada [4] (section 4. Example), we write the Ahlfors function at Weierstrass points $P_j \in R$ $(j=1,\dots,g+1)$ as of the form

$$e^{i\theta}\,\frac{y}{\prod_{j=1}^{g+1}(1-\overline{\alpha_j}x)} \qquad (\theta \in \boldsymbol{R}),$$

where $x(P_j)=\alpha_j$ $(j=1,\dots,g+1)$.

3. Conformal automorphisms and Ahlfors functions.

LEMMA 4. *Let Riemann surfaces R and S be conformally equivalent and let $\Phi : R \to S$ be a conformal mapping. Let f_P be the Ahlfors function at $P \in R$ and let g_Q be the Ahlfors function at $Q \in S$. Then we have $f_P = e^{i\theta}g_{\Phi(P)} \circ \Phi$ for some $\theta \in \boldsymbol{R}$.*

PROOF. This follows from the fact that the Ahlfors functions do not depend on the choice of local parameters. □

Let $\hat{S}_1$ and $\hat{S}_2$ be hyperelliptic Riemann surfaces of genus g (≥ 2) defined by

$$\hat{S}_1 : y^2 = \prod_{j=1}^{g+1}(x-\alpha_j)(1-\overline{\alpha_j}x)$$

and

$$\hat{S}_2 : Y^2 = \prod_{j=1}^{g+1}(X - \beta_j)(1 - \overline{\beta_j}X),$$

respectively. Put $S_1 = \{P \in \hat{S}_1 : |x(P)| < 1\}$ and $S_2 = \{P \in \hat{S}_2 : |X(P)| < 1\}$.

LEMMA 5. *The Riemann surfaces S_1 and S_2 are conformally equivalent if and only if there is a linear transformation T of the unit disk such that $T(\{\alpha_1, \dots, \alpha_{g+1}\}) = \{\beta_1, \dots, \beta_{g+1}\}$.*

PROOF. Suppose that there is a linear transformation T of the unit disk such that $T(\{\alpha_1, \dots, \alpha_{g+1}\}) = \{\beta_1, \dots, \beta_{g+1}\}$. Then T maps $\{1/\overline{\alpha_1}, \dots, 1/\overline{\alpha_{g+1}}\}$ to $\{1/\overline{\beta_1}, \dots, 1/\overline{\beta_{g+1}}\}$. Thus, if we write the linear transformation T as the form $T(x) = e^{i\theta}(x - \alpha)/(1 - \bar{\alpha}x)$, then we have a birational relation

$$\begin{cases} X = e^{i\theta}\dfrac{x - \alpha}{1 - \bar{\alpha}x} \\ Y = \dfrac{\sqrt{\prod_{j=1}^{g+1}(e^{i\theta} + \bar{\alpha}\beta_j)(1 + e^{i\theta}\alpha\overline{\beta_j})}}{(1 - \bar{\alpha}x)^{g+1}}y. \end{cases}$$

Thus $\hat{S}_1$ and $\hat{S}_2$ are conformally equivalent and a point P of $\hat{S}_1$ with $|x(P)| < 1$ is mapped to a point Q with $|X(Q)| < 1$. Hence S_1 and S_2 are conformally equivalent.

Conversely, suppose that S_1 and S_2 are conformally equivalent. Then $\hat{S}_1$ and $\hat{S}_2$ are conformally equivalent, too. Let Φ be the conformal mapping. By Farkas and Kra [**2**] (Theorem III.7.3), $X \circ \Phi$ is a linear transformation of x. We denote it by $T(x)$. Then T maps $\{\alpha_1, \dots, \alpha_{g+1}\}$ to $\{\beta_1, \dots, \beta_{g+1}\}$. □

For any point $P \in S_1$ with $x(P) = \alpha$, we put $\beta_j = (\alpha_j - \alpha)/(1 - \overline{\alpha_j}\alpha)$. Then S_1 and S_2 are conformally equivalent and P is mapped to $O \in S_2$ with $X(O) = 0$. Note that there are two points O_j with $X(O_j) = 0$ $(j = 1, 2)$, in general. Assume that the Ahlfors function $f_{O_1} = r_1(X)Y + r_2(X)$, where $r_j(X)$ $(j = 1, 2)$ are rational functions of X. There is the hyperelliptic involution J of S_2, by Lemma 4, we have that $f_{O_2} = f_{J(O_1)} = f_{O_1} \circ J = -r_1(X)Y + r_2(X)$. Thus the difference between f_{O_1} and f_{O_2} is small. We may take any one of them O. Hence we do not consider the Ahlfors function at an arbitrary point. Instead, we consider the Ahlfors function at O with $X(O) = 0$ on an arbitrary Riemann surface.

4. Distribution of degree of Ahlfors functions.

We shall investigate necessary and sufficient conditions that the Ahlfors function at O is of degree 2. Let $\hat{R}$ be the compact Riemann surface defined by

$$y^2 = \prod_{j=1}^{g+1}(x - \alpha_j)(1 - \overline{\alpha_j}x), \tag{2}$$

where $|\alpha_j| < 1$ $(j = 1, \dots g + 1)$ and $\alpha_j \neq \alpha_k$, and let $R = \{P \in \hat{R} : |x(P)| < 1\}$. We may assume that $\alpha_j \neq 0$ $(j = 1, \dots, g + 1)$. By Yamada [**4**] (section 4. Example.), the function x is the Ahlfors function at O if and only if there is a meromorphic differential ψ such that ψ is real and non-negative along ∂R and its divisor $D(\psi)$ is equal to $-O+J(O)+B+\phi(B)-\phi(O)+J\circ\phi(O)$, where B is a positive divisor of degree $g-1$ such that $O\notin B\subset R\cup\partial R$.

LEMMA 6. *Let* $D_1=-O+J(O)-\phi(O)+J\circ\phi(O)$ *and let* $D_2=J(O)+J\circ\phi(O)$. *Then we have*

$$\dim A(D_1)=g-1$$

and

$$\dim A(D_2)=g-2,$$

where $A(D_1)=\{\omega:\omega \text{ is a meromorphic differential with } D(\omega)\geq D_j\}\cup\{0\}$.

PROOF. By the Riemann-Roch theorem, we have

$$\begin{aligned}\dim A(D_1)&=\dim L(D_1)-\deg(D_1)+g-1\\&=\dim L(D_1)+g-1,\end{aligned}$$

where $L(D_1)=\{f:f \text{ is a meromorphic function with } D(f)\geq -D_1\}\cup\{0\}$. Since $J(O)$ is not a Weierstrass point, there is not a meromorphic function of degree 2 which has a pole only at $J(O)$. Thus $\dim L(D_1)=0$. Hence we have $\dim A(D_1)=g-1$. Similarly, we have that $L(D_2)=\boldsymbol{C}$ and $\dim L(D_2)=1$. Since $\deg(D_2)=2$, $\dim A(D_2)=g-2$. □

By Lemma 6, we see that there is a meromorphic differential ψ_1 such that $\psi_1\in A(D_1)\backslash A(D_2)$. The equation (2) can be written as

$$\begin{aligned}y^2&=\prod_{j=1}^{g+1}(x-\alpha_j)(1-\overline{\alpha_j}x)\\&=\prod_{j=1}^{g+1}(-\alpha_j)-\prod_{j=1}^{g+1}(-\alpha_j)\sum_{k=1}^{g+1}\left(\frac{1}{\alpha_k}+\overline{\alpha_k}\right)x+\cdots\\&\quad-\prod_{j=1}^{g+1}(-\overline{\alpha_j})\sum_{k=1}^{g+1}\left(\alpha_k+\frac{1}{\overline{\alpha_k}}\right)x^{2g+1}+\prod_{j=1}^{g+1}(-\overline{\alpha_j})x^{2g+2}.\end{aligned}\tag{3}$$

We put $a_0=\sqrt{\prod_{j=1}^{g+1}(-\alpha_j)}$ and $a_1=-a_0/2\sum_{k=1}^{g+1}(1/\alpha_k+\overline{\alpha_k})$, where we take a sign of the square root so that $y=a_0+a_1x+\cdots$ in a neighborhood of O.

LEMMA 7. *The differential*

$$\psi_1=\frac{y+a_0+a_1x+\overline{a_1}x^g+\overline{a_0}x^{g+1}}{ix}\frac{dx}{y}$$

satisfies the following conditions:

(i) $\psi_1\in A(D_1)$ *and* $\psi_1\notin A(D_2)$.

(ii) ψ_1 *is real along* ∂R.

PROOF. Since $y=a_0+a_1x+\cdots$ in a neighborhood of O, by the equation (3), $y=-a_0-a_1x-\cdots$ in a neighborhood of $J(O)$, $y=\overline{a_0}x^{g+1}+\overline{a_1}x^g+\cdots$ in a neighborhood of $\phi(O)$ and $y = -\overline{a_0}x^{g+1} - \overline{a_1}x^g - \cdots$ in a neighborhood $J \circ \phi(O)$. Thus the differential ψ_1 has simple poles at O and at $\phi(O)$. Clearly ψ_1 does not have poles at the other points and has zeros at $J(O)$ and at $J \circ \phi(O)$. Hence we have that $\psi_1 \in A(D_1)$ and $\psi_1 \notin A(D_2)$.

For the anti-conformal involution ϕ of $\hat{R}$, we have that $\overline{x \circ \phi} = 1/x$ and $\overline{y \circ \phi} = y/x^{g+1}$. Hence we see that $\overline{\psi_1 \circ \phi} = \psi_1$. This implies that the differential ψ_1 is real along ∂R. □

By Farkas and Kra [2] (III.7.5. Corollary 1), the g differentials

$$\frac{x^j\,dx}{y} \quad (j = 0, \ldots, g-1)$$

form a basis of the space of the holomorphic differentials on $\hat{R}$. By Ahlfors [1] (Theorem 2 and its corollary), we see that the space of holomorphic differentials which are real along ∂R is real g-dimensional. Simple calculation shows that we can take the following as a basis of holomorphic differentials which are real along ∂R:

$$\left\{\frac{(x^j + x^{g-1-j})\,dx}{iy}\right\}_{j=0,\ldots,n-1} \quad \left\{\frac{(x^j - x^{g-1-j})\,dx}{y}\right\}_{j=0,\ldots,n-1}$$

for even g, where $g = 2n$;

$$\left\{\frac{(x^j + x^{g-1-j})\,dx}{iy}\right\}_{j=0,\ldots,n} \quad \left\{\frac{(x^j - x^{g-1-j})\,dx}{y}\right\}_{j=0,\ldots,n-1}$$

for odd g, where $g = 2n+1$. Thus the subset of the differentials above which correspond to $j = 1, \ldots, n-1$ (or n) form a basis of $A(D_2)$. Let ψ_j $(j = 2, \ldots, g-1)$ be the basis. Then we know that the function x is the Ahlfors function at O on R if and only if there are $c_1, \ldots, c_{g-1} \in \boldsymbol{R}$ $(c_1 \neq 0)$ such that

$$\psi = \sum_{j=1}^{g-1} c_j \psi_j$$

is non-negative along ∂R.

Put $\mathfrak{S} = \{(\gamma_1, \ldots, \gamma_{g+1}) : |\gamma_j| < 1,\ \gamma_j \neq \gamma_k (j \neq k)\}$ for fixed $g \geq 2$. We regard $\mathfrak{S}$ as a topological subspace of $\boldsymbol{C}^{g+1}$. For any $(\gamma_1, \ldots, \gamma_{g+1}) \in \mathfrak{S}$, there is a Riemann surface $\hat{S}$ defined by

$$y^2 = \prod_{j=1}^{g+1} (x - \gamma_j)(1 - \overline{\gamma_j}x).$$

We consider the Ahlfors function at O on $S = \{P \in \hat{S} : |x(P)| < 1\}$ and we say that $(\gamma_1, \ldots, \gamma_{g+1})$ is of degree 2 (or $g+1$) if the degree of the Ahlfors function is 2 (or $g+1$). We denote by $\mathfrak{S}_2$ the set of all points of degree 2 and by $\mathfrak{S}_{g+1}$ the set of all points of degree $g+1$. Clearly, we have $\mathfrak{S}_2 \cap \mathfrak{S}_{g+1} = \varnothing$ and $\mathfrak{S}_2 \cup \mathfrak{S}_{g+1} = \mathfrak{S}$.

THEOREM 8. *Let $\mathfrak{S}_2$, $\mathfrak{S}_{g+1}$ and $\mathfrak{S}$ be defined as in the above. Then $\mathfrak{S}_2$ is closed in $\mathfrak{S}$, and so $\mathfrak{S}_{g+1}$ is open in $\mathfrak{S}$. Neither $\mathfrak{S}_2$ nor $\mathfrak{S}_{g+1}$ are empty.*

PROOF. We shall show that $\mathfrak{S}_{g+1}$ is not empty. Take a point $(0, \gamma_2, \ldots, \gamma_{g+1}) \in \mathfrak{S}$. Let $\hat{S}_1$ be the Riemann surface defined by $y^2 = x \prod_{j=2}^{g+1}(x - \gamma_j)(1 - \overline{\gamma_j}x)$ and let O be the point of $\hat{S}_1$ satisfying $x(O) = 0$. Then $S_1 = \{P \in \hat{S}_1 : |x(P)| < 1\}$ corresponds to the point $(0, \gamma_2, \ldots, \gamma_{g+1})$ and O is a Weierstrass point. Thus the Ahlfors function at O on S_1 is

$$e^{i\theta} \frac{y}{\prod_{j=2}^{g+1}(1 - \overline{\gamma_j}x)} \quad (\theta \in \boldsymbol{R})$$

and is of degree $g + 1$. Hence $\mathfrak{S}_{g+1}$ is not empty.

We shall show that $\mathfrak{S}_2$ is not empty. Put $\zeta = e^{(2\pi i)/(g+1)}$. Assume that $(1/2, 1/2\zeta, 1/2\zeta^2, \ldots, 1/2\zeta^{g-1}, 1/2\zeta^g) \in \mathfrak{S}$ is of degree $g + 1$. Then there is the Riemann surface $\hat{R}$ defined by $y^2 = \prod_{j=1}^{g+1}(x - 1/2\zeta^j)(1 - 1/2\bar{\zeta}^j x)$ and the Ahlfors function f at O on R, where $R = \{P \in \hat{R} : |x(P)| < 1\}$. The function f has $g + 1$ zeros on R and one of them is the point O. On the other hand, there is an automorphism Φ of R which maps (x, y) to $(\zeta x, y)$. The order of Φ is $g + 1$ and Φ fixes the point O. Thus functions f, $f \circ \Phi$, $f \circ \Phi^2, \ldots, f \circ \Phi^g$ must be the Ahlfors function at O. This contradicts the uniqueness of the Ahlfors function. Hence $(1/2, 1/2\zeta, \ldots, 1/2\zeta^g)$ is of degree 2.

Next we shall show that $\mathfrak{S}_2$ is closed in $\mathfrak{S}$. Put $x = e^{i\theta} (\theta \in \boldsymbol{R})$. Then the function y is a continuous function of θ and α_j $(j = 1, \ldots, g + 1)$. We note that the function y is a multi-valued function of θ. We may assume the differential $dx/ix = d\theta$ is positive. For a basis $\{\psi\}_{j=1}^{g-1}$ of $A(-O + J(O) - \phi(O) + J \circ \phi(O))$, put

$$\psi_j = f_j(\theta, \alpha_1, \ldots, \alpha_{g+1})\, d\theta \quad (j = 1, \ldots, g - 1).$$

Then the functions $f_j(\theta, \alpha_1, \ldots, \alpha_{g+1})$ $(j = 1, \ldots, g - 1)$ are (multi-valued) continuous functions. Accordingly, the point $(\alpha_1, \ldots, \alpha_{g+1})$ is of degree 2 if and only if there are $(c_1, \ldots, c_{g-1}) \in \boldsymbol{R}\ (c_1 \neq 0)$ such that

$$\inf_{\theta \in \boldsymbol{R}} \sum_{j=1}^{g-1} c_j f_j(\theta, \alpha_1, \ldots, \alpha_{g+1}) \geq 0.$$

Since $c_1 \neq 0$, we may assume $\sum_{j=1}^{g-1} c_j^2 = 1$, that is $(c_1, \ldots, c_{g-1}) \in S^{g-2}$, where S^{g-2} is the $(g - 2)$-sphere. Suppose that a sequence $\{(\alpha_1^{(k)}, \ldots, \alpha_{g+1}^{(k)})\}_{k=1}^{\infty}$ of $\mathfrak{S}_2$ converges to a point $(\alpha_1^{(0)}, \ldots, \alpha_{g+1}^{(0)}) \in \mathfrak{S}$. Then, for each k, there is a point $(c_1^{(k)}, \ldots, c_{g+1}^{(k)}) \in S^{g-2}$ which corresponds to $(\alpha_1^{(k)}, \ldots, \alpha_{g+1}^{(k)})$. Since S^{g-2} is compact, the sequence $\{(c_1^{(k)}, \ldots, c_{g+1}^{(k)})\}_{k=0}^{\infty}$ has a cluster point $(c_1^{(0)}, \ldots, c_{g+1}^{(0)}) \in S^{g-2}$. The continuity of the functions f_j $(j = 1, \ldots, g - 1)$ shows that

$$\inf_{\theta \in \boldsymbol{R}} \sum_{j=1}^{g-1} c_j^{(0)} f_j(\theta, \alpha_1^{(0)}, \ldots, \alpha_{g+1}^{(0)}) \geq 0.$$

Hence the set $\mathfrak{S}_2$ is closed. □

From this theorem, we see that on each Riemann surface $\hat{R}$ the set of points of degree 2 is closed on R and the set of points of degree $g + 1$ is open on R.

5. Examples.

In this section we assume that the genus of $\hat{R}$ is 2. Then we can write $\hat{R}$ as

$$y^2 = \prod_{j=1}^{3}(x - \alpha_j)(1 - \overline{\alpha_j}x)$$

and $R = \{P \in \hat{R} : |x(P)| < 1\}$. In this case, dimension of $A(D_1)$ and $A(D_2)$ in Lemma 6 are equal to 1 and 0, respectively. Thus the differential ψ_1 in Lemma 7 forms a basis of $A(D_1)$. We can calculate explicitly the necessary and sufficient condition that x is the Ahlfors function at O, where $x(O) = 0$. As in the proof of Lemma 3 of Yamada [**4**], we see that ψ_1 is real and non-negative along ∂R if and only if

$$\frac{y^2 - (a_0 + a_1 x + \overline{a_1}x^2 + \overline{a_0}x^3)^2}{x^3} \geq 0 \quad \text{for all } |x| = 1. \tag{4}$$

Put $s_1 = \alpha_1 + \alpha_2 + \alpha_3$, $s_2 = \alpha_1\alpha_2 + \alpha_2\alpha_3 + \alpha_3\alpha_1$ and $s_3 = \alpha_1\alpha_2\alpha_3$. Then $a_0^2 = -s_3$ and $2a_0a_1 = \overline{s_1}s_3 + s_2$. In the numerator of the left-hand side of the inequality (4), the terms of x of order 0, 1, 5 and 6 are equal to zero. Put $A = -(\overline{s_1}s_2 + s_2\overline{s_3} + s_1 + 2a_0\overline{a_1} + a_1^2)$ and $B = (1 + s_1\overline{s_1} + s_2\overline{s_2} + s_3\overline{s_3} - 2a_0\overline{a_0} - 2a_1\overline{a_1})$. Then the inequality (4) holds if and only if

$$\frac{A}{x} + B + \bar{A}x \geq 0 \quad \text{for all } |x| = 1.$$

When $x = -e^{i \arg A}$, $A/x + B + \bar{A}x$ takes a minimum. Thus $A/x + B + \bar{A}x \geq B - 2|A|$ for all $|x| = 1$. Hence x is the Ahlfors function at O if and only if

$$B - 2|A| \geq 0.$$

We show some figures below. Points contained in black regions of figures correspond to the projection of points of degree 3. Points contained in white regions of figures correspond to the projection of points of degree 2.

First, we put $\alpha_1 = r$, $\alpha_2 = re^{(2\pi i)/3}$ and $\alpha_3 = re^{(4\pi i)/3}$.

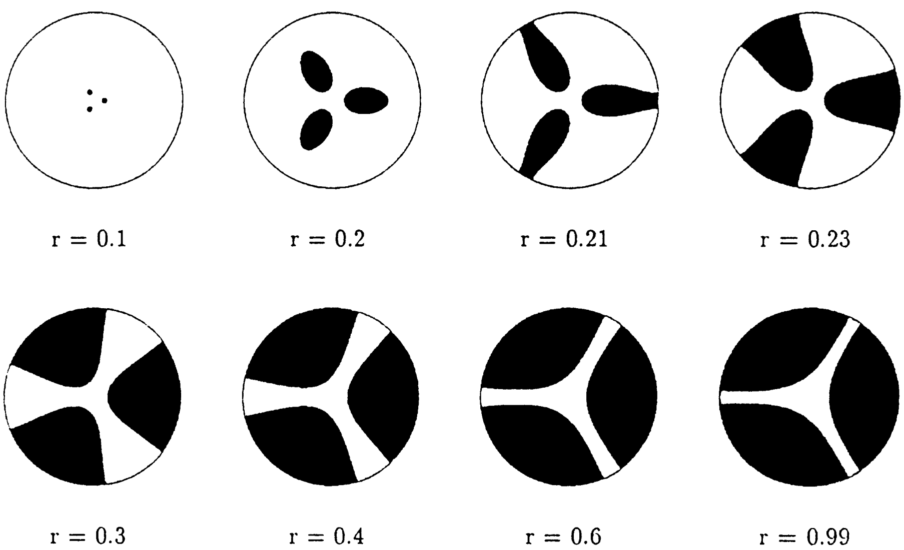

Fig. 1.

Second, we put $\alpha_1 = r$, $\alpha_2 = 0$ and $\alpha_3 = -r$.

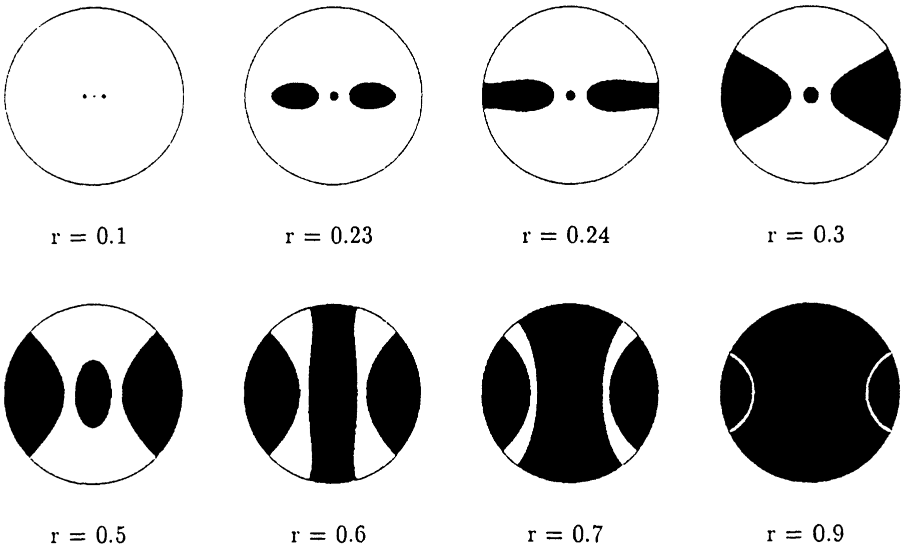

Fig. 2.

Finally, we show another types.

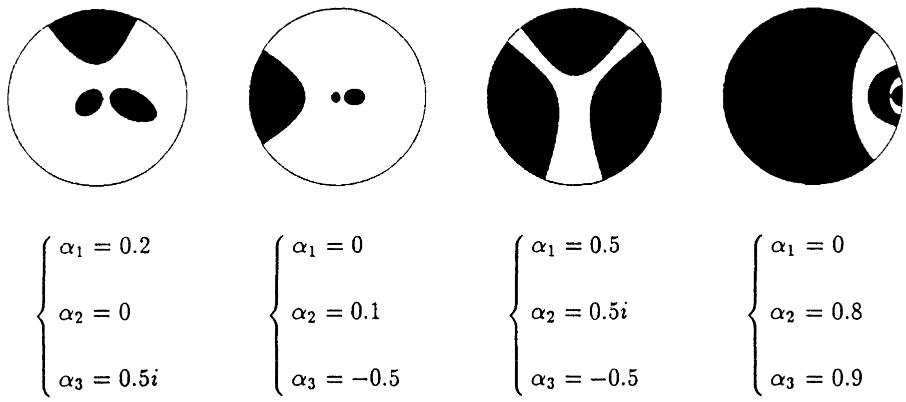

Fig. 3.

As in the case of $r = 0.1$ in Figure 1 and 2, if the three points α_1, α_2 and α_3 are sufficiently close, then the region of degree 3 is very small. We shall show this. Take a real number r $(0 < r < 1)$. Then

$$\lim_{\alpha_1,\alpha_2,\alpha_3 \to -r} \frac{y^2 - (a_0 + a_1 x + \overline{a_1} x^2 + \overline{a_0} x^3)^2}{x^3} = \frac{(r-1)^4}{4}\left(3rx + 4r^2 - 2r + 4 + \frac{3r}{x}\right).$$

The right-hand side of the equation takes a minimum when $x = -1$. So we have

$$\frac{(r-1)^4}{4}\left(3rx + 4r^2 - 2r + 4 + \frac{3r}{x}\right)\bigg|_{x=-1} = (r-1)^6 > 0.$$

This implies that for any fixed r $(0 < r < 1)$, for any α_j $(j = 1, 2, 3)$ which are sufficiently close to $-r$, the point O is of degree 2. This completes the proof.

6. Conjectures.

In this final section, the author gives two conjectures. First, in the case of $g = 2$, as in the figures above, the region of degree 3 consists of three simply connected components and the region of degree 2 is non-empty. The author could not settle this. More generally, it is plausible that the region of degree $g + 1$ consists of $g + 1$ simply connected components and the region of degree 2 is non-empty. Second, in the case of $g = 2$, we have seen that if the branch points α_j $(j = 1, 2, 3)$ are sufficiently close, the region of degree 3 is very small. The author guesses that if the $g + 1$ branch points α_j $(j = 1, \ldots, g + 1)$ are sufficiently close, then the region of degree $g + 1$ is very small.

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