J. Math. Soc. Japan
Vol. 50. No. 3, 1998

Zariski pairs, fundamental groups and Alexander polynomials

Enrique ARTAL BARTOLO* and Jorge CARMONA RUBER*

(Received Jan. 30, 1996)
(Revised July 2, 1996)

In this paper we present new examples of Zariski pairs and we compute some invariants of Zariski pairs which were already known. We discuss also the consequences of these new results in the theory of isolated singularities of surface.

We recall the notion of Zariski pair which was introduced in [A1]. We will say that two curves C and D are members of a Zariski pair if:

(i) There is a degree-preserving bijection α between the set of irreducible components of C and D and there exist regular neighbourhoods of $T(C)$ and $T(D)$ (of C and D, respectively) such that the pairs $(T(C), C)$ and $(T(D), D)$ are homeomorphic and the homeomorphism respects the bijection above.

(ii) The pairs $(\boldsymbol{P}^2, C)$ and $(\boldsymbol{P}^2, D)$ are not homeomorphic.

We recall that first condition means that there exists also a bijection β between the branches of the singular points of C and D such that:

(i1) If T is a branch at a singular point of C, T and $\beta(T)$ have the same topological type.

(i2) If T, T' are two different branches at singular points of C, then their intersection number equals the intersection number of $\beta(T)$ and $\beta(T')$.

(i3) If T is a branch at a singular point of C and C_T is the irreducible component of C which contains T, then $\alpha(C_T)$ is the irreducible component of D which contains $\beta(T)$.

The first Zariski pair appears in the works of Zariski, see [Z1], [Z2], [Z3]: the members of the pair are irreducible sextics with six ordinary cusps; in one case the cusps lie in a conic and they do not in the other one (explicit equations for this case appear in [O1] and [A1]). Some other examples can be found in [A1]. The invariant used to distinguish the members of these pairs is the same: we called it the b-invariant and one construct is as follows:

Let C be a reduced plane curve of degree d and $F(x,y,z)=0$ a defining equation of C. Let X be any desingularization of the projective hypersurface X_1 in $\boldsymbol{P}^3$ defined by $F(x,y,z)=t^d$; two such desingularizations are birationally equivalent, so the first Betti number of X is an invariant $b(C)$ of the pair $(\boldsymbol{P}^2, C)$. One can take a finer invariant; if one composes the desingularization of X_1 with the restriction to X_1 of the projection in the first three variables x,y,z, then one has a d-sheeted cyclic covering $\sigma: X \to \boldsymbol{P}^2$ which is unramified on $\boldsymbol{P}^2 \setminus C$. The monodromy operator $\tau: X \to X$ acts on $\mathrm{H}^1(X; \boldsymbol{C})$.

1991 *Mathematics Subject Classification.* 14H30, 14N10, 14J17.

Key words and phrases. Fundamental group, plane curve, Alexander polynomial.

*Partially supported by CAICYT PB91-0370-C02-01 and PB94-0291.

DEFINITION. The *Alexander polynomial* Δ_C of the curve C is the characteristic polynomial of the action of τ on $H^1(X; \boldsymbol{C})$ (its degree is $b(C)$).

This definition agrees with the usual one because of a result of Libgober, see [**Li**]. In [**Z2**], there is a formula giving Δ_C when the only singularities of C are nodes and ordinary cusps. A general formula can be found in [**A1**].

Recently, new examples of Zariski pairs have been discovered by Shimada [**Sh**], M. Oka [**O2**] and Tokunaga [**T**]. The two first authors distinguish the members of the pairs by the fundamental group of the complement. The third one distinguishes them by the existence or non-existence of dihedral coverings of $\boldsymbol{P}^2$ ramified along the curves. In any case, these examples can be distinguished by the Alexander polynomial.

DEFINITION. Let $C \subset \boldsymbol{P}^2$ a reduced curve. The *group* G_C *of the curve* is the fundamental group of $\boldsymbol{P}^2 \setminus C$.

We recall also that in Shimada's examples, one member of the pair has always abelian group.

Also recently, Degtyarev have classified all irreducible sextics with non-trivial Alexander polynomials, see [**D**]. By the way, he produces some new Zariski pairs. The members of one of them are sexitics D_1 and D_2 with three $\boldsymbol{E}_6$-singularities. There exists a conic tangent to the three singular points of D_1 and it is not the case for D_2. Degtyarev communicated us that the group of D_1 is $\boldsymbol{Z}/2\boldsymbol{Z} * \boldsymbol{Z}/3\boldsymbol{Z}$ (it is the same as the group of the sextic with six cusps on a conic, which was computed by Zariski in [**Z1**]). He wondered about the fundamental group of D_2. We will prove later that it is abelian; this question was the starting point of this paper but the method used to find an answer yields to some interesting results.

Let us take one of the Zariski pairs of [**A1**]. The members are curves A_1, A_2, with four irreducible components: one smooth cubic and three lines in general position which are tangent to the cubic at inflection points. The three inflection points are aligned in A_1, but it is not the case for A_2 (this example is also related to Zariski's example). It was stated without proof in [**A1**] that the group of A_2 was abelian. We will prove this fact and we will show that it is the way to prove that the group of Degtyarev's curve D_2 is abelian.

By the way we have found new examples of Zariski pairs with an interesting feature: they are not distinguished by the Alexander polynomial. In one of the pairs, the two members have non-abelian fundamental group. In other one, the curves have only rational irreducible components.

The existence of Zariski pairs which are not distinguished by the Alexander polynomial yields to the problem of finding invariants of the embedded topology of an isolated germ of singularity of surface in $\boldsymbol{C}^3$. There is an interesting family of such singularities, the so-called superisolated singularities, which was introduced by Luengo in [**Lu**]. We may view these singularities as follows: take a reduced plane algebraic curve and look at it as a homogeneous singularity of surface. Then make a generic deformation so as to get an isolated singularity such that the tangent cone remains invariant. We can interpret some results of Luengo in [**Lu**] as follows: *the superisolated singularities associated to the members of a Zariski pair have the same abstract topology.* As it was shown in [**A2**] and announced in [**St**], it is also the case for the characteristic polynomial of the monodromy. As we can find in [**A2**], if the members of a Zariski pair have not the same Alexander polynomial, then, the Jordan form of the monodromy is not the same, and then, the two superisolated singularities have not homeomorphic embeddings in $\boldsymbol{C}^3$. We conjecture that it is also the case for the new Zariski pairs, but up to now there is no way to distinguish them.

In § 1 we define an affine curve which is the base of all computations and we describe the examples of Zariski pairs. In § 2 we compute the fundamental group of the curve in § 1 and we take a subgroup of index 2, related to an unramified double covering. In § 3 we apply the results of § 2 in order to compute the fundamental groups of the members of some known Zariski pairs. In § 4 we present the first example of a Zariski pair where both members have non-abelian fundamental group and are not distinguished by the Alexander polynomial (the fundamental groups are not isomorphic). In § 5 we present another Zariski pair which is not distinguished by the Alexander polynomial; the example of § 4 is a degeneration of this one. In § 6, we study another degeneration of the example in § 5 which verifies that the irreducible components of each member are rational. In § 7, we sketch the relationship with the singularities of surface.

§ 1. A useful affine curve

Let us consider an affine curve with four smooth irreducible components:

$$C : y = x(x^2 - a^2),$$

$$D : y = 0,$$

$$P : y = 9(x - \sqrt{3})^2,$$

$$N : y = -9(x + \sqrt{3})^2,$$

where $a := 3\sqrt[4]{3}\sqrt{2 - \sqrt{3}}$.

Let us list the intersection of these irreducible components:

(a) $C \cap D = \{(0,0), (a,0), (-a,0)\}$ and the intersection is transversal.
(b) $C \cap P = \{(3, 54(2 - \sqrt{3}))\}$ and the contact order is 3.
(c) $C \cap N = \{(-3, -54(2 - \sqrt{3}))\}$ and the contact order is 3.
(d) $D \cap P = \{(\sqrt{3}, 0)\}$ and the contact order is 2.
(e) $D \cap N = \{(-\sqrt{3}, 0)\}$ and the contact order is 2.
(f) $P \cap N = \{(i\sqrt{3}, -54i), (-i\sqrt{3}, 54i)\}$ and the intersection is transversal.

We have drawn the real part of this curve in Figure 1. Let us explain why this curve is useful. We consider the double covering

$$\phi : \boldsymbol{C}^2 \to \boldsymbol{C}^2, \quad \phi(x, y) := (x, y^2).$$

We note that ϕ induces an unramified double covering $\phi_| : \boldsymbol{C}^2 \backslash D \to \boldsymbol{C}^2 \backslash D$.

Let us denote $C_1 := \phi^{-1}(C)$; it is a smooth cubic defined by the equation

$$y^2 = x(x^2 - a^2).$$

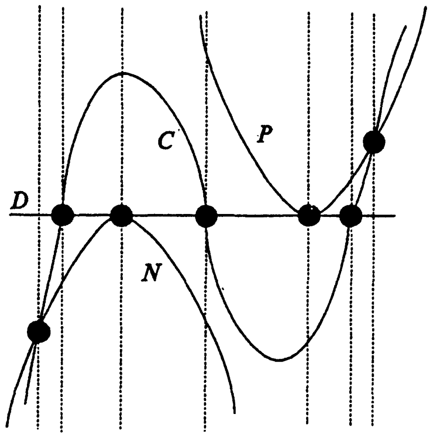

Figure 1.

It is easily seen that $\phi^{-1}(P)$ has two irreducible components, say P_1 and P_2; they are tangent lines at two inflection points of C_1, denoted I_1 and I_2 respectively. We suppose that their equations are given by:

$$P_1 : y = 3(x - \sqrt{3}) \quad \text{and} \quad P_2 : y = -3(x - \sqrt{3}).$$

Then, we have

$$I_1 = (3, 3\sqrt{3}(\sqrt{3} - 1)) \quad \text{and} \quad I_2 = (3, -3\sqrt{3}(\sqrt{3} - 1)).$$

In the same way, we find that $\phi^{-1}(N)$ is the union of two tangent lines L_1, L_2, at inflection points of C_1, J_1 and J_2 respectively. We may suppose that their equations are:

$$N_1 : y = 3i(x + \sqrt{3}) \quad \text{and} \quad N_2 : y = -3i(x + \sqrt{3})$$

Then, we have

$$J_1 = (-3, -3\sqrt{3}i(\sqrt{3} - 1)) \quad \text{and} \quad J_2 = (-3, 3\sqrt{3}i(\sqrt{3} - 1)).$$

Let us compactify $\boldsymbol{C}^2 \subset \boldsymbol{P}^2$ with homogeneous coordinates $[x : y : z]$.

CONVENTION. We denote in the same way the affine curves and their compactification in $\boldsymbol{P}^2$.

Denote L the line at infinity

$$L : z = 0.$$

We note that L is tangent to C_1 at an inflection point

$$O := [0 : 1 : 0]$$

(whenever we will use the abelian group structure of C_1 we will suppose that O is the zero element). We are going to find several Zariski pairs related with this construction:

EXAMPLE 1. We find the Zariski pair of **[A1]** which appears in the introduction. We take $A_1 = C_1 \cup L \cup P_1 \cup P_2$ because O, I_1, I_2 are in a vertical line. We take $A_2 = C_1 \cup L \cup P_1 \cup N_1$ as O, I_1, J_1 are not aligned. In both cases the lines are in general position.

Example 2. We find Zariski's example as in [**A1**]. Take $\boldsymbol{Z}/2\boldsymbol{Z} \times \boldsymbol{Z}/2\boldsymbol{Z}$-coverings of $\boldsymbol{P}^2$ ramified on the three lines of each member of Example 1. The preimage of C_1 in the first case (resp. second case) is a sextic with six cusps in a conic (resp. not in a conic).

Example 3. We find now the Zariski pair of [**D**] which appears in the introduction. Take the Cremona transformations associated to the three lines of each member of Example 1. The strict transform of C_1 is D_1 in the first case and D_2 in the second case.

Example 4. It is the first new example. We take $B_1 = C_1 \cup L \cup P_1 \cup P_2 \cup N_1$ and $B_2 = C_1 \cup P_1 \cup P_2 \cup N_1 \cup N_2$. In both cases the lines are in general position. In the first case three inflection points are aligned; in the second case the four inflection points are in general position.

The last two examples will be constructed by deformation. We begin their discussion with:

Question. Which is the condition on two points $R_1, R_2 \in C_1$ such that there exists a conic Q which intersects C_1 at R_1 and R_2 with contact order 3 at both points?

Using the group structure, see [**W**], we know that it is the case if and only if $3(R_1 + R_2) = O$. As the 3-torsion of C_1 is the set of inflection points:

Answer. There exists such a conic if and only if the line passing through R_1 and R_2 intersects C_1 at R_1, R_2 and an inflection point P_Q.

Then, given such a conic Q, we call P_Q the inflection point associated to Q. We note that Q is irreducible if and only if R_1 (or R_2) is not an inflection point.

Example 5. In this example we have three irreducible components: a smooth cubic and two irreducible conics. The conics intersect the cubic as above and they intersect each other transversally.

We denote such a curve E_1 when the two associated inflection points are not the same. If they coincide, we denote the curve E_2.

It is easily seen that these curves exist: each conic is parameterized by an inflection point and a generic point of the cubic. If the two generic points degenerate to inflection points in a suitable way we find members of Example 4, where the two conics degenerate onto four lines. As in Example 4 the lines are in general position, it is also the case for conics.

Example 6. Replace the cubic C_1 by a nodal cubic C_n in Example 5. In this case there is a group structure on C_n^* (C_n minus the singular point) which is isomorphic to $\boldsymbol{C}^*$. Here the 3-torsion has 3 elements and the discussion above is possible with slight modifications.

The curves of this Example are degenerations of the curves in Example 5.

§ 2. Fundamental group of the useful curve

We recall this definition which will be used everywhere.

DEFINITION. Let X be a smooth projective manifold and let H, $K \subset X$ be hypersurfaces. Let $* \in X \backslash (H \cup K)$. A *meridian* of H in the group $\pi_1(X \backslash (K \cup H), *)$ is the homotopy class of a loop μ defined as follows: take a point $P \in H$ which is smooth in $H \cup K$; take a small disk Δ around P transversal to H and disjoint from K; fix a point $*' \in \partial\Delta$ and let m be the loop based at $*'$ which turns once along $\partial\Delta$ in the positive direction. Choose any path ℓ from $*$ to $*'$ in $X \backslash (H \cup K)$ such that $\ell \cap \Delta = \{*'\}$. Then $\mu := \ell \cdot m \cdot \ell^{-1}$ (we note that two meridians of H are conjugate if H is irreducible).

Let us denote $\Gamma := D \cup C \cup P \cup N$. We fix $* := (\varepsilon, K) \in \boldsymbol{C}^2 \backslash \Gamma$ such that $0 < \varepsilon \ll 1$ and $K \gg 1$.

PROPOSITION. *Let* $G := \pi_1(\boldsymbol{C}^2 \backslash \Gamma; *)$. *Then, there exist loops* d, c, p, n *which are meridians of* D, C, P, N, *respectively, such that*:

$$G = \langle d, c, p, n : \mathscr{A}, \mathscr{B}, \mathscr{C}, \mathscr{D}, \mathscr{E}, \mathscr{F}, \mathscr{G}, \mathscr{H}, \mathscr{I} \rangle,$$

where, if we note $[x, y] := xyx^{-1}y^{-1}$, *we have*:

$$\mathscr{A} : [d, c] = 1, \quad \mathscr{B} : (dp)^2 = (pd)^2, \quad \mathscr{C} : [c, p^{-1}dp] = 1,$$

$$\mathscr{D} : (cp)^3 = (pc)^3, \quad \mathscr{E} : (dn)^2 = (nd)^2, \quad \mathscr{F} : [c, ndn^{-1}] = 1,$$

$$\mathscr{G} : (cn)^3 = (nc)^3, \quad \mathscr{H} : [p, n] = 1, \quad \mathscr{I} : [p, (dc)^{-1}n(dc)] = 1.$$

PROOF. We apply Zariski-Van Kampen method to the projection p_x in the x-variable. The key point is that the real picture in Figure 1 allow us to compute braid monodromy in almost every case.

Let us note that the non-generic fibers of p_x, with respect to Γ, are exactly those which correspond with $0, \sqrt{3}, -\sqrt{3}, a, -a, 3, -3, i\sqrt{3}$ and $-i\sqrt{3}$; we call them irregular values of p_x with respect to Γ and we note NT the set of these values. We take as generic fixed fiber $F := p_x^{-1}(\varepsilon)$; we choose there $* = (\varepsilon, K)$ as base point. It is easily seen that

$$\pi_1(F \backslash \Gamma; *) = \langle d, c, p, n : \text{—} \rangle,$$

where d, c, p, n are meridians of C, P, D, N, respectively, as it is shown in Figure 2.

Zariski-Van Kampen theorem says that these elements generate G. We are going to explain how to obtain the relations. Let us consider $\boldsymbol{C} \backslash NT$; picture in Figure 3 shows a set of generators of the free group $\pi_1(\boldsymbol{C} \backslash NT; \varepsilon)$.

For each $w \in NT$ we denote α_w the path from ε to the boundary of the small disk around w; we denote δ_w the counterclockwise boundary of this disk. Then $\pi_1(\boldsymbol{C} \backslash NT; \varepsilon)$

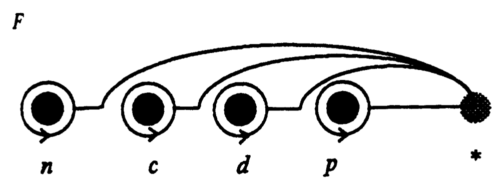

Figure 2.

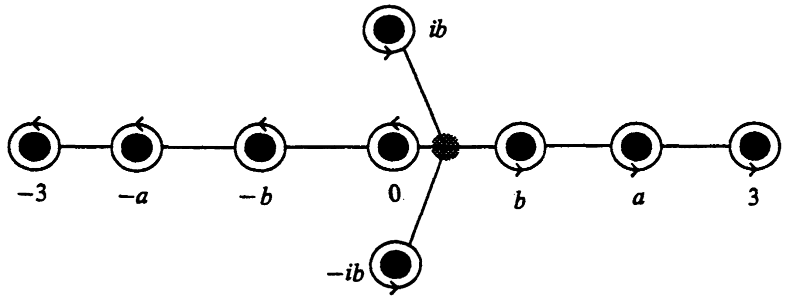

Figure 3. $b = \sqrt{3}$.

is the free group generated by the homotopy classes of

$$\mu_w := \alpha_w \cdot \delta_w \cdot \alpha_w^{-1}, \quad \text{for } w \in NT.$$

Relations are obtained by the action on $\pi_1(F\backslash\Gamma;*)$ of the braids determined by Γ on each μ_w (we can think of Γ as a multivalued function of $\boldsymbol{C}\backslash NT$ and the support of μ_w is outside the set of ramification points NT).

Let us fix $w \in NT$; the braid associated to μ_w may be decomposed as sts^{-1} where s (resp. t) is the braid associated to α_w (resp. δ_w). The conjugacy class of t depends only on the topological type of the singularities of the projection of p at w; so, it can be obtained if we know the singularities of p_x. In the general case, the difficult part of the Zariski-Van Kampen method is to determine the braid associated to s. In our case, we determine s for the real values of NT by means of the real picture; for the non-real values, we compute directly the braid.

Remark. Let us denote $w' := \alpha_w \cap \delta_w$ and $F_w := p_x^{-1}(w')$. Then, s defines an isomorphism from $\pi_1(F\backslash\Gamma;*)$ onto $\pi_1(F_w\backslash\Gamma;(w',K))$. We construct a set of generators of $\pi_1(F_w\backslash\Gamma;(w',K))$ as we have done in Figure 2 for F. We connect these two base points by the lifting of α_w in the line $y = K$ in order to regard the loops based on (w',K) as loops based on $*$. We consider another two generic fibers; let us call $3''$ the opposite element to $3'$ in δ_3 and let us note $F_\infty := p_x^{-1}(3'')$. We define $F_{-\infty}$ in the same way, near -3.

Let us consider the monodromy around 0; there is an ordinary double point on this fiber which is transversal to $p_x^{-1}(0)$. The braid associated to μ_0 is in Figure 4; it can be computed from the real picture.

We get the relation

$$\mathscr{A} : [d,c] = 1.$$

The singular point on $p_x^{-1}(\sqrt{3})$ is a tacnode not tangent to the fiber. We draw the braid determined by $\mu_{\sqrt{3}}$ from the real picture again (see Figure 5).

We obtain the relation

$$\mathscr{B} : (dp)^2 = (pd)^2.$$

For $w = 0, \sqrt{3}$, the braid s may be considered trivial. It will not be the case in the sequel.

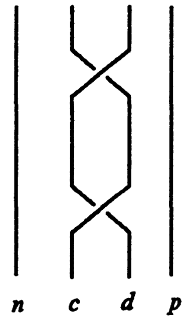

Figure 4. Braid is upwards.

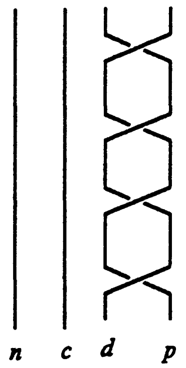

Figure 5.

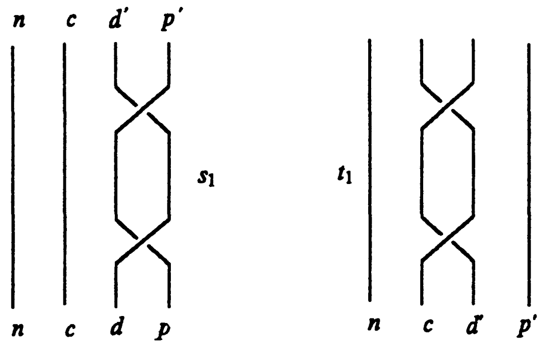

Figure 6.

Let us consider now the monodromy around a (we have again an ordinary double point which determine t_1). In this case, the braid obtained is of the form $s_1 t_1 s_1^{-1}$, see Figure 6.

We do not change the name of the loops (in F and F_a) if they coincide in G. Applying the braid s_1 and the relation $\mathscr{B}$, we have $d' = p^{-1}dp$ and $p' = dpd^{-1}$. Then, we have the relation:

$$\mathscr{C} : [c, p^{-1}dp] = 1.$$

The singularity on $p_x^{-1}(3)$ has local equations $u^2 - v^6 = 0$. The braid obtained when we turn around 3 is of the form $s_2 t_2 s_2^{-1}$, see Figure 7.

We proceed as before. In this case, we have $(cdpd^{-1})^3 = (dpd^{-1}c)^3$. From $\mathscr{A}$, this

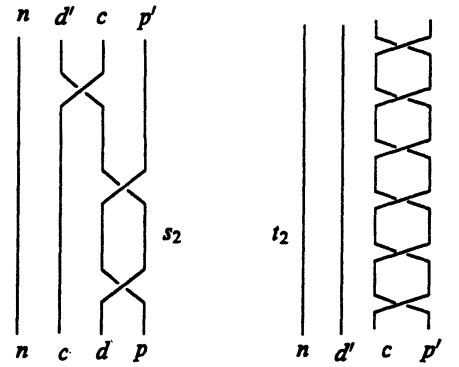

Figure 7.

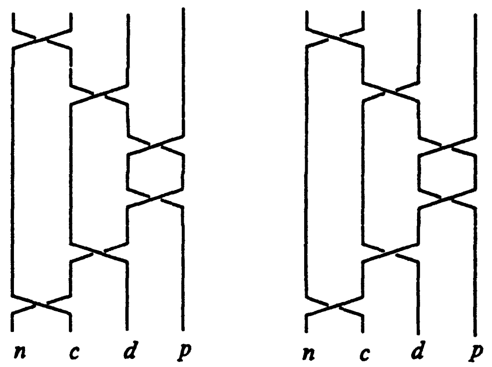

Figure 8.

is equivalent to

$$\mathscr{D} : (cp)^3 = (pc)^3.$$

In order to get relations $\mathscr{E}$, $\mathscr{F}$ and $\mathscr{G}$, we turn around $-\sqrt{3}$, $-a$ and -3. We find the relations in the same way as above.

The real picture does not give any information about the monodromy around $\pm i\sqrt{3}$. It is easily seen that in this case we obtain the braids in Figure 8.

These braids give the relations $\mathscr{H}$ and $\mathscr{I}$. □

Now, we are concerned with the group of the projective curve

$$\Theta := C_1 \cup P_1 \cup P_2 \cup N_1 \cup N_2 \cup L.$$

PROPOSITION. *Let us fix a point* $*_1 \in \phi^{-1}(*) \subset \boldsymbol{C}^2 \subset \boldsymbol{P}^2$. *Let* $H := \pi_1(\boldsymbol{P}^2 \backslash \Theta; *_1)$. *Then there exist loops* c_1, p_1, p_2, n_1, n_2 *which are meridians of* C_1, P_1, P_2, N_1, N_2 *respectively, such that*:

$$H = \langle c_1, p_1, p_2, n_1, n_2 : \mathscr{B}_1, \mathscr{C}_1, \mathscr{D}_1, \mathscr{E}_1, \mathscr{F}_1, \mathscr{G}_1, \mathscr{H}_1, \mathscr{H}_2, \mathscr{I}_1, \mathscr{I}_2 \rangle,$$

where:

$$\mathscr{B}_1 : [p_1, p_2] = 1, \quad \mathscr{C}_1 : [c_1, p_1^{-1} p_2] = 1, \quad \mathscr{D}_1 : (c_1 p_1)^3 = (p_1 c_1)^3, \quad \mathscr{E}_1 : [n_1, n_2] = 1,$$

$$\mathscr{F}_1 : [c_1, n_1^{-1} n_2] = 1, \quad \mathscr{G}_1 : (c_1 n_1)^3 = (n_1 c_1)^3, \quad \mathscr{H}_1 : [p_1, n_1] = 1,$$

$$\mathscr{H}_2 : [p_2, n_2] = 1, \quad \mathscr{I}_1 : [p_1, c_1^{-1} n_2 c_1] = 1, \quad \mathscr{I}_2 : [p_2, c_1^{-1} n_1 c_1] = 1.$$

PROOF. Let us consider the affine curve

$$\Theta_1 := C_1 \cup P_1 \cup P_2 \cup N_1 \cup N_2 \cup D.$$

There are two facts:

- The map $\phi : \boldsymbol{C}^2 \backslash \Theta_1 \to \boldsymbol{C}^2 \backslash \Gamma$ is the unramified double covering determined by the monodromy epimorphism $\varphi : G \to \boldsymbol{Z}/2\boldsymbol{Z}$ such that

$$\varphi(c) = \varphi(p) = \varphi(n) = 0 \bmod 2 \quad \text{and } \varphi(d) = 1 \bmod 2.$$

- $\boldsymbol{C}^2 \backslash \Theta_1 = \boldsymbol{P}^2 \backslash (\Theta \cup D)$.

We can compute a presentation for

$$\ker \varphi = \pi_1(\boldsymbol{C}^2 \backslash \Theta_1; *_1) = \pi_1(\boldsymbol{P}^2 \backslash (\Theta \cup D); *_1)$$

applying the Reidemeister-Schreier algorithm. It is easily seen that $\ker \varphi$ is generated by $d_1 := d^2$ (meridian of D), $c_1 := c$ (meridian of C_1), $p_1 := p$ (meridian of P_1), $p_2 := dpd^{-1}$ (meridian of P_2), $n_1 := n$ (meridian of N_1) and $n_2 : dnd^{-1}$ (meridian of N_2).

Let us consider the homomorphism

$$\sigma : \pi_1(\boldsymbol{P}^2 \backslash (\Theta \cup D); *_1) \to \pi_1(\boldsymbol{P}^2 \backslash \Theta; *_1)$$

induced by the open embedding

$$\boldsymbol{P}^2 \backslash (\Theta \cup D) \hookrightarrow \boldsymbol{P}^2 \backslash \Theta.$$

It is well-known, see [**Z1**] or [**F**], that σ is an epimorphism and the kernel is the subgroup generated by the meridians of D. Then it is enough to add to the set of relations obtained by the Reidemeister-Schreier algorithm, the relation $d_1 = 1$ (any meridian of D is conjugated to d_1). Simplifying the new set of relations, we get the one of the statement. □

LEMMA. *The loop* $l_1 := (n_2 c_1 n_1 c_1 p_2 c_1 p_1)^{-1}$ *is a meridian of* L *in the group* H.

PROOF. Let us look for a meridian of L. We proceed as follows. Take a curve H_1 with a transversal intersection with L at a point S — which is not a point at infinity of any irreducible (affine) component of Θ_1. Take a meridian of S in $H_1 \backslash \Theta$ and take a path from $*_1$ to a generic point of H_1; if we conjugate by this path, this loop in H_1 becomes a meridian of L.

Let us take the parabola H whose (real) equation is $y = -3(x/3 - 7/3)^2 + 64/3$. Let $H_1 = \phi^{-1}(H)$; H_1 is a hyperbola and its points at infinity are not the points at infinity of the irreducible components of Θ_1. It intersects $C \cup D \cup N \cup P$ as it is shown in figure 9. Take a temporary base point $*'$ in the vertex of H. Choose the shortest path ℓ from $*$ to $*'$ and consider the morphism defined by the inclusion and the path ℓ:

$$\pi_1(H \backslash (C \cup D \cup N \cup P); *') \to \pi_1(\boldsymbol{C}^2 \backslash \Gamma; *).$$

such that $* \in H$, $\delta \gg 0$, $\gamma \in (-\sqrt{3}, \sqrt{3})$, $\delta \gg 0$ and $0 < \beta < 3$.

Let us explain how to construct the meridians in $H \backslash \Gamma$. Let us fix an intersection point of Γ and H, say U (any such point is real). We choose the shortest path l_U from

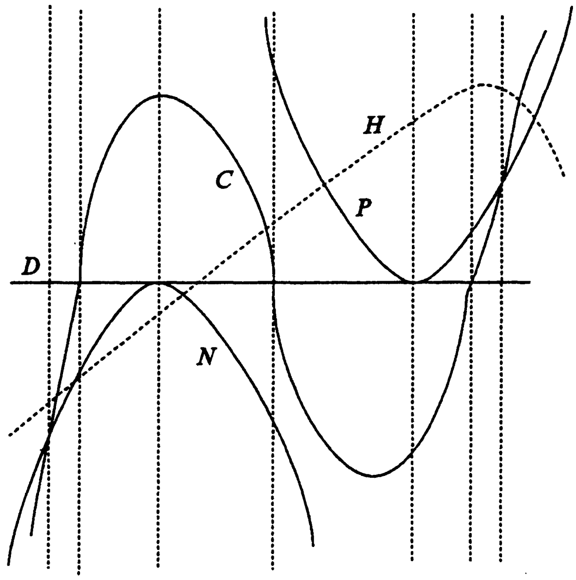

Figure 9.

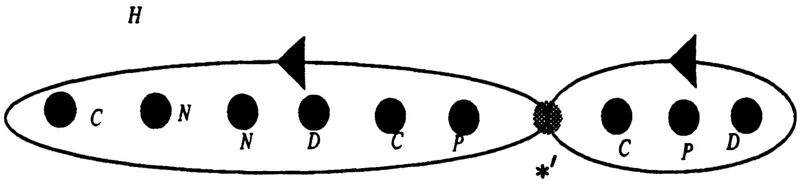

Figure 10.

$*'$ towards U contained in the real part of H. Take also a small disk Δ_U centered at U; let us call U_+ the intersection of $\partial\Delta_U$ with l_U and U_- the opposite point to U_+ in Δ_U. We construct a path m_U from $*'$ to U_+ as follows: start from l_U and for any $V \in H \cap \Gamma$ replace the segment $[V_+, V_-]$ by the the counterclockwise arc from V_+ to V_- in $\partial\Delta_V$. Let us note δ_U the loop based in U_+ which turns counterclockwise the circle $\partial\Delta_U$. The meridian associated to U is $m_U \cdot \delta_U \cdot m_U^{-1}$. We construct in this way meridians $\tilde{c}_1$, $\tilde{n}_1$, $\tilde{n}_2$, $\tilde{d}_1$, $\tilde{c}_2$, $\tilde{p}_1$, $\tilde{d}_2$, $\tilde{p}_2$ and $\tilde{c}_3$. We will use also this notation for the meridians based on $*$ and obtained from these ones applying the change of base point by ℓ.

From figure 10, we can choose a meridian l of the point at infinity of H such that $l = (\tilde{l}_1 \cdot \tilde{l}_2)^{-1}$, where:

$$\tilde{l}_1 := \tilde{c}_1\tilde{n}_1\tilde{n}_2\tilde{d}_1\tilde{c}_2\tilde{p}_1, \quad \tilde{l}_2 := \tilde{d}_2\tilde{p}_2\tilde{c}_3.$$

The loop $\tilde{l}_1$ (resp. $\tilde{l}_2$) turns around counterclockwise the intersection points of Γ and H in the *left-hand side* (resp. *right-hand side*) of $*$ in H.

It is easily seen that the meridian associated with each point in $\Gamma \cap H$ (except for the point belonging to C in the left-hand side) is homotopy equivalent to a meridian (based at $*$) of the corresponding irreducible component of Γ in a fiber F_w for a given $w \in NT \cup \{\pm\infty\}$. The choice of w is determined by the real sector which contains the given point in $\Gamma \cap H$. For $\tilde{c}_1$ we must conjugate with $\tilde{n}_1$.

We show that:

$$\tilde{c}_1 = d^{-1}ndcd^{-1}n^{-1}d, \quad \tilde{n}_1 = d^{-1}nd, \quad \tilde{n}_2 = n, \quad \tilde{d}_1 = d, \quad \tilde{c}_2 = c, \quad \tilde{p}_1 = p.$$

Then, we have:

$$\tilde{l}_1 := (d^{-1}nd)cndcp.$$

There is a homotopy which shows that $\tilde{l}_2 = cdp$. Then,

$$l := (d^{-1}ndcndcpcdp)^{-1}.$$

It is easily seen that $l \in \ker \varphi$ (d appears twice). Then, $l_1 := \sigma(l)$ is a meridian of L; we obtain l_1 as in the statement. □

§ 3. Old Zariski pairs

We will apply elsewhere a well-known result, see [**Z1**] or [**F**] (we have already used it):

LEMMA. *Let $A, B \subset \boldsymbol{P}^2$ be projective plane curves with no irreducible component in common and let $* \in \boldsymbol{P}^2 \backslash (A \cup B)$. Then the morphism*

$$\sigma : \pi_1(\boldsymbol{P}^2 \backslash (A \cup B); *) \to \pi_1(\boldsymbol{P}^2 \backslash A; *)$$

induced by the inclusion is an epimorphism. The kernel of σ is the subgroup generated by the meridians of the irreducible components of B.

EXAMPLE 1. Let us recall the members of the Zariski pair:

$$A_1 = C_1 \cup L \cup P_1 \cup P_2, \quad A_2 = C_1 \cup L \cup P_1 \cup N_1.$$

Consider the epimorphism

$$\sigma_1 : G \to \pi_1(\boldsymbol{P}^2 \backslash A_1; *_1).$$

Then, we obtain a presentation of $\pi_1(\boldsymbol{P}^2 \backslash A_1; *_1)$ from the given presentation of G: they have the same sets of generators and we have to add the relations

$$n_1 = 1, \quad n_2 = 1.$$

Simplifying the presentation we have:

$$\pi_1(\boldsymbol{P}^2 \backslash A_1; *_1) = \langle c_1, p_1, p_2 : [p_1, p_2] = 1, [c_1, p_1^{-1}p_2] = 1, (c_1p_1)^3 = (p_1c_1)^3 \rangle.$$

We find that $l_1(A_1) := (c_1p_2c_1p_1c_1)^{-1}$ is a meridian of L. Let us call $q = p_1^{-1}p_2$. We obtain:

$$\pi_1(\boldsymbol{P}^2 \backslash A_1; *_1) = \langle c_1, p_1, q : [p_1, q] = 1, [c_1, q] = 1, (c_1p_1)^3 = (p_1c_1)^3 \rangle.$$

Then

$$\pi_1(\boldsymbol{P}^2 \backslash A_1; *_1) = \langle q : \text{—} \rangle \times \langle c_1, p_1 : (c_1p_1)^3 = (p_1c_1)^3 \rangle = \boldsymbol{Z} \times G_3,$$

where G_3 is the fundamental group of the torus link of type $(2, 6)$ in the sphere of real dimension 3. In particular, G_3 and, therefore, $\pi_1(\boldsymbol{P}^2 \backslash A_1; *_1)$ are non-abelian.

Let us consider the epimorphism

$$\sigma_2 : G \to \pi_1(\boldsymbol{P}^2 \setminus A_2; *_1).$$

In this case we add the relations

$$p_2 = 1, \quad n_2 = 1.$$

Simplifying the presentation we have:

$$\pi_1(\boldsymbol{P}^2 \setminus A_2; *_1) = \langle c_1, p_1, n_1 : [c_1, p_1] = 1, [c_1, n_1] = 1, [p_1, n_1] = 1 \rangle.$$

Then, we have

$$\pi_1(\boldsymbol{P}^2 \setminus A_2; *_1) = \boldsymbol{Z}^3.$$

We have proved the claim in [**A1**].

EXAMPLE 2. We recall the construction of this example. Let $L_X, L_Y, L_Z \subset \boldsymbol{P}^2$ three lines in general position. We may suppose that their equations are $X = 0$, $Y = 0$ and $Z = 0$ respectively. Let us consider the map $\tau : \boldsymbol{P}^2 \to \boldsymbol{P}^2$ given by

$$\tau([x : y : z]) := [x^2 : y^2 : z^2];$$

then τ is a $\boldsymbol{Z}/2\boldsymbol{Z} \times \boldsymbol{Z}/2\boldsymbol{Z}$-covering ramified along $L_X \cup L_Y \cup L_Z$. If $A \subset \boldsymbol{P}^2$ is a curve which does not contain any of the lines L_X, L_Y, L_Z, the curve $\tau^{-1}(A)$ is called the *$\boldsymbol{Z}/2\boldsymbol{Z} \times \boldsymbol{Z}/2\boldsymbol{Z}$-covering of A with respect to* $\{L_X, L_Y, L_Z\}$ (this curve is well-defined up to linear automorphisms of $\boldsymbol{P}^2$). Let $* \in \boldsymbol{P}^2 \setminus (A \cup L_X \cup L_Y \cup L_Z)$ and $*_1 \in \tau^{-1}(*)$. We have a monomorphism

$$\tau_* : \pi_1(\boldsymbol{P}^2 \setminus (\tau^{-1}(A) \cup L_X \cup L_Y \cup L_Z); *_1) \to \pi_1(\boldsymbol{P}^2 \setminus (A \cup L_X \cup L_Y \cup L_Z); *)$$

and an epimorphism

$$\sigma : \pi_1(\boldsymbol{P}^2 \setminus (\tau^{-1}(A) \cup L_X \cup L_Y \cup L_Z); *_1) \to \pi_1(\boldsymbol{P}^2 \setminus (\tau^{-1}(A); *_1)$$

induced by the inclusion.

Identify τ_* with an inclusion; we have a subgroup of index 4 and we can recover a presentation of the subgroup from a presentation of the whole group applying Reidemeister-Schreier algorithm.

Denote m_x, m_y, m_z some meridians of the lines (in the base of the covering), we can see that the kernel of σ is the normal subgroup generated by m_x^2, m_y^2, m_z^2.

We return to the example. Let us denote Z_1 (resp. Z_2) the $\boldsymbol{Z}/2\boldsymbol{Z} \times \boldsymbol{Z}/2\boldsymbol{Z}$-covering of C_1 with respect to $\{L, P_1, P_2\}$ (resp. $\{L, P_1, N_1\}$). We have shown in [**A1**] that Z_1 (resp. Z_2) is a sextic having six ordinary cusps on a conic (resp. not on a conic).

From the method explained above it is not difficult but long to compute the group of Z_1. Of course, one obtains that G_{Z_1} is the free product of $\boldsymbol{Z}/2\boldsymbol{Z}$ and $\boldsymbol{Z}/3\boldsymbol{Z}$, as it was computed by Zariski in [**Z1**].

Let us consider now Z_2. We must apply the method above to the curve A_2 of Example 1. We have shown that its group is abelian; it is also the case for G_{Z_2} which is a quotient of a subgroup of G_{A_2}. By homological arguments, G_{Z_2} is cyclic of order 6.

Then we have found a sextic with six cusps not on a conic such that its fundamental group is abelian. M. Oka found another such sextic in [O1]. Zariski sketched an argument in the same direction if the sextic degenerates onto a sextic with nine cusps, see [Z3]. We observe that these results do not imply, up to now, that the fundamental group of any sextic with six cusps not on a conic is cyclic of order six: There is no result about the connectivity of the space of such curves.

EXAMPLE 3. We are concerned now with Degtyarev's example. We begin with some general facts.

Let $L_X, L_Y, L_Z \subset \boldsymbol{P}^2$ three lines in general position. We suppose as above that their equations are $X = 0$, $Y = 0$ and $Z = 0$ respectively. Let us consider the rational map $\gamma : \boldsymbol{P}^2 \to \boldsymbol{P}^2$ given by

$$\gamma([x : y : z]) := [yz : xz : xy];$$

γ is a Cremona transformation of $\boldsymbol{P}^2$ which is an automorphism outside $L_\gamma := L_X \cup L_Y \cup L_Z$. If $A \subset \boldsymbol{P}^2$ is a curve which does not contain any of the lines L_X, L_Y, L_Z, the curve $A_\gamma := \overline{\gamma^{-1}(A \backslash L_\gamma)}$ is called the *strict transform of A with respect to* $\{L_X, L_Y, L_Z\}$ (A_γ is well-defined up to linear automorphisms of $\boldsymbol{P}^2$). Let $* \in \boldsymbol{P}^2 \backslash (A \cup L_X \cup L_\gamma)$ and $*_1 \in \gamma^{-1}(*)$. We have an isomorphism

$$\gamma_* : \pi_1(\boldsymbol{P}^2 \backslash (A_\gamma \cup L_\gamma); *_1) \to \pi_1(\boldsymbol{P}^2 \backslash (A \cup L_\gamma); *)$$

and an epimorphism

$$\sigma : \pi_1(\boldsymbol{P}^2 \backslash (A_\gamma \cup L_\gamma); *_1) \to \pi_1(\boldsymbol{P}^2 \backslash A_\gamma; *_1)$$

induced by the inclusion.

We recall that the members of this pair are sextics with three singular points of type $\boldsymbol{E}_6$. We define D_1 as the strict transform of C_1 with respect to $\{L, P_1, P_2\}$ (there exists a conic tangent to C at the singular points). We set D_2 as the strict transform of C_1 with respect to $\{L, P_1, N_1\}$ (there is no such a conic). These two curves are projectively rigid.

In order to compute G_{D_1}, we must add three relations to the presentation of G_{A_1}; it is not difficult (but rather long) to find these relations. With this method, we find the result which was communicated by Degtyarev, $G_{D_1} = \boldsymbol{Z}/2\boldsymbol{Z} * \boldsymbol{Z}/3\boldsymbol{Z}$.

We recall that G_{D_2} is a quotient of G_{A_2}, which is abelian. Then, $G_{D_2} = \boldsymbol{Z}/6\boldsymbol{Z}$.

§ 4. A new Zariski pair

EXAMPLE 4. We recall this example: $B_1 = C_1 \cup L \cup P_1 \cup P_2 \cup N_1$ and $B_2 = C_1 \cup P_1 \cup P_2 \cup N_1 \cup N_2$.

For a given degree d we consider the space $\boldsymbol{P}_d$ of all curves of degree d, which is a projective space of dimension $d(d+3)/2$.

PROPOSITION. *Let* $\mathscr{M}_1$ (*resp.* $\mathscr{M}_2$) *the space of curves of degree 7 having five reducible components* C, L_1, L_2, L_3, L_4 *such that*:

(a) *C is a smooth cubic and L_i is a line, $i = 1, \dots, 4$.*

(b) *The arrangement of lines $\{L_1, \dots, L_4\}$ has only double points.*

(c) *L_i is tangent to C at an inflection point $P_i, i = 1, \dots, 4$.*
(d) *There are (resp. are not) three aligned points in $\{P_1, \dots, P_4\}$.*
Then, the spaces $\mathcal{M}_1$ and $\mathcal{M}_2$ are connected.

We leave to the reader the verification of this statement. We observe that $B_1 \in \mathcal{M}_1$ and $B_2 \in \mathcal{M}_2$. We recall another fact which can be deduced from [**A1**]:

PROPOSITION. *The Alexander polynomial of any curve in $\mathcal{M}_1$ and $\mathcal{M}_2$ is trivial.*

We can compute the fundamental groups of B_1 and B_2. We denote $G_i := G_{B_i}$, $i = 1, 2$. It is easily seen that:

$$G_1 = \langle c_1, p_1, p_2, n_1 : \mathcal{B}_1, \mathcal{C}_1, \mathcal{D}_1, \mathcal{F}_2, \mathcal{H}_1, \mathcal{I}_3 \rangle,$$

where

$$\mathcal{B}_1 : [p_1, p_2] = 1, \quad \mathcal{C}_1 : [c_1, p_1^{-1} p_2] = 1, \quad \mathcal{D}_1 : (c_1 p_1)^3 = (p_1 c_1)^3,$$

$$\mathcal{F}_2 : [c_1, n_1] = 1, \quad \mathcal{H}_1 : [p_1, n_1] = 1, \quad \mathcal{I}_3 : [p_2, n_1] = 1.$$

As n_1 is central, we deduce from the computations in Example 1 that:

PROPOSITION. *We have $G_1 = \boldsymbol{Z}^2 \times G_3$, where $\boldsymbol{Z}^2$ is generated by n_1 and $p_1^{-1} p_2$ and $G_3 = \langle p_1, c_1 : (p_1 c_1)^3 = (c_1 p_1)^3 \rangle$. The loop c_1 is a meridian of C_1, p_1 is a meridian of P_1, p_2 is a meridian of P_2 and $(c_1 n_1 c_1 p_2 c_1 p_1)^{-1}$ is a meridian of L.*

Let us compute now G_2. We have:

PROPOSITION. *We have*

$$G_2 = \langle c_1, p_1, p_2, n_1, n_2 : \mathcal{B}_1, \mathcal{C}_1, \mathcal{D}_1, \mathcal{E}_1, \mathcal{F}_1, \mathcal{G}_1, \mathcal{H}_1, \mathcal{H}_2, \mathcal{I}_1, \mathcal{I}_2, \mathcal{J} \rangle,$$

where

$$\mathcal{B}_1 : [p_1, p_2] = 1, \quad \mathcal{C}_1 : [c_1, p_1^{-1} p_2] = 1, \quad \mathcal{D}_1 : (c_1 p_1)^3 = (p_1 c_1)^3, \quad \mathcal{E}_1 : [n_1, n_2] = 1,$$

$$\mathcal{F}_1 : [c_1, n_1^{-1} n_2] = 1, \quad \mathcal{G}_1 : (c_1 n_1)^3 = (n_1 c_1)^3, \quad \mathcal{H}_1 : [p_1, n_1] = 1,$$

$$\mathcal{H}_2 : [p_2, n_2] = 1, \quad \mathcal{I}_1 : [p_1, c_1^{-1} n_2 c_1] = 1, \quad \mathcal{I}_2 : [p_2, c_1^{-1} n_1 c_1] = 1,$$

$$\mathcal{J} : n_2 c_1 n_1 c_1 p_2 c_1 p_1 = 1.$$

These loops are meridians corresponding to the curves with capitalized letters.

By counting epimorphisms onto the third symmetric group we have:

THEOREM. *The groups G_1 and G_2 are both non-abelian but non-isomorphic. Then (B_1, B_2) is a Zariski pair which is not distinguished by the Alexander polynomial and such that the two members have non-abelian fundamental group.*

We can distinguish these groups also by counting epimorphisms onto the fourth symmetric group. Computations have been performed by GAP which is a software for computation in groups, [**GAP**]

§5. A Zariski pair obtained by deformation

EXAMPLE 5. Let us fix a smooth cubic $\tilde{C}$. Let us consider a conic $\tilde{Q}$ such that $\tilde{Q} \cap \tilde{C} = \{R_1, R_2\}$ and $(\tilde{Q} \cdot \tilde{C})_{R_i} = 3$. We have seen that the line determined by R_1 and R_2 cuts $\tilde{C}$ also in an inflection point $P_{\tilde{Q}}$ of $\tilde{C}$, which is called *the inflection point of* $\tilde{C}$ *associated to* $\tilde{Q}$.

We recall that $\tilde{Q}$ is reducible if and only if R_1 (or/and R_2) is an inflection point of $\tilde{C}$. We call $\tilde{Q}$ *a double-inflection conic of* $\tilde{C}$.

PROPOSITION. *Let $\mathcal{N}_1$ (resp. $\mathcal{N}_2$) the space of curves of degree 7 having three reducible components C, Q_1, Q_2 such that:*
(a) *C is a smooth cubic and Q_i is an irreducible conic, $i = 1, 2$.*
(b) *Q_1 and Q_2 intersect transversally at four points.*
(c) *Q_i is a double-inflection conic of C; let P_{Q_i} be the associated inflection point, $i = 1, 2$.*
(d) *$P_{Q_1} \neq P_{Q_2}$ (resp. $P_{Q_1} = P_{Q_2}$).*
Then, the spaces $\mathcal{N}_1$ and $\mathcal{N}_2$ are connected and non-empty.

PROOF. We will prove that $\mathcal{N}_2$ is connected and non-empty; the statement for $\mathcal{N}_1$ is proven in the same way. Let $\mathcal{M}$ be the space of smooth cubics; for a given $C \in \mathcal{M}$, let I_C be the set of inflection points of C. We set

$$\mathcal{M}_I := \{(C, P) \in \mathcal{M} \times \boldsymbol{P}^2 \mid P \in I_C\}$$

and

$$\tilde{\mathcal{M}} := \{((C, P), R_1, R_2) \in \mathcal{M}_I \times \boldsymbol{P}^2 \times \boldsymbol{P}^2 \mid R_1, R_2 \in C\}.$$

We note that there exists a dominant rational map of

$$\delta : \tilde{\mathcal{M}} \dashrightarrow \tilde{\mathcal{N}_2},$$

where $\tilde{\mathcal{N}_2}$ is the closure of the space of curves verifying (a), (c) and the first part of (d) for P. Given R_i, $i = 1, 2$, let R_i' be the other point of C aligned with R_i and P; take the double-inflection conic Q_i associated to P and passing through R_i and R_i'. Then,

$$\delta((C, P), R_1, R_2) := C \cup Q_1 \cup Q_2.$$

We deduce that $\tilde{\mathcal{N}_2}$ is irreducible. It is easily seen that $\mathcal{N}_2$ is a Zariski-open subset of $\tilde{\mathcal{N}_2}$. Then, it is enough to see that $\mathcal{N}_2$ is non-empty.

Take the curve B_2 of the last example; it is the image of

$$\Omega := ((C_1, O), I_1, J_1) \in \tilde{\mathcal{M}}.$$

The fact that the four lines of B_2 are in general position implies that the conics corresponding to the image by δ of a point in $\tilde{\mathcal{M}}$ close to Ω verify (b). Then, $\mathcal{N}_2$ is non-empty. $\square$

DEFINITION. Let $\gamma : I \to: \boldsymbol{P}_d$ a continuous path of reduced curves, I is an interval and $C_t := \gamma(t)$ for $t \in I$. We say that the family $\{C_t \mid t \in I\}$ is *equisingular* if there exist continuous paths $\gamma_i : I \to \boldsymbol{P}^2, i = 1, \ldots, n$, such that
(a) For all $t \in I, \{\gamma_i(t) \mid 1 \leq i \leq n\}$ is the set of singular points of C_t.
(b) For all $i = 1, \ldots, n$, the family $\{(C_t, \gamma_i(t)) \mid t \in I\}$ is equisingular.

The next result is well-known, see [C] for a proof:

Lemma. *The curves in an equisingular path of curves are isotopic in* $\boldsymbol{P}^2$.

Then, any two curves in $\mathcal{N}_1$ are isotopic; the same statement is true for $\mathcal{N}_2$. Let us fix $M_1 \in \mathcal{N}_1$ (resp. $M_2 \in \mathcal{N}_2$) close to B_1 (resp. B_2).

Definition. Let $\gamma : [0,1] \to \boldsymbol{P}_d$ a continuous path of reduced curves, such that the family induced by $\gamma_{|(0,1]}$ is equisingular. Note $C_t := \gamma(t)$ if $t \in [0,1]$. We say that C_0 is a *degeneration* of C_1 and γ is the *degeneration path.*

We can also find a proof of this well-known result in [C]:

Lemma. *Let* $C_0, C_1 \subset \boldsymbol{P}^2$ *such that* C_0 *is a degeneration of* C_1. *Then there exists an epimorphism* $\sigma : G_{C_0} \to G_{C_1}$.

We can describe also the kernel of this epimorphism from the Zariski-Van Kampen method.

Set $H_i := G_{M_i}$, $i = 1,2$. Let us fix a degeneration map from M_i to B_i. For each $i = 1,2$ there exist two ordinary double points $P_1^i, P_2^i \in B_i$ which are not limit of singular points in the degeneration path. If $\tilde{P}_i$ is another singular point of B_i, then there is a small neighbourhood of $\tilde{P}_i$ which contains exactly one singular point $\hat{P}_i$ of M_i; moreover, $(B_i, \tilde{P}_i)$ and $(M_i, \hat{P}_i)$ have the same topological type.

In the Zariski-Van Kampen method, each double point produces a relation. Suppose that the relation induced by P_j^i is $[g_j^i, h_j^i] = 1$, where g_j^i, h_j^i are meridians of the irreducible components of B_i which meet at P_j^i.

Lemma. *Consider the epimorphism* $\sigma_i : G_i \to H_i$ *of the degeneration path. Then,* $\ker \sigma_i$ *is the smallest normal subgroup of* G_i *containing* $g_j^i (h_j^i)^{-1}$.

Proof. Consider the Zariski-Van Kampen method. We choose a generic fiber, where we can identify the generators of G_i and H_i. We can suppose that the monodromy for B_i around each singular point $\tilde{P}_i \neq P_1^i, P_2^i$ agrees with the monodromy for M_i around $\hat{P}_i$.

We find differences near P_1^i and P_2^i, where $[g_j^i, h_j^i] = 1$ (relation in G_i) is replaced by $g_j^i = h_j^i$ (in H_i). □

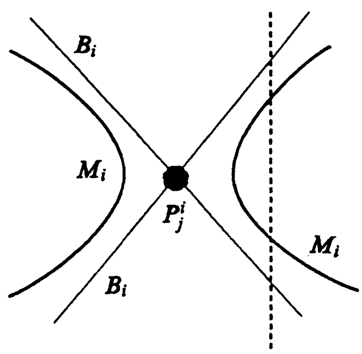

Figure 11.

THEOREM. *The group H_1 is free abelian of rank* 2.

PROOF. We can suppose that a meridian of N_1 equals a meridian of P_1. As n_1 is central in H_1, we deduce that $p_1 = n_1$ in H_1 and is also central in H_1. By homological reasons, $H_1 = \boldsymbol{Z}^2$. □

THEOREM. *The group H_2 is the middle term of a short exact sequence*:

$$0 \to H_2' \xrightarrow{i} H_2 \xrightarrow{\mu} \boldsymbol{Z}^2 \to 0, \tag{1}$$

where H_2' (the first derived group of H_2) is also the middle term of a short exact sequence:

$$0 \to H_2'' \xrightarrow{j} H_2' \xrightarrow{\nu} \boldsymbol{Z}^2 \to 0, \tag{2}$$

where H_2'' (the second derived group of H_2) is cyclic of order two. There exist $b_1, b_2 \in H_2'$ such that $\nu(b_1), \nu(b_2)$ generate $\boldsymbol{Z}^2$ and $[b_1, b_2] = j(t)$ where t is the nonzero element of H_2''. We can choose $a_1, a_2 \in H_2$ such that $\mu(a_1)$, $\mu(a_2)$ generate $\boldsymbol{Z}^2$ and

$$[a_1, a_2] = i(b_1), \quad a_1 i(b_1) a_1^{-1} = i(b_2), \quad a_1 i(b_2) a_1^{-1} = i(b_2^{-1} b_1^{-1}),$$

$$a_2 i(b_1) a_2^{-1} = i(b_2), \quad a_2 i(b_2) a_2^{-1} = i(b_1^{-1} b_2^{-1}).$$

This group is non-abelian but admits a subgroup of index 12 *which is abelian.*

PROOF. We can realize the degeneration path by moving P down and moving N up. One shows immediately that the relations we must add are $p_1 = p_2$ and $n_1 = n_2$. We find:

$$H_2 = \langle c_1, p_1, n_1 \mid (c_1 p_1)^3 = (p_1 c_1)^3, \quad (c_1 n_1)^3 = (n_1 c_1)^3,$$

$$[p_1, n_1] = 1, \quad [p_1, c_1^{-1} n_1 c_1] = 1, \quad n_1 c_1 n_1 (c_1 p_1)^2 = 1 \rangle.$$

Using GAP, we show the existence of an epimorphism of H_2 onto the fourth alternating group; then H_2 is not abelian.

We construct the exact sequences from the map onto the abelianized groups of H_2 and H_2'. It is a long but easy computation. One can show that the subgroup generated by $a_1^3, (a_2 a_1^{-1})^2, b_1, b_2^2$ and t is isomorphic to $\boldsymbol{Z}^4 \times \boldsymbol{Z}/2\boldsymbol{Z}$ and it is of index 12. □

COROLLARY. *The groups H_1 and H_2 are non-isomorphic. Then (M_1, M_2) is a Zariski pair which is not distinguished by the Alexander polynomial.*

§ 6. A Zariski pair whose members have only rational components.

EXAMPLE 6. Take C an irreducible nodal cubic. We can define double-inflection conics as in the smooth case; it is also possible to associate an inflection point to each double-inflection conic.

The next proposition is analogous to the proposition in Example 5.

PROPOSITION. *Let $\mathscr{P}_1$ (resp. $\mathscr{P}_2$) the space of curves of degree 7 having three reducible components C, Q_1, Q_2 such that*:

(a) *C is an irreducible nodal cubic and Q_i is an irreducible conic, $i = 1, 2$.*
(b) *Q_1 and Q_2 intersect transversally at four points.*

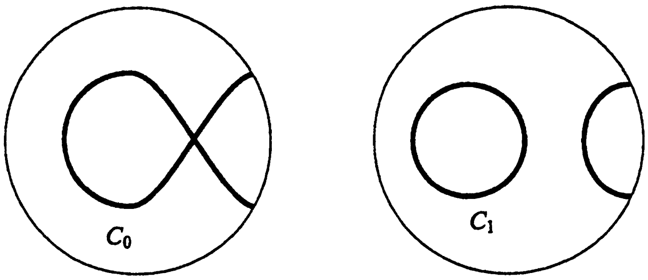

Figure 12.

(c) *Q_i is a double-inflection conic of C; let P_{Q_i} be the associated inflection point, $i = 1, 2$.*
(d) *$P_{Q_1} \neq P_{Q_2}$ (resp. $P_{Q_1} = P_{Q_2}$).*

Then, the spaces $\mathscr{P}_1$ and $\mathscr{P}_2$ are connected and non-empty. Moreover, each curve in $\mathscr{P}_i$ is a degeneration of curves in $\mathscr{N}_i$, $i = 1, 2$.

In order to get the fundamental groups of curves in $\mathscr{P}_1$ and $\mathscr{P}_2$, we prove next lemma:

Lemma. *Let $\gamma : [0,1] \to \boldsymbol{P}_d$ a continuous path of curves, set $C_t := \gamma(t)$. Suppose that:*
(a) *C_0 is a degeneration of C_1 and γ is a degeneration path.*
(b) *C_t has real equations for all $t \in [0,1]$.*
(c) *There exist $P \in C_0$ and a neighbourhood B_P of P in $\boldsymbol{P}^2$ verifying:*
  (c1) *P is a nodal point of C_0, and there is no singular point of C_t in B_P, $t \in [0,1]$.*
  (c2) *Let E_P the closure of $\boldsymbol{P}^2 \backslash B_P$; there exists an isotopy of E_P which sends $C \cap E_P$ onto $D \cap E_P$ (γ is equisingular outside B_P).*
  (c3) *The real part of C_1* degenerates to *D in B_P as in the Figure* 12.

Then G_{C_0} is isomorphic to G_{C_1}.

Proof. It is enough to apply Zariski-Van Kampen method. For a given projection anything is similar for C_0 and C_1 but in the neighbourhood B_P of P. Let us call m_t and n_t the meridians of C_t whose circle is contained in B_P, $t \in [0,1]$ (we can choose them such that they coincide outside B_P).

There is only one relation for $C_1 : m_1 = n_1$. For C_0 we find two relations: $[m_0, n_0] = 1$ and $m_0 = n_0$. Let us consider the epimorphism $\sigma : G_{C_0} \to G_{C_1}$. It is easily seen that $\sigma(m_0) = m_1$ and $\sigma(n_0) = n_1$.

We deduce also that the kernel of σ is the smallest normal subgroup of G_{C_0} containing $m_0(n_0)^{-1}$. From the above arguments, the kernel is trivial. □

Theorem. *For any $S_i \in \mathscr{P}_i$, $G_{S_i} = H_i$, $i = 1, 2$.*

Proof. The key of the proof is to find real models of S_1 and S_2 such that there is no singular point of the vertical projection between the singular point of the cubic and a point of vertical tangent in the cubic. It will be clear that a suitable deformation produces curves in $\mathscr{N}_1$ and $\mathscr{N}_2$ respectively, such that the two degenerations match with the previous lemma.

Observe that in the model of § 1, the goal is to obtain curves where a maximum

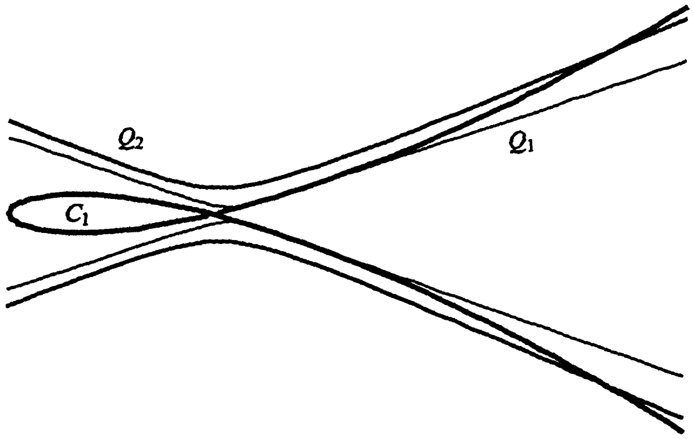

Figure 13.

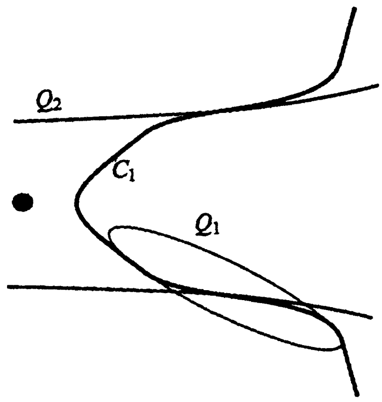

Figure 14.

number of non-transversal vertical lines should have real equations. In this case, we already know one fundamental group and we do not need a so complicated model.

Let us consider an example of a curve $S_2 := C \cup Q_1 \cup Q_2$ whose equations are:

$$C : y^2z = x^2(x+3z), \quad Q_1 : y^2 = 6x^2 - 3xz + z^2, \quad Q_2 : y^2 = 9x^2 - 2xz + 8z^2.$$

It is easily seen that there exists a degeneration path of curves $\mathcal{N}_2$ towards S_2 satisfying the previous lemma. We can see the real affine part of S_2 in Figure 13.

Let us consider a curve S_1 given by:

$$C : y^2z = x^2(x-3z),$$
$$Q_1 : y^2 = 3(\alpha-1)x^2 - 3\alpha^2xz + \alpha^3z^2, \quad \alpha \in \boldsymbol{R} \text{ big enough},$$

and the curve Q_2:

$$3(3-3t+t^2)x^2 + y^2 + 3(t-1)xy + (t-7)(t-1)^2yz - 3(t+1)(t-1)^2xz + 8(t-1)^3z^2$$
$$= 0, \quad t \in (3,4) \subset \boldsymbol{R}.$$

The real affine part of S_1 is in figure 14.

There exists a degeneration path of curves in $\mathcal{N}_1$ towards S_1 which verifies the previous lemma if we replace (c3) by (c3′) and (c3″):

(c3′): *The real part of C_t in B_P degenerates to C_0 as in Figure* 15.

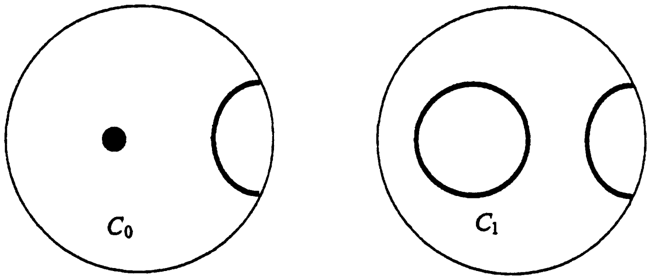

Figure 15.

(c3″): *There exists a ramified double covering ramified $p : B_P \longrightarrow p(B_P)$ along the line D such that the real part of $p(C_t)$ degenerates to $p(C_0)$ in $p(B_P)$ as in Figure* 16.

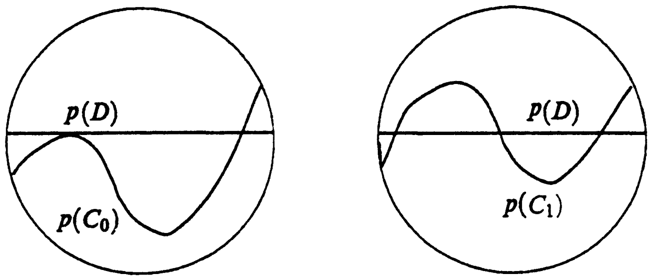

Figure 16.

$\square$

COROLLARY. *Let $S_1 \in \mathscr{P}_1$ and $S_2 \in \mathscr{P}_2$. Then (S_1, S_2) is a Zariski pair which is not distinguished by the Alexander polynomial and such that the irreducible components of each member are rational.*

§7. Zariski pairs and superisolated singularities

We refer to **[Lu]** and **[A2]** and references therein. Let $(V, 0) \subset (\boldsymbol{C}^3, 0)$ a germ of isolated singularity of multiplicity d; let $f \in \boldsymbol{C}\{x, y, z\}$ a convergent power series such that $V = f^{-1}(0)$. Let $f := f_d + f_{d+1} + \cdots$, where f_m is a homogeneous polynomial of degree m, $m \geq d$. Let us call $C_m \subset \boldsymbol{P}^2$ the projective curve defined by f_m ($C_m = \boldsymbol{P}^2$ if $f_m \equiv 0$). We recall that C_d is the tangent cone of V ($C_d \neq \boldsymbol{P}^2$).

DEFINITION. $(V, 0)$ *is a superisolated singularity* if the singular points of C_d are not in C_{d+1}.

This is not the usual definition but it is convenient for us. We note that the tangent cone of a superisolated singularity is always reduced.

We introduce some aspects of the Milnor theory, see **[M]**. Let V be a germ of isolated singularity in $\boldsymbol{C}^3$. For a small $\varepsilon > 0$, the intersection of the euclidean sphere centered at 0 of radius ε (denoted S_ε^5) with V is a compact oriented 3-manifold without boundary, denoted K_ε. The topological type of $(S_\varepsilon^5, K_\varepsilon)$ does not depend on ε; it will be denoted (S^5, K) and K (K is the *abstract link* of the singularity and (S^5, K) is its *link*). There is a locally trivial fibration $\varphi : S^5 \backslash K \to S^1$ (called the *Milnor fibration*) of V. The fiber F (the *Milnor fiber*) of φ has the homotopy type of a *bouquet* of spheres of dimension 2. The fibration φ is determined (up to isotopy and conjugation) by a homeomorphism $\sigma : F \to F$ (the *geometric monodromy*). The *complex monodromy* is the linear automorphism $\sigma^* : H^2(F; \boldsymbol{C}) \to H^2(F; \boldsymbol{C})$.

PROPOSITION. **[Lu]** *Let $\{V_t\}_{t \in [0,1]}$ a continuous family of superisolated singularities. Let us suppose that the induced family of tangent cones is equisingular. Then:*
(1) *The family of singularities is equisingular.*
(2) *If K_t is the link of the singularity V_t in the sphere S^5 of dimension* 5, *there exists an isotopy of S^5 which sends K_0 onto K_1.*

In particular, for a given reduced curve $C \subset \boldsymbol{P}^2$ there exists a superisolated singularity whose tangent cone is C; two such singularities determine the same link in S^5 up to isotopy.

PROPOSITION. *Let $C_1, C_2 \subset \boldsymbol{P}^2$ members of a Zariski pair. Let us take superisolated singularities V_1, V_2 whose tangent cones are C_1 and C_2, respectively. Then:*

(1) (*see* **[Lu]**) *The abstract links are homeomorphic.*

(2) (*see* **[St]** *and* **[A2]**) *The characteristic polynomials of the complex monodromies are equal.*

(3) (*see* **[A2]**) *The Jordan forms of the complex monodromies are equal if and only if the two curves have the same Alexander polynomial.*

COROLLARY. *Let $C_1, C_2 \subset \boldsymbol{P}^2$ members of a Zariski pair which is distinguished by the Alexander polynomial. Then, there is no homeomorphism of S^5 which sends one link onto the other one.*

This corollary suggests the next question:

QUESTION. *Let us consider a Zariski pair (C_1, C_2) as in Examples* 4, 5 *or* 6. *Let us take superisolated singularities V_1, V_2 whose tangent cones are C_1 and C_2, respectively. Then, does there exist a homeomorphism of S^5 which sends the link of V_1 onto the link of V_2?*

References

[A1] E. Artal, Les couples de Zariski, J. Algebraic Geom. **3** (1994), 223–247.

[A2] ———, Forme de Jordan de la monodromie des singularitées superisolées de surfaces, Mem, Amer. Math. Soc., vol. 109, no. 525, American Mathematical Society, Providence RI.

[C] J.-I. Cogolludo, Grupo fundamental del complementario de curvas proyectivas planas con nodos *y* cúspides, Preprint (1995).

[D] A. Degtyarev, Alexander polynomial of a curve of degree six, J. Knot Theory Ramifications **3** (1994), 439–454.

[F] T. Fujita, On the topology of non-complete surfaces, J. Fac. Sci. Univ. Tokyo Sect. 1A Math. **29** (1982), 503–566.

[GAP] M. Schönert et al., GAP version 3.4, 4th edition, Lehrstuhl D für Mathematik, RWTH Aachen, 1995.

[Li] A. Libgober, Alexander polynomials of plane algebraic curves and cyclic multiple planes, Duke Math. J. **49** (1982), 833–851.

[Lu] I. Luengo, The μ-constant stratum is not smooth, Invent. Math. **90** (1987), 139–152.

[M] J. Milnor, Singular Points of Complex Hypersurfaces, Annals of Mathematics Sudies 61, Princeton Univ. Press, Princeton N.Y., 1968.

[O1] M. Oka, Symmetric plane curves with nodes and cusps, J. Math. Soc. Japan **44** (1992), 375–414.

[O2] ———, Two transforms of plane curves and their fundamental groups, Preprint (1995).

[Sh] I. Shimada, A note on Zariski pairs, Preprint MPI für Mathematik (1995).

[St] J. Stevens, On the μ-constant Stratum and the V-filtration: an Example, Math. Z. **201** (1989), 139–144.

[T] H.-O Tokunaga, Some examples of Zariski pairs arising from certain elliptic $K3$ surfaces, Preprint (1995).

[W] R.-J. Walker, Algebraic curves, Princeton Math. Series 13, Princeton University Press, Princeton NJ.

[Z1] O. Zariski, On the problem of existence of algebraic functions of two variables possessing a given branch curve, Amer. J. Math. **51** (1929), 305–328.

[Z2] ———, On the irregularity of cyclic multiple planes, Ann. Math. **32** (1931), 445–489.

[Z3] ———, The topological discriminant group of a Riemann surface of genus p, Amer. J. Math. **59** (1937), 335–358.

Enrique ARTAL BARTOLO and Jorge CARMONA RUBER
Departamento de Matemáticas
Campus Plaza de San Francisco
Universidad de Zaragoza
E-50009 Zaragoza. Spain
E-mail address: artal@posta.unizar.es, carmona@posta.unizar.es