J. Math. Soc. Japan
Vol. 60, No. 3 (2008) pp. 865–903
doi: 10.2969/jmsj/06030865

Continued fractions and certain real quadratic fields of minimal type

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(Received Oct. 1, 2007)

Abstract. The main purpose of this article is to introduce the notion of real quadratic fields of minimal type in terms of continued fractions with period ℓ. We show that fundamental units of real quadratic fields that are not of minimal type are relatively small. So, we see by a theorem of Siegel that such fields have relatively large class numbers. Also, we show that there exist exactly 51 real quadratic fields of class number 1 that are not of minimal type, with one more possible exception. All such fields are listed in the table of Section 8.2. Therefore we study real quadratic fields with period ℓ of minimal type in order to find real quadratic fields of class number 1, and first examine the case where $\ell \leq 4$. In particular we obtain a result on Yokoi invariants m_d and class numbers h_d of real quadratic fields $\boldsymbol{Q}(\sqrt{d})$ with period 4 of minimal type.

1. Introduction.

Let $\boldsymbol{Q}(\sqrt{d})$ be a real quadratic field where d is a square-free positive integer with $d > 1$. We put $\omega = \omega(d) := \sqrt{d}$ or $\omega := (1+\sqrt{d})/2$ according to whether $d \equiv 2, 3$ or $d \equiv 1 \bmod 4$. The canonical integral basis of $\boldsymbol{Q}(\sqrt{d})$ is given by $\{1, \omega\}$, and the simple continued fraction expansion of ω becomes of the form

$$\omega = [a_0, \overline{a_1, \ldots, a_\ell}]. \tag{$*$}$$

Here, ℓ is the (minimal) period of ω. For brevity, we say that ℓ is *a period of a real quadratic field* $\boldsymbol{Q}(\sqrt{d})$. It is known by the classical theory of continued fractions (Euler, Lagrange, Legendre, Galois) that (i) the symmetric property: $a_n = a_{\ell-n}$, $1 \leq n \leq \ell - 1$ is satisfied, and that (ii) the last positive integer a_ℓ appeared in the right hand side of equality $(*)$ is uniquely determined by the integral part a_0 of ω. It is also known that the fundamental unit $\varepsilon_d > 1$ of it is calculated by using partial quotients $a_0, a_1, \ldots, a_\ell$.

2000 *Mathematics Subject Classification.* Primary 11R29; Secondary 11A55, 11R11, 11R27.
Key Words and Phrases. continued fractions, real quadratic fields, fundamental units, class numbers.

We can write uniquely $\varepsilon_d = (t + u\sqrt{d})/2$ with positive integers t, u. Yokoi [**26**] introduced an integer $m_d := [u^2/t]\,(\geq 0)$ as an invariant of $\boldsymbol{Q}(\sqrt{d})$, which is one of Yokoi's d-invariants n_d, m_d, a_d, b_d for a real quadratic field. Here, $[x]$ denotes the largest integer $\leq x$ for a real number x. Yokoi [**22**], [**23**] defined the first form of n_d, a_d, b_d to study Gauss conjecture (class number one problem) on real quadratic fields $\boldsymbol{Q}(\sqrt{p})$, p being a prime number for which $p \equiv 1 \mod 4$. In Yokoi [**24**], he reformed these invariants and first introduced the definition of m_d, where he treated real quadratic fields whose fundamental units have norm -1, that is, ones with odd period. Finally, he defined Yokoi's d-invariants for general real quadratic fields in [**26**]. These invariants give an estimate for the class number of a real quadratic field ([**26**], Yokoi [**25**]), and a necessary and sufficient condition for Ankney-Artin-Chowla conjecture [**1**] to hold ([**24**]). Recently, they have given also a necessary and sufficient condition for Pell equation $x^2 - dy^2 = \pm 2$ to have a solution in positive integers, and in particular the least solution is naturally expressed by using m_d (see Yokoi [**27**] and Yuan [**28**]).

Here, we call simply m_d among Yokoi's d-invariants *the Yokoi invariant of a real quadratic field* $\boldsymbol{Q}(\sqrt{d})$. Let $n_d := [t/u^2]\,(\geq 0)$. Yokoi proved ([**26**, Proposition 4.1 (2)] and [**24**, Corollary 1.4]) that there exist finitely many real quadratic fields $\boldsymbol{Q}(\sqrt{d})$ of class number 1 with $n_d > 0$. (In fact, it is immediate from [**26**, Proposition 4.1 (2)] that there exist exactly 39 such fields with one more possible exception.) When $d \neq 3, 5$, since the condition that $m_d > 0$ is equivalent to the condition that $n_d = 0$, as its consequence, he introduced the Yokoi invariant m_d to study real quadratic fields with $n_d = 0$ (cf. [**24**, Introduction]). Furthermore, he proved that if $d > 13$ then $m_d d < \varepsilon_d < (m_d + 1)d$, so that the quantity m_d gives a size of the fundamental unit for d. When the value of m_d is large, we may consider that the fundamental unit is large. This paper is mainly about real quadratic fields with $m_d > 0$.

Conversely, let ℓ be a fixed positive integer and $a_1, \ldots, a_{\ell-1}$ any symmetric positive integers. Then, we construct a non-square positive integer d such that the continued fraction expansion of $\omega = \omega(d)$ with period ℓ is of the form $(*)$, and if it is square-free then we consider a real quadratic field $\boldsymbol{Q}(\sqrt{d})$. A construction of non-square positive integers d such that the continued fraction expansion of ω has the given symmetric part $a_1, \ldots, a_{\ell-1}$ is obtained by Friesen [**4**] in the case where $\omega = \sqrt{d}$, and by Halter-Koch [**5**] in the case where $\omega = (1 + \sqrt{d})/2$ and $d \equiv 1 \mod 4$. We improve their result by considering three cases separately (Theorem 3.1). By using an integer s_0 appeared in this theorem of Friesen and Halter-Koch, we can define the notion of *a positive integer with period* ℓ *of minimal type* and *a real quadratic field with period* ℓ *of minimal type* (Definition 3.1). Their result implies that there exist infinitely many real quadratic fields which are not of minimal type such that the continued fraction expansion of ω has the given symmetric part (cf. Example 3.2 and Proposition 8.2 of Section 8.1). As a consequence of Theorem 3.1, we shall prove that the Yokoi invariant of a real quadratic field with arbitrary period that is not of minimal is equal to at most 3. Therefore the fundamental unit of such a field is relatively small. So, we see by a theorem of Siegel [**19**] concerning the approximate behavior of the product of class number and regulator that the class number is relatively large. Louboutin [**10**, Section 5] has dealt with real quadratic fields such that the continued fraction expansion of ω has a given constant symmetric part, and reported that a few field has class number 1 among such fields, which become not of minimal type (Example 3.5). Furthermore, we shall prove that there exist exactly 51 real quadratic fields of class number 1 that are not of minimal type, with one more possible exception (Proposition 4.4). Hence we have to examine a construction of real quadratic fields of minimal type in order to find real quadratic fields of class number 1. For any positive integers ℓ and h, Sasaki [**18**, Theorem 1] and Lachaud [**9**, Theorem 2.2] proved by using a theorem of Siegel that there exist at most finitely many real quadratic fields with period ℓ of class number h.

Thus, we are interested in a construction of real quadratic fields of minimal type. In the present paper, we shall give real quadratic fields with period 4 of minimal type. Let h_d denote the class number (in the wide sense) of real quadratic field $\boldsymbol{Q}(\sqrt{d})$.

THEOREM 1.1.

(i) *Let $\delta = 2$ or 3, and a be a positive integer such that $2a+1$ is square-free. Then, for any positive integer h, there exist infinitely many real quadratic fields $\boldsymbol{Q}(\sqrt{d})$, $d \equiv \delta \mod 4$ with period 4 of minimal type such that $h_d > h$ and $m_d = 16a$.*

(ii) *Let a be a positive odd integer such that $a+2$ is square-free. Then, for any positive integer h, there exist infinitely many real quadratic fields $\boldsymbol{Q}(\sqrt{d})$, $d \equiv 1 \mod 4$ with period 4 of minimal type such that $h_d > h$ and $m_d = a$.*

Our family of real quadratic fields obtained explicitly has two parameters of nonnegative integers. Since the values of Yokoi invariants are bounded, a theorem of Siegel yields that the class numbers are relatively large (Lemma 4.5). We use a theorem of Nagell to show that our family contains infinite ones.

This paper is organized as follows. After preparations on continued fractions in Section 2, we prove Theorem 3.1 in Section 3 which is our basic tool. The notations $A, B, C, s_0, g(x), h(x)$ and $f(x)$ as explained there are used throughout this paper. In Definition 3.1, we define the notion of a positive integer with period ℓ of minimal type and a real quadratic field with period ℓ of minimal type by using Theorem 3.1. Then, $\boldsymbol{Q}(\sqrt{5})$ is only a real quadratic field with period 1 of minimal type. Also, we see that there does not exist a real quadratic field with period $2, 3$

of minimal type. In Section 4, we calculate Yokoi invariants of some real quadratic fields by using Theorem 3.1. We show in Proposition 4.2 that the Yokoi invariant of a real quadratic field with arbitrary period that is not of minimal is equal to at most 3. We see also that the Yokoi invariant of a real quadratic field with period ≤ 3 is equal to at most 2. On the other hand, Yokoi invariants of real quadratic fields with period 4 of minimal type obtained in Section 5 are large (Proposition 5.2). In Section 6, we improve a theorem of Nagell (Proposition 6.1). Though it seems known, a proof of it will be given here, as no reference for it is known to the authors. In Section 7 we prove Theorem 1.1. See Remark 8.1 for a numerical example of it. All numerical examples in this paper are calculated by using PARI-GP [**3**]. In Section 8.1, following an idea of Friesen, we show that there exist infinitely many real quadratic fields with period ℓ which are not of minimal type such that the continued fraction expansion of ω has a given symmetric part.

In [**8**] we will examine a construction of real quadratic fields with large even period of minimal type.

We denote by $\boldsymbol{N}$, $\boldsymbol{Z}$ and $\boldsymbol{Q}$ the set of positive integers, the ring of rational integers and the field of rational numbers, respectively. For a set S, $\sharp S$ denotes the cardinal of S.

2. Preparations on continued fractions.

We begin with basic properties of continued fractions, and refer the reader to excellent books of Ono [**15**] and Rosen [**17**] for them. If a_0 is any positive integer and $\{a_n\}_{n\geq 1}$ is a sequence of positive integers, then we define nonnegative integers p_n, q_n, r_n by using recurrence equations:

$$\begin{cases} p_0 = 1,\ p_1 = a_0,\ p_n = a_{n-1}p_{n-1} + p_{n-2}, & \\ q_0 = 0,\ q_1 = 1,\ q_n = a_{n-1}q_{n-1} + q_{n-2}, & n \geq 2 \\ r_0 = 1,\ r_1 = 0,\ r_n = a_{n-1}r_{n-1} + r_{n-2}. & \end{cases} \tag{2.1}$$

Let λ be a variable. Then the following are known:

$$[a_0, \ldots, a_n, \lambda] = \frac{\lambda p_{n+1} + p_n}{\lambda q_{n+1} + q_n}, \quad [a_0, \ldots, a_n] = \frac{p_{n+1}}{q_{n+1}}, \quad n \geq 0, \tag{2.2}$$

$$p_n q_{n-1} - p_{n-1} q_n = (-1)^n, \quad n \geq 1, \tag{2.3}$$

$$p_n = a_0 q_n + r_n, \quad n \geq 0. \tag{2.4}$$

We see easily (2.4) by induction in n. (Recurrence equations and partial quotients of a continued fraction are both numbered beginning with 0.) We let $a_1, \ldots, a_\ell$ be any ℓ positive integers, and assume that $\ell-1$ positive integers $a_1,\dots,a_{\ell-1}$ satisfy the symmetric property: $a_n = a_{\ell-n}$, $1 \le n \le \ell-1$ if $\ell \ge 2$. Then we define a sequence $\{a_n\}_{n\ge1}$ of positive integers: for each integer $n \ge 1$, we put $a_n := a_r$ if $r > 0$, and otherwise $a_n := a_\ell$ where r is the remainder of the division of n by ℓ. Thus, we construct periodically $\{a_n\}_{n\ge1}$ from $a_1,\dots,a_\ell$ in what follows and throughout this paper.

LEMMA 2.1. *Let k be a positive integer. Under the above setting, the following hold.*

$$q_{k\ell-1} = r_{k\ell}, \tag{2.5}$$

$$q_{k\ell-1}^2 - (-1)^{k\ell} = q_{k\ell} r_{k\ell-1}. \tag{2.6}$$

PROOF. Let $M_0 := E$ be the unit matrix of degree 2, and put for each integer $n \ge 1$,

$$M_n := \begin{pmatrix} a_1 & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} a_n & 1 \\ 1 & 0 \end{pmatrix}.$$

We see easily that

$$M_n = \begin{pmatrix} q_{n+1} & q_n \\ r_{n+1} & r_n \end{pmatrix}, \quad n \ge 0 \tag{2.7}$$

by induction. Since $a_1,\dots,a_{\ell-1}$ have the symmetric property, $M_{\ell-1}$ is a symmetric matrix. Furthermore, $M_{k\ell-1}$ is also a symmetric matrix by the definition of the sequence $\{a_n\}_{n\ge1}$. Therefore we have $q_{k\ell-1} = r_{k\ell}$ from (2.7). Note that $q_n r_{n-1} - q_{n-1} r_n = (-1)^{n-1}$ for each $n \ge 1$. (See the determinants of both sides of (2.7).) Hence,

$$q_{k\ell-1}^2 = q_{k\ell-1} r_{k\ell} = q_{k\ell} r_{k\ell-1} - (-1)^{k\ell-1},$$

which yields (2.6). □

From now on, we let d be a non-square positive integer and put $\omega := \sqrt{d}$ or $(1+\sqrt{d})/2$. Also, we assume $d \equiv 1 \mod 4$ if $\omega = (1+\sqrt{d})/2$. (This assumption is necessary for the proof of Lemma 2.2.) It is known by Euler-Lagrange Theorem ([**15**, p. 184, Theorem 4.10], [**17**, p. 493, Theorem 12.21]) that the continued fraction expansion of ω is periodic:

$$\omega = [a_0, \overline{a_1, \dots, a_\ell}].$$

Here, ℓ is the period of ω. Furthermore, we see by a theorem of Galois that $a_1, \dots, a_{\ell-1}$ are symmetric positive integers if $\ell \geq 2$. And it holds that $a_\ell = 2a_0$ in the case where $\omega = \sqrt{d}$, and that $a_\ell = 2a_0 - 1$ in the case where $\omega = (1+\sqrt{d})/2$. We can find the proof for $\omega = \sqrt{d}$ in [**17**, p. 501], and for $\omega = (1+\sqrt{d})/2$ it is given similarly. (See also Perron [**16**, p. 79, Satz 3.9] for $\omega = \sqrt{d}$, and [**16**, p. 105, Satz 3.30] for $\omega = (1+\sqrt{d})/2$.) We put $a := [\sqrt{d}]$ and write $\omega = (P_0 + \sqrt{d})/Q_0$ for brevity. Then,

$$P_0 < \sqrt{d}, \quad 0 < Q_0, \quad d \equiv P_0^2 \mod Q_0. \tag{2.8}$$

It is well known that the continued fraction expansion of ω is obtained by the calculations in integers. We put

$$Q_{-1} := (d - P_0^2)/Q_0, \quad a_0 := [(P_0 + a)/Q_0], \quad \omega_0 := \omega.$$

Then Q_{-1} is a positive integer by (2.8). Since

$$[(P+\sqrt{d})/Q] = [(P+a)/Q] \qquad \text{for } P,\, Q \in \boldsymbol{Z},\ Q > 0,$$

we have $[\omega_0] = [\frac{P_0+a}{Q_0}] = a_0$. Also, $a_0 > 0$ as $\omega_0 > 1$. Let $n \geq 0$. By (2.9), (2.11) and (2.13), we shall determine positive integers $P_{n+1}, Q_{n+1}, a_{n+1}$ and a (positive) quadratic irrational ω_{n+1} from positive integers P_n, Q_n, Q_{n-1}, a_n and a quadratic irrational $\omega_n = (P_n + \sqrt{d})/Q_n$ (> 1) in what follows. There are uniquely integers a_n, b_n such that

$$P_n + a = a_n Q_n + b_n, \quad 0 \leq b_n < Q_n.$$

So, $a_n = [\frac{P_n+a}{Q_n}] = [\omega_n]$. Consequently, $a_n > 0$ as $\omega_n > 1$. First, we put

$$P_{n+1} := a - b_n, \quad \text{that is, } P_{n+1} = a_n Q_n - P_n. \tag{2.9}$$

Since $a \geq P_n$ by the assumption "$\sqrt{d} > P_n$" and $a_n > 0$, we have

$$2a \geq P_n + a = a_n Q_n + b_n \geq Q_n + b_n > 2b_n,$$

so that $a > b_n$. We see by (2.9) that

$$0 < P_{n+1} \le a < \sqrt{d}. \tag{2.10}$$

Next, we put

$$Q_{n+1} := Q_{n-1} + a_n(P_n - P_{n+1}). \tag{2.11}$$

Then it holds by induction in $n \ge 0$ that

$$Q_n Q_{n+1} = d - P_{n+1}^2. \tag{2.12}$$

By (2.10) and (2.12), $Q_n Q_{n+1} > 0$. As $Q_n > 0$, we have $Q_{n+1} > 0$. So, we put

$$a_{n+1} := [(P_{n+1} + a)/Q_{n+1}], \quad \omega_{n+1} := (P_{n+1} + \sqrt{d})/Q_{n+1}. \tag{2.13}$$

Then, $[\omega_{n+1}] = a_{n+1}$. Since $\omega_n = (P_n + \sqrt{d})/Q_n$, it follows from (2.9) and (2.12) that

$$\frac{1}{\omega_n - a_n} = \frac{Q_n}{\sqrt{d} - P_{n+1}} = \frac{Q_n(\sqrt{d} + P_{n+1})}{d - P_{n+1}^2} = \omega_{n+1}.$$

By $\omega_n \notin \boldsymbol{Q}$ and $[\omega_n] = a_n$, we have $\omega_{n+1} > 1$. (Also, $(P_{n+1} + \sqrt{d})/Q_{n+1} > 1$ and (2.10) imply that $0 < Q_{n+1} < 2\sqrt{d}$.) By repeating this process, we calculate P_n, Q_n, a_n and ω_n for each integer $n \ge 1$. Since $\omega_n = a_n + (1/\omega_{n+1})$ and $a_n = [\omega_n]$ hold for all $n \ge 0$, we get the continued fraction expansion $\omega = [a_0, a_1, a_2, \ldots]$. As there is the least positive integer ℓ for which $\omega_{\ell+1} = \omega_1$, we see that $\omega = [a_0, \overline{a_1, \ldots, a_\ell}]$.

LEMMA 2.2. *Under the above setting, we put $a := [\sqrt{d}]$ and let $2 \le \ell$, $1 \le n \le \ell - 1$. Then, Q_n/Q_0 is a positive integer ≥ 2 and the following are true.*

(i) *In the case where $\omega = \sqrt{d}$, we have $Q_n a_n \le 2a$, and in particular $a_n \le a$.*
(ii) *In the case where $\omega = (1 + \sqrt{d})/2$, if a is even then $Q_n a_n \le 2a - 1$, and otherwise $Q_n a_n \le 2a$. In particular, $a_n < a/2$.*

PROOF. From (2.9) and (2.11) we have $Q_{n+1} \equiv Q_{n-1} + a_n Q_n \mod 2$ for all $n \ge 0$. When $\omega = (1+\sqrt{d})/2$, Q_{-1} is even as $d \equiv 1 \mod 4$ and $Q_0 = 2$, so that each Q_n is also even. Thus, Q_n/Q_0 is always a positive integer. Let $1 \le n \le \ell - 1$. We see by (2.10) that

$$a_n = \left[\frac{P_n + a}{Q_n}\right] \le \frac{P_n + a}{Q_n} \le \frac{2a}{Q_n}.$$

Therefore, $Q_n a_n \leq 2a$. If $\omega = (1+\sqrt{d})/2$ then $P_n - P_{n+1} \equiv 0 \mod 2$ by (2.9). As $P_0 = 1$, each P_n is odd. When a is even, since $(P_n + a)/Q_n$ is not an integer, we obtain $Q_n a_n < 2a$. We assume $Q_n \leq Q_0$ to prove $Q_n > Q_0$. We claim that

$$Q_n = Q_0, \quad a_n Q_0 = P_n + P_1, \quad P_{n+1} = P_1. \tag{2.14}$$

As $Q_n \leq Q_0$,

$$0 \leq b_n < Q_n \leq Q_0. \tag{2.15}$$

First, let $\omega = \sqrt{d}$. As $Q_0 = 1$, (2.15) yields that $Q_n = Q_0 = 1$ and $b_n = 0$, so that $P_n + a = a_n Q_0$. Also, we see from (2.9) that $P_{n+1} = a$ and $P_1 = a_0 Q_0 - P_0 = a_0 = a$. Hence (2.14) holds. Next, let $\omega = (1+\sqrt{d})/2$. As $Q_0 = 2$, (2.15) yields that $b_n = 0$, or 1. As Q_n is even, we have $Q_n = Q_0 = 2$. Therefore, $P_n + a = a_n Q_0$, or $a_n Q_0 + 1$. Also, we see from (2.9) that $P_{n+1} = a$, or $a - 1$. Since P_{n+1} is odd, a is odd or even according to whether $b_n = 0$ or 1, so that $a_0 = [(1+a)/2] = (a+1)/2$, or $a/2$. Hence, $P_1 = a_0 Q_0 - P_0 = a$, or $a-1$. Thus we obtain (2.14) and our claim is proved. It follows from $Q_n = Q_0$ and $(2.12)_{n-1}$ that $Q_{n-1} = (d - P_n^2)/Q_0$. We see by (2.11) and (2.14) that

$$Q_{n+1} = \frac{d - P_n^2}{Q_0} + \frac{P_n + P_1}{Q_0}(P_n - P_1) = (d - P_1^2)/Q_0 = Q_1.$$

This and $P_{n+1} = P_1$ imply that $\omega_{n+1} = \omega_1$. Hence $\ell \mid n$, and this contradicts $n \leq \ell - 1$. Therefore, $Q_n > Q_0$. Since Q_n/Q_0 is an integer, we have $Q_n/Q_0 \geq 2$. This proves our lemma. □

3. A theorem of Friesen and Halter-Koch.

We let ℓ be a fixed positive integer, and assume that positive integers $a_1, \ldots, a_{\ell-1}$ have the symmetric property: $a_n = a_{\ell-n}$, $1 \leq n \leq \ell - 1$. (When $\ell = 1$, we consider that this condition is trivially satisfied.) By using recurrence equations (2.1), we define nonnegative integers $q_0, \ldots, q_\ell$, $r_0, \ldots, r_{\ell-1}$. For brevity, we put $A := q_\ell$, $B := q_{\ell-1}$, $C := r_{\ell-1}$, and define polynomials $g(x), h(x)$ of degree 1 and a quadratic polynomial $f(x)$ in $\boldsymbol{Z}[x]$ by putting

$$\begin{aligned} g(x) &:= Ax - (-1)^\ell BC, \qquad h(x) := Bx - (-1)^\ell C^2, \\ f(x) &:= g(x)^2 + 4h(x) = A^2x^2 + 2(2B - (-1)^\ell ABC)x + (B^2 - (-1)^\ell 4)C^2. \end{aligned}$$

Furthermore, we let s_0 be the least integer s for which $g(s) > 0$, that is, $s > (-1)^\ell BC/A$. The quadratic function $f(x)$ becomes strictly, monotonously increasing in the interval $[s_0, \infty)$. We consider three cases separately:

(I) $A \equiv 1 \mod 2$, (II) $(A, C) \equiv (0, 0) \mod 2$,
(III) $(A, C) \equiv (0, 1) \mod 2$.

It is not difficult to separate such cases by using recurrence equations (2.1) (cf. [**8**, Lemmas 2.2, 2.3]). Under the above setting, Theorem 3.1 was shown in Friesen [**4**, Theorem] and Halter-Koch [**5**, Theorem 1A and Corollary 1A], which we improve by considering the above three cases separately.

Theorem 3.1 (Friesen, Halter-Koch).

[A] *Let ℓ be a fixed positive integer and $a_1, \ldots, a_{\ell-1}$ symmetric positive integers.*

(i) *In Case* (III), *there is no positive integer d such that*

$$\sqrt{d} = \left[[\sqrt{d}], \overline{a_1, \ldots, a_{\ell-1}, 2[\sqrt{d}]}\right] \tag{3.1}$$

is the continued fraction expansion of $\sqrt{d}$. Also, when Case (I) *or Case* (II) *occurs, we let s be any integer with $s \geq s_0$, and put $d := f(s)/4$ and $a_0 := g(s)/2$. Here, we choose an even integer s in Case* (I), *and assume that*

$$g(s) > a_1, \ldots, a_{\ell-1} \tag{3.2}$$

if $\ell \geq 2$. Then, d and a_0 are positive integers, d is non-square, $a_0 = [\sqrt{d}]$, and (3.1) *is the continued fraction expansion with period ℓ of $\sqrt{d}$.*

(ii) *In Case* (II), *there is no positive integer d such that $d \equiv 1 \mod 4$ and*

$$(1+\sqrt{d})/2 = \left[[(1+\sqrt{d})/2], \overline{a_1, \ldots, a_{\ell-1}, 2[(1+\sqrt{d})/2]-1}\right] \tag{3.3}$$

is the continued fraction expansion of $(1+\sqrt{d})/2$. Also, when Case (I) *or Case* (III) *occurs, we let s be any integer with $s \geq s_0$, and put $d := f(s)$ and $a_0 := (g(s)+1)/2$. Here, we choose an odd integer s in Case* (I), *and assume that* (3.2) *holds if $\ell \geq 2$. Then, d and a_0 are positive integers, d is non-square, $d \equiv 1 \mod 4$, $a_0 = [(1+\sqrt{d})/2]$, and* (3.3) *is the continued fraction expansion with period ℓ of $(1+\sqrt{d})/2$.*

[B] *Conversely, we let d be any non-square positive integer. By using a quadratic polynomial $f(x)$ obtained as above from the symmetric part of the continued* *fraction expansion of* $\sqrt{d}$, *d becomes uniquely of the form* $d = f(s)/4$ *with some integer* $s \geq s_0$, *and* (3.2) *holds if* $\ell \geq 2$. *If* $d \equiv 1 \mod 4$ *in addition then the same thing is true for* $(1+\sqrt{d})/2$.

Proof. We begin by proving the assertion on d, a_0, before we show the non-existence of d in the assertion [A]. For brevity, we put

$$\omega = \begin{cases} \sqrt{d}, \\ (1+\sqrt{d})/2, \end{cases} \qquad \alpha = \begin{cases} a_0, \\ a_0 - 1, \end{cases} \qquad \delta = \begin{cases} d, \\ (d-1)/4, \end{cases} \qquad a_\ell = \begin{cases} 2a_0, \\ 2a_0 - 1 \end{cases}$$

according to whether we consider the assertion [A-i] or [A-ii]. Then we see in both assertions that

$$a_0 + \alpha = \begin{cases} 2a_0 = g(s), \\ 2a_0 - 1 = g(s), \end{cases} \tag{3.4}$$

and by the definitions of d and a_0 that

$$\delta - a_0\alpha = h(s). \tag{3.5}$$

Since $AC = B^2 - (-1)^\ell$ by $(2.6)_{k=1}$ of Lemma 2.1, we have $AC \equiv B + 1 \mod 2$. When Case (I) (resp. (II), (III)) occurs, the definition of $g(x)$ implies that

$$g(s) \equiv s \text{ (resp.,} \equiv 0,\ 1) \mod 2. \tag{3.6}$$

In the assertion [A-i], since we assume that s is even when Case (I) occurs, $g(s)$ is always even by (3.6). Therefore we see from (3.4) that a_0 is an integer. As $d = h(s) + a_0^2$ by (3.5), d is also an integer. In the assertion [A-ii], since we assume that s is odd when Case (I) occurs, $g(s)$ is always odd by (3.6). Therefore we see from (3.4) that a_0 is an integer. As $d = 1 + 4h(s) + 4a_0(a_0 - 1)$ by (3.5), d is also an integer and $d \equiv 1 \mod 4$. As $g(s) > 0$ by the definition of s_0, (3.4) yields that $a_0 > 0$.

First, we assume $\ell \geq 2$ to show

$$h(s) > 0, \qquad g(s) - h(s) \geq 0. \tag{3.7}$$

Then, $B > 0$. By using $s \geq s_0 > (-1)^\ell B\{B^2 - (-1)^\ell\}/A^2$ and $C = \{B^2 - (-1)^\ell\}/A$, we see that

$$h(s) = Bs - (-1)^\ell C^2 > \frac{B^2 - (-1)^\ell}{A^2}. \tag{3.8}$$

Also, when $A - B > 0$,

$$g(s) - h(s) = (A - B)s + (-1)^\ell C(C - B) > -\frac{B^2 - (-1)^\ell}{A^2}.$$

When $A - B = 0$, we have $\ell = 2, a_1 = 1$ so that $A = B = 1, C = 0$. Consequently, $g(s) - h(s) = 0 = -(B^2 - (-1)^\ell)/A^2$. Therefore we see that

$$g(s) - h(s) \geq -\frac{B^2 - (-1)^\ell}{A^2}. \tag{3.9}$$

If ℓ is even then $(B^2 - 1)/A^2 < 1$ by $A \geq B$. If ℓ is odd, as $\ell \geq 3$, then $A > B$ so that $A \geq B + 1$. Therefore, $(B^2 + 1)/A^2 < 1$. Thus it always holds that $(0 \leq)\ \{B^2 - (-1)^\ell\}/A^2 < 1$. Hence, (3.8) and (3.9) imply (3.7). Next, we assume $\ell = 1$. Since $A = C = 1$ and $B = 0$, Case (I) occurs and we obtain

$$s \geq s_0 = 1, \quad g(x) = x, \quad h(x) = 1.$$

Hence, (3.7) holds.

First, we consider the assertion [A-i]. It follows from (3.7), (3.5) and (3.4) that $d > a_0^2$ and $2a_0 \geq d - a_0^2$. Consequently, $a_0^2 < d < (a_0+1)^2$ so that $a_0 < \sqrt{d} < a_0+1$. Hence $a_0 = [\sqrt{d}]$, and this inequality yields that d is non-square. Next, we consider the assertion [A-ii]. By the similar argument, we have

$$(d-1)/4 > a_0(a_0 - 1), \quad 2a_0 - 1 \geq (d-1)/4 - a_0(a_0 - 1).$$

Consequently, $(2a_0 - 1)^2 < d < (2a_0 + 1)^2$ so that $2a_0 - 1 < \sqrt{d} < 2a_0 + 1$. Hence, $a_0 = [(1 + \sqrt{d})/2]$. If we assume that $\sqrt{d}$ is an integer, then we see by the last inequality that $\sqrt{d} = 2a_0$. Therefore, $d \equiv 0 \mod 4$ and this is a contradiction. Thus, d is non-square.

Since

$$h(x)A - g(x)B = (-1)^\ell C(-AC + B^2) = C,$$

(3.4) and (3.5) imply that

$$(\delta - a_0\alpha)A - (a_0 + \alpha)B = C. \tag{3.10}$$

Now we get the continued fraction expansion of ω. By using the positive integer a_0 and the recurrence equation (2.1), we define other positive integers $p_0, \dots, p_\ell$. By (2.4) and $(2.5)_{k=1}$, we have

$$p_\ell = a_0 q_\ell + q_{\ell-1}. \tag{3.11}$$

Therefore it follows from $(2.3)_{n=\ell}$, (3.11) and $(2.6)_{k=1}$ that

$$\begin{aligned} p_{\ell-1} &= (p_\ell q_{\ell-1} - (-1)^\ell)/q_\ell = \{(a_0 q_\ell + q_{\ell-1})q_{\ell-1} - (-1)^\ell\}/q_\ell \\ &= a_0 q_{\ell-1} + (q_{\ell-1}^2 - (-1)^\ell)/q_\ell = a_0 B + C. \end{aligned} \tag{3.12}$$

By putting $\lambda = \alpha + \omega$ in $(2.2)_{n=\ell-1}$, we see that

$$\omega = [a_0, \dots, a_{\ell-1}, \alpha + \omega] \tag{3.13}$$

if and only if

$$\omega = \frac{(\alpha+\omega)p_\ell + p_{\ell-1}}{(\alpha+\omega)A + B}.$$

(Note that the denominator of the right hand side of this equation is not equal to 0 as $\omega \notin \boldsymbol{Q}$, or it is positive.) This is equivalent to the following condition. (Note that $\omega^2 = \delta + \omega$ in the assertion [A-ii].)

$$\begin{aligned} &(\alpha p_\ell + p_{\ell-1}) + p_\ell \omega = \omega^2 A + (\alpha A + B)\omega \\ &\quad \Longleftrightarrow (\alpha p_\ell + p_{\ell-1}) + p_\ell \omega = \delta A + (a_0 A + B)\omega \\ &\quad \Longleftrightarrow \alpha p_\ell + p_{\ell-1} = \delta A \quad \text{(by (3.11))}. \end{aligned}$$

Hence we find that the condition (3.13) is equivalent to

$$\delta A = \alpha p_\ell + p_{\ell-1}. \tag{3.14}$$

By the way, as

$$\alpha p_\ell + p_{\ell-1} = a_0 \alpha A + (a_0 + \alpha)B + C$$

by (3.11) and (3.12), it follows from (3.10) that $\alpha p_\ell + p_{\ell-1} = \delta A$. Since this is equivalent to (3.13) as we have seen above, we have $1/(\omega - a_0) = [a_1, \ldots, a_{\ell-1}, \alpha + \omega]$. Hence we see that

$$\alpha + \omega = a_\ell + \frac{1}{1/(\omega - a_0)} = [a_\ell, a_1, \ldots, a_{\ell-1}, \alpha + \omega] = [\overline{a_\ell, a_1, \ldots, a_{\ell-1}}].$$

By substituting it for (3.13), we obtain $\omega = [a_0, \overline{a_1, \ldots, a_\ell}]$. This gives the continued fraction expansion of ω by the uniqueness of continued fraction expansion ([**15**, Theorem 1.8]). As $a_\ell = g(s)$ by (3.4), we see from (3.2) that $a_1, \ldots, a_{\ell-1} < a_\ell$. Since the positive integer a_ℓ is not in the symmetric part of the continued fraction expansion, ℓ becomes the period of ω.

[B] Conversely, we let d be a non-square positive integer and put $\omega := \sqrt{d}$ or $(1 + \sqrt{d})/2$. Also, we assume $d \equiv 1 \mod 4$ if $\omega = (1 + \sqrt{d})/2$. So, then $(d-1)/4$ is an integer. As we mention in Section 2, the continued fraction expansion of ω is periodic: $\omega = [a_0, \overline{a_1, \ldots, a_\ell}]$. Furthermore, $a_1, \ldots, a_{\ell-1}$ are symmetric positive integers if $\ell \geq 2$, and it holds that $a_\ell = 2a_0$ in the case where $\omega = \sqrt{d}$, and that $a_\ell = 2a_0 - 1$ in the case where $\omega = (1 + \sqrt{d})/2$. By using the partial quotients and recurrence equations (2.1), we define nonnegative integers p_n, q_n, $0 \leq n \leq \ell$, and various values $A, B, C, f(x), \ldots, s_0, \alpha, \delta, \ldots$. Then, in particular (3.11) and (3.12) hold. Since $1/(\omega - a_0) = [\overline{a_1, \ldots, a_\ell}]$, the definition of α yields that

$$\alpha + \omega = a_\ell + \frac{1}{1/(\omega - a_0)} = [a_\ell, \overline{a_1, \ldots, a_\ell}] = [\overline{a_\ell, a_1, \ldots, a_{\ell-1}}].$$

Therefore,

$$\omega = [a_0, a_1, \ldots, a_{\ell-1}, \overline{a_\ell, a_1, \ldots, a_{\ell-1}}] = [a_0, a_1, \ldots, a_{\ell-1}, \alpha + \omega].$$

Thus (3.13) holds. Since this condition is equivalent to (3.14), substituting (3.11) and (3.12) for (3.14) implies that $\delta A = \alpha(a_0 A + B) + a_0 B + C$. Hence we obtain

$$(\delta - a_0\alpha)A - (a_0 + \alpha)B = C.$$

The pair $(\delta - a_0\alpha, a_0 + \alpha)$ is a solution of a Diophantine equation $Ax - By = C$. On the other hand, as

$$\{-(-1)^\ell C^2\}A - \{-(-1)^\ell BC\}B = (-1)^\ell C(-AC + B^2) = C,$$

the pair $(-(-1)^\ell C^2, -(-1)^\ell BC)$ is also a solution. Since $(A, B) = 1$ by (2.3), there is some integer s such that

$$\delta - a_0\alpha = Bs - (-1)^\ell C^2 \ (= h(s)), \quad a_0 + \alpha = As - (-1)^\ell BC \ (= g(s)). \quad (3.15)$$

As $a_0 > 0$, the second equation of (3.15) yields that $g(s) > 0$, so that $s \geq s_0$. We see by (3.4) and (3.5) that

$$f(s) = g(s)^2 + 4h(s) = (a_0 + \alpha)^2 + 4(\delta - a_0\alpha) = 4d, \text{ or } d$$

according to whether $\omega = \sqrt{d}$, or $\omega = (1+\sqrt{d})/2$. Also, the definition of α and the second equation of (3.15) yield that $a_0 = g(s)/2$, or $(g(s)+1)/2$. By using $AC = B^2 - (-1)^\ell$, we can obtain by simple calculations

$$f(x) = A^2\left\{x + \frac{(2-(-1)^\ell AC)B}{A^2}\right\}^2 - \frac{(-1)^\ell 4^2}{4A^2} \text{ and } s_0 > -\frac{(2-(-1)^\ell AC)B}{A^2}.$$

Hence the quadratic function $f(x)$ is strictly, monotonously increasing in the interval $[s_0, \infty)$. (The discriminant $d(f)$ of quadratic polynomial $f(x)$ is equal to $(-1)^\ell 4^2$.) This implies the uniqueness of s.

We prove the parity of s and the non-existence of d in the assertion [A]. First, we consider the case where $\omega = \sqrt{d}$. As $\alpha = a_0$, the second equation of (3.15) yields that $g(s)$ is even. Therefore we see by (3.6) that s have to be even when Case (I) occurs, and that Case (III) never occurs. In Case (III), hence there is no positive integer d such that (3.1) is the continued fraction expansion of $\sqrt{d}$. Next, we consider the case where $\omega = (1+\sqrt{d})/2$. As $\alpha = a_0 - 1$, $g(s)$ is odd. Therefore we see by (3.6) that s have to be odd when Case (I) occurs, and that Case (II) never occurs. In Case (II), hence there is no positive integer d such that $d \equiv 1 \mod 4$ and (3.3) is the continued fraction expansion of $(1+\sqrt{d})/2$.

Finally, we show (3.2) if $\ell \geq 2$, by using Lemma 2.2. For brevity, we put $a := [\sqrt{d}]$, and let $1 \leq n \leq \ell - 1$. When $\omega = \sqrt{d}$, we see by Lemma 2.2 (i) that $a_n \leq a = a_0 < 2a_0 = g(s)$. When $\omega = (1+\sqrt{d})/2$, we have $a_n < a/2$ and $a_0 = [(1+a)/2]$. If a is even then, as $a_0 = a/2$ we have $a_n < a_0 \leq 2a_0 - 1 = g(s)$. If a is odd then, as $a_0 = (1+a)/2$ we have $a_n < a_0 - 1/2 < 2a_0 - 1 = g(s)$. Thus (3.2) holds, and the positive integer d is obtained by our construction. □

Definition 3.1. Let d be any non-square positive integer. We see by Theorem 3.1 [B] that d is uniquely of the form $d = f(s)/4$ with some integer $s \geq s_0$. Here, the quadratic polynomial $f(x)$ and the integer s_0 are obtained as above from the symmetric part of the continued fraction expansion with period ℓ of $\sqrt{d}$. If $s = s_0$ then we say that d is *a positive integer with period ℓ of minimal type for $\sqrt{d}$*. When $d \equiv 1 \mod 4$ in addition, we see that d is uniquely of the form $d = f(s)$ with some integer $s \geq s_0$. Here, the quadratic polynomial $f(x)$ and the integer s_0

are obtained as above from the symmetric part of the continued fraction expansion with period ℓ of $(1+\sqrt{d})/2$. If $s = s_0$ then we say that d is *a positive integer with period ℓ of minimal type for* $(1+\sqrt{d})/2$.

Let $\boldsymbol{Q}(\sqrt{d})$ be a real quadratic field. Here, d is a square-free positive integer. We say that $\boldsymbol{Q}(\sqrt{d})$ is *a real quadratic field with period ℓ of minimal type*, if d is a positive integer with period ℓ of minimal type for $\sqrt{d}$ when $d \equiv 2, 3 \mod 4$, and if d is a positive integer with period ℓ of minimal type for $(1+\sqrt{d})/2$ when $d \equiv 1 \mod 4$.

EXAMPLE 3.2. We let $\ell \geq 2$ be any positive integer, and consider a symmetric string of $\ell - 1$ positive integers $a_1, \ldots, a_{\ell-1}$. Let $L := [\ell/2]$. We shall see that there exist infinitely many real quadratic fields with period ℓ which are not of minimal type such that the continued fraction expansion of ω has the symmetric part $a_1, \ldots, a_{\ell-1}$.

By Theorem 3.1 [A-i], when Case (III) occurs for $a_1, \ldots, a_{\ell-1}$, there is no positive integer d such that (3.1) is the continued fraction expansion of $\sqrt{d}$. So, we first assume that Case (III) does not occur. For an example, if ℓ and a_L are both even then Case (II) occurs ([**8**, Remark 2.1]). For $\delta = 1, 2, 3$, let $\mathscr{D}_\delta = \mathscr{D}_\delta(\ell; a_1, \ldots, a_{\ell-1})$ denote the set of all square-free positive integers d which are not of minimal type for $\sqrt{d}$ such that $d \equiv \delta \mod 4$ and $\sqrt{d} = [[\sqrt{d}], \overline{a_1, \ldots, a_{\ell-1}, 2[\sqrt{d}]}]$ is the continued fraction expansion with period ℓ of $\sqrt{d}$. Then, Friesen [**4**, Section III] shows that $\mathscr{D}_1 \cup \mathscr{D}_2 \cup \mathscr{D}_3$ is infinite. Furthermore, we can prove that $\mathscr{D}_2 \cup \mathscr{D}_3$ is infinite (Proposition 8.2 of Section 8.1). (When $\ell = 1$, we can see that $\mathscr{D}_3$ is empty, and both $\mathscr{D}_1$ and $\mathscr{D}_2$ are infinite.)

By Theorem 3.1 [A-ii], when Case (II) occurs for $a_1, \ldots, a_{\ell-1}$, there is no positive integer d such that $d \equiv 1 \mod 4$ and (3.3) is the continued fraction expansion with period ℓ of $(1+\sqrt{d})/2$. So, we next assume that Case (II) does not occur. For an example, if "ℓ is even and a_L is odd", or ℓ is odd, then Case (II) does not occur ([**8**, Remark 2.1]). Then, Halter-Koch [**5**, Theorem 2A (ii)] shows that there exist infinitely many square-free positive integers d which are not of minimal type for $(1+\sqrt{d})/2$ such that $d \equiv 1 \mod 4$ and $(1+\sqrt{d})/2 = [a_0, \overline{a_1, \ldots, a_{\ell-1}, 2a_0 - 1}]$ is the continued fraction expansion with period ℓ of $(1+\sqrt{d})/2$. Here, $a_0 := [(1+\sqrt{d})/2]$. (When $\ell = 1$, the same thing is true.)

EXAMPLE 3.3. When $d \equiv 1 \mod 4$, we give positive integers d of minimal type for $\sqrt{d}$ and for $(1+\sqrt{d})/2$ in Table 1. (When $d \equiv 1 \mod 4$, it is known that the parity of period of $\sqrt{d}$ coincides with that of $(1+\sqrt{d})/2$. For an example, see Kaplan and K. S. Williams [**7**, Introduction].)

EXAMPLE 3.4 ($\ell = 1$). As we have seen in the proof of Theorem 3.1, when $\ell = 1$, Case (I) occurs and $s_0 = 1$. As $g(x) = x$ and $h(x) = 1$, Theorem 3.1 [A-ii]

Table 1. Positive integers d, $2 \le d \le 200$ of dual minimal type.

	$\omega = \sqrt{d}$			$\omega = (1+\sqrt{d})/2$		
d	ℓ	s_0	Case	ℓ	s_0	Case
57	6	4	II	6	1	III
73	7	-20	I	9	-115	III
89	5	-4	I	7	-1	III
97	11	-348	I	9	-87	III
109	15	-72688	I	7	-9	I
113	9	-18	I	7	-79	III
129	10	68	II	10	17	III
133	16	33904	II	4	1	I
137	9	-52	I	7	-13	III
153	8	9	II	8	113	III
157	17	-57402	I	5	-7	I
161	10	303	II	10	1117	III
177	8	132	II	12	33	III
181	21	-7709944	I	5	-1	I
193	13	-90260	I	15	-22565	III

implies that $d = f(s_0) = 1 + 4 = 5$ is a positive integer with period 1 of minimal type for $(1+\sqrt{d})/2$. Also, we see by the assertions [A-i] and [B] of it that there exists no positive integer d with period 1 of minimal type for $\sqrt{d}$.

Example 3.5 ($\ell = 2$, 3). We let $\ell \ge 2$ be any positive integer, and consider a (symmetric) constant string of $\ell - 1$ positive integers $a, \ldots, a$. Since we see easily that

$$r_n = q_{n-1}, \quad 1 \le n \le \ell,$$

we have $(A, B, C) = (q_\ell, r_\ell, q_{\ell-2})$. For brevity, we put $D := Ar_{\ell-2} - BC$, and then

$$\begin{aligned} D &= q_\ell r_{\ell-2} - r_\ell q_{\ell-2} = (aq_{\ell-1} + q_{\ell-2})r_{\ell-2} - (ar_{\ell-1} + r_{\ell-2})q_{\ell-2} \\ &= a(q_{\ell-1}r_{\ell-2} - q_{\ell-2}r_{\ell-1}) = (-1)^{\ell-2}a = (-1)^\ell a. \end{aligned}$$

Therefore,

$$(-1)^\ell r_{\ell-2} - (-1)^\ell BC/A = (-1)^\ell D/A = a/q_\ell.$$

If $\ell > 2$ then, as $q_{\ell-2} > 0$ we have $q_\ell > aq_{\ell-1} \ge a$. Consequently, $0 < a/q_\ell < 1$.

By the definition of s_0, we obtain $s_0 = (-1)^{\ell} r_{\ell-2}$. We see easily that this holds when $\ell = 2$. Since

$$g(s_0) = q_\ell \cdot (-1)^{\ell} r_{\ell-2} - (-1)^{\ell} r_\ell q_{\ell-2} = (-1)^{\ell} D = a,$$

the condition (3.2) of Theorem 3.1 for $s = s_0$ does not hold. As a continued fraction expansion with period $2, 3$ has a constant symmetric part, in particular, this implies that there exists no positive integer d with period $2, 3$ of minimal type for $\sqrt{d}$, and for $(1+\sqrt{d})/2$.

In Section 5 we shall construct positive integers with period 4 of minimal type.

4. Yokoi invariant.

In this section, we let d be a square-free positive integer with $d > 1$, and consider a real quadratic field $\boldsymbol{Q}(\sqrt{d})$. Let $\varepsilon > 1$ be the fundamental unit of it, and we write uniquely $\varepsilon = (t + u\sqrt{d})/2$ with positive integers t, u. Then, we define the Yokoi invariant m_d of a real quadratic field $\boldsymbol{Q}(\sqrt{d})$ by putting $m_d := [u^2/t]$. Also, we put $\omega = \omega(d) := \sqrt{d}$ or $\omega := (1+\sqrt{d})/2$ according to whether $d \equiv 2$, 3 or $d \equiv 1 \bmod 4$.

Lemma 4.1. *We express d as in Theorem* 3.1 [B] *by using the quadratic polynomial $f(x)$ and the integer s, and put*

$$\lambda := \frac{A^2}{g(s)A + 2B}.$$

Then, if $d \equiv 2, 3 \mod 4$ then $m_d = [4\lambda]$, and if $d \equiv 1 \mod 4$ then $m_d = [\lambda]$.

Proof. Let x be a variable. For each integer $n \geq 0$, it holds that $[a_1, \dots, a_n, x] = (xq_{n+1} + q_n)/(xr_{n+1} + r_n)$. Let α be the same meaning as in the proof of Theorem 3.1. The continued fraction expansion of ω and (2.5) imply that

$$\omega_1 = 1/(\omega - a_0) = [a_1, \dots, a_{\ell-1}, \alpha + \omega] = \frac{(\alpha+\omega)q_\ell + q_{\ell-1}}{(\alpha+\omega)r_\ell + r_{\ell-1}} = \frac{(\alpha+\omega)A + B}{(\alpha+\omega)B + C}.$$

Therefore, $\{(\alpha+\omega)B + C\}\omega_1 - B = (\alpha+\omega)A$. On the other hand, as

$$a_\ell B + C = (\alpha+\omega)B + C - B(\omega - a_0)$$

by $a_\ell = a_0 + \alpha$, we see from $(\omega - a_0)\omega_1 = 1$ that

$$(a_\ell B + C)\omega_1 = \{(\alpha + \omega)B + C\}\omega_1 - B.$$

Hence we obtain $(a_\ell B + C)\omega_1 = (\alpha + \omega)A$. Also, it holds that

$$\omega_1 = [a_1, \dots, a_\ell, \omega_1] = \frac{q_{\ell+1}\omega_1 + q_\ell}{r_{\ell+1}\omega_1 + r_\ell},$$

and it is known ([**15**, Proposition 4.16]) that the denominator of the right hand side of it is equal to ε. (Also, $N_{\boldsymbol{Q}(\sqrt{d})/\boldsymbol{Q}}(\varepsilon) = (-1)^\ell$.) Therefore we see by (2.5) that

$$\varepsilon = (a_\ell r_\ell + r_{\ell-1})\omega_1 + r_\ell = (a_\ell q_{\ell-1} + r_{\ell-1})\omega_1 + q_{\ell-1} = (a_\ell B + C)\omega_1 + B.$$

Consequently, $\varepsilon = \alpha A + B + A\omega$. First, we assume $d \equiv 2, 3 \mod 4$. As $\alpha = g(s)/2$ from (3.4), we have $\varepsilon = (g(s)A + 2B + 2A\sqrt{d})/2$, so that $t = g(s)A + 2B$ and $u = 2A$. Hence, $m_d = [4\lambda]$. Next, we assume $d \equiv 1 \mod 4$. As $\alpha = (g(s) - 1)/2$ from (3.4), we have $\varepsilon = (g(s)A + 2B + A\sqrt{d})/2$, so that $t = g(s)A + 2B$ and $u = A$. Hence, $m_d = [\lambda]$. Our lemma is proved. □

Proposition 4.2. *Let $\boldsymbol{Q}(\sqrt{d})$ be a real quadratic field with arbitrary period that is not of minimal type. Then, if $d \equiv 2, 3 \mod 4$ then $0 \le m_d \le 3$, and if $d \equiv 1 \mod 4$ then $m_d = 0$.*

Proof. As $A > 0$, the linear function $g(x)$ is strictly, monotonously increasing. Since $s > s_0$ by the assumption, we have

$$\begin{aligned} g(s) &\ge g(s_0 + 1) = A(s_0 + 1) - (-1)^\ell BC \\ &> A \cdot (-1)^\ell BC/A + A - (-1)^\ell BC = A. \end{aligned}$$

Therefore, $g(s) \ge A + 1$. Hence,

$$\lambda \le \frac{A^2}{(A+1)A + 2B} = V^{-1}, \text{ where } V := \frac{(A+1)A + 2B}{A^2} = 1 + \frac{1}{A} + \frac{2B}{A^2}.$$

If $d \equiv 2, 3 \mod 4$ then Lemma 4.1 implies that $m_d = [4\lambda] \le [4V^{-1}]$. As $V > 1$, we have $4V^{-1} < 4$. Hence, $m_d \le 3$. If $d \equiv 1 \mod 4$ then $m_d = [\lambda] \le [V^{-1}]$. As $V^{-1} < 1$, we obtain $m_d = 0$. This proves our proposition. □

We suppose that d is a square-free positive integer, and let ℓ be the period of $\omega = \omega(d)$.

Example 4.1 ($\ell = 1$). We have $m_d = 2$ if $d = 2$, $m_d = 1$ if $d = 5$, and otherwise $m_d = 0$.

Proof. When $\ell = 1$, Case (I) occurs, $s_0 = 1$, and $f(x) = x^2 + 4$. First, we assume $d \equiv 2, 3 \mod 4$. We see by the assertions [B] and [A-i] of Theorem 3.1 that there is some even integer $s \geq 2$ for which $d = (s^2 + 4)/4$. Since $A = 1$ and $B = 0$, we have $\lambda = 1/s$, and Lemma 4.1 implies that $m_d = [4/s]$. Therefore, $m_d = 2$ if $s = 2$ ($d = 2$). As $d \equiv 2, 3 \mod 4$, we have $s \neq 4$. If $s \geq 6$ then $m_d = 0$. Next, we assume $d \equiv 1 \mod 4$. We see by the assertions [B] and [A-ii] of it that there is some odd integer $s \geq 1$ for which $d = s^2 + 4$. Lemma 4.1 implies that $m_d = [1/s]$. Therefore, $m_d = 1$ if $s = 1$ ($d = 5$). If $s \geq 3$ then $m_d = 0$. $\square$

We shall calculate the value of m_d when $\ell = 2, 3$. As we have seen in Example 3.5, then d is a positive integer that is not of minimal type. Hence, Proposition 4.2 yields that $m_d = 0$ if $d \equiv 1 \mod 4$. So, we may assume $d \equiv 2, 3 \mod 4$.

Example 4.2 ($\ell = 2$). The integer d is of the form $d = (as/2)^2 + s$. Here, "a is a positive odd integer and s is a positive even integer", or "a is a positive even integer and $s \geq 2$ is a positive integer". Then, we have $m_d = 1$ if $s = 2$ or 3, and otherwise $m_d = 0$.

Proof. We consider a "symmetric" positive integer a. Since $A = a$, $B = 1$ and $C = 0$, if a is odd then Case (I) occurs, and if a is even then Case (II) occurs. As we have seen in Example 3.5, $s_0 = (-1)^2 r_0 = 1$. Also, as $g(x) = ax$ and $h(x) = x$, we have $f(x) = a^2x^2 + 4x$. For any integer $s \geq 1$, the condition (3.2) holds if and only if $s \geq 2$. Since $d \equiv 2, 3 \mod 4$, we see by the assertions [B] and [A-i] of Theorem 3.1 that there is some integer $s \geq 2$ for which $d = f(s)/4 = (as/2)^2 + s$. Here, s becomes even if a is odd, and

$$\sqrt{d} = \left[\frac{as}{2}, \overline{a, as}\right]$$

is the continued fraction expansion with period 2 of $\sqrt{d}$. Furthermore,

$$\lambda = \frac{a^2}{(as)a + 2} = \frac{a^2}{a^2 s + 2}.$$

If $s = 2$, as $4\lambda = 1 + (a^2 - 1)/(a^2 + 1)$, then Lemma 4.1 implies that $m_d = [4\lambda] = 1$.

If $s = 3$ then $4\lambda = 1 + (a^2 - 2)/(3a^2 + 2)$, and $a \geq 2$ as a is even. Therefore, $m_d = 1$. If $s \geq 4$ then, as $(a^2 s + 2) - 4a^2 > 0$ we have $4\lambda < 1$, so that $m_d = 0$. Thus we obtain our assertion. $\square$

Example 4.3 ($\ell = 3$). The integer d is of the form $d = (a^2+1)^2(s/2)^2 + a(a^2+3)s/2 + (a/2)^2 + 1$. Here, a and s are both positive even integers. Then, we have $m_d = 1$ if $s = 2$, and $m_d = 0$ if $s \geq 4$.

Proof. We consider a (symmetric) constant string of 2 positive integers a, a. Since $A = a^2+1$, $B = a$ and $C = 1$, if a is even then Case (I) occurs, and if a is odd then Case (III) occurs. As we have seen in Example 3.5, $s_0 = (-1)^3 r_1 = 0$. Also, as $g(x) = (a^2+1)x + a$ and $h(x) = ax + 1$, we have $f(x) = (a^2+1)^2 x^2 + 2a(a^2+3)x + a^2 + 4$. For any integer $s \geq 0$, the condition (3.2) holds if and only if $s > 0$. As $d \equiv 2, 3 \mod 4$, we see by the assertions [B] and [A-i] of Theorem 3.1 that Case (I) occurs, that is, a is even, and that there is some even integer $s \geq 2$ for which $d = f(s)/4 = (a^2+1)^2(s/2)^2 + a(a^2+3)s/2 + (a/2)^2 + 1$. Then,

$$\sqrt{d} = \left[\frac{(a^2+1)s + a}{2}, \overline{a,\, a,\, (a^2+1)s + a}\right]$$

is the continued fraction expansion with period 3 of $\sqrt{d}$. Furthermore,

$$\lambda = \frac{(a^2+1)^2}{\{(a^2+1)s + a\}(a^2+1) + 2a} = \frac{(a^2+1)^2}{(a^2+1)^2 s + a(a^2+3)}.$$

If $s = 2$, since the second term of the right hand side of

$$4\lambda = 1 + \frac{2(a^2+1)^2 - a(a^2+3)}{2(a^2+1)^2 + a(a^2+3)}$$

is positive and less than 1, Lemma 4.1 implies that $m_d = [4\lambda] = 1$. If $s \geq 4$ then, as $0 < 4\lambda < 1$ we have $m_d = 0$. Thus we obtain our assertion. $\square$

Example 4.4. When $d = 55$, as $s_0 = 1, s = 2$, $\boldsymbol{Q}(\sqrt{d})$ is a real quadratic field (with period 4) that is not of minimal type, and we have $m_d = 3$ ($h_d = 2$). Also, when $d = 58$, as $s_0 = -3, s = -2$, $\boldsymbol{Q}(\sqrt{d})$ is a real quadratic field (with period 7) that is not of minimal type, and $m_d = 3$ ($h_d = 2$). Thus there exist real quadratic fields that are not of minimal type satisfying $m_d = 3$.

Finally, we see a usefulness of Yokoi invariant. Let $\varepsilon_d > 1$ and m_d denote the fundamental unit and the Yokoi invariant of a real quadratic field $\boldsymbol{Q}(\sqrt{d})$, respectively. Yokoi [**26**, Theorem 1.1] proved the following:

Lemma 4.3. *If $d > 13$ then $m_d d < \varepsilon_d < (m_d + 1)d$.*

Proposition 4.4. *There exist exactly* 51 *real quadratic fields of class number* 1 *that are not of minimal type, with one more possible exception. All such fields are listed in the table of Section* 8.2.

Proof. Let h_d denote the class number of a real quadratic field $\boldsymbol{Q}(\sqrt{d})$. It is known by a theorem of Tatuzawa [**20**, Theorem 2] and Dirichlet's class number formula that

$$h_d > \frac{0.3275}{s} \cdot \frac{d^{(s-2)/2s}}{\log \varepsilon_d}$$

for any real number $s \geq 11.2$, and any square-free positive integer $d \geq e^s$ with one possible exception. (For an example, see [**26**, p. 187].) We let $\boldsymbol{Q}(\sqrt{d})$ be a real quadratic field that is not of minimal type, take $s := 16$, and assume that $d \geq e^{16} = 8886110.5\cdots$. Since $m_d \leq 3$ by Proposition 4.2, Lemma 4.3 yields that $\varepsilon_d < 4d$. Hence,

$$h_d > \frac{0.3275}{16} \cdot \frac{d^{7/16}}{\log 4d} \geq \frac{0.3275}{16} \cdot \frac{e^7}{\log 4e^{16}} = 1.2\cdots > 1.$$

Here, we use the fact that the function $(x^{7/16})/\log 4x$ is monotonously increasing in the interval $[(e^{16/7})/4, \infty)$. On the other hand, when $d < 10^7$, there are exactly 51 real quadratic fields $\boldsymbol{Q}(\sqrt{d})$ of class number 1 that are not of minimal type by the table of Section 8.2. This proves our proposition. □

Lemma 4.5 is used in Section 7.

Lemma 4.5. *We suppose that a sequence $\{d_n\}_{n\geq 1}$ of square-free positive integers is strictly monotonously increasing. Let $\varepsilon_{d_n} > 1$, m_{d_n} and h_{d_n} denote the fundamental unit, the Yokoi invariant and the class number of a real quadratic field $\boldsymbol{Q}(\sqrt{d_n})$, respectively. We assume that $m_{d_n} \geq 1$ for all $n \geq 1$ and the sequence $\{m_{d_n}\}_{n\geq 1}$ of positive integers is bounded. Then, the sequence $\{h_{d_n}\}_{n\geq 1}$ of positive integers is not bounded. Namely, for any positive integer h, there exist infinitely many numbers $n \geq 1$ such that $h_{d_n} > h$.*

Proof. By our assumption, there is some real number $c' > 0$ such that $m_{d_n} \leq c'$ for any number $n \geq 1$. Also, as $\{d_n\}_{n\geq 1}$ is strictly monotonously increasing, there is some number $n_0 \geq 1$ for which $d_n > 13$ if $n \geq n_0$. Consequently, Lemma 4.3 implies that

$$m_{d_n} d_n < \varepsilon_{d_n} < (m_{d_n} + 1) d_n$$

for all $n \geq n_0$. Put $c := c' + 1$, and then $c > 1$. As $m_{d_n} \geq 1$, we have

$$d_n < \varepsilon_{d_n} < c d_n, \quad \forall n \geq n_0. \tag{4.1}$$

Let $n \geq n_0$. Note that $cd_n > 1$ as $d_n > 1$. We see by (4.1) that

$$\frac{\log \log d_n}{\log d_n} < \frac{\log \log \varepsilon_{d_n}}{\log d_n} < \frac{\log \log (c d_n)}{\log d_n}.$$

Hence,

$$\lim_{n \to \infty} \frac{\log \log \varepsilon_{d_n}}{\log d_n} = 0. \tag{4.2}$$

Let D_{d_n} denote the discriminant of $\boldsymbol{Q}(\sqrt{d_n})$. We put $\delta := 4$ or $\delta := 1$ according to whether $d_n \equiv 2, 3$ or $d_n \equiv 1 \bmod 4$. Then, $D_{d_n} = \delta d_n$. It follows from (4.2) that

$$\frac{\log \log \varepsilon_{d_n}}{\log D_{d_n}} = \frac{\log \log \varepsilon_{d_n} / \log d_n}{(\log \delta / \log d_n) + 1} \longrightarrow 0 \quad (n \longrightarrow \infty). \tag{4.3}$$

On the other hand, we see by a theorem of Siegel (Narkiewicz [**14**, Theorem 8.14]) that

$$\lim_{n \to \infty} \frac{\log (h_{d_n} \log \varepsilon_{d_n})}{\log D_{d_n}} = \frac{1}{2}. \tag{4.4}$$

Since

$$\frac{\log (h_{d_n} \log \varepsilon_{d_n})}{\log D_{d_n}} = \frac{\log h_{d_n}}{\log D_{d_n}} + \frac{\log \log \varepsilon_{d_n}}{\log D_{d_n}},$$

if we assume that $\{h_{d_n}\}_{n \geq 1}$ is bounded then (4.4) yields that

$$\lim_{n \to \infty} \frac{\log \log \varepsilon_{d_n}}{\log D_{d_n}} = \frac{1}{2}.$$

This contradicts (4.3). Therefore, $\{h_{d_n}\}_{n\geq 1}$ is not bounded. □

5. Real quadratic fields with period 4 of minimal type.

In this section, we shall construct positive integers with period 4 of minimal type to give real quadratic fields with period 4 of minimal type.

LEMMA 5.1. *We consider a symmetric string of* 3 *positive integers* a_1, a_2, a_1. *Then the following hold.*

(i) *If* $(a_1, a_2) \equiv (1, 1) \mod 2$ *then Case* (I) *occurs, if* a_2 *is even then Case* (II) *occurs, and if* $(a_1, a_2) \equiv (0, 1) \mod 2$ *then Case* (III) *occurs.*
(ii) *The condition* (3.2) *of Theorem* 3.1 *for* $s = s_0$ *holds if and only if "*$a_1 > a_2$*", or "*$a_1 < a_2$ *and* $a_1 \nmid a_2$*". Furthermore, if this condition holds then* $s_0 = [a_2/a_1] + 1$.

PROOF. The assertion (i) follows from $A = a_1(a_1a_2 + 2)$, $B = a_1a_2 + 1$ and $C = a_2$. To show the assertion (ii), we note that

$$0 < (-1)^\ell BC/A = \frac{a_2}{a_1} - \frac{C}{A} < \frac{a_2}{a_1}$$

and $g(s_0) = a_1(a_1a_2 + 2)s_0 - a_2(a_1a_2 + 1)$.

(A) The case where $a_1 \geq a_2$. As $0 < (-1)^\ell BC/A < 1$, the definition of s_0 implies that $s_0 = 1$. When $a_1 = a_2$, we see by Example 3.5 that $(3.2)_{s=s_0}$ does not hold. When $a_1 > a_2$, as $g(s_0) = (a_1 - a_2)(a_1a_2 + 1) + a_1$, we have $g(s_0) > a_1$, so that $(3.2)_{s=s_0}$ holds.

(B) The case where $a_1 < a_2$. Let q and r be the quotient and the remainder of the division of a_2 by a_1, respectively. Then,

$$(-1)^\ell BC/A = q + \frac{r}{a_1} - \frac{C}{A}. \tag{5.1}$$

If $r = 0$, as $0 < C/A < 1$, we have $s_0 = q = a_2/a_1$ by (5.1). Therefore $g(s_0) = a_2$, and then $(3.2)_{s=s_0}$ does not hold. If $r > 0$ then

$$rA - a_1C = a_1\{r(a_1a_2 + 2) - a_2\} = a_1\{(ra_1 - 1)a_2 + 2r\} \geq 2a_1r > 0,$$

which yields that $0 < \frac{r}{a_1} - \frac{C}{A}$. On the other hand, we have $\frac{r}{a_1} - \frac{C}{A} < 1 - \frac{C}{A} < 1$. From (5.1) we obtain $s_0 = q + 1 = [a_2/a_1] + 1$. Hence, since

$$g(s_0) = a_1(a_1a_2+2)q + (a_1-a_2)(a_1a_2+2) + a_2$$
$$= (a_1a_2+2)(a_1q+a_1-a_2) + a_2 = (a_1a_2+2)(a_1-r) + a_2 > a_2,$$

$(3.2)_{s=s_0}$ holds. This proves our lemma. □

Proposition 5.2.

(i) *Let a be a positive integer. For any integer $t \geq 0$, we put*

$$d = d(t) := \{(8a^2+6a+1)t + 8a^2+4a+1\}^2 + (4a+2)t + 4a + 1.$$

Then, d is a positive integer with period 4 of minimal type for $\sqrt{d}$ and

$$\sqrt{d} = \Big[(8a^2+6a+1)t + 8a^2+4a+1, \overline{4a+1,\ (4a+1)t+4a,\ 4a+1,\ (16a^2+12a+2)t+16a^2+8a+2}\Big]. \tag{5.2}$$

If t is even then $d \equiv 2 \mod 4$, and if t is odd then $d \equiv 3 \mod 4$. Furthermore, if d is square-free then $m_d = 16a$.

(ii) *Let a be a positive odd integer. For any integer $t \geq 0$, we put*

$$d = d(t) := \{(a^2+3a+2)t + a^2+2a+2\}^2 + 4\{(a+2)t + a + 1\}.$$

Then, d is a positive integer with period 4 of minimal type for $(1+\sqrt{d})/2$, $d \equiv 1 \mod 4$, and

$$(1+\sqrt{d})/2 = \left[\frac{(a^2+3a+2)t + a^2+2a+3}{2}, \overline{a+1,\ (a+1)t+a,\ a+1,\ (a^2+3a+2)t+a^2+2a+2}\right]. \tag{5.3}$$

Furthermore, if d is square-free then $m_d = a$.

Proof. We let b and t be integers with $b \geq 1, t \geq 0$, and consider a symmetric string of 3 positive integers $b+1$, $(b+1)t+b$, $b+1$, to make the integral part of λ in Lemma 4.1 larger. Then, Lemma 5.1 (ii) implies that the condition $(3.2)_{s=s_0}$ holds and $s_0 = t+1$. Since

$$A = (b+1)\{(b+1)^2t + b^2+b+2\},\ B = (b+1)^2t + b^2+b+1,\ C = (b+1)t + b,$$

simple calculations yield that

$$g(s_0) = (b^2 + 3b + 2)t + b^2 + 2b + 2, \quad h(s_0) = (b + 2)t + b + 1.$$

Consequently,

$$\begin{aligned} f(s_0) &= g(s_0)^2 + 4h(s_0) \\ &= \{(b^2 + 3b + 2)t + b^2 + 2b + 2\}^2 + 4\{(b + 2)t + b + 1\}. \end{aligned} \tag{5.4}$$

For brevity, we put $a_1 := b + 1$, $a_2 := (b + 1)t + b$, and obtain

$$\begin{aligned} g(s_0)A + 2B &= a_1^2(a_1a_2 + 2)^2 s_0 - a_1a_2(a_1a_2 + 2)(a_1a_2 + 1) + 2(a_1a_2 + 1) \\ &= a_1^2(a_1a_2 + 2)^2 s_0 - a_1a_2(a_1a_2 + 2)^2 + a_1a_2(a_1a_2 + 2) \\ &\quad + 2(a_1a_2 + 2) - 2 \qquad (\because a_1a_2 + 1 = a_1a_2 + 2 - 1) \\ &= a_1^2(a_1a_2 + 2)^2 s_0 - a_1a_2(a_1a_2 + 2)^2 + (a_1a_2 + 2)^2 - 2 \\ &= (a_1a_2 + 2)^2(a_1^2 s_0 - a_1a_2 + 1) - 2. \end{aligned}$$

As $a_1^2 s_0 - a_1a_2 + 1 = b + 2$, we have

$$\lambda := \frac{A^2}{g(s_0)A + 2B} = \frac{(b + 1)^2\{(b + 1)^2 t + b^2 + b + 2\}^2}{(b + 2)\{(b + 1)^2 t + b^2 + b + 2\}^2 - 2}.$$

Hence,

$$\begin{aligned} \lambda &= \frac{(b^2 + 2b)\{(b + 1)^2 t + b^2 + b + 2\}^2 - 2b + 2b + \{(b + 1)^2 t + b^2 + b + 2\}^2}{(b + 2)\{(b + 1)^2 t + b^2 + b + 2\}^2 - 2} \\ &= b + \frac{2b + \{(b + 1)^2 t + b^2 + b + 2\}^2}{(b + 2)\{(b + 1)^2 t + b^2 + b + 2\}^2 - 2}. \end{aligned} \tag{5.5}$$

The second term of the right hand side of it is positive and less than 1.

(i) Take $b = 4a$. Since $(b + 1)t + b \equiv t \mod 2$, we see by Lemma 5.1 (i) that if t is odd then Case (I) occurs for the given 3 symmetric positive integers and s_0 is even. If t is even then Case (II) occurs. Put $d := f(s_0)/4$, and then (5.4) yields that

$$d = \{(8a^2 + 6a + 1)t + 8a^2 + 4a + 1\}^2 + (4a + 2)t + 4a + 1.$$

By Theorem 3.1 [A-i], we obtain the continued fraction expansion (5.2). As $d \equiv t^2 + 2 \mod 4$, if t is even then $d \equiv 2 \mod 4$, and if t is odd then $d \equiv 3 \mod 4$. As $b \geq 4$, the second term of the right hand side of (5.5) is less than 1/4. Hence, when d is square-free, Lemma 4.1 implies that $m_d = [4\lambda] = 16a$.

(ii) Take $b = a$. Since a is odd, we see by Lemma 5.1 (i) that Case (III) occurs for the given 3 symmetric positive integers. Put

$$d := f(s_0) = \{(a^2 + 3a + 2)t + a^2 + 2a + 2\}^2 + 4\{(a + 2)t + a + 1\},$$

and then, from Theorem 3.1 [A-ii], we obtain $d \equiv 1 \mod 4$ and the continued fraction expansion (5.3). When d is square-free, it follows from Lemma 4.1 and (5.5) that $m_d = [\lambda] = a$. Our proposition is proved. □

Remark 5.1. Let d be a positive square-free integer. Let D and ℓ be the discriminant and the period of a real quadratic field $\boldsymbol{Q}(\sqrt{d})$, respectively. When $\ell \leq 4$, Hendy [**6**, Theorem 1] gave explicit representations of d and partial quotients of the continued fraction expansion of $\omega(d)$ to study the class number of $\boldsymbol{Q}(\sqrt{d})$. By using the continued fraction expansion of a reduced quadratic irrational with discriminant D, Azuhata [**2**] also conduced to another representation of d, which is different from one of [**6**], when $\ell \leq 4$ (also in some case when $\ell = 5$). We assume that $\ell = 4, 5$ and $d \equiv 1 \mod 4$. Then, Tomita [**21**] investigated the connection between the form of d and partial quotients of the continued fraction expansion of $\omega(d) = (1 + \sqrt{d})/2$, and [**21**, Theorem, Remark 2] gave a representation of d by using the partial quotients and some parameters determined by them. Especially, in the case where $[\sqrt{d}]$ is even, it becomes a concrete example that gives the representation of d by Yokoi's d-invariant n_d (see [**26**, Theorem 2.1]). Also, in this case, it was shown ([**21**, Corollary 1]) that $m_d = 0$ holds when $\ell = 4$, and a necessary and sufficient condition for $m_d = 1$ to hold was given when $\ell = 5$ ([**21**, Corollary 2]). In the case where $[\sqrt{d}]$ is odd, a necessary and sufficient condition for $m_d \neq 0$ to hold was given ([**21**, Corollaries 1, 3]). However, in this case, we did not suggest explicitly a representation of m_d. When $\ell = 4$, Proposition 5.2 is an example that gives explicit forms of d and m_d by using partial quotients of the continued fraction expansion of $\omega(d)$.

Remark 5.2. Let d be a non-square positive integer such that $d \equiv 1 \mod 4$, the period of $(1 + \sqrt{d})/2$ is equal to 4, and $[\sqrt{d}]$ is even. Then we see by [**6**, p. 269] or [**21**, Remark 2 (i)] that the symmetric part of the continued fraction expansion of $(1 + \sqrt{d})/2$ is of the form $1, a_2, 1$. Hence, Theorem 3.1 [B] and Lemma 5.1 (ii) imply that d is not of minimal type for $(1 + \sqrt{d})/2$. (Also, it follows from Theorem 3.1 [A-ii] and Lemma 5.1 (i) that a_2 is odd and Case (I) occurs.)

6. Arithmetic of quadratic polynomials.

In Nagell [**12**, Section 2], it is proved under some conditions on a quadratic polynomial $f(x)$ (indeed, of general degree) that there exist infinitely many positive integers t for which $f(t)$ is square-free. In order to construct an infinite family of real quadratic fields in Section 7, we make his conditions easy to handle in Proposition 6.1. Though it seems known, a proof of it will be given here, as no reference for it is known to the authors. The proof is essentially due to Nagell.

PROPOSITION 6.1. *Let $f(x) = ax^2 + bx + c$ be a quadratic polynomial in $\boldsymbol{Z}[x]$ with $a > 0$. As $a > 0$, there is some integer t_1 such that*

$$t \in \boldsymbol{Z},\ t \geq t_1 \Longrightarrow f(t) > 0. \tag{6.1}$$

We suppose that the discriminant $d(f) = b^2 - 4ac$ of $f(x)$ is not equal to 0, the greatest common divisor (a, b, c) is square-free, and

$$\text{“there is some integer } t \text{ for which } f(t) \not\equiv 0 \mod 4\text{”}. \tag{6.2}$$

Then, the set $\{f(t) \mid t \in \boldsymbol{Z},\ t \geq t_1\}$ contains infinite square-free elements.

REMARK 6.1. If we can find some integer $t_2 \geq t_1$ such that $f(t_2)$ is square-free, then (a, b, c) is square-free. Also, since $f(t_2)$ is not divisible by 2^2, (6.2) holds.

PROOF. For any real number $x > t_1$, we define

$$A(x) := \sharp\{t \in \boldsymbol{Z} \mid t_1 \leq t \leq x,\ f(t) \text{ is square-free}\}.$$

We shall prove $A(x) \to \infty$ as $x \to \infty$. By $a > 0$, since the function $f(x)$ is strictly monotonously increasing for sufficiently large x, this implies that the set $\{f(t) \mid t \in \boldsymbol{Z},\ t \geq t_1\}$ contains infinite square-free elements. It is known by Nagell [**13**, p. 82, Theorem 45] that there are infinitely many prime numbers p which is a divisor of $f(t)$ with some integer $t \geq t_1$. We arrange such prime numbers in order of size: $p_1,\ p_2,\ p_3,\ \ldots$. As $\sum_{i=1}^{\infty} \frac{1}{p_i^2} < \sum_{k=1}^{\infty} \frac{1}{k^2} < \infty$, there is some number $m \geq 2$ such that

$$\sum_{i=m}^{\infty} \frac{1}{p_i^2} < \frac{1}{2}, \tag{6.3}$$

$$i \geq m \Longrightarrow p_i \nmid ad(f), \tag{6.4}$$

and put

$$P := p_1^2 \cdots p_{m-1}^2.$$

We let $\mathrm{ord}_p(*)$ denote the additive valuation on the rationals $\boldsymbol{Q}$ with $\mathrm{ord}_p(p) = 1$ for a prime number p, and claim that there is some integer $t_{0,i}$ such that

$$\mathrm{ord}_{p_i}(f(t_{0,i})) < 2,$$

for each $i\,(1 \leq i \leq m-1)$. Let $1 \leq i \leq m-1$, $p = p_i$. When $p = 2$, it follows from (6.2). When $p \geq 3$, we can choose 3 integers u_1, u_2, u_3, any two of which are distinct modulo p. We assume $\mathrm{ord}_p(f(t)) \geq 2$ for all integers t. Then it holds that $c + bu_j + au_j^2 = f(u_j) \equiv 0 \mod p^2$ for each u_j, that is,

$$\begin{pmatrix} 1 & u_1 & u_1^2 \\ 1 & u_2 & u_2^2 \\ 1 & u_3 & u_3^2 \end{pmatrix} \begin{pmatrix} c \\ b \\ a \end{pmatrix} \equiv \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \mod p^2.$$

The determinant of the matrix in the left hand side of it is equal to $(u_2-u_1)(u_3-u_1)$ (u_3-u_2), which is co-prime to p. Since the matrix is invertible modulo p^2, all c, b, a are divisible by p^2. This contradicts that (a, b, c) is square-free, and our claim is proved. Also, Chinese remainder theorem implies that there is some integer $t_0 \geq t_1$ such that

$$t_0 \equiv t_{0,i} \mod p_i^2, \quad 1 \leq \forall i \leq m-1.$$

We consider a quadratic polynomial

$$g(y) := f(Py + t_0)$$

in $\boldsymbol{Z}[y]$, and define

$$B(y) := \sharp\{t \in \boldsymbol{Z} \mid 0 \leq t \leq y,\, g(t) \text{ is square-free}\}$$

for any real number $y > 0$. If $0 \leq t \leq y$ and $g(t)$ is square-free, then the definition of $g(y)$ yields that $f(Pt+t_0)$ is square-free and $t_1 \leq t_0 \leq Pt+t_0 \leq Py+t_0$. Thus, we find a rough comparison of the cardinals

$$A(Py + t_0) \geq B(y) \tag{6.5}$$

for $y > 0$. For a prime number p and a real number $y > 0$, we define

$$\hat{B}_p(y) := \sharp\{t \in \boldsymbol{Z} \mid 0 \leq t \leq y,\, g(t) \equiv 0 \mod p^2\}.$$

That $g(t)$ is square-free means that $g(t)$ is not divisible by p^2 for any prime number p. So, we obtain a rough estimate for $B(y)$ from below:

$$B(y) \geq y - \sum_p \hat{B}_p(y). \tag{6.6}$$

Here, p ranges over all prime numbers. By considering 4 cases separately, we show that $\hat{B}_p(y) = 0$ for almost all prime numbers p when y is sufficiently large. Let p be any prime number.

(I) If p is different from p_i's, $i \geq 1$, then p does not divide $f(t)$ for all $t \geq t_1$. The definition of $g(y)$ implies that p does not divide $g(t)$ for all $t \geq 0$. Hence, $\hat{B}_p(y) = 0$ for $y > 0$.

(II) The case where $p = p_i$ with some i, $1 \leq i \leq m-1$. Let t be any integer. Since $Pt + t_0 \equiv t_0 \equiv t_{0,i} \mod p_i^2$, we see by the definition of $t_{0,i}$ that $g(t) \equiv f(t_{0,i}) \not\equiv 0 \mod p_i^2$. Consequently, $\hat{B}_p(y) = 0$ for $y > 0$.

We let $G := (a, b, c)$, and

$$f(x) = G \prod_{k=1}^{\nu} f_k(x)$$

be a factorization of $f(x)$ into irreducible polynomials in $\boldsymbol{Z}[x]$ ($\nu = 1$ or 2). Then we have the factorization

$$g(y) = G \prod_{k=1}^{\nu} g_k(y), \quad g_k(y) := f_k(Py + t_0). \tag{6.7}$$

Put $n_k := \deg f_k(x)$. Each factor $g_k(y)$ is a polynomial of degree 1 or 2 in $\boldsymbol{Z}[y]$, and the leading term of $g_k(y)$ is greater than or equal to 1 since it is a positive integer by $P > 0$. Hence there are some real numbers $y_k > 0$ and $c_k > 1$ such that

$$y \geq y_k \Longrightarrow |g_k(y)| < c_k y^{n_k}, \tag{6.8}$$

$$y \geq y_k,\ y \geq t \geq 0 \Longrightarrow |g_k(t)| \leq |g_k(y)|. \tag{6.9}$$

We put $y_0 := \max\{y_k \mid 1 \leq k \leq \nu\}$ and $c := \max\{c_k \mid 1 \leq k \leq \nu\}$, and assume that $y \geq y_0$ throughout the present proof.

(III) The case where $p = p_i$ and $p_i > cy$ with some $i \geq m$. Let $t \in \boldsymbol{Z}$, $0 \leq t \leq y$. We show that $g(t) \not\equiv 0 \mod p^2$. For each k $(1 \leq k \leq \nu)$, it follows from (6.9), (6.8) and $c_k > 1$ that

$$|g_k(t)| \leq |g_k(y)| < c_k y^{n_k} \leq c_k^{n_k} y^{n_k} \leq c^{n_k} y^{n_k} < p^{n_k}.$$

As $n_k \leq 2$, we obtain $|g_k(t)| < p^2$. On the other hand, $Pt + t_0 \geq t_0 \geq t_1$ and (6.1) yield that $g(t) = f(Pt + t_0) > 0$. Consequently we have $|g_k(t)| > 0$ from (6.7). Therefore each $g_k(t)$ is not divisible by p^2. First, we assume $\nu = 1$. Since G is co-prime to p by (6.4), the factorization (6.7) implies that $g(t)$ (> 0) is not divisible by p^2. Next, we assume $\nu = 2$. If we assume that $p \mid g_1(t)$ and $p \mid g_2(t)$, then $p \mid f_1(t')$ and $p \mid f_2(t')$. Here we put $t' := Pt + t_0$. Let θ_1 be a solution of equation $f_1(x) = 0$, and a_1 the leading term of linear polynomial $f_1(x)$. Then, $a_1\theta_1 \in \boldsymbol{Z}$. Since a_1 is a divisor of a, we have $p \nmid a_1$ by (6.4). Also, we see from $p \mid f_1(t')$ that $\mathrm{ord}_p(a_1(t' - \theta_1)) = \mathrm{ord}_p(f_1(t')) > 0$. Therefore, $\mathrm{ord}_p(t' - \theta_1) > 0$. Similarly, let θ_2 be a solution of equation $f_2(x) = 0$, and we obtain $\mathrm{ord}_p(t' - \theta_2) > 0$. Hence, $\mathrm{ord}_p(\theta_1 - \theta_2) = \mathrm{ord}_p(\theta_1 - t' + t' - \theta_2) > 0$. Therefore, $d(f) = a^2(\theta_1 - \theta_2)^2$ implies that $\mathrm{ord}_p(d(f)) > 0$, and this contradicts (6.4). Thus, only one of $g_1(t)$ and $g_2(t)$ can be divisible by p(, and not divisible by p^2). We see by the same reason as above that $g(t)$ is not divisible by p^2. Thus we have always $\hat{B}_p(y) = 0$.

(IV) The case where $p = p_i$ and $p_i \leq cy$ with some $i \geq m$. Note that $y < ([y/p^2] + 1)p^2$ as $y/p^2 < [y/p^2] + 1$. Since the interval $[0, y]$ is contained in the union of intervals $[kp^2, (k+1)p^2]$, $0 \leq k \leq [y/p^2]$, we obtain

$$\hat{B}_p(y) = \sum_{\substack{0 \leq t \leq y \\ g(t) \equiv 0 \bmod p^2}} 1 \leq \left(\left[\frac{y}{p^2} \right] + 1 \right) \sum_{\substack{t \bmod p^2 \\ g(t) \equiv 0 \bmod p^2}} 1.$$

Let u be any solution of congruence $f(x) \equiv 0 \mod p$ in integers. We let θ_1, θ_2 be all solutions of equation $f(x) = 0$ to show $p \nmid f'(u)$. Here, $f'(x)$ is the derivate of $f(x)$. We denote by $\boldsymbol{C}_p$ the completion of an algebraic closure of the p-adic field $\boldsymbol{Q}_p$, and denote again by the same symbol the unique extension of $\mathrm{ord}_p(*)$ to $\boldsymbol{C}_p$. Also, we fix an embedding of an algebraic closure of $\boldsymbol{Q}$ in $\boldsymbol{C}_p$. Since $p \nmid a$ by (6.4) and $f(x) = a(x - \theta_1)(x - \theta_2)$, we have $\mathrm{ord}_p(u - \theta_1) + \mathrm{ord}_p(u - \theta_2) = \mathrm{ord}_p(f(u)) > 0$. So, there is some number i for which $\mathrm{ord}_p(u - \theta_i) > 0$, say $i = 1$. If we assume that $p \mid f'(u)$, then $f'(u) = a(u - \theta_1) + a(u - \theta_2)$ and $\mathrm{ord}_p(u - \theta_1) > 0$ yield that $\mathrm{ord}_p(u - \theta_2) > 0$. Therefore, $\mathrm{ord}_p(\theta_1 - \theta_2) > 0$. This and the same argument as in (III) imply that $\mathrm{ord}_p(d(f)) > 0$, and it contradicts (6.4). Thus, we obtain $p \nmid f'(u)$. It is known under this condition that the congruence $f(x) \equiv 0 \mod p^2$

has at most 2 solutions. As $p \nmid P$ by $i \geq m$, the definition of $g(y)$ yields that $g(y) \equiv 0 \mod p^2$ has also at most 2 solutions. Hence,

$$\hat{B}_p(y) \leq 2\left(\left[\frac{y}{p^2}\right]+1\right) \leq \frac{2y}{p^2}+2.$$

The above assertions (I)–(IV) and (6.6) imply that

$$\begin{aligned}B(y) &\geq y - \sum_{p_m \leq p_i \leq cy} B_{p_i}(y) \geq y - \sum_{p_m \leq p_i \leq cy} \left(\frac{2y}{p_i^2}+2\right)\\ &= y\left(1-2\sum_{p_m \leq p_i \leq cy}\frac{1}{p_i^2}\right) - 2\sum_{p_m \leq p_i \leq cy} 1 > y\left(1-2\sum_{i=m}^{\infty}\frac{1}{p_i^2}\right) - 2\pi(cy)\end{aligned}$$

for $y \geq y_0$. Here, $\pi(x)$ is the number of prime numbers $\leq x$. We see by (6.3) that $K_1 := 1 - 2\sum_{i=m}^{\infty}\frac{1}{p_i^2} > 0$. Also, the prime number theorem says that $\pi(x) = o(x)$, where $f = o(g)$ means that $f(x)/g(x) \to 0$ as $x \to \infty$. Therefore $B(y) > K_1 y + o(y)$, so that $B(y) \to \infty$ as $y \to \infty$. From (6.5), since $A(x) \geq B((x-t_0)/P)$ for $x > t_0$, we obtain $A(x) \to \infty$ as $x \to \infty$. This proves our proposition. □

7. Proof of Theorem 1.1.

Proof of Theorem 1.1 (i). We suppose that a is a positive integer such that $2a+1$ is square-free. For any integer $t \geq 0$, we let $d(t)$ be a non-square integer as in Proposition 5.2 (i). For brevity, we write $d(t) = A_0t^2 + A_1t + A_2$. By using PARI-GP [**3**], it is easy to see

$$\begin{aligned}A_0 &= (8a^2+6a+1)^2 = (2a+1)^2(4a+1)^2,\\ A_1 &= 2(8a^2+6a+1)(8a^2+4a+1)+4a+2 = 4(2a+1)^2(8a^2+2a+1),\\ A_2 &= (8a^2+4a+1)^2+4a+1 = 2(2a+1)(16a^3+8a^2+4a+1).\end{aligned}$$

Therefore we obtain $A_1^2 - 4A_0A_2 = 8(2a+1)^3 \neq 0$ and $(A_0, A_1, A_2) = (2a+1)g$. Here we put

$$g := \big((2a+1)(4a+1)^2,\ 4(2a+1)(8a^2+2a+1),\ 2(16a^3+8a^2+4a+1)\big).$$

Since g is a divisor of an odd integer $(2a+1)(4a+1)^2$, g is also odd. Hence,

$$g = \big((2a+1)(4a+1)^2,\ (2a+1)(8a^2+2a+1),\ 16a^3+8a^2+4a+1\big),$$

and we denote by g' the right hand side of it. If we assume that $g' > 1$ then there is some odd prime p such that $p \mid g'$, so that

$$16a^3 + 8a^2 + 4a + 1 \equiv 0 \mod p. \tag{7.1}$$

As $p \mid (2a+1)(4a+1)^2$, we have $p \mid (2a+1)$, or $p \mid (4a+1)$. When $p \mid (2a+1)$, $2a \equiv -1 \mod p$ and (7.1) yield that $-1 \equiv 0 \mod p$, and this is a contradiction. When $p \mid (4a+1)$, $4a \equiv -1 \mod p$ and (7.1) yield that $a \equiv 0 \mod p$. As $1 \equiv 0 \mod p$ by (7.1), this is also a contradiction. Thus we get $g' = 1$, and obtain $(A_0, A_1, A_2) = 2a+1$. When $d(t)$ is square-free, we see by Proposition 5.2 (i) that $m_{d(t)} = 16a$, if t is even then $d(t) \equiv 2 \mod 4$, and if t is odd then $d(t) \equiv 3 \mod 4$. In particular, the condition (6.2) of Proposition 6.1 holds.

(A) The case where $\delta = 2$. We take an even integer t, and write $t = 2u$ with some integer $u \geq 0$. Since the discriminant of a quadratic polynomial $d(2u)$ in $\boldsymbol{Z}[u]$ is equal to the product of 2^2 and that of $d(t)$, it is not equal to 0. As $d(2u) = 4A_0u^2 + 2A_1u + A_2$, the greatest common divisor of coefficients of it is equal to

$$\begin{aligned}
&(4A_0, 2A_1, A_2)\\
&\quad = 2(2a+1)(2(2a+1)(4a+1)^2,\ 4(2a+1)(8a^2+2a+1),\ 16a^3+8a^2+4a+1)\\
&\quad = 2(2a+1)g' \quad (\because 16a^3+8a^2+4a+1 \text{ is odd.})\\
&\quad = 2(2a+1). \quad (\because g' = 1)
\end{aligned}$$

Since the odd integer $2a+1$ is square-free by the assumption, this is square-free. Hence, Proposition 6.1 implies that the set $\{d(2u) \mid u \in \boldsymbol{Z},\, u \geq 0\}$ contains infinite square-free elements. Consequently, we can choose a sequence $\{d_n\}_{n\geq 1}$ of square-free positive integers which is strictly monotonously increasing such that $d_n \equiv 2 \mod 4$ and $m_{d_n} = 16a$. Since the sequence $\{m_{d_n}\}_{n\geq 1}$ of positive integers is constant, we see by Lemma 4.5 that $\{h_{d_n}\}_{n\geq 1}$ is not bounded. Therefore we obtain the assertion (i).

(B) The case where $\delta = 3$. We take an odd integer t, and write $t = 2u - 1$ with some $u \in \boldsymbol{N}$. Since the discriminant of a quadratic polynomial $d(2u-1)$ in $\boldsymbol{Z}[u]$ is equal to the product of 2^2 and that of $d(t)$, it is not equal to 0. As

$$d(2u-1) = 4A_0u^2 + (2A_1 - 4A_0)u + A_0 - A_1 + A_2,$$

the greatest common divisor of coefficients of it is equal to

$$\begin{aligned}&(4A_0, 2A_1 - 4A_0, A_0 - A_1 + A_2)\\ &\quad= (4A_0, 2A_1, A_0 - A_1 + A_2)\\ &\quad= (A_0, A_1, A_0 - A_1 + A_2) \quad (\because A_0 - A_1 + A_2 \text{ is odd.})\\ &\quad= (A_0, A_1, A_2) = 2a + 1,\end{aligned}$$

which is square-free by our assumption. Hence, Proposition 6.1 implies that the set $\{d(2u-1) \mid u \in \boldsymbol{N}\}$ contains infinite square-free elements. Consequently, we can choose a sequence $\{d_n\}_{n\geq 1}$ of square-free positive integers which is strictly monotonously increasing such that $d_n \equiv 3 \mod 4$ and $m_{d_n} = 16a$. Similarly, Lemma 4.5 yields the assertion (i). $\square$

The proof of Theorem 1.1 (ii). We suppose that a is a positive odd integer such that $a+2$ is square-free. For any integer $t \geq 0$, we let $d(t)$ be a non-square integer as in Proposition 5.2 (ii). For brevity, we write $d(t) = A_0t^2 + A_1t + A_2$, and then

$$\begin{aligned}A_0 &= (a^2+3a+2)^2 = (a+1)^2(a+2)^2,\\ A_1 &= 2(a^2+3a+2)(a^2+2a+2) + 4a + 8 = 2(a+2)^2(a^2+a+2),\\ A_2 &= (a^2+2a+2)^2 + 4a + 4 = (a+2)(a^3+2a^2+4a+4).\end{aligned}$$

As a is odd, we obtain $A_1^2 - 4A_0A_2 = 16(a+2)^3 \neq 0$ and $(A_0, A_1, A_2) = (a+2)g$. Here we put

$$g := ((a+1)^2(a+2),\ 2(a+2)(a^2+a+2),\ a^3+2a^2+4a+4).$$

Since g is a divisor of an odd integer a^3+2a^2+4a+4, g is also odd. If we assume that $g > 1$ then there is some odd prime p such that $p \mid g$, so that

$$a^3 + 2a^2 + 4a + 4 \equiv 0 \mod p. \tag{7.2}$$

As $p \mid (a+1)^2(a+2)$, we have $p \mid (a+1)$, or $p \mid (a+2)$. When $p \mid (a+1)$, $a \equiv -1 \mod p$ and (7.2) yield that $1 \equiv 0 \mod p$, and this is a contradiction. When $p \mid (a+2)$, $a \equiv -2 \mod p$ and (7.2) yield that $-4 \equiv 0 \mod p$, so that $p = 2$. This is also a contradiction. Thus we get $g = 1$, and obtain $(A_0, A_1, A_2) = a + 2$, which is square-free by our assumption. When $d(t)$ is square-free, we see by Proposition

5.2 (ii) that $m_{d(t)} = a$ and $d(t) \equiv 1 \mod 4$. In particular, the condition (6.2) holds. Hence, Proposition 6.1 implies that the set $\{d(t) \mid t \in \boldsymbol{Z},\ t \geq 0\}$ contains infinite square-free elements. Consequently, we can choose a sequence $\{d_n\}_{n\geq 1}$ of square-free positive integers which is strictly monotonously increasing such that $d_n \equiv 1 \mod 4$ and $m_{d_n} = a$. Since the sequence $\{m_{d_n}\}_{n\geq 1}$ of positive integers is constant, we see by Lemma 4.5 that $\{h_{d_n}\}_{n\geq 1}$ is not bounded. Therefore we obtain the assertion (ii). $\square$

Remark 7.1. We see in the following table that there exist real quadratic fields $\boldsymbol{Q}(\sqrt{d})$ with period 4 of minimal type such that $1 \leq \mathrm{ord}_2(m_d) \leq 3$, which are different from ones as in Theorem 1.1.

d	Case	s_0	m_d	h_d
1397	I	1	2	1
2222	II	1	8	2
4515	II	2	4	8
15411	I	2	8	16
18829	I	1	4	5
19346	II	1	14	8
22243	II	2	6	20
26598	II	1	6	8
37333	I	1	2	10
40458	II	3	8	12
66253	I	5	2	6

d	Case	s_0	m_d	h_d
69227	II	2	8	14
72402	II	1	10	16
76839	II	6	4	32
77363	I	4	8	12
120458	II	1	6	16
134981	I	7	2	10
139211	II	2	4	24
185818	II	1	8	40
186747	I	6	8	24
191113	III	1	2	36
218138	II	1	24	16

Remark 7.2. For any positive integer t, we put $d(t) := 4t^2 + 2$, and then see by Example 4.2 ($a = 2t$, $s = 2$) that the period of $\sqrt{d(t)}$ is equal to 2, $d(t) \equiv 2 \mod 4$, and $m_{d(t)} = 1$. By using Proposition 6.1 and Lemma 4.5, the same argument as in the proof of Theorem 1.1 implies that for any positive integer h, there exist infinitely many real quadratic fields $\boldsymbol{Q}(\sqrt{d})$, $d \equiv 2 \mod 4$ with period 2 which are not of minimal type such that $h_d > h$ and $m_d = 1$. Also, if we put $d(t) := 4t^2 - 4t + 3$ ($a = 2t - 1$, $s = 2$) for any positive integer t, then we can obtain an infinite family of real quadratic fields $\boldsymbol{Q}(\sqrt{d})$, $d \equiv 3 \mod 4$ with the same property.

8. Appendix.

8.1. Results of Friesen and of Halter-Koch.

As mentioned in Example 3.2, there exist infinitely many real quadratic fields with period ℓ which are not of minimal type such that the continued fraction expansion of ω has a given symmetric part. In the present section we show this fact by using an idea of Friesen. We utilize the following corollary to prove Proposition 8.2.

COROLLARY 8.1. *Let $\varphi(x) = ax^2 + bx + c$ be a quadratic polynomial in $\boldsymbol{Z}[x]$ with $a > 0$. As $a > 0$, there is some integer s_1 such that $\varphi(s) > 0$ for all integers $s \geq s_1$. We suppose that the discriminant of $\varphi(x)$ is of the form $d(\varphi) = \pm 2^e$ with some $e \in \boldsymbol{N}$, and that there is some integer s for which $\varphi(s) \not\equiv 0 \mod 4$. Then, the greatest common divisor (a, b, c) is square-free and the set $\{\varphi(s) \mid s \in \boldsymbol{Z},\ s \geq s_1\}$ contains infinite square-free elements.*

PROOF. By our assumption, there is some integer s_2 for which $\varphi(s_2) \not\equiv 0 \mod 4$. If we assume that (a, b, c) is not square-free, then there is some prime number q such that $q^2 \mid (a, b, c)$. Therefore, $\varphi(s_2) \equiv 0 \mod q^2$. On the other hand, as $q \mid b^2 - 4ac = d(\varphi) = \pm 2^e$, we obtain $q = 2$. Consequently $\varphi(s_2) \equiv 0 \mod 4$, and this is a contradiction. Thus, (a, b, c) is square-free. Hence, Proposition 6.1 implies our assertion. □

PROPOSITION 8.2. *We let $\ell \geq 2$ be any positive integer, and consider a symmetric string of $\ell - 1$ positive integers $a_1, \ldots, a_{\ell-1}$.*

(i) *Suppose that Case* (I) *or Case* (II) *occurs for $a_1, \ldots, a_{\ell-1}$. Then, there exist infinitely many real quadratic fields $\boldsymbol{Q}(\sqrt{d})$, $d \equiv 2$ or $3 \mod 4$ with period ℓ which are not of minimal type such that*

$$\omega = \sqrt{d} = \left[[\sqrt{d}], \overline{a_1, \ldots, a_{\ell-1}, 2[\sqrt{d}]}\,\right].$$

(ii) *Suppose that Case* (I) *or Case* (III) *occurs for $a_1, \ldots, a_{\ell-1}$. Then, there exist infinitely many real quadratic fields $\boldsymbol{Q}(\sqrt{d})$, $d \equiv 1 \mod 4$ with period ℓ which are not of minimal type such that*

$$\omega = (1 + \sqrt{d})/2 = [a_0, \overline{a_1, \ldots, a_{\ell-1}, 2a_0 - 1}], \quad a_0 := [(1 + \sqrt{d})/2].$$

PROOF. We let s_0 be an integer as in the beginning of Section 3, and define a quadratic polynomial

$$f(x) := g(x)^2 + 4h(x) = A^2x^2 + 2(2B - (-1)^\ell ABC)x + (B^2 - (-1)^\ell 4)C^2$$

in $\boldsymbol{Z}[x]$. As we have seen in the proof of Theorem 3.1, the discriminant of $f(x)$ is equal to $(-1)^\ell 2^4$. Therefore, for any $u, v, w \in \boldsymbol{Z}$, $uw \neq 0$,

$$d(f(ux+v)/w) = u^2 d(f)/w^2 = (-1)^\ell 2^4 u^2/w^2.$$

Since $AC = B^2 - (-1)^\ell$ by $(2.6)_{k=1}$ of Lemma 2.1, we have $AC \equiv B+1 \mod 2$.

(i) We shall define a quadratic polynomial $\varphi(x)$ in the following assertions (A), (B), and see that the leading term of it is positive, the discriminant of $\varphi(x)$ is of the form $d(\varphi) = \pm 2^e$ with some $e \in \boldsymbol{N}$, and $\varphi(s) \equiv 2$ or $3 \mod 4$ for any integer s. In particular there is some integer s for which $\varphi(s) \not\equiv 0 \mod 4$.

(A) The case where Case (I) occurs. Since A is odd, we have $C \equiv B+1 \mod 2$. If $(B,C) \equiv (0,1)$ (resp., $\equiv (1,0)$) mod 2, then we note that

$$\begin{aligned} f(2x)/4 &= A^2x^2 + (2B - (-1)^\ell ABC)x + (B^2 - (-1)^\ell 4)C^2/4 \\ &\equiv x^2 + Bx + (B/2)^2 - (-1)^\ell \text{ (resp., } \equiv x^2 + (2+C)x + (C/2)^2) \mod 4. \end{aligned}$$

We see easily the following: first, we assume $B \equiv 2 \mod 4$. If ℓ is even then we put $\varphi(x) := f(4x+2)/4$, and obtain $\varphi(s) \equiv 3 \mod 4$. If ℓ is odd then we put $\varphi(x) := f(4x)/4$, and obtain $\varphi(s) \equiv 2 \mod 4$. Next, we assume $B \equiv 0 \mod 4$. If ℓ is even then we put $\varphi(x) := f(4x)/4$, and obtain $\varphi(s) \equiv 3 \mod 4$. If ℓ is odd then we put $\varphi(x) := f(4x+2)/4$, and obtain $\varphi(s) \equiv 2 \mod 4$. Finally, we assume that B is odd. Put $\varphi(x) := f(4x+2)/4$. If $C \equiv 2 \mod 4$ then $\varphi(s) \equiv 2 \mod 4$, and if $C \equiv 0 \mod 4$ then $\varphi(s) \equiv 3 \mod 4$.

(B) The case where Case (II) occurs. Since A is even, B is odd. If $A \equiv 2$ (resp., $\equiv 0$) mod 4, then we note that

$$\begin{aligned} f(x)/4 &= (A/2)^2x^2 + \left(B - (-1)^\ell \frac{ABC}{2}\right)x + (B^2 - (-1)^\ell 4)\cdot(C/2)^2 \\ &\equiv (A/2)^2x^2 + \left(B - (-1)^\ell \frac{ABC}{2}\right)x + (C/2)^2 \\ &\equiv x^2 + (B+C)x + (C/2)^2 \quad \text{(resp., } \equiv Bx + (C/2)^2) \mod 4. \end{aligned}$$

We see easily the following: first, we assume $A \equiv 2 \mod 4$. Put $\varphi(x) := f(4x+2)/4$. If $C \equiv 2 \mod 4$ then $\varphi(s) \equiv 3 \mod 4$, and if $C \equiv 0 \mod 4$ then $\varphi(s) \equiv 2 \mod 4$. Next, we assume $A \equiv 0 \mod 4$. If $C \equiv 2$ (resp., $\equiv 0$) mod 4 then we put $\varphi(x) := f(4x+B)/4$ (resp., $:= f(4x+2)/4$), and obtain $\varphi(s) \equiv 2 \mod 4$. Also, if $C \equiv 2$ (resp., $\equiv 0$) mod 4 then we put $\varphi(x) := f(4x+2)/4$ (resp., $:= f(4x+3B)/4$), and obtain $\varphi(s) \equiv 3 \mod 4$.

We take an integer s_1 such that

$$s_1 > s_0 \quad \text{and} \quad g(s_1) > a_1, \ldots, a_{\ell-1}.$$

The above definition of φ and Corollary 8.1 imply that the set $\{\varphi(s) \mid s \in \boldsymbol{Z}, s \geq s_1\}$ contains infinite square-free elements. Consequently, we can choose a sequence $\{d_n\}_{n\geq 1}$ of square-free positive integers which is strictly monotonously increasing such that $d_n \equiv 2$ or $3 \bmod 4$. By the definition of φ, we see that each d_n is of the form $d_n = f(4s+u_0)/4$ with some integer $s \geq s_1$. Here, $u_0 = 0, 2, B$, or $3B$, and u_0 is even when Case (I) occurs. As $4s+u_0 \geq s \geq s_1$, the condition (3.2) of Theorem 3.1 for $4s+u_0$ holds. Also, $4s+u_0 > s_0$. Hence, Theorem 3.1 [A-i] implies that the continued fraction expansion of $\sqrt{d_n}$ has the desired form, and that each $\boldsymbol{Q}(\sqrt{d_n})$ is a real quadratic field with period ℓ which is not of minimal type.

(ii) Since either of A, B and C is even, we obtain $f(x) \equiv A^2x^2 + B^2C^2 \bmod 4$. Consequently, when Case (I) (resp., (III)) occurs, we have $f(2x+1) \equiv 1$ (resp., $f(x) \equiv 1$) mod 4. So, if we put $\varphi(x) := f(2x+1)$ (resp., $:= f(x)$), then $\varphi(s) \equiv 1 \mod 4$ for any integer s. In particular there is some integer s for which $\varphi(s) \not\equiv 0 \mod 4$. Also, the discriminant of φ is of the form $d(\varphi) = \pm 2^e$ with some $e \in \boldsymbol{N}$. By using Corollary 8.1 and Theorem 3.1 [A-ii], the same argument as in the assertion (i) implies the assertion (ii). Our proposition is proved. □

8.2. List of real quadratic fields of class number 1 that are not of minimal type.

When $d < 10^7$, we see in the following table that there are exactly 51 real quadratic fields $\boldsymbol{Q}(\sqrt{d})$ of class number 1 that are not of minimal type.

Remark 8.1. In Mollin and H. C. Williams [**11**, Theorem 3.1], it is shown that there are exactly 20 real quadratic fields of period 4 satisfying $h_d = 1$, with one more possible exception, and all such fields are listed in Table 3.1 of it. On the other hand, the table below says that 13 fields are not of minimal type among such 20 fields: $d = 7, 14, 23, 47, 62, 69, 167, 213, 398, 413, 717, 1077, 1757$. The 4 fields among 7 fields $\boldsymbol{Q}(\sqrt{d})$ of minimal type can be obtained from Proposition 5.2 (ii): $d = 33$ $(a = 1, t = 0)$, $d = 141$ $(a = 1, t = 1)$, $d = 573$ $(a = 1, t = 3)$, $d = 1293$ $(a = 1, t = 5)$. (The 3 rest fields are as follows. For $d = 133$, $m_d = h_d = 1$, $(1+\sqrt{133})/2 = [6, \overline{3, 1, 3, 11}]$. For $d = 1397$, $m_d = 2, h_d = 1$, $(1+\sqrt{1397})/2 = [19, \overline{5, 3, 5, 37}]$. For $d = 3053$, $m_d = h_d = 1$, $(1+\sqrt{3053})/2 = [28, \overline{7, 1, 7, 55}]$.)

d	ℓ	s_0	s	Case	m_d
2	1	1	2	I	2
3	2	1	2	I	1
6	2	1	2	II	1
7	4	1	2	I	2
11	2	1	2	I	1
13	1	1	3	I	0
14	4	2	3	II	2
17	3	0	1	III	0
21	2	1	3	I	0
23	4	3	4	I	2
29	1	1	5	I	0
37	3	0	2	III	0
38	2	1	2	II	1
47	4	5	6	I	2
53	1	1	7	I	0
61	3	0	1	I	0
62	4	6	7	II	2
69	4	1	3	I	0
77	2	1	7	I	0
83	2	1	2	I	1
93	2	1	3	I	0
101	3	0	4	III	0
149	5	-1	1	I	0
167	4	11	12	I	2
173	1	1	13	I	0
197	3	0	6	III	0

d	ℓ	s_0	s	Case	m_d
213	4	3	5	I	0
227	2	1	2	I	1
237	2	1	3	I	0
269	5	-3	-2	III	0
293	1	1	17	I	0
317	3	0	3	I	0
341	6	6	7	I	0
398	4	18	19	II	2
413	4	1	7	I	0
437	2	1	19	I	0
453	2	1	3	I	0
461	3	0	1	I	0
557	3	0	2	III	0
677	3	0	12	III	0
717	4	7	9	I	0
773	3	0	5	I	0
797	7	-3	-1	I	0
1013	5	-2	-1	I	0
1077	4	9	11	I	0
1133	2	1	11	I	0
1253	2	1	7	I	0
1757	4	3	4	III	0
1877	3	0	1	I	0
2477	5	0	3	I	0
3533	5	0	1	I	0

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