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On reducible hyperplane sections of 4-folds

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Abstract. We describe 4-dimensional complex projective manifolds X admitting a simple normal crossing divisor of the form $A + B$ among their hyperplane sections, both components A and B having sectional genus zero. Let L be the hyperplane bundle. Up to exchanging the two components, (X, L, A, B) is one of the following: 1) (X, L) is a scroll over $\boldsymbol{P}^1$ with A itself a scroll and B a fibre, 2) $(X, L) = (\boldsymbol{P}^2 \times \boldsymbol{P}^2, \mathcal{O}_{\boldsymbol{P}^2 \times \boldsymbol{P}^2}(1, 1))$ with $A \in |\mathcal{O}_{\boldsymbol{P}^2 \times \boldsymbol{P}^2}(1, 0)|, B \in |\mathcal{O}_{\boldsymbol{P}^2 \times \boldsymbol{P}^2}(0, 1)|$, 3) $X = \boldsymbol{P}_{\boldsymbol{P}^2}(\mathscr{V})$ where $\mathscr{V} = \mathcal{O}_{\boldsymbol{P}^2}(1)^{\oplus 2} \oplus \mathcal{O}_{\boldsymbol{P}^2}(2)$, L is the tautological line bundle, $A = \boldsymbol{P}_{\boldsymbol{P}^2}(\mathcal{O}_{\boldsymbol{P}^2}(1)^{\oplus 2})$, and $B \in \pi^*|\mathcal{O}_{\boldsymbol{P}^2}(2)|$, where $\pi : X \to \boldsymbol{P}^2$ is the scroll projection. This supplements a recent result of Chandler, Howard, and Sommese.

Introduction and statement of the result.

The study of the structure of projective manifolds admitting some special variety among their hyperplane sections is a classical subject in algebraic geometry. Recently, Chandler, Howard, and Sommese [**CHS**] started to study projective manifolds $X \subset \boldsymbol{P}^N$ in terms of a reducible hyperplane section, which is a union of distinct smooth irreducible components $A_1, \dots, A_r$, $r \geq 2$, meeting transversally. In particular they studied the case in which every component A_i has minimal sectional genus $g(A_i, L_{A_i}) = h^1(\mathcal{O}_{A_i})$, $i = 1, \dots, r$, where $L = \mathcal{O}_X(1)$. They proved [**CHS**, Corollary 4.6] that if X has dimension $n \geq 5$, then only one possibility is allowed: namely (X, L) is a scroll over a smooth curve and all components are fibres, except one, which meets every fibre along a hyperplane. For $n \leq 4$ the situation is quite richer and hence more interesting than in higher dimensions. In fact, restricting to the case $n = 4$, $r = 2$, in [**CHS**, Corollary 4.8] a short list of possibilities is given for pairs (X, L) such that $g(A_i, L_{A_i}) = 0$, $i = 1, 2$. In this paper we rule out one of them; namely we show that (X, L) cannot be a Mukai 4-fold in the above situation. As a final output the result of [**CHS**, Corollary 4.8] is improved as follows.

THEOREM. *Let L be a very ample line bundle on a smooth complex projective 4-fold X. Assume that $|L|$ contains an element $A + B$, with A, B smooth irreducible*

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divisors meeting transversally. If $g(A, L_A) = g(B, L_B) = 0$, then (X, L) is one of the following:

(1) *a scroll over* $\boldsymbol{P}^1$;
(2) $(\boldsymbol{P}^2 \times \boldsymbol{P}^2, \mathcal{O}_{\boldsymbol{P}^2 \times \boldsymbol{P}^2}(1,1))$; *or*
(3) $(\boldsymbol{P}_{\boldsymbol{P}^2}(\mathcal{V}), \xi_{\mathcal{V}})$, *where* $\mathcal{V} = \mathcal{O}_{\boldsymbol{P}^2}(1)^{\oplus 2} \oplus \mathcal{O}_{\boldsymbol{P}^2}(2)$ *and* $\xi_{\mathcal{V}}$ *is the tautological line bundle.*

Moreover, up to exchanging components, A itself is a scroll and B is a fibre of the scroll projection in case (1), $A \in |\mathcal{O}_{\boldsymbol{P}^2 \times \boldsymbol{P}^2}(1,0)|$, $B \in |\mathcal{O}_{\boldsymbol{P}^2 \times \boldsymbol{P}^2}(0,1)|$ *in case* (2), *and* $A = \boldsymbol{P}_{\boldsymbol{P}^2}(\mathcal{O}_{\boldsymbol{P}^2}(1)^{\oplus 2}) \cong \boldsymbol{P}^2 \times \boldsymbol{P}^1$, $B \in \pi^*|\mathcal{O}_{\boldsymbol{P}^2}(2)|$, *where* $\pi : X \to \boldsymbol{P}^2$ *denotes the scroll projection, in case* (3).

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1. The proof.

In [**CHS**] it is shown that if (X, L) is not as in cases (1)–(3) of the Theorem with A, B as above, then

$$(X, L) \text{ is a Mukai 4-fold, i.e. } -K_X = 2L. \tag{1.1}$$

We retake the proof at this point, showing that case (1.1) cannot occur. We use the standard notation and terminology from algebraic geometry; moreover, we adopt the same symbols as in [**CHS**] simply omitting hat accents, since we do not need reductions. In particular, we denote by h the smooth surface $A \cap B$. We recall that case (1.1) arises from the following situation: $K_X + 3L$ is nef and

$$(h, L_h) = (\boldsymbol{P}^1 \times \boldsymbol{P}^1, \mathcal{O}_{\boldsymbol{P}^1 \times \boldsymbol{P}^1}(2,2)), \tag{1.2}$$

which implies that

$$g(h, L_h) = 1. \tag{1.3}$$

First, by adjunction we have

$$-2 = 2g(A, L_A) - 2 = (K_A + 2L_A)L_A^2 = (K_X + L)AL^2 + L^3A + A^2L^2,$$

and

$$-2 = 2g(B, L_B) - 2 = (K_B + 2L_B)L_B^2 = (K_X + L)BL^2 + L^3B + B^2L^2.$$

Summing up these two expressions we get

$$\begin{aligned} -4 &= (K_X + L)L^3 + L^4 + (A^2 + B^2)L^2 \\ &= (K_X + L)L^3 + 2L^4 - 2ABL^2 \\ &= (K_X + 3L)L^3 - 2ABL^2 \\ &= 2g(X, L) - 2 - 2L_h^2. \end{aligned}$$

Now, if (X, L) is as in (1.1) then $L_h^2 = 8$ by (1.2) and so $g(X, L) = 7$. Thus the genus formula gives

$$L^4 = 12. \tag{1.4}$$

We claim that the second Betti number of X is ≥ 2. Assume otherwise; then $\mathrm{Pic}(X) \cong \boldsymbol{Z}$ generated by an ample line bundle, say Λ. Thus $[A] = a\Lambda$, $[B] = b\Lambda$ with a, b positive integers; hence $L = t\Lambda$ for some integer $t \geq 2$. On the other hand $-K_X = 2t\Lambda$ by (1.1) and since the Fano index of X cannot exceed $\dim X + 1 = 5$ we get $t \leq 2$. Therefore $t = 2$, i.e. $-K_X = 4\Lambda$, which implies by the Kobayashi–Ochiai theorem that $(X, L) = (\boldsymbol{Q}^4, \mathcal{O}_{\boldsymbol{Q}^4}(2))$. But this gives $L^4 = 2^4 2 = 32$, which contradicts (1.4). Therefore $b_2(X) \geq 2$. Now note that the fundamental linear system of X is L, which is very ample; hence the assumption (0.1) in [**W**] is trivially satisfied. Then, by the classification result of Wiśniewski [**W**, Theorem 0.2] (see also [**IP**, Theorem 7.2.15, p. 148 and Table 12.7, p. 225]) there are only two possibilities:

i) $X = \boldsymbol{P}^1 \times V$, where $V = V_3 \subset \boldsymbol{P}^4$ is a smooth cubic hypersurface and $L = p^*\mathcal{O}_{\boldsymbol{P}^1}(1) \otimes q^*\mathcal{O}_V(1)$, p, q denoting the projections of X onto the factors;

ii) there is a morphism $\pi : X \to \boldsymbol{P}^2 \times \boldsymbol{P}^2$ expressing X as a double cover branched along a smooth divisor in the linear system $|\mathcal{O}_{\boldsymbol{P}^2 \times \boldsymbol{P}^2}(2, 2)|$, and $L = \pi^*\mathcal{O}_{\boldsymbol{P}^2 \times \boldsymbol{P}^2}(1, 1)$.

We need a case-by-case analysis.

In case i) we have

$$A \in |p^*\mathcal{O}_{\boldsymbol{P}^1}(a_1) \otimes q^*\mathcal{O}_V(a_2)| \quad \text{and} \quad B \in |p^*\mathcal{O}_{\boldsymbol{P}^1}(b_1) \otimes q^*\mathcal{O}_V(b_2)|,$$

for some integers a_i, b_i, $i = 1, 2$ such that

$$a_1 + b_1 = a_2 + b_2 = 1. \tag{1.5}$$

Let l and γ denote a fibre of q and a curve section in a fibre of p respectively. Of course we can choose these curves in such a way that they are contained neither in A nor in B. Hence

$$a_1 = A \cdot l \geq 0, \quad a_2 = A \cdot \gamma \geq 0$$

and

$$b_1 = B \cdot l \geq 0, \quad b_2 = B \cdot \gamma \geq 0.$$

Up to exchanging A and B we thus conclude that $a_1 = b_2 = 1$, $a_2 = b_1 = 0$, i.e.,

$$A \in |p^* \mathcal{O}_{\boldsymbol{P}^1}(1)| \quad \text{and} \quad B \in |q^* \mathcal{O}_V(1)|.$$

In case ii) we have

$$A \in |\pi^* \mathcal{O}_{\boldsymbol{P}^2 \times \boldsymbol{P}^2}(a_1, a_2)| \quad \text{and} \quad B \in |\pi^* \mathcal{O}_{\boldsymbol{P}^2 \times \boldsymbol{P}^2}(b_1, b_2)|,$$

for some integers a_i, b_i, $i = 1, 2$ satisfying (1.5) again. Let $l_i \subset \boldsymbol{P}^2 \times \boldsymbol{P}^2$ be a line contained in a fibre of the j-th projection, where $j \neq i$. Of course we can choose l_1 and l_2 in such a way that they are contained neither in $\pi(A)$ nor in $\pi(B)$. Hence

$$a_i = A \cdot \pi^{-1}(l_i) \geq 0 \quad \text{and} \quad b_i = B \cdot \pi^{-1}(l_i) \geq 0 \quad \text{for } i = 1, 2.$$

Thus, up to exchanging A and B, we conclude that $a_1 = b_2 = 1$, $a_2 = b_1 = 0$ as before, i.e.,

$$A \in |\pi^* \mathcal{O}_{\boldsymbol{P}^2 \times \boldsymbol{P}^2}(1, 0)| \quad \text{and} \quad B \in |\pi^* \mathcal{O}_{\boldsymbol{P}^2 \times \boldsymbol{P}^2}(0, 1)|.$$

This shows that in both cases i) and ii) A and B are nef divisors. But this combined with (1.3) leads to a contradiction, in view of the following

LEMMA. *Let (X, L) be as in* (1.1) *and suppose that $|L| \ni A + B$, with A, B smooth divisors meeting along a smooth surface h. If A is nef, then $g(h, L_h) \leq g(A, L_A)$.*

PROOF. By adjunction we have

$$(*) \qquad \begin{aligned} 2g(A, L_A) - 2 &= (K_A + 2L_A)L_A^2 \\ &= (K_X + A + 2L)L^2 A \\ &= (K_X + 3L - B)LA(A + B) \\ &= (K_X + 3L)A^2 L + (K_X + 2L)ABL \\ &= (K_X + 3L)A^2 L + 2g(h, L_h) - 2. \end{aligned}$$

On the other hand, since A is nef, we get from (1.1)

$$(K_X + 3L)A^2 L = A^2 L^2 \geq 0,$$

and then the assertion follows from $(*)$. □

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