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The homotopy principle for maps with singularities of given $\mathscr{K}$-invariant class

By Yoshifumi ANDO

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Abstract. Let N and P be smooth manifolds of dimensions n and p respectively such that $n \geq p \geq 2$ or $n < p$. Let $\mathscr{O}(N,P)$ denote an open subspace of $J^\infty(N,P)$ which consists of all regular jets and singular jets of certain given $\mathscr{K}$-invariant class (including fold jets if $n \geq p$). An $\mathscr{O}$-regular map $f : N \to P$ refers to a smooth map such that $j^\infty f(N) \subset \mathscr{O}(N,P)$. We will prove that a continuous section s of $\mathscr{O}(N,P)$ over N has an $\mathscr{O}$-regular map f such that s and $j^\infty f$ are homotopic as sections. As an application we will prove this homotopy principle for maps with $\mathscr{K}$-simple singularities of given class.

Introduction.

Let N and P be smooth (C^∞) manifolds of dimensions n and p respectively. Let $J^k(N,P)$ denote the k-jet space of the manifolds N and P with the projections π^k_N and π^k_P onto N and P mapping a jet onto its source and target respectively. Let $J^k(n,p)$ denote the k-jet space of C^∞-map germs $(\boldsymbol{R}^n, 0) \to (\boldsymbol{R}^p, 0)$. Let $\mathscr{K}$ denote the contact group defined in [**MaIII**]. Let $\mathscr{O}(N,P)$ denote an open subbundle of $J^k(N,P)$ associated to a given $\mathscr{K}$-invariant open subset $\mathscr{O}(n,p)$ of $J^k(n,p)$. In this paper a smooth map $f : N \to P$ is called an *$\mathscr{O}$-regular map* if $j^k f(N) \subset \mathscr{O}(N,P)$.

We will study a homotopy theoretic condition for finding an $\mathscr{O}$-regular map in a given homotopy class. Let $C^\infty_{\mathscr{O}}(N,P)$ denote the space consisting of all $\mathscr{O}$-regular maps equipped with the C^∞-topology. Let $\Gamma_{\mathscr{O}}(N,P)$ denote the space consisting of all continuous sections of the fiber bundle $\pi^k_N|\mathscr{O}(N,P) : \mathscr{O}(N,P) \to N$ equipped with the compact-open topology. Then there exists a continuous map

$$j_{\mathscr{O}} : C^\infty_{\mathscr{O}}(N,P) \longrightarrow \Gamma_{\mathscr{O}}(N,P)$$

defined by $j_{\mathscr{O}}(f) = j^k f$. If any section s in $\Gamma_{\mathscr{O}}(N,P)$ has an $\mathscr{O}$-regular map f such that s and $j^k f$ are homotopic as sections in $\Gamma_{\mathscr{O}}(N,P)$, then we say that the homotopy principle holds for $\mathscr{O}$-regular maps. The terminology "homotopy principle" has been used in [**G2**]. It follows from the well-known theorem due to Gromov [**G1**] that if N is a connected open manifold, then $j_{\mathscr{O}}$ is a weak homotopy equivalence. If N is a closed manifold, then the homotopy principle is a hard problem. As the primary investigation preceding [**G1**], we

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must refer to the Smale-Hirsch Immersion Theorem ([**Sm**], [**H**]), the k-mersion Theorem due to Feit [**F**], the Phillips Submersion Theorem for open manifolds ([**P**]). In [**E1**] and [**E2**], Èliašberg has proved the well-known homotopy principle on the 1-jet level for fold-maps. Succeedingly there have appeared the homotopy principles for maps with the extensibility condition in [**duP2**], for maps without certain Thom-Boardman singularities in [**duP1**] (see [**T**], [**B**] and [**L**] for Thom-Boardman singularities) and for maps with $\mathscr{K}$-simple singularities in [**duP3**]. Although these du Plessis's homotopy principles are parametric and useful, one can not apply them in many cases, in particular, in the dimensions $n \geq p$. We refer to the relative homotopy principle for maps with prescribed Thom-Boardman singularities in [**An6**], which is available in the dimensions $n \geq p \geq 2$.

In this paper we will study a general condition on $\mathscr{O}(n,p)$ for the relative homotopy principle on the existence level. We say that a nonempty $\mathscr{K}$-invariant open subset $\mathscr{O}(n,p)$ is *admissible* if $\mathscr{O}(n,p)$ consists of all regular jets and a finite number of disjoint $\mathscr{K}$-invariant submanifolds $V^i(n,p)$ of codimension ρ_i $(1 \leq i \leq \iota)$ such that the following properties (H-i to v) are satisfied.

(H-i) $V^i(n,p)$ consists of singular k-jets of rank r_i, namely, $V^i(n,p) \subset \Sigma^{n-r_i}(n,p)$.

(H-ii) For each i, the set $\mathscr{O}(n,p)\backslash\{\bigcup_{j=i}^{\iota} V^j(n,p)\}$ is an open subset.

(H-iii) For each i with $\rho_i \leq n$, there exists a $\mathscr{K}$-invariant submanifold $V^i(n,p)^{(k-1)}$ of $J^{k-1}(n,p)$ such that $V^i(n,p)$ is open in $(\pi^k_{k-1})^{-1}(V^i(n,p)^{(k-1)})$. Here, $\pi^k_{k-1} : J^k(n,p) \to J^{k-1}(n,p)$ is the canonical projection.

(H-iv) If $n \geq p$, then $p \geq 2$ and $V^1(n,p) = \Sigma^{n-p+1,0}(n,p)$.

Here, $\Sigma^{n-p+1,0}(n,p)$ denotes the Thom-Boardman manifold in $J^k(n,p)$, which consists of $\mathscr{K}$-orbits of fold jets. Let $\boldsymbol{d} : (\pi^k_N)^*(TN) \longrightarrow (\pi^k_{k-1})^*(T(J^{k-1}(N,P)))$ denote the bundle homomorphism defined by $\boldsymbol{d}(z,\boldsymbol{v}) = (z, d_x(j^{k-1}f)(\boldsymbol{v}))$ where $z = j^k_x f \in J^k(N,P)$ and $d_x(j^{k-1}f) : T_xN \to T_{\pi^k_{k-1}(z)}(J^{k-1}(N,P))$ is the differential. Let $V^i(N,P)$ denote the subbundle of $J^k(N,P)$ associated to $V^i(n,p)$. Let $\boldsymbol{K}(V^i)$ be the kernel bundle in $(\pi^k_N)^*(TN)|_{V^i(N,P)}$ defined by $\boldsymbol{K}(V^i)_z = (z, \mathrm{Ker}(d_xf))$.

(H-v) For each i with $\rho_i \leq n$ and any $z \in V^i(N,P)$, we have

$$\boldsymbol{d}\big(\boldsymbol{K}(V^i)_z\big) \cap \big(\pi^k_{k-1}|V^i(N,P)\big)^*\big(T(V^i(N,P)^{(k-1)})\big)_z = \{0\}.$$

For example, let $\mathscr{O}^{sim}(n,p)$ be an nonempty open subset in $J^k(n,p)$ which consists of a finite number of $\mathscr{K}$-k-simple $\mathscr{K}$-orbits, and of $\Sigma^{n-p+1,0}(n,p)$ in addition in the case $n \geq p$. Then if $k \geq p+2$, then we will prove in Section 7 that $\mathscr{O}^{sim}(n,p)$ is admissible.

We will prove the following relative homotopy principle on the existence level for $\mathscr{O}$-regular maps.

THEOREM 0.1. *Let k be an integer with $k \geq 3$. Let $\mathscr{O}(n,p)$ denote a nonempty admissible open subspace of $J^k(n,p)$. We assume that if $n \geq p$, then $p \geq 2$ and $\mathscr{O}(n,p)$ contains $\Sigma^{n-p+1,0}(n,p)$ at least. Let N and P be connected manifolds of dimensions n and p respectively with $\partial N = \varnothing$. Let C be a closed subset of N. Let s be a section in $\Gamma_{\mathscr{O}}(N,P)$ which has an $\mathscr{O}$-regular map g defined on a neighborhood of C to P, where $j^kg = s$.*

Then there exists an $\mathscr{O}$-regular map $f : N \to P$ such that j^kf is homotopic to s relative to a neighborhood of C by a homotopy s_λ in $\Gamma_{\mathscr{O}}(N,P)$ with $s_0 = s$ and $s_1 = j^kf$.

In particular, we have $f = g$ on a neighborhood of C.

In the proof of Theorem 0.1 the relative homotopy principles on the existence level for fold-maps in [**An3**, Theorem 4.1] and [**An4**, Theorem 0.5] in the case $n \geq p \geq 2$ and the Smale-Hirsch Immersion Theorem in the case $n < p$ together with [**G1**] will play important roles.

The relative homotopy principle on the existence level for maps and singular foliations having only what are called A, D and E singularities has been given in [**An1**]–[**An5**]. Recently it turns out that this kind of homotopy principle has many applications. First of all, Theorem 0.1 is very important even for fold-maps in proving the relations between fold-maps, surgery theory and stable homotopy groups of spheres in [**An3**, Corollary 2, Theorems 3 and 4] and [**An7**]. In [**Sady**] Sadykov has applied [**An1**, Theorem 1] to the elimination of higher A_r singularities ($r \geq 3$) for Morin maps when $n - p$ is odd. This result is a strengthened version of the Chess conjecture proposed in [**C**]. In [**An8**] it has been proved that the cobordism group of $\mathscr{O}$-regular maps to a given connected manifold P is isomorphic to the stable homotopy group of a certain space related to $\mathscr{O}(n, p)$.

In Section 1 we will explain the notations which are used in this paper. In Section 2 we will review the definitions and the fundamental properties of $\mathscr{K}$-orbits, from which we deduce several further results. In Section 3 we will announce a special form of a homotopy principle in Theorem 3.2 and reduce the proof of Theorem 0.1 to the proof of Theorem 3.2 by induction. Furthermore, we will introduce a certain rotation of the tangent spaces defined around the singularities of a given type in N for a preliminary deformation of the section s. In Section 4 we will prepare two lemmas which are used to deform the section s in a nice position. In Section 5 we will construct an $\mathscr{O}$-regular map around the singularities of a given type in N. We will prove Theorem 3.2 in Section 6. In Section 7 we will apply Theorem 0.1 to maps with $\mathscr{K}$-k-simple singularities of given class.

1. Notations.

Throughout the paper all manifolds are Hausdorff, paracompact and smooth of class C^∞. Maps are basically continuous, but may be smooth (of class C^∞) if necessary. Given a fiber bundle $\pi : E \to X$ and a subset C in X, we denote $\pi^{-1}(C)$ by E_C or $E|_C$. Let $\pi' : F \to Y$ be another fiber bundle. A map $\tilde{b} : E \to F$ is called a fiber map over a map $b : X \to Y$ if $\pi' \circ \tilde{b} = b \circ \pi$ holds. The restriction $\tilde{b}|(E|_C) : E|_C \to F$ (or $F|_{b(C)}$) is denoted by $\tilde{b}_C$ or $\tilde{b}|_C$. We denote, by b^F, the induced fiber map $b^*(F) \to F$ covering b. A fiberwise homomorphism $E \to F$ is simply called a homomorphism. For a vector bundle E with a metric and a positive function δ on X, let $D_\delta(E)$ be the associated disk bundle of E with radius δ. If there is a canonical isomorphism between two vector bundles E and F over $X = Y$, then we write $E \cong F$.

When E and F are smooth vector bundles over $X = Y$, $\mathrm{Hom}(E, F)$ denotes the smooth vector bundle over X with fiber $\mathrm{Hom}(E_x, F_x)$, $x \in X$ which consists of all homomorphisms $E_x \to F_x$.

Let $J^k(N, P)$ denote the k-jet space of manifolds N and P. The map $\pi_N^k \times \pi_P^k$: $J^k(N, P) \to N \times P$ induces a structure of a fiber bundle with structure group $L^k(p) \times L^k(n)$, where $L^k(m)$ denotes the group of all k-jets of local diffeomorphisms of $(\boldsymbol{R}^m, 0)$.

The fiber $(\pi_N^k \times \pi_P^k)^{-1}(x,y)$ is denoted by $J^k_{x,y}(N,P)$.

Let π_N and π_P be the projections of $N \times P$ onto N and P respectively. We set

$$J^k(TN,TP) = \bigoplus_{i=1}^{k} \mathrm{Hom}\big(S^i(\pi_N^*(TN)), \pi_P^*(TP)\big) \tag{1.1}$$

over $N \times P$. Here, for a vector bundle E over X, let $S^i(E)$ be the vector bundle $\bigcup_{x\in X} S^i(E_x)$ over X, where $S^i(E_x)$ denotes the i-fold symmetric product of E_x. If we provide N and P with Riemannian metrics, then the Levi-Civita connections induce the exponential maps $\exp_{N,x} : T_xN \to N$ and $\exp_{P,y} : T_yP \to P$. In dealing with exponential maps we always consider convex neighborhoods ([**K-N**]). We define the smooth bundle map

$$J^k(N,P) \longrightarrow J^k(TN,TP) \quad \text{over } N \times P \tag{1.2}$$

by sending $z = j^k_x f \in J^k_{x,y}(N,P)$ to the k-jet of $(\exp_{P,y})^{-1} \circ f \circ \exp_{N,x}$ at $\mathbf{0} \in T_xN$, which is regarded as an element of $J^k(T_xN, T_yP) (= J^k_{x,y}(TN,TP))$ (see [**K-N**, Proposition 8.1] for the smoothness of exponential maps). More strictly, (1.2) gives a smooth equivalence of the fiber bundles under the structure group $L^k(p) \times L^k(n)$. Namely, it gives a smooth reduction of the structure group $L^k(p) \times L^k(n)$ of $J^k(N,P)$ to $O(p) \times O(n)$, which is the structure group of $J^k(TN,TP)$.

Under the projection $\pi_N^k \times \pi_P^k : J^k(N,P) \to N \times P$, let $T^{\mathfrak{f}}(J^k(N,P))$ denote the tangent bundle along the fiber of $J^k(N,P)$, whose fiber over (x,y) is $T(J^k_{x,y}(N,P))$. By using the Levi-Civita connections we can define the projection

$$T(J^k(N,P)) \longrightarrow T^{\mathfrak{f}}(J^k(N,P)) \tag{1.3}$$

as follows. Let U and V be the convex neighborhoods of x and y. Let $\ell(x,x')$ (respectively $\ell(y',y)$) denote the parallel translation of U (respectively V) mapping x to x' (respectively y' to y). Define the trivialization

$$t_{x,y} : J^k(U,V) \longrightarrow J^k_{x,y}(U,V)$$

by $t_{x,y}(z_{x',y'}) = \ell(y',y) \circ z_{x',y'} \circ \ell(x,x')$, where $z_{x',y'} \in J^k_{x',y'}(U,V)$ and $\ell(x,x')$ and $\ell(y',y)$ are identified with their k-jets. We define the projection in (1.3) by

$$d(t_{x,y})_z : T_z(J^k(U,V)) \longrightarrow T_z\big(J^k_{x,y}(U,V)\big)$$

at $z \in J^k_{x,y}(U,V)$, where we should note $T_z(J^k_{x,y}(U,V)) = T^{\mathfrak{f}}_z(J^k(N,P))$.

Let $(x_1, \dots, x_n)$ and $(y_1, \dots, y_p)$ be the normal coordinates on the convex neighborhoods of (N,x) and (P,y) associated to orthonormal bases of T_xN and T_yP respectively. Then a jet $z \in J^k_{x,y}(N,P)$ is often identified with the germ of the polynomial map of degree k with variables $x_1, \dots, x_n$.

2. Singularities of $\mathscr{K}$-invariant class.

Let us begin by recalling the results in [**MaIII**], [**MaIV**] and [**MaV**]. Let C_x and C_y denote the rings of smooth function germs on (N,x) and (P,y) respectively. Let $\mathfrak{m}_x$ and $\mathfrak{m}_y$ denote the maximal ideals of C_x and C_y respectively. Let $f:(N,x)\to(P,y)$ be a germ of a smooth map. Let $f^*:C_y\to C_x$ denote the homomorphism defined by $f^*(a)=a\circ f$. Let $\theta(N)_x$ denote the C_x module of all germs at x of smooth vector fields on N. Let $\theta(f)_x$ denote the C_x module of germs at x of smooth vector fields along f, namely which consists of all germs $\varsigma:(N,x)\to TP$ such that $p_P\circ\varsigma=f$. Here, $p_P:TP\to P$ is the canonical projection. Then we have the homomorphism

$$tf:\theta(N)_x\longrightarrow\theta(f)_x \tag{2.1}$$

defined by $tf(u_N)=df\circ u_N$ for $u_N\in\theta(N)_x$.

Let us review the $\mathscr{K}$-equivalence of two smooth map germs $f,g:(N,x)\to(P,y)$, which has been introduced in [**MaIII**, (2.6)], by following [**Mar1**, II, 1]. The above two map germs f and g are *$\mathscr{K}$-equivalent* if there exists a smooth map germ $h_1:(N,x)\to GL(\boldsymbol{R}^p)$ and a local diffeomorphism $h_2:(N,x)\to(N,x)$ such that $f(x)=h_1(x)g(h_2(x))$. In this paper we also say that j^k_xf and j^k_xg are *$\mathscr{K}$-equivalent* in this case. It is known that this $\mathscr{K}$-equivalence is nothing but the contact equivalence introduced in [**MaIII**]. The contact group $\mathscr{K}$ is defined as a some subgroup of the group of germs of local diffeomorphisms $(N,x)\times(P,y)$. Let $\mathscr{K}z$ denote the orbit submanifold of $J^k_{x,y}(N,P)$ consisting of all k-jets w which are $\mathscr{K}$-equivalent to z. This fact is also observed from the above definition.

In the case $n\geq p$ let $\Sigma^{n-p+1,0}(n,p)$ denote the Thom-Boardman submanifold in $J^k(n,p)$ consisting of all fold jets. The union $\Omega^{n-p+1,0}(n,p)$ of all regular jets and $\Sigma^{n-p+1,0}(n,p)$ is open (see, for example, [**duP1**]).

We define the bundle homomorphism

$$\boldsymbol{d}_1:\left(\pi^k_N\right)^*(TN)\longrightarrow\left(\pi^k_P\right)^*(TP). \tag{2.2}$$

Let $z=j^k_xf$. We set $(\boldsymbol{d}_1)_z(z,\boldsymbol{v})=(z,df(\boldsymbol{v}))$. Let $V^i(n,p)$ be a $\mathscr{K}$-invariant smooth submanifold of $J^k(n,p)$ which consists of singular jets with given rank r $(0\leq r\leq\min(n,p))$. Namely, we have $V^i(n,p)\subset\Sigma^{n-r}(n,p)$. Let $V^i(N,P)$ denote the subbundle of $J^k(N,P)$ associated to $V^i(n,p)$. We define the kernel bundle $\boldsymbol{K}(V^i)$ in $(\pi^k_N|V^i(n,p))^*(TN)$ and the cokernel bundle $\boldsymbol{Q}(V^i)$ of $(\pi^k_P|V^i(n,p))^*(TP)$ by, for $z\in V^i(N,P)$,

$$\boldsymbol{K}(V^i)_z=(z,\mathrm{Ker}(d_xf))\quad\text{and}\quad\boldsymbol{Q}(V^i)_z=(z,\mathrm{Coker}(d_xf))$$

respectively. The dimension of $\boldsymbol{K}(V^i)$, as a vector bundle, is $n-r$.

Let $\mathscr{O}(n,p)$ be an admissible open subset in $J^k(n,p)$ defined in Introduction whose singularities are decomposed into a finite number of disjoint $\mathscr{K}$-invariant submanifolds $V^i(n,p)$ of codimension ρ_i $(1\leq i\leq\iota)$ satisfying (H-i to v). We note that $V^i(n,p)$ may not be connected and that even if $i<j$, then ρ_i is not necessarily smaller than ρ_j. We denote, by $\mathscr{O}^i(n,p)$, the open subset $\mathscr{O}(n,p)\backslash\{\bigcup_{j=i+1}^{\iota}V^j(n,p)\}$ and, by $\mathscr{O}^i(N,P)$, the

open subbundle of $J^k(N,P)$ associated to $\mathscr{O}^i(n,p)$ for each i $(0 \leq i \leq \iota)$.

Let $z = j^k_x f \in J^k_{x,y}(N,P)$ be of rank r and $w = \pi^k_{k-1}(z)$. Let $\mathscr{K}^w(N,P)$ denote the subbundle of $J^{k-1}(N,P)$ associated to the $\mathscr{K}$-orbit $\mathscr{K}w$. We call $\mathscr{K}^w(N,P)$ the $\mathscr{K}$-orbit bundle of w in this paper. The fiber of $\mathscr{K}^w(N,P)$ over (x,y) is denoted by $\mathscr{K}^w_{x,y}(n,p)$. Let us recall the description of the tangent space of $\mathscr{K}^w_{x,y}(N,P)$ in [**MaIII**, (7.3)]. There have been defined the isomorphism, expressed in this paper by $\pi^{k-1}_{\theta,T}$,

$$T_w\big(J^{k-1}_{x,y}(N,P)\big) \longrightarrow \mathfrak{m}_x\theta(f)_x/\mathfrak{m}^k_x\theta(f)_x. \tag{2.3}$$

We do not give the definition. According to [**MaIII**, (7.4)], $T_w(\mathscr{K}^w_{x,y}(N,P))$ corresponds by $\pi^{k-1}_{\theta,T}$ to

$$\big(tf(\mathfrak{m}_x\theta(N)_x) + f^*(\mathfrak{m}_y)\theta(f)_x + \mathfrak{m}^k_x\theta(f)_x\big)/\mathfrak{m}^k_x\theta(f)_x, \tag{2.4}$$

which we denote by $I(w)$ for simplicity.

We choose Riemannian metrics on N and P. Let Q_y denote $T_y(P)/\mathrm{Im}(d_xf)$. We always identify $T_y(P)/\mathrm{Im}(d_xf)$ with the orthogonal complement of $\mathrm{Im}(d_xf)$ in $T_y(P)$. In the convex neighborhoods of x and y where f is defined, let $e(K_x)$ and $e(Q_y)$ denote $\exp_{N,x}(\mathrm{Ker}(d_xf))$ and $\exp_{P,y}(T_y(P)/\mathrm{Im}(d_xf))$ with the normal coordinates $x^\bullet = (x_{r+1},\dots,x_n)$ and $y^\bullet = (y_{r+1},\dots,y_p)$ associated to the orthonormal bases of K_x and Q_y respectively. Let $(y_1,\dots,y_r)$ be the normal coordinates of $\exp_{P,y}(\mathrm{Im}(d_xf))$ associated to the orthonormal basis of $\mathrm{Im}(d_xf)$. Setting $x_i = y_i \circ f$ for $1 \leq i \leq r$, we have the coordinates $(x_1,\dots,x_n)$ and $(y_1,\dots,y_p)$ of (N,x) and (P,y) respectively. Let p_{Q_y} : $(P,y) \to (e(Q_y),y)$ be the germ of the orthogonal projection. Let $f^\bullet : e(K_x) \to e(Q_y)$ be the map defined by $f^\bullet = p_{Q_y} \circ f|e(K_x)$. In the module $\mathfrak{m}_{x^\bullet}\theta(f^\bullet)_{x^\bullet}/\mathfrak{m}^k_{x^\bullet}\theta(f^\bullet)_{x^\bullet}$, let $I^\bullet(w)$ denote the submodule of

$$\big(tf^\bullet(\mathfrak{m}_{x^\bullet}\theta(e(K_x))_{x^\bullet}) + (f^\bullet)^*(\mathfrak{m}_{y^\bullet})\theta(f^\bullet)_{x^\bullet} + \mathfrak{m}^k_{x^\bullet}\theta(f^\bullet)_{x^\bullet}\big)/\mathfrak{m}^k_{x^\bullet}\theta(f^\bullet)_{x^\bullet}.$$

In this situation, since $f^\bullet(x^\bullet) = (y_{r+1}\circ f(x^\bullet),\dots,y_p\circ f(x^\bullet))$, the submodule $I^\bullet(w)$ is generated by

$$\begin{cases} \mathfrak{m}_{x^\bullet} \displaystyle\sum_{i=r+1}^{p} \frac{\partial y_i \circ f^\bullet}{\partial x_j}\left(\frac{\partial}{\partial y_i}\circ f^\bullet\right) & \text{for } r < j \leq n, \\ \langle y_{r+1}\circ f^\bullet,\dots,y_p\circ f^\bullet\rangle \dfrac{\partial}{\partial y_i}\circ f^\bullet & \text{for } r < i \leq p, \end{cases} \tag{2.5}$$

where $\partial/\partial y_i$ is the vector field on (P,y) and the notation $\langle * \rangle$ refers to an ideal.

If $z = j^k_x f \in V^i_{x,y}(N,P)$, then $w \in \mathscr{K}^w_{x,y}(N,P) \subset V^i_{x,y}(N,P)^{(k-1)}$ by (H-iii) and $T_w(\mathscr{K}^w_{x,y}(N,P)) \subset T_w(V^i_{x,y}(N,P)^{(k-1)})$. Under the above local coordinates $(x_1,\dots,x_n)$ and $(y_1,\dots,y_p)$, let $\mathscr{M}(V^i)^{(k-1)}$ and $\mathscr{M}(V^i)^{\bullet(k-1)}$ denote the vector bundles over $V^i(N,P)$ with fibers

$$\mathfrak{m}_x\theta(f)_x/\mathfrak{m}^k_x\theta(f)_x \quad \text{and} \quad \mathfrak{m}_{x^\bullet}\theta(f^\bullet)_{x^\bullet}/\mathfrak{m}^k_{x^\bullet}\theta(f^\bullet)_{x^\bullet}$$

over z respectively. These vector bundles are well defined as far as the Riemannian metrics on N and P are chosen and fixed. We use the same notation $\pi_{\theta,T}^{k-1}$ for the bundle isomorphism over $V^i(N,P)$ as follows.

$$\pi_{\theta,T}^{k-1} : \left(\pi_{k-1}^k\right)^* \left(T^{\mathfrak{f}}(J^{k-1}(N,P))\right)|_{V^i(N,P)} \longrightarrow \mathscr{M}(V^i)^{(k-1)}.$$

Furthermore, we define the canonical projection

$$p_{\mathscr{M}^\bullet} : \mathscr{M}(V^i)^{(k-1)} \longrightarrow \mathscr{M}(V^i)^{\bullet(k-1)} \tag{2.6}$$

by

$$(p_{\mathscr{M}^\bullet})_z\left(\sum_{i=1}^{r} h_i t f\left(\frac{\partial}{\partial x_i}\right) + \sum_{i=r+1}^{p} k_i\left(\frac{\partial}{\partial y_i}\circ f\right)\right) = \sum_{i=r+1}^{p} k_i^\bullet\left(\frac{\partial}{\partial y_i}\circ f^\bullet\right).$$

This definition is the global version of the homomorphism defined in [**MaIV**, Section 1].

We canonically identify $\nu(V^i(N,P)) = (\pi_{k-1}^k|V^i(N,P))^*(\nu(V^i(N,P)^{(k-1)}))$. It is not difficult to see that $(p_{\mathscr{M}^\bullet})_z$ induces the isomorphism of $\nu(\mathscr{K}^w(N,P))_w$ onto the vector spaces of dimension ρ

$$\mathfrak{m}_x\theta(f)_x/\left(I(w)+\mathfrak{m}_x^k\theta(f)_x\right) \approx \mathfrak{m}_{x^\bullet}\theta(f^\bullet)_{x^\bullet}/\left(I^\bullet(w)+\mathfrak{m}_{x^\bullet}^k\theta(f^\bullet)_{x^\bullet}\right). \tag{2.7}$$

The epimorphism $\nu(\mathscr{K}^w(N,P))_w \to \nu(V^i(N,P))_w^{(k-1)}$ canonically induces the epimorphism

$$p_\nu^{\mathscr{M}} : \mathscr{M}(V^i)^{\bullet(k-1)} \longrightarrow \nu(V^i(N,P)) \tag{2.8}$$

over $V^i(N,P)$.

Let

$$\Pi_{\mathfrak{f}}^k : T(J^k(N,P)) \to \left(\pi_{k-1}^k\right)^*\left(T(J^{k-1}(N,P))\right) \to \left(\pi_{k-1}^k\right)^*\left(T^{\mathfrak{f}}(J^{k-1}(N,P))\right)$$

denote the composite of canonical projections and let

$$p_{\nu(V^i)} : T(J^k(N,P))|_{V^i(N,P)} \longrightarrow \nu(V^i(N,P))$$

denote the canonical projection.

Lemma 2.1. *Let $z \in V_{x,y}^i(N,P)$. Under the above notation the epimorphism $p_{\nu(V^i)}|_z$ coincides with the composite $p_\nu^{\mathscr{M}} \circ p_{\mathscr{M}^\bullet} \circ \pi_{\theta,T}^{k-1} \circ (\Pi_{\mathfrak{f}}^k)_z$:*

$$T_z(J^k(N,P)) \to \mathscr{M}(V^i)_z^{(k-1)} \to \mathscr{M}(V^i)_z^{\bullet(k-1)} \to \nu(V^i(N,P))_z. \tag{2.9}$$

Recall the homomorphism $\boldsymbol{d}$ in Introduction. Let us study the composite

$$\pi^{\mathfrak{f}} \circ \boldsymbol{d} : \left(\pi^k_N\right)^*(TN)|_{V^i(N,P)} \longrightarrow \left(\pi^k_{k-1}\right)^*\left(T^{\mathfrak{f}}(J^{k-1}(N,P))\right)|_{V^i(N,P)}$$

and the isomorphism in (2.3). For $z = j^k_x f \in V^i(N,P)$ and $\boldsymbol{v} \in T_xU$, let $v(t) = \exp_{N,x}(t\boldsymbol{v})$ be the geodesic curve. Then the composite $t_{x,y} \circ j^{k-1}f \circ v : I \to J^{k-1}_{x,y}(N,P)$ yields that

$$\begin{aligned}\left(d(t_{x,y} \circ j^k f \circ v)|_{t=0}\right)(d/dt) &= \left((d(t_{x,y}) \circ d(j^k f) \circ dv)|_{t=0}\right)(d/dt) \\ &= d(t_{x,y}) \circ d(j^k f)(\boldsymbol{v}) \\ &= d(t_{x,y}) \circ \boldsymbol{d}(\boldsymbol{v}) \\ &= \pi^{\mathfrak{f}} \circ \boldsymbol{d}(\boldsymbol{v}),\end{aligned} \tag{2.10}$$

where $\pi^{\mathfrak{f}} \circ \boldsymbol{d}(\boldsymbol{v})$ is regarded as an element of $J^{k-1}_{x,y}(N,P)$. Let $F : U \times [0,1] \to P$ be the following map

$$\begin{aligned}F(x',t) &= \ell(f(v(t)), f(x)) \circ f \circ \ell(x, v(t))(x') \\ &= \ell(f(v(t)), f(x)) \circ f(x' + v(t) - x) \\ &= f(x' + v(t) - x) + f(x) - f(v(t)).\end{aligned}$$

In particular, we have $F(x,t) = f(x) = y$. Let $F_{x'}(t) = F_t(x') = F(x',t)$ and $G(t) = f(x' + v(t) - x)$.

Remark 2.2. It follows that $\pi^{k-1}_{\theta,T} \circ \pi^{\mathfrak{f}} \circ \boldsymbol{d}(\boldsymbol{v})$ is represented by the vector fields $\zeta^z_{\boldsymbol{v}} : (N,x) \to TP$ defined by $\zeta^z_{\boldsymbol{v}}(x') = (dF_{x'}|_{t=0})(d/dt)$. Let us briefly prove this fact. We note that

$$j^{k-1}_x F_t = \ell(f(v(t)), f(x)) \circ j^{k-1}_{v(t), f(v(t))} f \circ \ell(x, v(t)) \in J^{k-1}_{x,y}(N,P).$$

By (2.10) we have $\pi^{\mathfrak{f}} \circ \boldsymbol{d}(\boldsymbol{v}) = (d(j^{k-1}_x F_t)|_{t=0})(d/dt)$. By the definition of the isomorphism $\pi^{k-1}_{\theta,T}$ in (2.3) in [**MaIII**, (7.3)] we obtain the assertion.

In Remark 2.2 $\zeta^z_{\boldsymbol{v}} = (dF_{x'}|_{t=0})(d/dt)$ is equal to

$$\begin{aligned}&(dG|_{t=0})(d/dt) - (d(f \circ v)|_{t=0})(d/dt) \\ &\quad = \left(\left[\cdots, \sum_{\ell=1}^{p} \left(\frac{\partial y_\ell \circ G(t)}{\partial x_j} - \frac{\partial y_\ell \circ f}{\partial x_j}(v(t))\right)\frac{\partial}{\partial y_\ell}, \cdots\right]_{t=0}\right) \bullet \boldsymbol{v}\end{aligned} \tag{2.11}$$

where "$\bullet$" refers to the inner product. If $\boldsymbol{v} = \sum_{j=1}^n a_j \partial/\partial x_j \in \boldsymbol{K}(V^i)_z$, then $(d(f \circ v)|_{t=0})(d/dt) = df(\boldsymbol{v}) = 0$ and

$$\zeta^z_{\boldsymbol{v}}(x') = \sum_{\ell=1}^{p}\left(\sum_{j=1}^{p} a_j \frac{\partial y_\ell \circ f}{\partial x_j}(x')\right)\frac{\partial}{\partial y_\ell} \tag{2.12}$$

and $\zeta_{\boldsymbol{v}}^z(x) = \mathbf{0}$. Therefore, if $\boldsymbol{v} \in \boldsymbol{K}(V^i)_z$, then $\zeta_{\boldsymbol{v}}^z$ lies in $\mathfrak{m}_x\theta(f)_x$.

Under the trivialization $TU = U \times T_xU$, there is the vector field $\boldsymbol{v}_U$ on U defined by $\boldsymbol{v}_U(x') = (x', \boldsymbol{v})$. Therefore, we have the following lemma.

LEMMA 2.3. *Let $z = j_x^k f \in V_{x,y}^i(N,P)$. Let $\boldsymbol{v} \in \boldsymbol{K}(V^i)_z$. Under the above notation, $\pi_{\theta,T}^{k-1} \circ \pi^{\mathfrak{f}} \circ \boldsymbol{d}(\boldsymbol{v})$ is represented by $\zeta_{\boldsymbol{v}}^z = tf(\boldsymbol{v}_U)$.*

3. Primary obstruction.

Let $\mathfrak{s} \in \Gamma_{\mathscr{O}}(N,P)$ be smooth around $\mathfrak{s}^{-1}(V^i(N,P))$ and transverse to $V^i(N,P)$. We set

$$S^{V^i}(\mathfrak{s}) = \mathfrak{s}^{-1}(V^i(N,P)), \qquad S^{n-p+1,0}(\mathfrak{s}) = \mathfrak{s}^{-1}(\Sigma^{n-p+1,0}(N,P)),$$
$$\big(\mathfrak{s}|S^{V^i}(\mathfrak{s})\big)^*(\boldsymbol{K}(V^i)) = K\big(S^{V^i}(\mathfrak{s})\big), \quad \big(\mathfrak{s}|S^{V^i}(\mathfrak{s})\big)^*\boldsymbol{Q}(V^i) = Q\big(S^{V^i}(\mathfrak{s})\big).$$

We often write $S^{V^i}(\mathfrak{s})$ as S^{V^i} if there is no confusion.

Let $\Gamma_{\mathscr{O}}^{tr}(N,P)$ denote the subspace of $\Gamma_{\mathscr{O}}(N,P)$ consisting of all smooth sections of $\pi_N^k|\mathscr{O}(N,P) : \mathscr{O}(N,P) \to N$ which are transverse to $V^j(N,P)$ for every j. Let C be a closed subset of N. For $s \in \Gamma_{\mathscr{O}}^{tr}(N,P)$ let C_{i+1} refer to the union $C \cup s^{-1}(\mathscr{O}(n,p)\backslash\mathscr{O}^i(n,p))$ $(C_{\iota+1} = C)$.

The following theorem has been proved in [**An3**, Theorem 4.1] and [**An4**, Theorem 0.5] in which [**E1**, 2.2 Theorem] and [**E2**, 4.7 Theorem] have played important roles.

THEOREM 3.1. *Let $n \geq p \geq 2$. Let $\mathscr{O}(n,p)$ denote $\Omega^{(n-p+1,0)}(n,p)$. Let N and P be connected manifolds of dimensions n and p respectively with $\partial N = \varnothing$. Let C be a closed subset of N. Let s be a section of $\Gamma_{\mathscr{O}}(N,P)$ such that there exists a fold-map g defined on a neighborhood of C into P, where $j^2g = s$. Then there exists a fold-map $f : N \to P$ such that j^2f is homotopic to s relative to C by a homotopy s_λ in $\Gamma_{\mathscr{O}}(N,P)$ with $s_0 = s$ and $s_1 = j^2f$. In particular, $f = g$ on a neighborhood of C.*

We show in this section that it is enough for the proof of Theorem 0.1 to prove the following theorem together with Theorem 3.1.

THEOREM 3.2. *Let $k \geq 3$. Let N and P be connected manifolds of dimensions n and p respectively with $\partial N = \varnothing$. We assume the same assumption for $\mathscr{O}(n,p)$ as in Theorem 0.1. Let C_{i+1} and $V^i(n,p)$ be as above for $1 \leq i \leq \iota$. We assume that if $n \geq p \geq 2$, then $V^i(n,p) \neq \Sigma^{n-p+1,0}(n,p)$ $(i > 1)$. Let s be a section in $\Gamma_{\mathscr{O}}^{tr}(N,P)$ which has an $\mathscr{O}$-regular map g_{i+1} $(g_{\iota+1} = g)$ defined on a neighborhood of C_{i+1} to P, where $j^kg_{i+1} = s$. Then there exists a homotopy $s_\lambda \in \Gamma_{\mathscr{O}}(N,P)$ of $s_0 = s$ relative to a neighborhood of C_{i+1} with the following properties.*

(3.2.1) *$s_1 \in \Gamma_{\mathscr{O}}^{tr}(N,P)$ and $s_1(N\backslash C_{i+1}) \subset \mathscr{O}(N,P)^i$.*

(3.2.2) *$S^{V^i}(s_\lambda) = S^{V^i}(s)$ for any λ.*

(3.2.3) *There exists an $\mathscr{O}$-regular map g_i defined on a neighborhood of C_i, where $j^kg_i = s_1$ holds. In particular, $g_i = g_{i+1}$ on a neighborhood of C_{i+1}.*

PROOF OF THEOREM 0.1. We first deform s to be transverse outside a small neighborhood of C. By the downward induction on i using Theorem 3.2 we next deform s keeping g near C to the jet extension of an $\mathscr{O}$-regular map defined around $\bigcup_{j=1}^{\iota} S^{V^j}(s)$ for $n < p$ and around $\bigcup_{j=2}^{\iota} S^{V^j}(s)$ for $n \geq p \geq 2$. In the final step we apply the Smale-Hirsch Immersion Theorem ([**H**, Theorem 5.7]) for $n < p$ and Theorem 3.1 for $n \geq p \geq 2$ to obtain the required $\mathscr{O}$-regular map f.

Take a closed neighborhood $U(C)$ of C where the given $\mathscr{O}$-regular map g is defined. Let $U_j(C)$ $(j = 1,2,3,4)$ be closed neighborhoods of C such that $U_4(C) \subset \mathrm{Int}U(C)$ and $U_j(C) \subset \mathrm{Int}U_{j+1}(C)$ $(j = 1,2,3)$. By [**G-G**, Ch. II, Corollary 4.11] there exists a homotopy of $\mathscr{O}$-regular maps $g_\lambda : U(C) \to P$ relative to $U_1(C)$ such that $g_0 = g$ and $j^k g_1|U(C)\backslash \mathrm{Int}U_2(C)$ is transverse to $V^j(N,P)$ for all j. By applying the homotopy extension property we obtain a homotopy μ_λ in $\Gamma_{\mathscr{O}}(N,P)$ such that $\mu_0 = s$, $\mu_\lambda|U_4(C) = j^k g_\lambda|U_4(C)$ and $\mu_1|(N\backslash U_2(C)) \in \Gamma^{tr}_{\mathscr{O}}(N\backslash U_2(C),P)$. Let $S(\mu_1)$ denote the subspace of all points $x \in N$ such that $\mu_1(x)$ are singular jets.

Let $N' = N\backslash U_2(C)$, $C' = U_3(C) \cap N'$ and $g' = g_1|(U_4(C)\backslash U_2(C))$. Let us choose the largest integer i such that $S^{V^i}(\mu_1)\backslash C' \neq \varnothing$. We first apply Theorem 3.2 to the case of $\mu_1|N'$, C', g' and $\mathscr{O}(N',P)$ in $J^k(N',P)$. There exist a homotopy s'_λ in $\Gamma_{\mathscr{O}}(N',P)$ of $s'_0 = \mu_1|N'$ relative to a neighborhood of C' and an $\mathscr{O}$-regular map g'_i defined on a neighborhood of C'_i in N' satisfying the properties (3.2.1) to (3.2.3) for N', C', g', g'_i and s'_λ.

Then we can prove by downward induction on integers i that there exists a homotopy s''_λ of $s''_0 = s'_1$ in $\Gamma^{tr}_{\mathscr{O}}(N',P)$ relative to $U_3(C)$ and an $\mathscr{O}$-regular map f' defined on a neighborhood of $(U_3(C) \cup S(\mu_1))\backslash U_2(C)$ for $n < p$ and of $(U_3(C) \cup (S(\mu_1)\backslash S^{(n-p+1,0)}(\mu_1)))\backslash U_2(C)$ for $n \geq p \geq 2$, such that

(i) $s''_1 \in \Gamma^{tr}_{\mathscr{O}}(N',P)$,
(ii) $s''_1(N\backslash C_1) \subset \mathscr{O}^0(N,P)$ for $n < p$ and $s''_1(N\backslash C_2) \subset \mathscr{O}^1(N,P)$ for $n \geq p \geq 2$,
(iii) $S^{V^j}(s''_\lambda) = S^{V^j}(s)$ except for $j = 1$ in the case $n \geq p \geq 2$.

Let

$$N'' = \begin{cases} N'/S(\mu_1) & \text{for the case } n < p, \\ (N'/S(\mu_1)) \cup S^{(n-p+1,0)}(\mu_1) & \text{for the case } n \geq p \geq 2. \end{cases}$$

It follows from the Smale-Hirsch Immersion Theorem for the case $n < p$ that there exist an immersion $f'' : N'' \to P$ and a homotopy $u_\lambda \in \Gamma_{\mathscr{O}}(N'',P)$ relative to the neighborhood of $U(C \cup S(\mu_1)) \cap N''$ such that $u_0 = s''_1|N''$ and $u_1 = j^k f''$. It follows from Theorem 3.2 for the case $n \geq p \geq 2$ that there exist an $\Omega^{(n-p+1,0)}$-regular map $f'' : N'' \to P$ and a homotopy $u_\lambda \in \Gamma_{\mathscr{O}}(N'',P)$ relative to a neighborhood of

$$\big\{(U(C \cup S(\mu_1))\backslash S(\mu_1)) \cup S^{(n-p+1,0)}(\mu_1)\big\} \cap N''$$

such that $u_0 = s''_1|N''$ and $u_1 = j^k f''$. Define $s'''_\lambda \in \Gamma_{\mathscr{O}}(N',P)$ by $s'''_\lambda|N'' = u_\lambda$ and $s'''_\lambda|(N'\backslash N'') = s''_1|(N'\backslash N'')$.

Now we have the homotopy $\overline{\mu}_\lambda$ in $\Gamma_{\mathscr{O}}(N,P)$ defined by

$$\overline{\mu}_\lambda|N' = \begin{cases} s'_{3\lambda} & (0 \le \lambda \le 1/3), \\ s''_{3\lambda-1} & (1/3 \le \lambda \le 2/3), \\ s'''_{3\lambda-2} & (2/3 \le \lambda \le 1) \end{cases}$$

and $\overline{\mu}_\lambda|U_3(C) = j^k g_1|U_3(C)$. Thus we obtain the required homotopy s_λ in Theorem 0.1 by pasting μ_λ and $\overline{\mu}_\lambda$. $\square$

We begin by preparing several notions and results, which are necessary for the proof of Theorem 3.2. For the map g_{i+1}, we take a closed neighborhood $U(C_{i+1})'$ of C_{i+1} around which g_{i+1} is defined and $j^k g_{i+1} = s$. Without loss of generality we may assume that $N \setminus U(C_{i+1})'$ is nonempty. Let us take a closed neighborhood $U(C_{i+1})$ of C_{i+1} in $\mathrm{Int}U(C_{i+1})'$ such that $U(C_{i+1})$ is a submanifold of dimension n with boundary $\partial U(C_{i+1})$. By virtue of Gromov's theorem ([**G1**, Theorem 4.1.1]), it suffices to consider the special case where

(C1) $N \setminus \mathrm{Int}U(C_{i+1})$ is compact, connected and nonempty,

(C2) $s \in \Gamma^{tr}_{\mathscr{O}}(N, P)$ and $S^{V^i}(s) \setminus \mathrm{Int}U(C_{i+1}) \neq \varnothing$,

(C3) $S^{V^i}(s)$ is transverse to $\partial U(C_{i+1})$.

For a manifold X and its submanifold Y let $\nu(Y)$ denote the normal bundle $(TX|_Y)/TY$ of Y. In what follows we set $r = r_i$ and $\rho = \rho_i$ for simplicity. Let $\nu(V^i(N,P))$ be the normal bundle of dimension $\rho \le n$. Then $p_{\nu(V^i)} \circ \boldsymbol{d}|\boldsymbol{K}(V^i) : \boldsymbol{K}(V^i) \to \nu(V^i(N,P))$ is a monomorphism over $V^i(N,P)$ by (H-v) under the identification $\nu(V^i(N,P))_z = (z, \nu(V^i(N,P)^{(k-1)})_{\pi^k_{k-1}(z)})$. The composite

$$p_{\nu(V^i)} \circ \boldsymbol{d}|\boldsymbol{K}(V^i) \circ \big(s|S^{V^i}\big)^{\boldsymbol{K}(V^i)} : K\big(S^{V^i}(s)\big) \to \boldsymbol{K}(V^i) \to \nu(V^i(N,P))$$

is also a monomorphism. Let $s \in \Gamma_{\mathscr{O}}(N,P)$ be the given section in Theorem 3.2. Let us provide N with a Riemannian metric. Let $\mathfrak{n}(s, V^i)$ be the orthogonal normal bundle of $S^{V^i}(s)$ in N. We have the bundle map

$$ds|\mathfrak{n}(s,V^i) : \mathfrak{n}(s,V^i) \longrightarrow \nu(V^i(N,P))$$

covering $s|S^{V^i} : S^{V^i}(s) \to V^i(N,P)$. Let $\boldsymbol{i}_{\mathfrak{n}(s,V^i)} : \mathfrak{n}(s,V^i) \subset TN|_{S^{V^i}}$ denote the inclusion. We define $\Psi(s,V^i) : K(S^{V^i}(s)) \to \mathfrak{n}(s,V^i) \subset TN|_{S^{V^i}}$ to be the composite

$$\begin{aligned} &\boldsymbol{i}_{\mathfrak{n}(s,V^i)} \circ \big((s|S^{V^i})^*(ds|\mathfrak{n}(s,V^i))\big)^{-1} \circ \big((s|S^{V^i})^*\big(p_{\nu(V^i)} \circ \boldsymbol{d}|\boldsymbol{K}(V^i) \circ (s|S^{V^i})^{\boldsymbol{K}(V^i)}\big)\big) \\ &\quad : K\big(S^{V^i}(s)\big) \to (s|S^{V^i})^*\nu(V^i(N,P)) \to \mathfrak{n}(s,V^i) \to TN|_{S^{V^i}}. \end{aligned} \tag{3.1}$$

Let $i_{K(S^{V^i}(s))} : K(S^{V^i}(s)) \to TN|_{S^{V^i}}$ be the inclusion.

Remark 3.3. If f is an $\mathscr{O}$-regular map such that $j^k f$ is transverse to $V^i(N,P)$, then it follows from the definition of $\boldsymbol{d}$ that $i_{K(S^{V^i}(j^k f))} = \Psi(j^k f, V^i)$ if we choose a Riemannian metric such that $K(S^{V^i}(j^k f))$ is orthogonal to $S^{V^i}(j^k f)$.

Here we give an outline of the proof of Theorem 3.2. We first deform the given section s in Theorem 3.2 so that $K(S^{V^i}(s))$ is normal to $S^{V^i}(s)$ and $i_{K(S^{V^i}(s))} = \Psi(s, V^i)$ (Lemma 4.1). Next we deform the section so that $\pi_P \circ s|S^{V^i}(s)$ is an immersion by applying the Smale-Hirsch Immersion Theorem (Lemma 4.2). In Section 5, using the transversality of the deformed section we construct an $\mathscr{O}^i$-regular map $\boldsymbol{q}$ defined around $S^{V^i}(s)$ by applying the versal unfolding developed in [**MaIV**] and modify $\boldsymbol{q}$ around C_{i+1} to be compatible with g_{i+1}. This is the required $\mathscr{O}$-regular map g_i. In section 6 we finally extend the homotopy between s and $j^k g_i$ defined around $S^{V^i}(s)$ to the homotopy defined on the whole space N and obtain a required section.

In what follows let $M = S^{V^i}(s) \setminus \mathrm{Int}(U(C_{i+1}))$. Let

$$\mathrm{Mono}\big(K(S^{V^i}(s))|_M, TN|_M\big)$$

denote the subset of $\mathrm{Hom}(K(S^{V^i}(s))|_M, TN|_M)$ which consists of all monomorphisms $K(S^{V^i}(s))_c \to T_cN$, $c \in M$. We denote the bundle of local coefficients $\mathscr{B}(\pi_j(\mathrm{Mono}(K(S^{V^i}(s))_c, T_cN)))$, $c \in M$, by $\mathscr{B}(\pi_j)$, which is a covering space over M with fiber $\pi_j(\mathrm{Mono}(K(S^{V^i}(s))_c, T_cN))$ defined in [**Ste**, 30.1]. By the obstruction theory due to [**Ste**, 36.3], the obstructions for $i_{K(S^{V^i}(s))}|_M$ and $\Psi(s, V^i)|_M$ to be homotopic relative to ∂M are the primary differences $d(i_{K(S^{V^i}(s))}|_M, \Psi(s, V^i)|_M)$, which are defined in $H^j(M, \partial M; \mathscr{B}(\pi_j))$ with local coefficients. We show that unless $n \geq p \geq 2$ and $V^i(n,p) = \Sigma^{n-p+1,0}(n,p)$, all of them vanish by [**Ste**, 38.2]. In fact, if $n \geq p \geq 2$ and $V^i(n,p) \neq \Sigma^{n-p+1,0}(n,p)$, then we have

$$\begin{aligned} \dim M &< n - \mathrm{codim}\Sigma^{n-p+1} = n - (n-r) = r, &\quad \text{for } r = p-1, \\ \dim M &\leq n - \mathrm{codim}\Sigma^{n-r} = n - (n-r)(p-r) < r, &\quad \text{for } r < p-1. \end{aligned}$$

If $n < p$, then

$$\dim M \leq n - \mathrm{codim}\Sigma^{n-r} = n - (n-r)(p-r) \leq n - 2(n-r) < r.$$

Since $\mathrm{Mono}(\boldsymbol{R}^{n-r}, \boldsymbol{R}^n)$ is identified with $GL(n)/GL(r)$, it follows from [**Ste**, 25.6] that $\pi_j(\mathrm{Mono}(\boldsymbol{R}^{n-r}, \boldsymbol{R}^n)) \cong \{\boldsymbol{0}\}$ for $j < r$. Hence, there exists a homotopy $\psi^M(s, V^i)_\lambda : K(S^{V^i}(s))|_M \to TN|_M$ relative to $M \cap U(C_{i+1})'$ in $\mathrm{Mono}(K(S^{V^i}(s))|_M, TN|_M)$ such that

$$\psi^M(s, V^i)_0 = i_{K(S^{V^i}(s))}|_M \text{ and } \psi^M(s, V^i)_1 = \Psi(s, V^i)|_M.$$

Let $\mathrm{Iso}(TN|_M, TN|_M)$ denote the subspace of $\mathrm{Hom}(TN|_M, TN|_M)$ which consists of all isomorphisms of T_cN, $c \in M$. The restriction map

$$r_M : \mathrm{Iso}(TN|_M, TN|_M) \longrightarrow \mathrm{Mono}\big(K(S^{V^i}(s))|_M, TN|_M\big)$$

defined by $r_M(h) = h|(K(S^{V^i}(s))_c)$, for $h \in \mathrm{Iso}(T_cN, T_cN)$, induces a structure of a fiber bundle with fiber $\mathrm{Iso}(\boldsymbol{R}^r, \boldsymbol{R}^r) \times \mathrm{Hom}(\boldsymbol{R}^r, \boldsymbol{R}^{n-r})$. By applying the covering homotopy property of the fiber bundle r_M to the sections $id_{TN|_M}$ and the homotopy $\psi^M(s, V^i)_\lambda$, we obtain a homotopy $\Psi(s, V^i)_\lambda : TN|_{S^{V^i}} \to TN|_{S^{V^i}}$ such that $\Psi(s, V^i)_0 = id_{TN|_{S^{V^i}}}$, $\Psi(s, V^i)_\lambda|_c = id_{T_cN}$ for all $c \in S^{V^i} \cap U(C_{i+1})$ and $r_M \circ \Psi(s, V^i)_\lambda|(K(S^{V^i}(s))|_M) = \psi^M(s, V^i)_\lambda$. We define $\Phi(s, V^i)_\lambda : TN|_{S^{V^i}} \to TN|_{S^{V^i}}$ by $\Phi(s, V^i)_\lambda = (\Psi(s, V^i)_\lambda)^{-1}$.

4. Lemmas.

The section s given in Theorem 3.2 may not satisfy $i_{K(S^{V^i}(s))} = \Psi(S^{V^i}(s))$ and $K(S^{V^i}(s))$ may not even transverse to $S^{V^i}(s)$ either. Therefore, we first have to deform the section s so that $K(S^{V^i}(s))$ is normal to $S^{V^i}(s)$ and $i_{K(S^{V^i}(s))} = \Psi(S^{V^i}(s))$. We next deform s so that $\pi_P \circ s|S^{V^i}(s)$ is an immersion by the Smale-Hirsch Immersion Theorem. The arguments of these two steps are quite similar to those in [**An6**, Lemmas 5.1 and 5.2]. So we only show important steps in the proofs.

In the proof of the following lemma, $\Phi(s, V^i)_\lambda|_c$ ($c \in S^{V^i}$) is regarded as a linear isomorphism of T_cN. We set $d_1(s, V^i) = (s|S^{V^i}(s))^*(\boldsymbol{d}_1)$. Let us take closed neighborhoods $W(C_{i+1})_j$ ($j = 1, 2$) of $U(C_{i+1})$ in $U(C_{i+1})'$ such that $W(C_{i+1})_1 \subset \mathrm{Int}W(C_{i+1})_2$, $W(C_{i+1})_j$ are submanifolds of dimension n with boundary $\partial W(C_{i+1})_j$ and that $\partial W(C_{i+1})_j$ meet transversely with $S^{V^i}(s)$.

Lemma 4.1. *Let $s \in \Gamma^{tr}_{\mathscr{O}}(N, P)$ be a section satisfying the hypotheses of Theorem 3.2. Assume that if $n \geq p \geq 2$, then $V^i(n, p) \neq \Sigma^{n-p+1,0}(n, p)$. Then there exists a homotopy s_λ relative to $W(C_{i+1})_1$ in $\Gamma^{tr}_{\mathscr{O}}(N, P)$ with $s_0 = s$ satisfying*

(4.1.1) *for any λ, $S^{V^i}(s_\lambda) = S^{V^i}(s)$ and $\pi^k_P \circ s_\lambda|S^{V^i}(s_\lambda) = \pi^k_P \circ s|S^{V^i}(s)$,*

(4.1.2) *we have $i_{K(S^{V^i}(s_1))} = \Psi(s_1, V^i)$, and in particular, $K(S^{V^i}(s_1))_c \subset \mathfrak{n}(s, V^i)_c$ for any point $c \in S^{V^i}(s_1)$.*

Proof. We write an element of $\mathfrak{n}(\sigma, V^i)_c$ as $\boldsymbol{v}_c$. There exists a small positive number δ such that the map

$$e : D_\delta(\mathfrak{n}(\sigma, V^i))|_M \longrightarrow N$$

defined by $e(\boldsymbol{v}_c) = \exp_{N,c}(\boldsymbol{v}_c)$ is an embedding, where $c \in M$. Let $\rho : [0, \infty) \to \boldsymbol{R}$ be a decreasing smooth function such that $0 \leq a(t) \leq 1$, $a(t) = 1$ if $t \leq \delta/10$ and $a(t) = 0$ if $t \geq \delta$.

Let $\ell(\boldsymbol{v})$ denote the parallel translation defined by $\ell(\boldsymbol{v})(\boldsymbol{a}) = \boldsymbol{a} + \boldsymbol{v}$. If we represent a jet of $J^k(N, P)$ by $j^k_x\iota_x$ for a germ $\iota_x : (N, x) \to (P, y)$, then we define the homotopy $b_\lambda : J^k(N, P) \to J^k(N, P)$ ($0 \leq \lambda \leq 1$) of the bundle maps over $N \times P$ as follows.

(i) If $x = e(\boldsymbol{v}_c)$, $c \in M$ and $\|\boldsymbol{v}_c\| \leq \delta$, then

$$b_\lambda\big(j^k_x\iota_x\big) = j^k_x\big(\iota_x \circ \exp_{N,c} \circ \ell(\boldsymbol{v}_c) \circ \Phi(s, V^i)_{a(\|\boldsymbol{v}_c\|)\lambda}|_c \circ \ell(-\boldsymbol{v}_c) \circ \exp^{-1}_{N,c}\big).$$

(ii) If $x \notin \mathrm{Im}(e)$, then $b_\lambda(j^k_x\iota_x) = j^k_x\iota_x$.

If δ is sufficiently small, then we may suppose that

$$e\big(D_\delta(\mathfrak{n}(\sigma,V^i))|_M\big)\cap W(C_{i+1})_1\subset e\big(D_\delta(\mathfrak{n}(\sigma,V^i))|_{M\cap W(C_{i+1})_2}\big).$$

If $c\in S^{V^i}\cap U(C_{i+1})$ or if $\|\boldsymbol{v}_c\|\geq\delta$, then $\Phi(s,V^i)_\lambda|_c$ or $\Phi(s,V^i)_{a(\|\boldsymbol{v}_c\|)\lambda}|_c$ is equal to $\Phi(s,V^i)_0|_c=id_{T_cN}$ respectively. Hence, b_λ is well defined. We define the homotopy s_λ of $\Gamma^{tr}_{\mathscr{O}}(N,P)$ using b_λ by $s_\lambda(x)=b_\lambda\circ s(x)$. By (i) and (ii) we have (4.1.1).

We have that $\mathfrak{n}(s,V^i)_c\supset K(S^{V^i}(s_1))_c$ and $i_{K(S^{V^i}(s_1))}=\Psi(s_1,V^i)$ for $c\in S^{V^i}(s)$. Indeed, let $\Psi(s,V^i)_c(\boldsymbol{v})=\boldsymbol{w}$ with $\boldsymbol{v}\in K(S^{V^i}(s))_c$ and $\boldsymbol{w}\in\mathfrak{n}(s,V^i)_c$. Setting $s(c)=j^k_c\iota_c$ we have by (i) and (ii) that

$$s_1(c)=s(c)\circ j^k_c\big(\exp_{N,c}\circ\Phi(s,V^i)_1|_c\circ\exp^{-1}_{N,c}\big).$$

Since $d_1(s_1,V^i)_c=d_1(s,V^i)_c\circ\Phi(s,V^i)_1|_c$ vanishes on $\Psi(s,V^i)(K(S^{V^i}(s))_c)$, we have $\Psi(s,V^i)(K(S^{V^i}(s))_c)=K(S^{V^i}(s_1))_c$. By (3.1), we have $\Psi(s_1,V^i)(\boldsymbol{w})=\boldsymbol{w}$. □

Lemma 4.2. *Let s be a section in $\Gamma^{tr}_{\mathscr{O}}(N,P)$ satisfying the property* (4.1.2) *for s (in place of s_1) of Lemma* 4.1 *and $V^i(n,p)$ be given in Theorem* 3.2. *Then there exists a homotopy α_λ relative to $W(C_{i+1})_1$ in $\Gamma_{\mathscr{O}}(N,P)$ with $\alpha_0=s$ such that*

(4.2.1) *α_λ is transverse to $V^i(N,P)$ and $S^{V^i}(\alpha_\lambda)=S^{V^i}(s)$ for any λ,*

(4.2.2) *we have $i_{K(S^{V^i}(\alpha_1))}=\Psi(\alpha_1,V^i)$, and in particular, $K(S^{V^i}(\alpha_1))_c\subset\mathfrak{n}(s,V^i)_c$ for any point $c\in S^{V^i}(\alpha_1)$,*

(4.2.3) *$\pi^k_P\circ\alpha_1|S^{V^i}(\alpha_1)$ is an immersion to P such that*

$$d\big(\pi^k_P\circ\alpha_1|S^{V^i}(\alpha_1)\big)=\big(\pi^k_P\circ\alpha_1\big)^{TP}\circ d_1(\alpha_1,V^i)|T(S^{V^i}(\alpha_1)):T\big(S^{V^i}(\alpha_1)\big)\to TP,$$

where $(\pi^k_P\circ\alpha_1)^{TP}:(\pi^k_P\circ\alpha_1)^*(TP)\to TP$ is the canonical induced bundle map,

(4.2.4) *$\alpha_\lambda(N\backslash(S^{V^i}(s)\cup\mathrm{Int}W(C_{i+1})_1))\subset\mathscr{O}^{i-1}(N,P)$.*

Proof. In the proof we set $S^{V^i}=S^{V^i}(s)$. We choose a Riemannian metric of P and identify $Q(S^{V^i})$ with the orthogonal complement of $\mathrm{Im}(d_1(s,V^i))$ in $(\pi^k_P\circ s|S^{V^i})^*(TP)$. Since $K(S^{V^i})\cap T(S^{V^i})=\{\mathbf{0}\}$, it follows that $(\pi^k_P\circ s)^{TP}\circ d_1(s,V^i)|T(S^{V^i})$ is a monomorphism. By the Smale-Hirsch Immersion Theorem there exists a smooth homotopy of monomorphisms $m'_\lambda:T(S^{V^i})\to TP$ covering a homotopy $m_\lambda:S^{V^i}\to P$ relative to $W(C_{i+1})_1$ such that $m'_0=(\pi^k_P\circ s)^{TP}\circ d_1(s,V^i)|T(S^{V^i})$ and m_1 is an immersion with $d(m_1)=m'_1$. Then we can extend m'_λ to a smooth homotopy $\widetilde{m'_\lambda}:TN|_{S^{V^i}}\to TP$ of homomorphisms of constant rank r relative to $S^{V^i}\cap W(C_{i+1})_1$ so that $\widetilde{m'_0}=(\pi^k_P\circ s)^{TP}\circ d_1(s,V^i)$.

Recall the submanifold $\Sigma^{n-r}(N,P)^{(1)}$ of $J^1(N,P)=J^1(TN,TP)$, which consists of all jets of rank r. Then

$$\pi^k_1|V^i(N,P):V^i(N,P)\longrightarrow\Sigma^{n-r}(N,P)^{(1)}$$

becomes a fiber bundle. We regard $\widetilde{m'_\lambda}$ as a homotopy $S^{V^i}\to\Sigma^{n-r}(N,P)^{(1)}$. By the covering homotopy property to $s|S^{V^i}$ and $\widetilde{m'_\lambda}$, we obtain a smooth homotopy $\alpha^\Sigma_\lambda:S^{V^i}\to$

$V^i(N,P)$ covering $\widetilde{m'_\lambda}$ relative to $W(C_{i+1})_1$ such that $\alpha_0^\Sigma = s|S^{V^i}$.

We have a smooth metric of $\mathfrak{n}(s,V^i)$ over S^{V^i}. For a sufficiently small positive function $\varepsilon : S^{V^i} \to \boldsymbol{R}$, let $E(S^{V^i})$ denote $\exp_N D_\varepsilon(\mathfrak{n}(s,V^i))$. By using the transversality of s and the homotopy extension property of bundle maps for $s|E(S^{V^i})$ and α_λ^Σ, we first extend α_λ^Σ to a smooth homotopy β_λ of $E(S^{V^i})$ to a tubular neighborhood of $V^i(N,P)$, say U_{V^i}, covering α_λ^Σ relative to $E(S^{V^i})\cap W(C_{i+1})_1$ such that $\beta_0 = s|E(S^{V^i})$ and β_λ is transverse to $V^i(N,P)$. Next extend β_λ to a homotopy $\alpha_\lambda \in \Gamma_{\mathscr{O}}(N,P)$ so that $\alpha_0 = s$, $\alpha_\lambda|E(S^{V^i}) = \beta_\lambda$, $\alpha_\lambda|W(C_{i+1})_1 = s|W(C_{i+1})_1$ and that

$$\alpha_\lambda\big(N\backslash \mathrm{Int}(E(S^{V^i})\cup W(C_{i+1})_1)\big) \subset \mathscr{O}^{i-1}(N,P). \tag{4.1}$$

This is the required homotopy α_λ. □

5. $\mathscr{O}^i$-regular map around singularities.

In what follows we denote, by σ, the section $\alpha_1 \in \Gamma_{\mathscr{O}}(N,P)$ in Lemma 4.2 which satisfies (4.2.1) to (4.2.4). In this section we construct an $\mathscr{O}^i$-regular map $\mathfrak{q}(\sigma,V^i)$ defined around $S^{V^i}(\sigma)$ by applying the versal unfolding developed in [**MaIV**]. Next we prepare lemmas which are used in Section 6 in the deformation of $\mathfrak{q}(\sigma,V^i)$ to an $\mathscr{O}$-regular map compatible with g_{i+1}.

We take a Riemannian metric on P, which induces the Riemannian metric on $S^{V^i}(\sigma)$. Let us choose a Riemannian metric on N which induces a metric of the normal bundle $\mathfrak{n}(\sigma,V^i)$ over $S^{V^i}(\sigma)$ such that

(i) $S^{V^i}(\sigma)$ is a Riemannian submanifold,
(ii) $K(S^{V^i}(\sigma))$ is orthogonal to $S^{V^i}(\sigma)$ in N.

For the section $\sigma \in \Gamma^{tr}_{\mathscr{O}}(N,P)$, we set $\mathscr{M}(S^{V^i}(\sigma)) = (\sigma|S^{V^i}(\sigma))^*(\mathscr{M}(V^i)^{(k-1)})$ and $\mathscr{M}(S^{V^i}(\sigma))^\bullet = (\sigma|S^{V^i}(\sigma))^*(\mathscr{M}(V^i)^{\bullet(k-1)})$. Let $c\in S^{V^i}(\sigma)$, $\sigma(c) = j^k_c f$ and $\pi^k_P(\sigma(c)) = y(c)$. Then an element of $\mathscr{M}(S^{V^i}(\sigma))^\bullet_c$ is expressed as

$$a_{r+1}(x^\bullet)\partial/\partial y_{r+1} + \cdots + a_p(x^\bullet)\partial/\partial y_p \tag{5.1}$$

where $a_i(x^\bullet) \in \mathfrak{m}_{x^\bullet}/\mathfrak{m}^k_{x^\bullet}$.

Let K and Q refer to $K(S^{V^i}(\sigma))$ and $Q(S^{V^i}(\sigma))$ respectively. Let $\mathfrak{n}(\sigma,V^i)/K$ refer to the orthogonal complement of K in $\mathfrak{n}(\sigma,V^i)$. We write $\mathfrak{n}(\sigma,V^i) = (\mathfrak{n}(\sigma,V^i)/K)\oplus K$. Let $E(S^{V^i})$ denote $\exp_N D_\varepsilon(\mathfrak{n}(\sigma,V^i))$.

Let us first define the smooth fiber map

$$q(\sigma,V^i)^{(1)} : E(S^{V^i}) \longrightarrow \mathrm{Im}(d_1(\sigma,V^i)|\mathfrak{n}(\sigma,V^i)) \quad \text{over } S^{V^i}(\sigma)$$

by $q(\sigma,V^i)^{(1)} = d_1(\sigma,V^i)\circ(\exp_N)^{-1}|E(S^{V^i})$. Note that $d_1(\sigma,V^i)$ vanishes on K and gives an isomorphism of $\mathfrak{n}(\sigma,V^i)/K$ onto $\mathrm{Im}(d_1(\sigma,V^i)|\mathfrak{n}(\sigma,V^i))$.

For a point $c\in S^{V^i}(\sigma)$ let $x^\# = (x_{n-\rho+1},\ldots,x_n)$ denote the normal coordinates of $E(S^{V^i})_c$ such that $\{\partial/\partial x_i\}$ for $n-\rho+1\le i\le r$ and $\{\partial/\partial x_i\}$ for $r+1\le i\le n$

constitute the orthonormal bases of $\mathfrak{n}(\sigma, V^i)_c/K_c$ and K_c respectively. Let $e(Q_c)$ denote $\exp_{P,y}(Q_c)$ and let $(y_{r+1}, \ldots, y_p)$ be the normal coordinates of $e(Q_c)$ such that $\{\partial/\partial y_i\}$ constitute the orthonormal basis of Q_c.

Let $\mathscr{D}\sigma$ denote the composite

$$\big(\sigma|S^{V^i}(\sigma)\big)^*\big(p_{\mathscr{M}^\bullet} \circ \pi^{k-1}_{\theta,T} \circ \Pi^k_{\mathfrak{f}} \circ d\sigma|\mathfrak{n}(\sigma, V^i)\big) : \mathfrak{n}(\sigma, V^i) \longrightarrow \mathscr{M}\big(S^{V^i}(\sigma)\big)^\bullet$$

which is a monomorphism over $S^{V^i}(\sigma)$ by the transversality of σ to $V^i(N,P)$.

Then we define $q(\sigma, V^i)^{(2)} : E(S^{V^i}) \to Q$ over $S^{V^i}(\sigma)$ by

$$q(\sigma, V^i)^{(2)}_c(x^\#) = j^k f^\bullet_c(x^\bullet) + \sum_{j=n-\rho+1}^{r} x_j \mathscr{D}\sigma\bigg(\frac{\partial}{\partial x_j}\bigg)_c(x^\bullet). \tag{5.2}$$

We have defined $q(\sigma, V^i)^{(2)}$ by using the orthonormal bases of $\mathfrak{n}(\sigma, V^i)$ and Q_c. However, the coordinate changes of $\mathfrak{n}(\sigma, V^i)$ and Q_c are linear and so, $q(\sigma, V^i)^{(2)}$ is a well defined smooth fiber map. Let us consider the direct sum decomposition $(\pi^k_P \circ \sigma|S^{V^i})^*(TP) = T(S^{V^i}) \oplus d_1(\sigma, V^i)(\mathfrak{n}(\sigma, V^i)) \oplus Q$. Define the smooth fiber map $q(\sigma, V^i) : E(S^{V^i}) \to d_1(\sigma, V^i)(\mathfrak{n}(\sigma, V^i)) \oplus Q(S^{V^i}(\sigma))$ by

$$q(\sigma, V^i) = q(\sigma, V^i)^{(1)} + q(\sigma, V^i)^{(2)} \quad \text{over } S^{V^i}(\sigma). \tag{5.3}$$

We define the smooth map $\mathfrak{q}(\sigma, V^i) : E(S^{V^i}) \to P$ by

$$\mathfrak{q}(\sigma, V^i)_c(x^\#) = \exp_{P,c} \circ \big(\pi^k_P \circ \sigma|S^{V^i}\big)^{TP} \circ q(\sigma, V^i)(x^\#). \tag{5.4}$$

Lemma 5.1. *Let $\varepsilon : S^{V^i}(\sigma) \to \boldsymbol{R}$ be a sufficiently small positive function. Let $V^i(n,p)$ be given as in Theorem* 3.2. *Under the above notation, the map $\mathfrak{q}(\sigma, V^i)$ is an $\mathscr{O}^i$-regular map such that $j^k\mathfrak{q}(\sigma, V^i)$ is transverse to $V^i(E(S^{V^i}), P)$ and $S^{V^i}(\sigma) = S^{V^i}(j^k\mathfrak{q}(\sigma, V^i))$.*

Proof. In the proof we write $\mathfrak{q}$ for $\mathfrak{q}(\sigma, V^i)$. Let us compare the local ring $Q_k(\sigma(c))$ and $Q_k(j^k_c\mathfrak{q})$. By the definition of $f^\bullet$, $Q_k(j^k_c f)$ and $Q_k(j^k_c\mathfrak{q})$ are isomorphic to $Q_k(j^k_c f^\bullet)$. Hence, $Q_k(j^k_c f)$ and $Q_k(j^k_c\mathfrak{q})$ are isomorphic. It follows from [**MaIV**, Theorem 2.1] that $\mathfrak{q}(c) \in \mathscr{K}^{\sigma(c)}(E(S^{V^i}), P) \subset V^i(E(S^{V^i}), P)$ for any point $c \in S^{V^i}$. Since $\mathscr{O}(n,p)$ is open, it follows that if ε is sufficiently small, then $\mathfrak{q}(E(S^{V^i})) \subset \mathscr{O}^i(N,P)$.

It is enough for the transversality of $j^k\mathfrak{q}(\sigma, V^i)$ to show that for $n - \rho + 1 \le j \le n$,

$$\big(j^k\mathfrak{q}|S^{V^i}(j^k\mathfrak{q})\big)^*\big(p_{\nu(V^i)} \circ d(j^k\mathfrak{q})\big)(\partial/\partial x_j) = \big(\sigma|S^{V^i}(\sigma)\big)^*(p_{\nu(V^i)} \circ d\sigma)(\partial/\partial x_j)$$

($j^k\mathfrak{q}|V^i(N,P)$ and $\sigma|V^i(N,P)$ are different in general). By Lemmas 2.1, 2.3 and (2.12) this follows from the following. For $r + 1 \le j \le n$, we have that

$$\begin{aligned}
\mathscr{D}\sigma_c\left(\frac{\partial}{\partial x_j}\right)(x^\bullet) &= \left(\sigma|S^{V^i}(\sigma)\right)^*\left(p_{\mathscr{M}^\bullet}\circ\pi^{k-1}_{\theta,T}\circ\Pi^k_{\mathfrak{f}}\circ d\sigma\left(\frac{\partial}{\partial x_j}\right)\right)(x^\bullet)\\
&= \left(\sigma|S^{V^i}(\sigma)\right)^*\left(p_{\mathscr{M}^\bullet}\circ\pi^{k-1}_{\theta,T}\circ\pi^{\mathfrak{f}}\circ \boldsymbol{d}\left(\sigma(c),\frac{\partial}{\partial x_j}\right)\right)(x^\bullet)\\
&= \left(\sigma|S^{V^i}(\sigma)\right)^*\left(p_{\mathscr{M}^\bullet}\circ tf\left(\frac{\partial}{\partial x_j}\right)\right)(x^\bullet)\\
&= tf^\bullet\left(\frac{\partial}{\partial x_j}\right)(x^\bullet)\\
&= \sum_{\ell=r+1}^{p}\left(\frac{\partial y_\ell\circ f^\bullet(x^\bullet)}{\partial x_j}\right)\frac{\partial}{\partial y_\ell}\\
&= \mathscr{D}(j^k\mathfrak{q})_c\left(\frac{\partial}{\partial x_j}\right)(x^\bullet).
\end{aligned}$$

For $n-\rho+1\le j\le r$, we have by (5.2) that

$$\mathscr{D}(j^k\mathfrak{q})_c\left(\frac{\partial}{\partial x_j}\right)(x^\bullet) = \frac{\partial}{\partial x_j}\left(x_j\mathscr{D}\sigma_c\left(\frac{\partial}{\partial x_j}\right)(x^\bullet)\right) = \mathscr{D}\sigma_c\left(\frac{\partial}{\partial x_j}\right)(x^\bullet). \qquad \square$$

Here we give a lemma necessary in the process of modifying $\mathfrak{q}(\sigma,V^i)$ to be compatible with g_{i+1}. Let $\pi_E: E(S^{V_i})\to S^{V_i}$ be the canonical projection.

LEMMA 5.2. *Let $f_j: E(S^{V_i})\to P$ $(j=1,2)$ be $\mathscr{O}^i$-regular maps such that, for any $c\in S^{V_i}$,*

(i) *$f_1|S^{V_i}=f_2|S^{V_i}$, which are immersions and $(df_1)_c=(df_2)_c$,*
(ii) *j^kf_j is transverse to $V^i(E(S^{V_i}),P)$ and $S^{V_i}=S^{V^i}(j^kf_1)=S^{V^i}(j^kf_2)$,*
(iii) *$K(S^{V^i}(j^kf_1))_c=K(S^{V^i}(j^kf_2))_c$, which are tangent to $\pi_E^{-1}(c)$,*
(iv) *$Q(S^{V^i}(j^kf_1))_c=Q(S^{V^i}(j^kf_2))_c$,*
(v) *$j^k_cf^\bullet_1(x^\bullet)=j^k_cf^\bullet_2(x^\bullet)$,*
(vi) *the two homomorphisms*

$$\mathscr{D}\big(j^kf_j\big):\mathfrak{n}(\sigma,V^i)\longrightarrow\mathscr{M}\big(S^{V^i}(j^kf_j)\big)^\bullet$$

for $j=1,2$ coincide with each other.

Let $\eta: S^{V_i}\to[0,1]$ be any smooth function. Let $\varepsilon: S^{V_i}\to\boldsymbol{R}$ in the definition of $E(S^{V_i})$ be a sufficiently small positive smooth function. We define $\boldsymbol{f}^\eta: E(S^{V_i})\to P$ by

$$\boldsymbol{f}^\eta(x_c)=\exp_{P,f_1(c)}\left((1-\eta(c))\exp^{-1}_{P,f_1(c)}(f_1(x_c))+\eta(c)\exp^{-1}_{P,f_2(c)}(f_2(x_c))\right)$$

for any $x_c\in\pi_E^{-1}(c)$ with $\|x_c\|\le\varepsilon(c)$. Then the map $\boldsymbol{f}^\eta$ is a well-defined $\mathscr{O}^i$-regular map such that for $j=1,2$, and for any $c\in S^{V_i}$,

(5.2.1) $\boldsymbol{f}^\eta|S^{V_i} = f_j|S^{V_i}$ *and* $(d\boldsymbol{f}^\eta)_c = (df_i)_c$,
(5.2.2) $j^k\boldsymbol{f}^\eta$ *is transverse to* $V^i(E(S^{V_i}), P)$ *and* $S^{V_i} = S^{V^i}(j^k\boldsymbol{f}^\eta)$,
(5.2.3) $K(S^{V^i}(j^k\boldsymbol{f}^\eta))_c = K(S^{V^i}(j^k f_j))_c$, *which is tangent to* $\pi_E^{-1}(c)$,
(5.2.4) $Q(S^{V^i}(j^k\boldsymbol{f}^\eta))_c = Q(S^{V^i}(j^k f_j))_c$,
(5.2.5) $j_c^k(\boldsymbol{f}^\eta)^\bullet(x^\bullet) = j_c^k f_j^\bullet(x^\bullet)$,
(5.2.6) *the homomorphism*

$$\mathscr{D}(j^k\boldsymbol{f}^\eta) : \mathfrak{n}(\sigma, V^i) \longrightarrow \mathscr{M}\big(S^{V^i}(j^k\boldsymbol{f}^\eta)\big)^\bullet$$

coincides with the homomorphisms $\mathscr{D}(j^k f_j|S^{V_i})$ $(j = 1, 2)$ *in* (vi).

Proof. The local coordinates of

$$\exp_{E(S^{V_i}),c}\big(K(S^{V^i}(j^k f_j)_c)\big) \quad \text{and} \quad \exp_{P,f_j(c)}\big(Q(S^{V^i}(j^k f_j)_c)\big)$$

are independent of coordinates of S^{V_i}, where $Q(S^{V^i}(j^k f_j)_c)$ is regarded as the orthogonal complement of $\mathrm{Im}(d_1(j^k f_j, V^i)_c)$ in $T_{f_j(c)}P$. For $\boldsymbol{v}_c \in \mathfrak{n}(\sigma, V^i)_c$, $d\boldsymbol{f}^\eta(\boldsymbol{v}_c)$ is equal to

$$\begin{aligned} & d(\exp_{P,f_1(c)}) \circ \big((1-\eta(c))d(\exp^{-1}_{P,f_1(c)} \circ f_1) + \eta(c)d(\exp^{-1}_{P,f_2(c)} \circ f_2)\big)(\boldsymbol{v}_c) \\ & = \big((1-\eta(c))df_1 + \eta(c)df_2\big)(\boldsymbol{v}_c) \\ & = (1-\eta(c))df_1(\boldsymbol{v}_c) + \eta(c)df_2(\boldsymbol{v}_c) \\ & = df_j(\boldsymbol{v}_c). \end{aligned}$$

Hence, we have (5.2.1), (5.2.3) and (5.2.4). From (v), (5.2.5) is evident.

We have the normal coordinates $(x_1, \ldots, x_{n-\rho})$ and $x^\# = (x_{n-\rho+1}, \ldots, x_n)$ of (S^{V_i}, c) and $(E(S^{V_i})_c, c)$ respectively. Let $(x_1, \ldots, x_r, y_{r+1}, \ldots, y_p)$ be the normal coordinates of (P, c) as before. Let $\mathbf{0}_n$ and $\mathbf{0}_p$ be the coordinates of c and $y(c)$ respectively. Let $v(t)$ be the geodesic curve of $\boldsymbol{v}_c$ in $E(S^{V_i})_c$ such that $(dv|_{t=0})(d/dt) = \boldsymbol{v}_c \in E(S^{V_i})_c$ and $v(0) = c$. For a map germ $g : (E(S^{V_i}), c) \to (P, f_j(c))$, set

$$F_t^g(x) = \ell(g(v(t)), \mathbf{0}_p) \circ g \circ \ell(\mathbf{0}_n, v(t))(x) = g(x + v(t)) - g(v(t)).$$

Since $F_t^g(\mathbf{0}_n) = \mathbf{0}_p$, F_t^g defines the map germs $(E(S^{V_i}), c) \to (P, y(c))$ with the parameter t and $F_x^g : ((-1,1), 0) \to P$ defined by $F_x^g(t) = F_t^g(x)$. Then we have $j_c^{k-1}F^g : ((-1,1),0) \to J^{k-1}_{c,f_j(c)}(N,P)$ defined by $j_c^{k-1}F^g(t) = j_c^{k-1}F_t^g$.

By the definition of $\pi^{\mathfrak{f}}$ we have that

$$\pi^{\mathfrak{f}}_{j^{k-1}\boldsymbol{f}^\eta(c)} \circ d_c(j^{k-1}\boldsymbol{f}^\eta)(\boldsymbol{v}_c) = \big(d(j_c^{k-1}F^{\boldsymbol{f}^\eta})|_{t=0}\big)(d/dt).$$

Furthermore, $\pi^{k-1}_{\theta,T} \circ \pi^{\mathfrak{f}}_{j^{k-1}\boldsymbol{f}^\eta(c)} \circ d_c(j^{k-1}\boldsymbol{f}^\eta)(\boldsymbol{v}_c)$ is represented by the germ

$$\big(dF_x^{\boldsymbol{f}^\eta}|_{t=0}\big)(d/dt) : (N, c) \longrightarrow TP$$

covering $\boldsymbol{f}^\eta$ as in Remark 2.2. The germ $(dF_x^{\boldsymbol{f}^\eta}|_{t=0})(d/dt)$ is equal to

$$\begin{aligned}
&\big(d(\boldsymbol{f}^\eta(x+v(t)) - \boldsymbol{f}^\eta(v(t)))\big)(dv(t)/dt)|_{t=0}\\
&\quad = \big((1-\eta(c))df_1(x+v(t))|_{t=0} + \eta(c)df_2(x+v(t))|_{t=0}\big)(\boldsymbol{v}_c)\\
&\qquad - \big((1-\eta(c))df_1(v(t))|_{t=0} + \eta(c)df_2(v(t))|_{t=0}\big)(\boldsymbol{v}_c)\\
&\quad = (1-\eta(c))\big((df_1(x+v(t)) - df_1(v(t)))|_{t=0}\big)(\boldsymbol{v}_c)\\
&\qquad + \eta(c)\big((df_2(x+v(t)) - df_2(v(t)))|_{t=0}\big)(\boldsymbol{v}_c)\\
&\quad = (1-\eta(c))\big(dF_x^{f_1}|_{t=0}\big)(d/dt)|_{t=0} + \eta(c)\big(dF_x^{f_2}|_{t=0}\big)(d/dt).
\end{aligned}$$

Then $p_{\mathscr{M}^\bullet}\circ\pi_{\theta,T}^{k-1}\circ\pi^{\mathfrak{f}}_{j^{k-1}\boldsymbol{f}^\eta(c)}\circ d_c(j^{k-1}\boldsymbol{f}^\eta)(\boldsymbol{v}_c)$ is represented by

$$\begin{aligned}
&\big(d(p_{Q_c}\circ F_x^{\boldsymbol{f}^\eta}|E(S^{V_i})_c)|_{t=0}\big)(d/dt)\\
&\quad = \big((1-\eta(c))d\big(p_{Q_c}\circ F_x^{f_1}|E(S^{V_i})_c\big)|_{t=0} + \eta(c)d\big(p_{Q_c}\circ F_x^{f_2}|E(S^{V_i})_c\big)|_{t=0}\big)(d/dt).
\end{aligned}$$

By the definition of $p_{\mathscr{M}^\bullet}\circ\pi_{\theta,T}^{k-1}\circ\pi^{\mathfrak{f}}$, we have

$$\mathscr{D}(j^k\boldsymbol{f}^\eta) = (1-\eta(c))\mathscr{D}\big(j^kf_1\big) + \eta(c)\mathscr{D}\big(j^kf_2\big) = \mathscr{D}\big(j^kf_j\big)$$

for $j=1,2$. This implies (5.2.2) and (5.2.6). This completes the proof. $\square$

Let $\mathfrak{q}$ denote $\mathfrak{q}(\sigma,V^i) : E(S^{V^i}) \to P$ in (5.4). Now we modify $\mathfrak{q}$ to be compatible with g_{i+1}. Let $\eta : S^{V^i} \to \boldsymbol{R}$ be a smooth function such that

(i) $0 \le \eta(c) \le 1$ for $c \in S^{V^i}$,
(ii) $\eta(c) = 0$ for c in a small neighborhood of $S^{V^i}\cap W(C_{i+1})_1$ within $S^{V^i}\setminus W(C_{i+1})_2$,
(iii) $\eta(c) = 1$ for $c \in S^{V^i}\setminus W(C_{i+1})_2$.

Then define the map $G : E(S^{V^i})\cup W(C_{i+1})_1 \to P$ by

- if $x \in W(C_{i+1})_1$, then $G(x) = g_{i+1}(x)$,
- if $x_c \in E(S^{V^i})|_{S^{V^i}\setminus \mathrm{Int}(W(C_{i+1})_2)}$, then $G(x_c) = \mathfrak{q}(x_c)$,
- if $x_c \in E(S^{V^i})|_{S^{V^i}\cap W(C_{i+1})_2}$, then $G(x_c)$ is equal to

$$\exp_{P,\mathfrak{q}(c)}\big((1-\eta(c))\exp^{-1}_{P,\mathfrak{q}(c)}(g_{i+1}(x_c)) + \eta(c)\exp^{-1}_{P,\mathfrak{q}(c)}(\mathfrak{q}(x_c))\big),$$

where δ is so small that $G(x)$ is well-defined and that $E(S^{V^i})\cap W(C_{i+1})_1 \subset \pi_E^{-1}(S^{V^i}\cap W(C_{i+1})_2)$ holds.

By Lemmas 5.1 and 5.2 we have the following corollary.

Corollary 5.3. *The above map G is an $\mathscr{O}$-regular map defined on $E(S^{V^i})\cup W(C_{i+1})_1$ such that*

(5.3.1) *j^kG is transverse to $V^i(N,P)$ and $(G|E(S^{V^i}))^{-1}(V^i(N,P)) = S^{V^i}$,*
(5.3.2) *$G|S^{V^i} = \mathfrak{q}|S^{V^i} = \pi^k_P \circ \sigma|S^{V^i}$ and $(dG)_c = (d\mathfrak{q})_c$,*
(5.3.3) *$G|E(S^{V^i})$ is $\mathscr{O}^i$-regular,*
(5.3.4) *$K(S^{V^i}(j^kG)) = K(S^{V^i}(j^k\mathfrak{q})) = K$, $Q(S^{V^i}(j^kG)) = Q(S^{V^i}(j^k\mathfrak{q})) = Q$,*
(5.3.5) *if we write $\sigma(c) = j^k_c(f_{\sigma(c)})$, then*

$$\big(j^k_c f^\bullet_{\sigma(c)}\big)(x^\bullet) = j^k_c\mathfrak{q}^\bullet(x^\bullet) = j^k_c G^\bullet(x^\bullet),$$

(5.3.6) *the following three homomorphisms coincide with each other.*

$$\mathscr{D}(j^kG) = \mathscr{D}(j^k\mathfrak{q}) = \mathscr{D}\sigma : \mathfrak{n}(\sigma, V^i) \to \mathscr{M}(S^{V^i}(\sigma))^\bullet.$$

Let us recall the additive structure of $J^k(N,P)$ in (1.2). Then we define the homotopy $\kappa_\lambda : S^{V^i} \to J^k(N,P)$ by

$$\kappa_\lambda(c) = (1-\lambda)\sigma(c) + \lambda j^kG(c) \quad \text{covering } \pi^k_P \circ \sigma|S^{V^i} : S^{V^i} \to P,$$

where $\pi^k_P \circ \sigma|S^{V^i}$ is the immersion.

Lemma 5.4. *The homotopy κ_λ is a map of S^{V^i} to $V^i(N,P)$.*

Proof. It follows from Corollary 5.3, (5.3.1) to (5.3.6) that $K(S^{V^i}(\kappa_\lambda)) = K$ and $Q(S^{V^i}(\kappa_\lambda)) = Q$ and that if we write $\kappa_\lambda(c) = j^k_c(f_\lambda)$, then $(j^k_c f^\bullet_\lambda)(x^\bullet) = (j^k_c f^\bullet_{\sigma(c)})(x^\bullet) = j^k_c G^\bullet(x^\bullet)$. By the definition of local rings we have $Q_k(j^k_c f) \approx Q_k(j^k_c f^\bullet)$, $Q_k(j^k_c f_\lambda) \approx Q_k(j^k_c f^\bullet_\lambda)$ and $Q_k(j^k_c G) \approx Q_k(j^k_c G^\bullet)$.

Since $V^i(N,P)$ is $\mathscr{K}$-invariant, it follows from [**MaIV**, Theorem 2.1] that $\kappa_\lambda(c)$ lies in $V^i_{c,y(c)}(N,P)$ for any λ and any $c \in S^{V^i}$, where $y(c) = \pi^k_P \circ \sigma(c)$. □

The proof of the following lemma is elementary, and so is left to the reader.

Lemma 5.5. *Let (Ω, Σ) be a pair consisting of a manifold and its submanifold of codimension ρ. Let $\varepsilon : S^{V_i} \to \boldsymbol{R}$ be a sufficiently small positive smooth function. Let $h : E(S^{V_i}) \to (\Omega, \Sigma)$ be a smooth map such that $S^{V_i} = h^{-1}(\Sigma)$ and that h is transverse to Σ. Then there exists a smooth homotopy $h_\lambda : (E(S^{V_i}), S^{V_i}) \to (\Omega, \Sigma)$ between h and $\exp_\Omega \circ dh \circ (\exp_N|\mathfrak{n}(\sigma, V^i))^{-1}|E(S^{V_i})$ such that*
(5.4.1) *$h_\lambda|S^{V_i} = h_0|S^{V_i}$, $S^{V_i} = h_\lambda^{-1}(\Sigma) = h_0^{-1}(\Sigma)$ for any λ,*
(5.4.2) *h_λ is smooth and is transverse to Σ for any λ,*
(5.4.3) *$h_0 = h$ and $h_1(x_c) = \exp_{\Omega,h(c)} \circ dh \circ (\exp_N|\mathfrak{n}(\sigma, V^i))^{-1}(x_c)$ for $c \in S^{V_i}$ and $x_c \in E(S^{V_i})_c$.*

6. Proof of Theorem 3.2.

In this section we deform $\mathfrak{q}(\sigma, V^i)$ to an $\mathscr{O}$-regular map G compatible with g_{i+1}. By the definition of the deformation we can construct a homotopy between σ and j^kG around $S^{V^i}(\sigma)$, which is extendable to a required homotopy to the whole space N.

Let us take closed neighborhoods $U(C_{i+1})_j$ $(j = 1, 2)$ of $U(C_{i+1})$ in the interior of $W(C_{i+1})_1$ with $U(C_{i+1})_1 \subset \mathrm{Int}U(C_{i+1})_2$ such that $U(C_{i+1})_j$ are submanifolds of dimension n with boundary $\partial U(C_{i+1})_j$ meeting transversely with $S^{V^i}(\sigma)$.

Proof of Theorem 3.2. Deform $s \in \Gamma^{tr}_{\mathscr{O}}(N, P)$ in Theorem 3.2 as before to a section $\sigma \in \Gamma_{\mathscr{O}}(N, P)$ as in Lemma 4.2 which satisfies (4.2.1), (4.2.2) and (4.2.3) where α_1 is replaced by σ. Set $S^{V^i} = S^{V^i}(\sigma)$, $K = K(S^{V^i}(\sigma))$ and $Q = Q(S^{V^i}(\sigma))$. Let $E(S^{V_i}) = \exp_N(D_{\delta\circ\sigma}(\mathfrak{n}(\sigma, V^i)))$, where $\delta : V^i(N, P) \to \boldsymbol{R}$ is a sufficiently small positive function which is constant on $\sigma(S^{V^i}(\sigma)\backslash\mathrm{Int}U(C_{i+1}))$.

It suffices for the proof of Theorem 3.2 to prove the following assertion (**A**). In fact, we obtain a required homotopy s_λ in Theorem 3.2 by pasting the homotopies α_λ in Lemma 4.2 and H_λ in (**A**).

(**A**) There exists a homotopy H_λ relative to $U(C_{i+1})_1$ in $\Gamma_{\mathscr{O}}(N, P)$ with $H_0 = \sigma$ and $H_1 \in \Gamma^{tr}_{\mathscr{O}}(N, P)$ satisfying the following (1), (2) and (3).

(1) H_λ is transverse to $V^i(N, P)$ and $S^{V^i}(H_\lambda) = S^{V^i}$ for any λ.

(2) We have an $\mathscr{O}$-regular map G which is defined on a neighborhood of $E(S^{V^i}) \cup U(C_{i+1})_1$ to P such that $j^kG = H_1$ on $E(S^{V^i}) \cup U(C_{i+1})_1$ and that $G(E(S^{V^i})) \subset \mathscr{O}^i(N, P)$.

(3) $H_\lambda(N\backslash\mathrm{Int}(E(S^{V^i}) \cup U(C_{i+1})_1)) \subset \mathscr{O}^{i-1}(N, P)$.

Let us prove (**A**). We use the Riemannian metrics which are chosen in the beginning of Section 5. The map $\exp_P \circ(\pi^k_P \circ \sigma|S^{V^i})^{TP}|D_\gamma(Q)$ is an immersion for some small positive function γ. We express a point of $E(S^{V^i})$ as x_c, where $c \in S^{V^i}$ and $\|x_c\| \leq \delta(\sigma(c))$.

It follows from Corollary 5.3 that G is an $\mathscr{O}$-regular map defined on $E(S^{V^i}) \cup W(C_{i+1})_1$. It is known that the Riemannian metrics on N and P induce the Riemannian metric on $J^k(N, P)$ by using (1.2) (see, for example, [**An6**, Section 3]). Let h^1_1 and h^3_0 be the maps $(E(S^{V^i}), S^{V^i}) \to (\mathscr{O}^i(N, P), V^i(N, P))$ defined by

$$\begin{aligned} h^1_1(x_c) &= \exp_{\mathscr{O}(N,P),\sigma(c)} \circ d_c\sigma \circ (\exp_{N,c})^{-1}(x_c), \\ h^3_0(x_c) &= \exp_{\mathscr{O}(N,P),j^kG(c)} \circ d_c(j^kG) \circ (\exp_{N,c})^{-1}(x_c). \end{aligned} \tag{6.1}$$

By applying Lemma 5.5 to the sections σ and h^1_1 (respectively h^3_0 and j^kG) we first obtain a homotopy h^1_λ (respectively h^3_λ) $\in \Gamma_{\mathscr{O}^i}(E(S^{V^i}), P)$ between $h^1_0 = \sigma$ and h^1_1 on $E(S^{V^i})$ (respectively between h^3_0 and $h^3_1 = j^kG$) satisfying the properties (5.5.1), (5.5.2) and (5.5.3) of Lemma 5.5.

Next we construct a homotopy of bundle maps $E(S^{V^i}) \to \nu(V^i(N, P))$ covering $\kappa_\lambda : S^{V^i} \to V^i(N, P)$ in Lemma 5.4 using a homotopy between $d\sigma|\mathfrak{n}(\sigma, V^i)$ and $d(j^kG)|\mathfrak{n}(\sigma, V^i)$. By the equalities of the homomorphisms in Corollary 5.3, (5.3.6), we obtain a homotopy of bundle maps

$$\kappa^{E,\mathscr{M}}_\lambda : \mathfrak{n}(\sigma, V^i) \to \mathscr{M}\big(S^{V^i}(\sigma)\big)^{\bullet} \xrightarrow{(\kappa_\lambda)^{\mathscr{M}(V^i)^{\bullet(k-1)}}} \mathscr{M}(V^i)^{\bullet(k-1)}$$

covering κ_λ as the composite $(\kappa_\lambda)^{\mathscr{M}(V^i)^{\bullet(k-1)}} \circ \mathscr{D}\sigma$. Let $\widetilde{\kappa_\lambda}$ denote the composite $p^{\mathscr{M}}_\nu \circ$

$\kappa_\lambda^{E,\mathscr{M}}$, where $p_\nu^{\mathscr{M}}$ is the projection in (2.9). Then $\widetilde{\kappa_\lambda}$ is a bundle map between the ρ-dimensional vector bundles covering κ_λ. Since the composite $p_\nu^{\mathscr{M}} \circ p_{\mathscr{M}^\bullet} \circ \pi_{\theta,T}^{k-1} \circ \pi^{\mathfrak{f}}$ is equal to the canonical projection $p_{\nu(V^i)}$ by Lemma 2.1, we have

$$\begin{aligned}\widetilde{\kappa_0} &= p_\nu^{\mathscr{M}} \circ \left(\sigma|S^{V^i}\right)^{\mathscr{M}(V^i)^{\bullet(k-1)}} \circ \mathscr{D}\sigma \\ &= p_\nu^{\mathscr{M}} \circ p_{\mathscr{M}^\bullet} \circ \pi_{\theta,T}^{k-1} \circ \Pi_{\mathfrak{f}}^k \circ d\sigma|\mathfrak{n}(\sigma, V^i) \\ &= p_{\nu(V^i)} \circ d\sigma|\mathfrak{n}(\sigma, V^i)\end{aligned}$$

and $\widetilde{\kappa_1} = p_{\nu(V^i)} \circ d(j^kG)|\mathfrak{n}(\sigma, V^i)$ similarly.

We define a homotopy $h_\lambda^2 : (E(S^{V^i}), S^{V^i}) \to (\mathscr{O}^i(N,P), V^i(N,P))$ covering κ_λ by

$$h_\lambda^2(x_c) = \exp_{\mathscr{O}(N,P),\sigma(c)} \circ \widetilde{\kappa_\lambda} \circ (\exp_{N,c})^{-1}(x_c),$$

where $h_0^2(x_c) = h_1^1(x_c)$, $h_1^2(x_c) = h_0^3(x_c)$ on $E(S^{V^i})$. Since $h_0^1(x_c) = h_1^3(x_c) = \sigma(x_c)$ for $x_c \in W(C_{i+1})_1$, we may assume in the construction of h_λ^1, h_λ^2 and h_λ^3 that if $x_c \in W(C_{i+1})_1$, then

$$h_\lambda^2(x_c) = h_0^2(x_c) = h_1^2(x_c) \text{ and } h_\lambda^1(x_c) = h_{1-\lambda}^3(x_c) \text{ for any } \lambda. \tag{6.2}$$

Let $h'_\lambda \in \Gamma_{\mathscr{O}^i}(E(S^{V^i}), P)$ be the homotopy which is obtained by pasting h_λ^1, h_λ^2 and h_λ^3. The homotopies h_λ^1 and h_λ^3 are not homotopies relative to $E(S^{V^i}) \cap W(C_{i+1})_1$ in general. By using the above properties and (6.2) about h_λ^1, h_λ^2 and h_λ^3, we can modify h'_λ to a smooth homotopy $h_\lambda \in \Gamma_{\mathscr{O}^i}(E(S^{V^i}), P)$ with $\pi_P^k \circ h_\lambda(c) = \pi_P^k \circ \sigma(c)$ such that

(4) $h_\lambda(x_c) = h_0(x_c) = \sigma(x_c)$ for any λ and $x_c \in E(S^{V^i}) \cap U(C_{i+1})_2$,

(5) $h_0(x_c) = \sigma(x_c)$ for any $x_c \in E(S^{V^i})$,

(6) $h_1(x_c) = j^kG(x_c)$ for any $x_c \in E(S^{V^i})$,

(7) h_λ is transverse to $V^i(N,P)$ and $h_\lambda^{-1}(V^i(N,P)) = S^{V^i}$.

Since $G(E(S^{V^i}) \cup W(C_{i+1})_1 \backslash C_{i+1}) \subset \mathscr{O}^i(N,P)$ and j^kG is transverse to $V^i(N,P)$, it follows from [**G-G**, Ch. II, Corollary 4.11] that there exists a homotopy G_λ of $\mathscr{O}$-regular maps $E(S^{V^i}) \cup U(C_{i+1})_2 \to P$ relative to $U(C_{i+1})_2$ with $G_0 = G$ such that

$$j^kG_\lambda^{-1}(\mathscr{O}(N,P) \backslash \mathscr{O}^i(N,P)) \subset \mathrm{Int}\big(\exp_N(D_{(1/2)\delta\circ\sigma}(\mathfrak{n}(\sigma, V^i))) \cup U(C_{i+1})_2\big),$$

that j^kG_λ is transverse to $V^i(N,P)$ for any λ and that j^kG_1 is transverse to $V^j(N,P)$ for all j.

By using (4)–(7), we can extend h_λ to the homotopy $H'_\lambda \in \Gamma_{\mathscr{O}}(E(S^{V^i}) \cup U(C_{i+1})_2, P)$ defined by

$$\begin{aligned}&H'_\lambda|E(S^{V^i}) = h_{2\lambda} && (0 \le \lambda \le 1/2), \\ &H'_\lambda|\big(E(S^{V^i}) \cup U(C_{i+1})_2\big) = j^kG_{2\lambda-1} && (1/2 \le \lambda \le 1), \\ &H'_\lambda|U(C_{i+1})_2 = \sigma|U(C_{i+1})_2 && (0 \le \lambda \le 1),\end{aligned}$$

such that $H'_\lambda(\partial(E(S^{V^i}) \cup U(C_{i+1})_2)) \subset \mathscr{O}^{i-1}(N,P)$. Furthermore, we slightly modify H'_λ to be smooth.

By the transversalities of H'_λ to $V^i(N,P)$ and of H'_1 to $V^j(N,P)$ for all j and the homotopy extension property to σ and H'_λ, we can extend H'_λ to a homotopy

$$H_\lambda : (N, S^{V^i}) \longrightarrow (\mathscr{O}(N,P), V^i(N,P))$$

relative to $U(C_{i+1})_1$ such that $H_0 = \sigma$, $H_1 \in \Gamma^{tr}_{\mathscr{O}}(N,P)$ and $H_1(N\backslash \mathrm{Int}(E(S^{V^i}) \cup U(C_{i+1})_2)) \subset \mathscr{O}^{i-1}(N,P)$. Then H_λ is the required homotopy in $\Gamma_{\mathscr{O}}(N,P)$ in the assertion (**A**). $\square$

7. $\mathscr{K}$-simple singularities.

Let z be a jet of $J^k(n,p)$. We say that z is *$\mathscr{K}$-k-simple* if there exists an open neighborhood U of z in $J^k(n,p)$ such that only a finite number of $\mathscr{K}$-orbits intersect with U. A $\mathscr{K}$-orbit $\mathscr{K}z$ of a $\mathscr{K}$-k-simple k-jet z is also called *$\mathscr{K}$-k-simple*.

Let W_j denote the subset consisting of all $z \in J^k(n,p)$ such that the codimensions of $\mathscr{K}z$ in $J^k(n,p)$ are not less than j. Let W^*_j denote the union of all irreducible components of W_j whose codimensions in $J^k(n,p)$ is less than j. The following lemma has been observed in [**MaV**, Section 7 and Proof of Theorem 8.1].

Lemma 7.1.
(i) *W_j is a closed algebraic subset of $J^k(n,p)$.*
(ii) *If we set $W'_j = W_j\backslash(W^*_j \cup W_{j+1})$, then W'_j is a Zariski locally closed subset of $J^k(n,p)$ of codimension j.*
(iii) *For any jet $z \in W'_j$, $\mathscr{K}z$ is open in W'_j.*
(iv) *W'_j consists of a finite number of $\mathscr{K}$-orbits.*

We define $\mathscr{K}$-k-simplicity for a jet in $J^k_{x,y}(N,P)$ similarly as in $J^k(n,p)$. A smooth map germ $f : (N,x) \to (P,y)$ is called $\mathscr{K}$-ℓ-determined if any smooth map germ $g : (N,x) \to (P,y)$ such that $j^\ell_x f = j^\ell_x g$ is $\mathscr{K}$-equivalent to f. If f is $\mathscr{K}$-ℓ-determined, then $j^\ell_x f$ is also called $\mathscr{K}$-ℓ-determined.

Proposition 7.2. *Let $k \geq p+1$ and $z \in J^k_{x,y}(N,P)$. If z is a singular $\mathscr{K}$-k-simple jet and* $\mathrm{codim}\mathscr{K}z \leq |n-p| + k - 2$, *then z is $\mathscr{K}$-$(k-1)$-determined.*

Proof. For $1 \leq \ell \leq k$, let $\pi^k_\ell : J^k_{x,y}(N,P) \to J^\ell_{x,y}(N,P)$ denote the canonical projection. Let $c_\ell(z)$ denote the codimension of the $\mathscr{K}$-orbit of $\pi^k_\ell(z)$ in $J^\ell_{x,y}(N,P)$. Since $\pi^k_\ell(z)$ is of rank $r < \min(n,p)$ and $\mathrm{codim}\Sigma^{n-r}(n,p) = (n-r)(p-r)$, we have $c_1 \geq (n-r)(p-r)$. Since $c_1 \leq c_2 \leq \cdots \leq c_k$, we have

$$|n-p| + 1 \leq c_1 \leq \cdots \leq c_k \leq |n-p| + k - 2.$$

There exists a number ℓ with $1 \leq \ell \leq k-2$ such that $c_\ell = c_{\ell+1}$. By applying [**MaIII**, Proposition 7.4] to the tangent spaces of $\mathscr{K}(\pi^k_\ell(z))$ and $\mathscr{K}(\pi^k_{\ell+1}(z))$, we have that

$$tf(\mathfrak{m}_x\theta(N)_x) + f^*(\mathfrak{m}_y)\theta(f)_x + \mathfrak{m}_x^{\ell+1}\theta(f)_x$$
$$= tf(\mathfrak{m}_x\theta(N)_x) + f^*(\mathfrak{m}_y)\theta(f)_x + \mathfrak{m}_x^{\ell+2}\theta(f)_x.$$

From the Nakayama Lemma it follows that

$$tf(\mathfrak{m}_x\theta(N)_x) + f^*(\mathfrak{m}_y)\theta(f)_x \supset \mathfrak{m}_x^{\ell+1}\theta(f)_x.$$

Therefore, z is $\mathscr{K}$-$(\ell+1)$-determined and so, $\mathscr{K}$-$(k-1)$-determined by [**W**, Theorem 1.2]. □

Corollary 7.3. *Let $k \geq p+2$. Let z be a singular $\mathscr{K}$-k-simple jet and* $\operatorname{codim}\mathscr{K}z \leq n$. *Then z is $\mathscr{K}$-$(k-1)$-determined and we have* $\mathscr{K}z = (\pi^k_{k-1})^{-1}(\mathscr{K}(\pi^k_{k-1}(z)))$.

Now we have the following Theorem.

Theorem 7.4. *Let $k \geq p+2$. Let $z = j^k_x f \in J^k_{x,y}(N,P)$ be $\mathscr{K}$-$(k-1)$-determined and $w = \pi^k_{k-1}(z)$. Then we have*

$$\boldsymbol{d}(\boldsymbol{K}(\mathscr{K}^z(N,P))_z) \cap \left(\pi^k_{k-1}|\mathscr{K}^z(N,P)\right)^*\left(T(\mathscr{K}^w(N,P))\right)_z = \{0\}.$$

Proof. For a vector $\boldsymbol{v} \neq \boldsymbol{0}$ let $\zeta^z_{\boldsymbol{v}}$ be the vector field in Lemma 2.3. Suppose that $\pi^{\mathfrak{f}} \circ \boldsymbol{d}(\boldsymbol{v}) \in T_w(\mathscr{K}^w_{x,y}(N,P))$. Then it follows from (2.4) and Corollary 7.3 that $tf(\boldsymbol{v}_U) \in tf(\mathfrak{m}_x\theta(N)_x) + f^*(\mathfrak{m}_y)\theta(f)_x$. It has been proved in the proof of [**MaIV**, Theorem 2.5] that $\boldsymbol{v}_U \in \mathfrak{m}_x\theta(N)_x$. This is a contradiction. □

The following theorem follows from Corollary 7.3, and Theorems 0.1 and 7.4.

Theorem 7.5. *Let k be an integer with $k \geq p+2$. Let $\mathscr{O}(n,p)$ be a nonempty open subset in $J^k(n,p)$ which consists of a finite number of $\mathscr{K}$-k-simple $\mathscr{K}$-orbits, and of $\Sigma^{n-p+1,0}(n,p)$ in addition in the case $n \geq p$. Then $\mathscr{O}(n,p)$ is an admissible open subset. In particular, Theorem 0.1 holds for $\mathscr{O}(n,p)$.*

Remark 7.6. In Theorem 7.5, if f is transverse to all singular $\mathscr{K}$-orbits, then the germ $f : (N,c) \to (P, f(c))$ is C^∞-stable in the sense of [**MaIV**]. This fact follows from [**Mar2**, Ch. XV, 5, Theorem].

Finally we give examples of open sets $\mathscr{O}(n,p)$ in $J^k(n,p)$ in Theorem 7.5. Let $k \gg n, p$.

(1) Let A_m, D_m and E_m denote the types of the singularities of function germs studied in [**Mo**] and [**Ar**]. We say that a smooth map germ $f : (\boldsymbol{R}^n, \boldsymbol{0}) \to (\boldsymbol{R}^p, \boldsymbol{0})$ has a singularity of type A_m, D_m or E_m, when f is $\mathscr{K}$-equivalent to one of the versal unfoldings $(\boldsymbol{R}^n, \boldsymbol{0}) \to (\boldsymbol{R}^p, \boldsymbol{0})$ of the following genotypes with respective singularities, where $n > p \geq 2$ in the case of types D_m and E_m.

$$(A_m) \quad \pm u^{m+1} \pm x_p^2 \pm \cdots \pm x_{n-1}^2 \ (m \geq 1),$$
$$(D_m) \quad u^2\ell \pm \ell^{m-1} \pm x_p^2 \pm \cdots \pm x_{n-2}^2 \ (m \geq 4),$$
$$(E_6) \quad u^3 \pm \ell^4 \pm x_p^2 \pm \cdots \pm x_{n-2}^2,$$
$$(E_7) \quad u^3 + u\ell^3 \pm x_p^2 \pm \cdots \pm x_{n-2}^2,$$
$$(E_8) \quad u^3 + \ell^5 \pm x_p^2 \pm \cdots \pm x_{n-2}^2.$$

Let $\mathfrak{a}_m$, $\mathfrak{d}_m$and $\mathfrak{e}_m$ denote the k-jets of the germs of types A_m, D_m and E_m of codimension $n-p+m \leq n$ in $J^k(n,p)$. Let $O(n,p)$ be a subset which consists of all regular jets and a number of $\mathscr{K}$-orbits $\mathscr{K}\mathfrak{a}_i$, $\mathscr{K}\mathfrak{d}_j$ and $\mathscr{K}\mathfrak{e}_h$ of codimensions $\leq n$. This subset $O(n,p)$ is an open subset of $J^k(n,p)$ if and only if the following three conditions are satisfied.

- (i) If $\mathscr{K}\mathfrak{a}_i \subset \mathscr{O}(n,p)$, then $\mathscr{K}\mathfrak{a}_\ell \subset \mathscr{O}(n,p)$ for all ℓ with $1 \leq \ell < i$.
- (ii) If $\mathscr{K}\mathfrak{d}_i \subset \mathscr{O}(n,p)$, then $\mathscr{K}\mathfrak{a}_\ell$ $(1 \leq \ell < i)$ and $\mathscr{K}\mathfrak{d}_\ell$ $(4 \leq \ell < i)$ are all contained in $\mathscr{O}(n,p)$.
- (iii) If $\mathscr{K}\mathfrak{e}_i \subset \mathscr{O}(n,p)$, then $\mathscr{K}\mathfrak{a}_\ell$ $(1 \leq \ell < i)$, $\mathscr{K}\mathfrak{d}_\ell$ $(4 \leq \ell < i)$ and $\mathscr{K}\mathfrak{e}_\ell$ $(6 \leq \ell < i)$ are all contained in $\mathscr{O}(n,p)$.

One can prove this assertion by the adjacency relation among the singularities of types A, D and E due to [**Ar**] (see, for example, the detailed proof in [**An5**]).

(2) Let $\mathscr{O}(n,p)$ denote the open subset in $J^k(n,p)$ which consists of all regular jets and $\mathscr{K}$-k-simple orbits.

(3) Let $n = p$. Let $\mathscr{O}(n,p)$ be the open subset in $J^k(n,p)$ which consists of all regular jets, the $\mathscr{K}$-orbits $\mathscr{K}\mathfrak{a}_m$ and the $\mathscr{K}$-orbits of the following types of codimensions $\leq n$ in [**MaVI**, Section 7].

$$\mathrm{I}_{a,b} \ : \boldsymbol{R}[[x,y]]/(xy, x^a + y^b), \quad b \geq a \geq 2,$$
$$\mathrm{II}_{a,b} : \boldsymbol{R}[[x,y]]/(xy, x^a - y^b), \quad b \geq a \geq 2,$$
$$\mathrm{III}_a \ : \boldsymbol{R}[[x,y]]/(x^2 + y^2, x^a), \quad a \geq 3.$$

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Yoshifumi Ando
Department of Mathematical Sciences
Faculty of Science
Yamaguchi University
Yamaguchi 753-8512, Japan
E-mail: andoy@yamaguchi-u.ac.jp