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The diffeomorphic types of the complements of arrangements in $\boldsymbol{CP}^3$ I: Point arrangements

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Abstract. For any arrangement of hyperplanes in $\boldsymbol{CP}^3$, we introduce the soul of this arrangement. The soul, which is a pseudo-complex, is determined by the combinatorics of the arrangement of hyperplanes. If the soul consists of a set of points (0-simplices) and a set of planes (2-simplices), then the arrangement is called point arrangement. In this paper, we give a sufficient combinatoric condition for two point arrangements of hyperplanes to be diffeomorphic to each other. In particular we have found sufficient condition on combinatorics for the point arrangement of hyperplanes whose moduli space is connected.

1. Introduction.

An arrangement of hyperplanes $\mathscr{A}^*$ in $\boldsymbol{CP}^n$ is a finite collection of hyperplanes of dimension $n-1$ in $\boldsymbol{CP}^n$. Associated with $\mathscr{A}^*$ is an open real $2n$-manifold, the complement $M(\mathscr{A}^*) = \boldsymbol{CP}^n - \bigcup_{H^* \in \mathscr{A}^*} H^*$. One of the central problems in this area is to decide to what extent the topology or differentiable structure of $M(\mathscr{A}^*)$ is determined by the combinatorial geometry of $\mathscr{A}^*$ and vice versa. It is well known that the combinatorial data of $\mathscr{A}^*$ is coded by $L(\mathscr{A}^*)$ which is the set of all intersections of elements of $\mathscr{A}^*$ partially ordered by reverse inclusion. In a series of papers, [**Fa1**], [**Fa2**] and [**Fa3**], Falk studied the question whether $L(\mathscr{A}^*)$ is a homotopic invariant. In [**Fa3**], Falk constructed two arrangements of hyperplanes in $\boldsymbol{CP}^2$, each of which has two triple points and nine double points, but their combinatorial data are different. The homotopic equivalence of their complements was shown in [**Fa3**]. Therefore $L(\mathscr{A}^*)$ is not a homotopic invariant. In 1993, Jiang and Yau ([**Ja-Ya2**], [**Ja-Ya4**]) proved that $L(\mathscr{A}^*)$ is indeed a topological invariant if $\mathscr{A}^*$ is an arrangement of hyperplanes in $\boldsymbol{CP}^2$. In their proof, they made use of some deep results of Waldhausen on three-manifolds. Indeed $L(\mathscr{A}^*)$ is no longer a topological invariant for arrangement of hyperplanes $\mathscr{A}^*$ in $\boldsymbol{CP}^n$, $n \geq 3$, (cf. [**Es-Fa**]).

The difficult and still unsolved problem is whether the topological or diffeomorphic type of complement $M(\mathscr{A}^*)$ of an arrangement is combinatorial in nature. In a famous preprint [**Ry**], G. Rybnikov announced the existence of two line arrangements $\mathscr{A}_1^*$ and $\mathscr{A}_2^*$ in $\boldsymbol{CP}^2$ which have the same combinatorics but whose complements $M(\mathscr{A}_1^*)$ and $M(\mathscr{A}_2^*)$ are not homeomorphic. Unfortunately there is no detail proof of the above result. Recently Bartolo, Ruber, Agustin and Buzunariz ([**B-R-A-B**]) prove the existence of complexified real arrangements with same combinatorics but different topology for

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complements of arrangements. The first step towards finding such pairs of arrangements involves finding combinatorics whose moduli space is not connected. On the other hand, if an arrangement $\mathscr{A}^*$ whose moduli space is connected, then Randell's lattice-isotopic theorem ([**Ra**]) implies that there is only one differentiable structure for any arrangement lying in this moduli space. For a central arrangement of hyperplanes $\mathscr{A}$ in $\boldsymbol{CP}^{n+1}$, one can define the underlying matroid $\mathscr{G}(\mathscr{A})$ of $\mathscr{A}$, (see for example [**Fa-Ra**]). Recall that the moduli space of arrangements is the same as the realization space of the underlying matroid (cf. [**Fa-Ra**]). In view of the result of Randell ([**Ra**]), the moduli space of Rybnikov arrangements ([**Ry**]) and the moduli space of Bartolo, Ruber, Agustin and Buzunariz ([**B-R-A-B**]) arrangements are nonconnected. Therefore there is enormous interest of finding combinatorics for which the moduli space is connected. In 1994, Jiang and Yau ([**Ji-Ya**]) first successfully described a large class of line arrangements in $\boldsymbol{CP}^2$ whose moduli space are connected. Recently we ([**Wa-Ya**]) have described a much larger class of line arrangements in $\boldsymbol{CP}^2$ whose moduli spaces are still connected.

In this paper we consider the above question for arrangements of hyperplanes in $\boldsymbol{CP}^3$, which is obviously a more difficult problem. For any such arrangement $\mathscr{A}^*$ in $\boldsymbol{CP}^3$, we introduce a soul $\mathscr{G}(\mathscr{A}^*)$ which is a pseudo-complex completely determined by the combinatoric data of the arrangement. If the soul consists of $\mathscr{G}(0)$ (a set of points or 0-simplices) and $\mathscr{G}(2)$ (a set of planes or 2-simplices), then the arrangement is called point arrangement. A point arrangement is called a nice arrangement if after removing disjoint stars of $\mathscr{G}$, the remaining pseudo-complex contains no loop (cf. Definition 2.7).

Reflection arrangements and Supersolvable arrangements have been studied extensively by many authors. Many beautiful results were obtained. Unfortunately the basic problem whether the diffeomorphic types of these arrangements are combinatorial is still unknown. We conjecture that the diffeomorphic type of Supersolvable arrangements are combinatorial in nature. As we can see from example 2.9 nice point arrangements form a big class of arrangements. Although reflection arrangements and Supersolvable arrangements may not be nice point arrangements, it is important to know whether the diffeomorphic types of this big class of nice point arrangements are combinatorial in nature. The following Theorem A gives an affirmative answer.

Theorem A. *Let $\mathscr{A}_0^*$ and $\mathscr{A}_1^*$ be two nice point arrangements of hyperplanes in $\boldsymbol{CP}^3$. If $L(\mathscr{A}_0^*)$ and $L(\mathscr{A}_1^*)$ are isomorphic, then $M(\mathscr{A}_0^*)$ and $M(\mathscr{A}_1^*)$ are diffeomorphic to each other.*

In the course of proving Theorem A, we have proved the following Theorem.

Theorem B. *Let $\mathscr{A}^*$ be a nice point arrangement of hyperplanes in $\boldsymbol{CP}^3$. The moduli space of $\mathscr{A}^*$ with fixed combinatorics $L(\mathscr{A}^*)$ is connected.*

Our paper is organized as follows. In section 2, for any arrangement $\mathscr{A}^*$ in $\boldsymbol{CP}^3$, we introduced a pseudo-complex $\mathscr{G}(\mathscr{A}^*)$ which is called the soul of $\mathscr{A}^*$. $\mathscr{G}(\mathscr{A}^*)$ is determined by the combinatorial data $L(\mathscr{A}^*)$. We also introduce the definition of nice point arrangement of hyperplanes. In section 3, we prove a sequence of lemmas which are needed to prove Theorem A and Theorem B. These parts are much harder than those in lower dimension obtained in [**Ji-Ya**]. In the final section, we shall prove Theorem A and

Theorem B. We thank the referee for many useful suggestion to improve the presentation of this paper.

2. Nice arrangements of hyperplanes in $\boldsymbol{CP}^3$.

In this paper we denote $\mathscr{A}^*$ arrangement of hyperplanes in $\boldsymbol{CP}^3$. Let $L(\mathscr{A}^*)$ be the set of all intersections of subsets of $\mathscr{A}^*$, partially ordered by reverse inclusion.

We give some definitions and examples of nice arrangements of hyperplanes in $\boldsymbol{CP}^3$ for the following sections.

Definition 2.1. A point p in $\boldsymbol{CP}^3$ is of multiplicity k, denoted by $m(p)$, in $\mathscr{A}^*$ if p is the intersection of exactly k hyperplanes in $\mathscr{A}^*$. A line l in $\boldsymbol{CP}^3$ is of multiplicity k, denoted by $m(l)$, in $\mathscr{A}^*$ if l is the intersection of exactly k hyperplanes in $\mathscr{A}^*$.

To study the combinatorial properties of $\mathscr{A}^*$ we need to consider all intersections (lines and points) of $\mathscr{A}^*$ in $\boldsymbol{CP}^3$. For an arrangement in $\boldsymbol{CP}^3$, any two planes must meet at a line. We only need to consider those intersection lines whose multiplicity is not less than 3. For any plane and line, if the line does not lie on the plane, they must intersect at a point with multiplicity 3 in the arrangement. We also know that a point may be an intersection of two lines. So, we need to consider those intersection points whose multiplicity is not less than 4. To get rid of the trivial situation that a point has multiplicity at least 4 which is obtained by a plan and a line with multiplicity at least 3, we need to add a condition for the intersection points: there are four planes passing through this point in the arrangement $\mathscr{A}^*$ such that every three of them are in general position. Now we can give the following definition naturally.

Definition 2.2. Let $p_k(\mathscr{A}^*)$ be the number of points of multiplicity $k(\geq 4)$ each of which has the property that there are four planes passing through this point in the arrangement $\mathscr{A}^*$ such that every three of them are in general position. Let $l_k(\mathscr{A}^*)$ be the number of lines of multiplicity $k(\geq 3)$ in the arrangement $\mathscr{A}^*$. Then the *complexity* $c(\mathscr{A}^*)$ of $\mathscr{A}^*$ is defined to be $\sum_{k\geq 4}(k-3)p_k(\mathscr{A}^*)+\sum_{k\geq 3}(k-2)l_k(\mathscr{A}^*)$.

Definition 2.3. A *soul* $\mathscr{G}$ of an arrangement $\mathscr{A}^*$ of hyperplanes in $\boldsymbol{CP}^3$ is a pseudo-complex which is defined as follows:

Let $\mathscr{G}(0)$ be the set of 0-simplices of $\mathscr{G}$ defined by $\{p \in \mathscr{A}^*$ is a point $|m(p) \geq 4$ and there are four planes passing through p in $\mathscr{A}^*$ from which any three of them are in general position.$\}$. An element of $\mathscr{G}(0)$ is called a *point*.

Let $\mathscr{G}(1)$ be the set of 1-simplices of $\mathscr{G}$ which is the set of lines of $\mathscr{A}^*$ with multiplicity $m(l) \geq 3$. An element of $\mathscr{G}(1)$ is called a *line*.

Let $\mathscr{G}(2)$ be the set of 2-simplices of $\mathscr{G}$. Each element of $\mathscr{G}(2)$ is a hyperplane of $\mathscr{A}^*$ that passes through an element of $\mathscr{G}(0)\cup\mathscr{G}(1)$. This means that it contains a point or line of $\mathscr{G}(0)\cup\mathscr{G}(1)$. An element of $\mathscr{G}(2)$ is called a *plane*.

We say that two different simplices of $\mathscr{G}$ *intersect to each other in* $\mathscr{G}$ if and only if they contain a same element of $\mathscr{G}(0)\cup\mathscr{G}(1)$ (See Example 2.8 below).

A *path* in $\mathscr{G}$ is defined to be a finite sequence of simplices $a_0, h_1, a_1, h_2, \ldots, a_{k-1}$, $h_k, a_k (k>0)$ of $\mathscr{G}$ where a_i and a_{i+1} are distinct elements in $\mathscr{G}(0)\cup\mathscr{G}(1)$, $h_{i+1}\in\mathscr{G}(2)$, which contains both a_i and a_{i+1} for $i = 0, 1, \ldots, k-1$ and h_j are distinct for $j = 1, \ldots, k$. k is called the *length of the path from* a_0 *to* a_k. When $a_0 = a_k, k \geq 3$, we call this path a *loop*.

For two elements a_1 and $a_2 \in \mathscr{G}(0) \cup \mathscr{G}(1)$, the *distance* from a_1 to a_2 is the minimum length of the path among all paths from a_1 to a_2.

Say a_1 to be a *k-element of* a_2 if the distance from a_1 to a_2 is k. If a_1 is a point, we call a_1 as a *k-point of* a_2. If a_1 is a line, we call a_1 as a *k-line of* a_2.

Remark 2.4. From the discussion and definitions above, we know that in $\boldsymbol{CP}^3$, each two planes must meet at a line and each plane and line must intersect at a point. Hence we do not need to consider these trivial cases in our definition of the pseudo-complex soul $\mathscr{G}$. Thus, it is easy to see that for two souls $\mathscr{G}_1$ and $\mathscr{G}_2$, if $\mathscr{G}_1$ is isomorphic to $\mathscr{G}_2$ and $|\mathscr{A}_1^*| = |\mathscr{A}_2^*|$, then $\mathscr{A}_1^*$ is isomorphic to $\mathscr{A}_2^*$.

Definition 2.5. For an arbitrary $u \in \mathscr{G}(0) \cup \mathscr{G}(1)$, a star $St(u)$ of u is $\{u\} \cup \{$2-simplices of $\mathscr{G}$ which contain $u\}$.

A point $v \in \mathscr{G}(0)(\neq u)$ is called an *end point* of the star $St(u)$ if $St(u)$ passes through v.

A line $l \in \mathscr{G}(1)(\neq u)$ is called an *end line* of the star $St(u)$ if $St(u)$ passes through l.

The end points and end lines of the star $St(u)$ are all called the *end elements* of the star $St(u)$.

For the stars $St(u_1), \ldots, St(u_m)$ in $\mathscr{G}$ $(m > 0)$, let $\mathscr{G}' = \mathscr{G} - \{St(u_1) \cup \cdots \cup St(u_m)\}$. $St(u_1), \ldots, St(u_m)$ are said to be *simple joint* in $\mathscr{G}$ if

(1) any end element of $St(u_1), \ldots, St(u_m)$ can connect to at most one another end element by a path in $\mathscr{G}'$,
(2) any two end elements of $St(u_1), \ldots, St(u_m)$ can be connected by at most one path in $\mathscr{G}'$.

Definition 2.6. An arrangement $\mathscr{A}^*$ of hyperplanes in $\boldsymbol{CP}^3$ is said to be *nice* if the soul $\mathscr{G}$ from $\mathscr{A}^*$ has the following properties:

(1) $\mathscr{G}(0)$ and $\mathscr{G}(1)$ are disjoint, i.e. for any $p \in \mathscr{G}(0)$ and any $q \in \mathscr{G}(1)$, p is not contained in q.
(2) $\mathscr{G}$ has no loop, or
(3) there are simple joint stars $St(u_1), \ldots, St(u_m)$ which are pairwise disjoint in $\mathscr{G}$ such that $\mathscr{G}' = \mathscr{G} - \{St(u_1) \cup \cdots \cup St(u_m)\}$ contains no loop where $u_1, \ldots, u_m$ in $\mathscr{G}(0) \cup \mathscr{G}(1)$.

Definition 2.7. An arrangement $\mathscr{A}^*$ of hyperplanes in $\boldsymbol{CP}^3$ is called a *point arrangement of hyperplanes* if the $\mathscr{G}(1)$ of $\mathscr{A}^*$ is empty. This means that $\mathscr{G}$ consists of the set of the points (0-simplices) and the set of the planes (2-simplices).

If a point arrangement is nice it is called a *nice point arrangement.*

An arrangement $\mathscr{A}^*$ of hyperplanes in $\boldsymbol{CP}^3$ is called a *line arrangement of hyperplanes* if the $\mathscr{G}(0)$ of $\mathscr{A}^*$ is empty. This means that $\mathscr{G}$ consists of the set of the lines (1-simplices) and the set of the planes (2-simplices).

If a line arrangement is nice it is called a *nice line arrangement.*

In the following we give some examples to show the nice line arrangement and the nice point arrangement in $\boldsymbol{CP}^3$.

EXAMPLE 2.8. Let $\mathscr{A}$ be an arrangement of hyperplanes in $\boldsymbol{C}^4$ consisting of the elements

$$H_1 : \{(x, y, z, w) \in C^4 : x = 0\},$$

$$H_2 : \{(x, y, z, w) \in C^4 : y = 0\},$$

$$H_3 : \{(x, y, z, w) \in C^4 : z = 0\},$$

$$H_4 : \{(x, y, z, w) \in C^4 : w = 0\},$$

$$H_5 : \{(x, y, z, w) \in C^4 : x = y\},$$

$$H_6 : \{(x, y, z, w) \in C^4 : w = z\}.$$

The corresponding projective arrangement $\mathscr{A}^*$ is a nice arrangement in $\boldsymbol{CP}^3$. As shown in Figure 1, the pseudo-complex soul $\mathscr{G}$ of $\mathscr{A}^*$ consists of six 2-simplices ABD, AED, ACD, ABC, FBC and DBC, and two 1-simplices AD and BC. We can see that AD incidents with ABD, AED and ACD, BC incidents with ABC, FBC and DBC. Also, we can see, two 2-simplices ABD and ADC intersect at a 1-simplex AD. Notice, there is no 0-simplices because no point in the Figure 1 satisfies the condition that any three of planes are in general position in Definition 2.3. $\mathscr{G}$ contains no loop. Hence, it is a nice line arrangement.

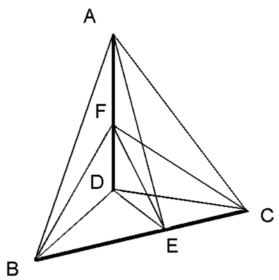

Figure 1. A nice line arrangement in $\boldsymbol{CP}^3$.

Example 2.8 is an example of a line arrangement and it is a nice arrangement. We give another example of nice point arrangement as follows:

EXAMPLE 2.9. Let $\mathscr{A}$ be an arrangement of hyperplanes in $\boldsymbol{C}^4$ consisting of the elements

$$AOD : \{(x, y, z, w) \in C^4 : x = 0\},$$

$$AOC : \{(x, y, z, w) \in C^4 : y = 0\},$$

$$ACD : \{(x, y, z, w) \in C^4 : x + y + z - w = 0\},$$

$$ABE : \{(x, y, z, w) \in C^4 : 2x + 6y + 7z - 7w = 0\},$$

$$AEF : \{(x, y, z, w) \in C^4 : 4x - 3y - z + w = 0\},$$
$$ABF : \{(x, y, z, w) \in C^4 : x + 3y + 11z - 11w = 0\},$$
$$BEF : \{(x, y, z, w) \in C^4 : x - 2y + z - 6w = 0\},$$
$$BCH : \{(x, y, z, w) \in C^4 : 3x + y - 5z - 3w = 0\},$$
$$BCG : \{(x, y, z, w) \in C^4 : x - 2y - 4z - w = 0\},$$
$$BGH : \{(x, y, z, w) \in C^4 : 4x - y - 2z - 11w = 0\},$$
$$COD : \{(x, y, z, w) \in C^4 : z = 0\},$$
$$CGH : \{(x, y, z, w) \in C^4 : 8x - 2y - 11z - 8w = 0\}.$$

The corresponding projective arrangement $\mathscr{A}^*$ is a point arrangement in $\boldsymbol{CP}^3$. In fact, we have written a computer program to check that the conditions of point arrangement are satisfied. As shown in Figure 2, the soul $\mathcal{G}$ of $\mathscr{A}^*$ consists of twelve 2-simplices:

$$AOD, AOC, ACD, ABE, AEF, ABF,$$
$$BEF, BCH, BCG, BGH, COD, CGH,$$

and three 0-simplices:

$$A, B, \text{ and } C.$$

Notice, there is no any 1-simplex because no line in the Figure 2 has the multiplicity greater than 2.

$$\mathscr{G} \text{ contains a loop:}$$
$$A, ABE, B, BCH, C, AOC, A.$$

It is also a nice point arrangement since deleting $St(A)$ (see Figure 3) gives a sub-pseudo-complex $\mathscr{G} - St(A)$ (see Figure 4) with no loop.

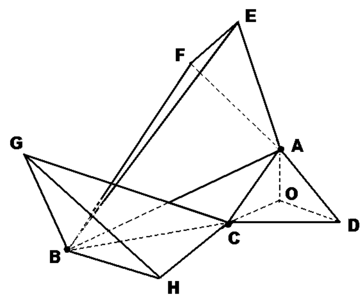

Figure 2. A nice point arrangement in $\boldsymbol{CP}^3$.

Here, $St(A) = \{AOD, AOC, ACD, ABE, AEF, ABF, A\}$.

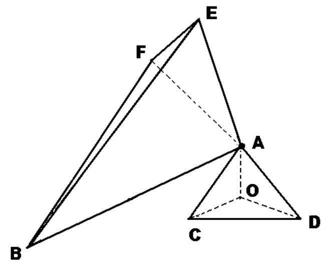

Figure 3. $St(A)$ in $\boldsymbol{CP}^3$.

$$\mathscr{G} - St(A) = \{BEF, BCH, BCG, BGH, COD, CGH, B, C\}.$$

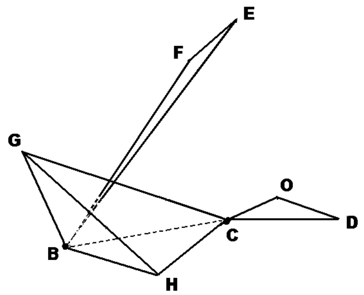

Figure 4. $\mathscr{G} - St(A)$ in $\boldsymbol{CP}^3$.

3. Regularity and lemmas for point arrangements of hyperplanes in $\boldsymbol{CP}^3$.

Basically our main theorem essentially asserts if two nice point arrangements of hyperplane $\mathscr{A}_0^* = \{G_1, G_2, \ldots, G_n\}$ and $\mathscr{A}_1^* = \{H_1, H_2, \ldots, H_n\}$ in $\boldsymbol{CP}^3$ have isomorphic $L(\mathscr{A}_0^*)$ and $L(\mathscr{A}_1^*)$, then $\mathscr{A}_0^*$ and $\mathscr{A}_1^*$ can be joined by a path in the moduli spaces of arrangements with fixed combinatorics $L(\mathscr{A}_0^*)$. For this purpose, we shall construct a one-parameter family of arrangements $\mathscr{A}^*(t)$ such that $\mathscr{A}^*(0) = \mathscr{A}_0^*$, $\mathscr{A}^*(1) = \mathscr{A}_1^*$ and $L(\mathscr{A}^*(t)) \equiv L(\mathscr{A}_0^*)$ for all $t \in [0,1]$. Assume that G_i corresponds to H_i, $1 \le i \le n$, under the combinatorics isomorphism. Consider arrangement $\mathscr{A}^*$ of the form $\mathscr{A}^* = \{F_1, F_2, \ldots, F_n\}$ where $F_i = x_i G_i + y_i H_i$ and $x_i, y_i \in \boldsymbol{C}$. Clearly $x_i G_i + y_i H_i$ and $x_i' G_i + y_i' H_i$ define the same hyperplane if (x_i, y_i) is a constant multiple of (x_i', y_i'). Therefore we can think of $(x_i : y_i)$ being a point in $\boldsymbol{CP}^1$. The condition that $L(\mathscr{A}^*)$ is isomorphic to $L(\mathscr{A}_0^*)$ can be translated to the condition that the parameters $(x_i : y_i)$ have to satisfy cubic equations of the form (3.1) below. Thus it remains to prove that the variety defined by these cubic equations in $(\boldsymbol{CP}^1)^n$ has an irreducible component which contains both $\mathscr{A}_0^*$ and $\mathscr{A}_1^*$. For this purpose we need to introduce the notions of regular point with respect to equation (3.1). Let

$$\begin{aligned} U = (\boldsymbol{CP}^1)^p - \big\{ \big((x_1 : y_1), \ldots, (x_p : y_p)\big) :\ &\text{for some } 1 \le i \le p, \\ &(x_i : y_i) \text{ is irregular of some equation of the form (3.1)} \big\}. \end{aligned}$$

Lemma 3.9 implies that U is an open connected in $(\boldsymbol{CP}^1)^p$. By solving the cubic equations of the form (3.1), one has an embedding $f : U \to (\boldsymbol{CP}^1)^n$ by Lemma 3.10. It turns out that $((1:0),\ldots,(1:0))$ which corresponds to $\mathscr{A}_0^*$ and $((0:1),\ldots,(0:1))$ which corresponds to $\mathscr{A}_1^*$ are in $f(U)$ and $f(U)$ is irreducible. Our main theorem follows immediately. Before showing the main theorem, we need the following definitions and lemmas.

Consider the following equation

$$\sum a_{\epsilon(1)\ldots\epsilon(4)} z_i^{\epsilon(1)} z_j^{\epsilon(2)} z_k^{\epsilon(3)} z_l^{\epsilon(4)} = 0 \tag{3.1}$$

where $\epsilon(1),\ldots,\epsilon(4)$ are either 0 or 1 satisfying $0 < \epsilon(1) + \cdots + \epsilon(4) < 4$, $z_i^0 = x_i$ and $z_i^1 = y_i$, $a_{\epsilon(1)\ldots\epsilon(4)} \neq 0$.

The left hand of the equation (3.1) has fourteen items. For convenience we write (3.1) as the following extensive form

$$\begin{aligned} & a y_i x_j x_k x_l + b x_i y_j x_k x_l + c x_i x_j y_k x_l + d x_i x_j x_k y_l \\ & + A x_i x_j y_k y_l + B x_i y_j x_k y_l + C x_i y_j y_k x_l + D y_i x_j x_k y_l + E y_i x_j y_k x_l + F y_i y_j x_k x_l \\ & + e x_i y_j y_k y_l + f y_i x_j y_k y_l + g y_i y_j x_k y_l + h y_i y_j y_k x_l \\ & \quad = 0 \end{aligned}$$

where $abcdABCDEFefgh \neq 0$.

Let $(x_i : y_i)$, $(x_j : y_j)$, $(x_k : y_k)$ and $(x_l : y_l)$ be the solution of (3.1). Because $(0:0)$ is always a solution of the homogeneous equation (3.1), we only consider non-zero solution of (3.1). We assume that $(x_s : y_s) \neq (0:0)$ for $s = i, j, k, l$ below.

DEFINITION 3.1. $(x_i : y_i) \in \boldsymbol{CP}^1$ is called *irregular* for the equation (3.1):

$$\begin{aligned} & (a y_i) x_j x_k x_l + (d x_i + D y_i) x_j x_k y_l + (c x_i + E y_i) x_j y_k x_l + (A x_i + f y_i) x_j y_k y_l \\ & + (b x_i + F y_i) y_j x_k x_l + (B x_i + g y_i) y_j x_k y_l + (C x_i + h y_i) y_j y_k x_l + (e x_i) y_j y_k y_l \\ & \quad = 0 \end{aligned}$$

if the following matrix of the coefficients has rank one.

$$\begin{pmatrix} a y_i & d x_i + D y_i & c x_i + E y_i & A x_i + f y_i \\ b x_i + F y_i & B x_i + g y_i & C x_i + h y_i & e x_i \end{pmatrix}$$

DEFINITION 3.2. Let $(x_k : y_k)$ and $(x_l : y_l) \in \boldsymbol{CP}^1$. The pair $((x_k : y_k), (x_l : y_l))$ is an *irregular pair* for the equation (3.1):

$$\begin{aligned} & \big[(c y_k) x_l + (d x_k + A y_k) y_l\big] x_i x_j + \big[(a x_k + E y_k) x_l + (D x_k + f y_k) y_l\big] y_i x_j \\ & + \big[(b x_k + C y_k) x_l + (B x_k + e y_k) y_l\big] x_i y_j + \big[(F x_k + h y_k) x_l + (g x_k) y_l\big] y_i y_j \\ & \quad = 0 \end{aligned}$$

if

$$|N| := \begin{vmatrix} (cy_k)x_l + (dx_k + Ay_k)y_l & (ax_k + Ey_k)x_l + (Dx_k + fy_k)y_l \\ (bx_k + Cy_k)x_l + (Bx_k + ey_k)y_l & (Fx_k + hy_k)x_l + (gx_k)y_l \end{vmatrix}$$
$$= 0.$$

Definition 3.3. Let $(x_k : y_k)$ and $(x_l : y_l) \in \boldsymbol{CP}^1$. The pair $((x_k : y_k), (x_l : y_l))$ is a *regular pair* for the equation (3.1) if the pair $((x_k : y_k), (x_l : y_l))$ is not an irregular pair. $(x_i : y_i) \in \boldsymbol{CP}^1$ is *regular* for the equation (3.1) if $(x_i : y_i)$ is neither irregular nor one of elements in an irregular pair.

Lemma 3.4.

$$P = ax_jx_kx_l + bx_jx_ky_l + cx_jy_kx_l + dx_jy_ky_l + ey_jx_kx_l + fy_jx_ky_l + gy_jy_kx_l + hy_jy_ky_l$$

is reducible if and only if

$$\frac{a}{e} = \frac{b}{f} = \frac{c}{g} = \frac{d}{h}.$$

i.e. the matrix

$$\begin{pmatrix} a & b & c & d \\ e & f & g & h \end{pmatrix}$$

has a rank 1.

Proof. First, we prove that it is necessary. Assume P is reducible. Notice that P is a homogeneous polynomial of degree three. Then we can write P as

$$P = (Ax_j + Bx_k + Cx_l + Dy_j + Ey_k + Fy_l)(Gx_kx_l + Hx_ky_l + Iy_kx_l + Jy_ky_l).$$

Then, we get

$$B = C = E = F = 0.$$

Hence,

$$\begin{aligned} P &= (Ax_j + Dy_j)(Gx_kx_l + Hx_ky_l + Iy_kx_l + Jy_ky_l) \\ &= AGx_jx_kx_l + AHx_jx_ky_l + AIx_jy_kx_l + AJx_jy_ky_l + DGy_jx_kx_l \\ &\quad + DHy_jx_ky_l + DIy_jy_kx_l + DJy_jy_ky_l. \end{aligned}$$

Comparing the coefficients of P, we have

$$AG = a,\ AH = b,\ AI = c,\ AJ = d,\ DG = e,\ DH = f,\ DI = g,\ DJ = h.$$

$$\frac{a}{e} = \frac{b}{f} = \frac{c}{g} = \frac{d}{h}.$$

It proves the necessity. Now, we prove the sufficiency.

Assume $\frac{a}{e} = \frac{b}{f} = \frac{c}{g} = \frac{d}{h}$, then

$$\begin{aligned} P &= (ax_j + ey_j)x_kx_l + (bx_j + fy_j)x_ky_l + (cx_j + gy_j)y_kx_l + (dx_j + hy_j)y_ky_l \\ &= e\left(\frac{a}{e}x_j + y_j\right)x_kx_l + f\left(\frac{b}{f}x_j + y_j\right)x_ky_l + g\left(\frac{c}{g}x_j + y_j\right)y_kx_l + h\left(\frac{d}{h}x_j + y_j\right)y_ky_l \\ &= \left(\frac{a}{e}x_j + y_j\right)(ex_kx_l + fx_ky_l + gy_kx_l + hy_ky_l). \end{aligned}$$ □

LEMMA 3.5. *$(x_i : y_i)$ is irregular for the equation (3.1) if and only if*

$$\begin{aligned} &(ay_i)x_jx_kx_l + (dx_i + Dy_i)x_jx_ky_l + (cx_i + Ey_i)x_jy_kx_l + (Ax_i + fy_i)x_jy_ky_l \\ &\quad + (bx_i + Fy_i)y_jx_kx_l + (Bx_i + gy_i)y_jx_ky_l + (Cx_i + hy_i)y_jy_kx_l + (ex_i)y_jy_ky_l \end{aligned} \tag{3.2}$$

is a reducible polynomial of the other three variables $(x_j : y_j)$, $(x_k : y_k)$ and $(x_l : y_l)$.

PROOF. If $(x_i : y_i)$ is irregular for the equation (3.1). By the definition,

$$\begin{pmatrix} ay_i & dx_i + Dy_i & cx_i + Ey_i & Ax_i + fy_i \\ bx_i + Fy_i & Bx_i + gy_i & Cx_i + hy_i & ex_i \end{pmatrix}$$

has rank one. This is equivalent to the following conditions:

$$\begin{vmatrix} ay_i & dx_i + Dy_i \\ bx_i + Fy_i & Bx_i + gy_i \end{vmatrix} = 0, \quad \begin{vmatrix} ay_i & cx_i + Ey_i \\ bx_i + Fy_i & Cx_i + hy_i \end{vmatrix} = 0,$$

$$\begin{vmatrix} ay_i & Ax_i + fy_i \\ bx_i + Fy_i & ex_i \end{vmatrix} = 0.$$

That is

$$\frac{ay_i}{bx_i + Fy_i} = \frac{dx_i + Dy_i}{Bx_i + gy_i} = \frac{cx_i + Ey_i}{Cx_i + hy_i} = \frac{Ax_i + fy_i}{ex_i}. \tag{3.3}$$

By Lemma 3.4, (3.2) is reducible.

On the other hand, if (3.2) is reducible, then (3.3) holds. This implies that the matrix of (3.2) has rank one. By the definition, $(x_i : y_i)$ is irregular for the equation (3.1). □

Corollary 3.6. *If $(x_i : y_i)$ is irregular, then $x_i \neq 0$ and $y_i \neq 0$.*

Proof. Assume $y_i = 0$. Then (3.1) becomes

$$\begin{aligned} &dx_ix_jx_ky_l + cx_ix_jy_kx_l + Ax_ix_jy_ky_l + bx_iy_jx_kx_l \\ &\quad + Bx_iy_jx_ky_l + Cx_iy_jy_kx_l + ex_iy_jy_ky_l = 0 \end{aligned}$$

which is irreducible by Lemma 3.3. Hence, $(x_i : y_i)$ is not irregular by Lemma 3.4. This is a contradiction. So, $y_i \neq 0$.

The proof of $x_i \neq 0$ is similar. □

Lemma 3.7. *Let $(x_k : y_k)$ and $(x_l : y_l) \in \boldsymbol{CP}^1$. The pair $((x_k : y_k), (x_l : y_l))$ is irregular for the equation* (3.1) *if and only if*

For some $(x_j, y_j) \neq (0, 0)$, either

$$(ay_i)x_kx_l + (dx_i + Dy_i)x_ky_l + (cx_i + Ey_i)y_kx_l + (Ax_i + fy_i)y_ky_l = 0 \quad \text{for } y_j \neq 0, \tag{3.4}$$

or

$$(bx_i + Fy_i)x_kx_l + (Bx_i + gy_i)x_ky_l + (Cx_i + hy_i)y_kx_l + (ex_i)y_ky_l = 0 \quad \text{for } x_j \neq 0. \tag{3.5}$$

Proof. Let the pair $((x_k : y_k), (x_l : y_l))$ be an irregular pair for the equation (3.1). By Definition 3.2,

$$\begin{vmatrix} (cy_k)x_l + (dx_k + Ay_k)y_l & (ax_k + Ey_k)x_l + (Dx_k + fy_k)y_l \\ (bx_k + Cy_k)x_l + (Bx_k + ey_k)y_l & (Fx_k + hy_k)x_l + (gx_k)y_l \end{vmatrix} = 0.$$

Hence, the equations

$$\big[(cy_k)x_l + (dx_k + Ay_k)y_l\big]x_i + \big[(ax_k + Ey_k)x_l + (Dx_k + fy_k)y_l\big]y_i = 0 \tag{3.6}$$

$$\big[(bx_k + Cy_k)x_l + (Bx_k + ey_k)y_l\big]x_i + \big[(Fx_k + hy_k)x_l + (gx_k)y_l\big]y_i = 0 \tag{3.7}$$

have non-zero solution (x_i, y_i). That is:

$$(ay_i)x_kx_l + (dx_i + Dy_i)x_ky_l + (cx_i + Ey_i)y_kx_l + (Ax_i + fy_i)y_ky_l = 0 \tag{3.8}$$

$$(bx_i + Fy_i)x_kx_l + (Bx_i + gy_i)x_ky_l + (Cx_i + hy_i)y_kx_l + (ex_i)y_ky_l = 0. \tag{3.9}$$

It proves the necessity. Now we prove the sufficiency.

Assume

$$(ay_i)x_kx_l + (dx_i + Dy_i)x_ky_l + (cx_i + Ey_i)y_kx_l + (Ax_i + fy_i)y_ky_l = 0$$

holds for some (x_j, y_j), $y_j \neq 0$, we have

$$\left[(ay_i)x_kx_l + (dx_i + Dy_i)x_ky_l + (cx_i + Ey_i)y_kx_l + (Ax_i + fy_i)y_ky_l\right]x_j = 0.$$

Hence, (3.1) becomes

$$\left[(bx_i + Fy_i)x_kx_l + (Bx_i + gy_i)x_ky_l + (Cx_i + hy_i)y_kx_l + ex_iy_ky_l\right]y_j = 0, \tag{3.10}$$

which implies

$$(bx_i + Fy_i)x_kx_l + (Bx_i + gy_i)x_ky_l + (Cx_i + hy_i)y_kx_l + ex_iy_ky_l = 0. \tag{3.11}$$

Thus, we imply the system of equations (3.6) and (3.7) has non-zero solution (x_i, y_i). Hence,

$$\begin{vmatrix} (cy_k)x_l + (dx_k + Ay_k)y_l & (ax_k + Ey_k)x_l + (Dx_k + fy_k)y_l \\ (bx_k + Cy_k)x_l + (Bx_k + ey_k)y_l & (Fx_k + hy_k)x_l + (gx_k)y_l \end{vmatrix} = 0.$$

This means that pair $((x_k : y_k), (x_l : y_l))$ is irregular for the equation (3.1).

Similarly one can prove that if

$$(bx_i + Fy_i)x_kx_l + (Bx_i + gy_i)x_ky_l + (Cx_i + hy_i)y_kx_l + (ex_i)y_ky_l = 0$$

for some (x_j, y_j), $x_j \neq 0$, then that pair $((x_k : y_k), (x_l : y_l))$ is an irregular pair for the equation (3.1). $\square$

Lemma 3.8. *Assume* $((x_1 : y_1), (x_2 : y_2), (x_3 : y_3), (x_4 : y_4)) \in (\boldsymbol{CP}^1)^4$ *is a solution of* (3.1). *If* $(x_1 : y_1)$ *is irregular, then there is at least one irregular or irregular pair in* $(x_2 : y_2), (x_3 : y_3), (x_4 : y_4)$ *for* (3.1). *If* $(x_1 : y_1)$ *is regular, then* $(x_2 : y_2), (x_3 : y_3)$ *and* $(x_4 : y_4)$ *are either all regular or at least two are irregular or one irregular pair for* (3.1). *In other words, the number of irregularity cannot be* 1.

Proof. Assume $(x_1 : y_1)$ is irregular. Write (3.1) as a polynomial of $(x_2 : y_2)$, $(x_3 : y_3)$ and $(x_4 : y_4)$

$$\begin{aligned} P = {} & (ay_1)x_2x_3x_4 + (dx_1 + Dy_1)x_2x_3y_4 + (cx_1 + Ey_1)x_2y_3x_4 \\ & + (Ax_1 + fy_1)x_2y_3y_4 + (bx_1 + Fy_1)y_2x_3x_4 + (Bx_1 + gy_1)y_2x_3y_4 \\ & + (Cx_1 + hy_1)y_2y_3x_4 + (ex_1)y_2y_3y_3. \end{aligned} \tag{3.12}$$

From Corollary 3.6, we know $x_1 \neq 0$ and $y_1 \neq 0$.

By Lemma 3.5, (x_1, y_1) is irregular if and only if (3.12) is reducible. By Lemma 3.4, (3.12) is reducible if and only if

$$\frac{ay_1}{bx_1+Fy_1}=\frac{dx_1+Dy_1}{Bx_1+gy_1}=\frac{cx_1+Ey_1}{Cx_1+hy_1}=\frac{Ax_1+fy_1}{ex_1}. \tag{3.13}$$

That is

$$\begin{aligned}(ay_1)(Bx_1+gy_1)&=(bx_1+Fy_1)(dx_1+Dy_1),\\(ay_1)(Cx_1+hy_1)&=(bx_1+Fy_1)(cx_1+Ey_1),\\(ay_1)(ex_1)&=(bx_1+Fy_1)(Ax_1+fy_1).\end{aligned}$$

We get

$$(bd)x_1^2+(bD+dF-aB)x_1y_1+(DF-ag)y_1^2=0, \tag{3.14}$$

$$(bc)x_1^2+(bE+Fc-aC)x_1y_1+(FE-ah)y_1^2=0, \tag{3.15}$$

$$(bA)x_1^2+(bf+AF-ae)x_1y_1+(fF)y_1^2=0, \tag{3.16}$$

which has at most two roots of $(x_1:y_1)$. Because (x_2,y_2) is a non-zero solution of (3.1), we assume $x_2\neq 0$ first. Then (3.1) can be written as

$$\begin{aligned}P&=\big[(ay_1)x_2+(bx_1+Fy_1)y_2\big]x_3x_4+\big[(dx_1+Dy_1)x_2+(Bx_1+gy_1)y_2\big]x_3y_4\\&\quad+\big[(cx_1+Ey_1)x_2+(Cx_1+hy_1)y_2\big]y_3x_4+\big[(Ax_1+fy_1)x_2+(ex_1)y_2\big]y_3y_4\\&=(bx_1+Fy_1)\left[\frac{(ay_1)}{(bx_1+Fy_1)}x_2+y_2\right]x_3x_4+(Bx_1+gy_1)\left[\frac{(dx_1+Dy_1)}{(Bx_1+gy_1)}x_2+y_2\right]x_3y_4\\&\quad+(Cx_1+hy_1)\left[\frac{(cx_1+Ey_1)}{(Cx_1+hy_1)}x_2+y_2\right]y_3x_4+(ex_1)\left[\frac{(Ax_1+fy_1)}{(ex_1)}x_2+y_2\right]y_3y_4\\&=\left[\frac{(ay_1)}{(bx_1+Fy_1)}x_2+y_2\right]\\&\quad\cdot\big[(bx_1+Fy_1)x_3x_4+(Bx_1+gy_1)x_3y_4+(Cx_1+hy_1)y_3x_4+(ex_1)y_3y_4\big]\\&=0.\end{aligned} \tag{3.17}$$

If $\frac{(ay_1)}{(bx_1+Fy_1)}x_2+y_2=0$, then

$$by_2x_1+(ax_2+Fy_2)y_1=0.$$

We have a solution

$$\frac{x_1}{y_1}=-\frac{ax_2+Fy_2}{by_2}. \tag{3.18}$$

Put (3.18) into (3.14), (3.15) and (3.16), it yields

$$
\begin{aligned}
&(bd)\big(a^2x_2^2+2aFx_2y_2+F^2y_2^2\big)\\
&\quad-(bD+dF-aB)\big(abx_2y_2+bFy_2^2\big)+(DF-ag)b^2y_2^2=0,\\
&(bc)\big(a^2x_2^2+2aFx_2y_2+F^2y_2^2\big)\\
&\quad-(bE+Fc-aC)\big(abx_2y_2+bFy_2^2\big)+(FE-ah)b^2y_2^2=0,\\
&(bA)\big(a^2x_2^2+2aFx_2y_2+F^2y_2^2\big)\\
&\quad-(bf+AF-ae)\big(abx_2y_2+bFy_2^2\big)+(fF)b^2y_2^2=0.
\end{aligned}
$$

Combining the like terms we get

$$
\begin{aligned}
(bda^2)x_2^2+(abdF+a^2bB-ab^2D)x_2y_2+(abBF-ab^2g)y_2^2&=0,\\
(a^2bc)x_2^2+(abcF+a^2bC-ab^2E)x_2y_2+(abFC-ab^2h)y_2^2&=0,\\
(a^2bA)x_2^2+(abAF+a^2be-ab^2f)x_2y_2+(abeF)y_2^2&=0,
\end{aligned}
$$

which is equivalent to

$$(ad)x_2^2+(dF+aB-bD)x_2y_2+(BF-bg)y_2^2=0, \tag{3.19}$$

$$(ac)x_2^2+(cF+aC-bE)x_2y_2+(FC-bh)y_2^2=0, \tag{3.20}$$

$$(aA)x_2^2+(AF+ae-bf)x_2y_2+(eF)y_2^2=0. \tag{3.21}$$

We claim that (3.19), (3.20) and (3.21) are necessary and sufficient conditions for $(x_2 : y_2)$ being irregular of (3.1). To see this, write (3.1) as a polynomial of $(x_1 : y_1)$, $(x_3 : y_3)$ and $(x_4 : y_4)$

$$
\begin{aligned}
P &= (by_2)x_1x_3x_4+(dx_2+By_2)x_1x_3y_4+(cx_2+Cy_2)x_1y_3x_4+(Ax_2+ey_2)x_1y_3y_4\\
&\quad+(ax_2+Fy_2)y_1x_3x_4+(Dx_2+gy_2)y_1x_3y_4+(Ex_2+hy_2)y_1y_3x_4+(fx_2)y_1y_3y_4\\
&=0.
\end{aligned} \tag{3.22}
$$

By Lemma 3.4 and Lemma 3.5, (x_2, y_2) is irregular for (3.1) if and only if (3.22) is reducible if and only if

$$\frac{by_2}{ax_2+Fy_2}=\frac{dx_2+By_2}{Dx_2+gy_2}=\frac{cx_2+Cy_2}{Ex_2+hy_2}=\frac{Ax_2+ey_2}{fx_2}. \tag{3.23}$$

That is

$$
\begin{aligned}
(by_2)(Dx_2+gy_2)&=(ax_2+Fy_2)(dx_2+By_2),\\
(by_2)(Ex_2+hy_2)&=(ax_2+Fy_2)(cx_2+Cy_2),\\
(by_2)(fx_2)&=(ax_2+Fy_2)(Ax_2+ey_2),
\end{aligned}
$$

which are exactly (3.19), (3.20) and (3.21).

If $[(bx_1+Fy_1)x_3x_4+(Bx_1+gy_1)x_3y_4+(Cx_1+hy_1)y_3x_4+(ex_1)y_3y_4]=0$, because $x_2\neq 0$, by Lemma 3.7, the pair $((x_3:y_3),(x_4:y_4))$ is irregular for (3.1).

Now, we consider $y_2\neq 0$. Then (3.1) can be written as

$$\begin{aligned}
P &= \big[(ay_1)x_2+(bx_1+Fy_1)y_2\big]x_3x_4+\big[(dx_1+Dy_1)x_2+(Bx_1+gy_1)y_2\big]x_3y_4\\
&\quad+\big[(cx_1+Ey_1)x_2+(Cx_1+hy_1)y_2\big]y_3x_4+\big[(Ax_1+fy_1)x_2+(ex_1)y_2\big]y_3y_4\\
&=(ay_1)\left[x_2+\frac{(bx_1+Fy_1)}{(ay_1)}y_2\right]x_3x_4+(dx_1+Dy_1)\left[x_2+\frac{(Bx_1+gy_1)}{(dx_1+Dy_1)}y_2\right]x_3y_4\\
&\quad+(cx_1+Ey_1)\left[x_2+\frac{(Cx_1+hy_1)}{(cx_1+Ey_1)}y_2\right]y_3x_4+(Ax_1+fy_1)\left[x_2+\frac{(ex_1)}{(Ax_1+fy_1)}y_2\right]y_3y_4\\
&=\left[x_2+\frac{(bx_1+Fy_1)}{(ay_1)}y_2\right]\\
&\quad\cdot\big[(ay_1)x_3x_4+(dx_1+Dy_1)x_3y_4+(cx_1+Ey_1)y_3x_4+(Ax_1+fy_1)y_3y_4\big]\\
&=0. \qquad (3.24)
\end{aligned}$$

If $x_2+\frac{(bx_1+Fy_1)}{(ay_1)}y_2=0$, then $by_2x_1+Fy_2y_1+ax_2y_1=0$. We get

$$\frac{x_1}{y_1}=-\frac{ax_2+Fy_2}{by_2},$$

which is (3.18). Same as above, we can prove that $(x_2:y_2)$ is irregular of (3.1).

If $(ay_1)x_3x_4+(dx_1+Dy_1)x_3y_4+(cx_1+Ey_1)y_3x_4+(Ax_1+fy_1)y_3y_4=0$, because $y_2\neq 0$, by Lemma 3.7, the pair $((x_3:y_3),(x_4:y_4))$ is irregular for (3.1). □

From the argument above we also have

LEMMA 3.9. *Assume $((x_i:y_i),(x_j:y_j),(x_k:y_k),(x_l:y_l))\in(\boldsymbol{CP}^1)^4$ is a solution of* (3.1). *Then there are at most finite irregular $(x_m:y_m)$ and irregular pair $((x_m:y_m)$, $(x_n:y_n))$ of* (3.1) *for each $m,n=i,j,k,l$. Therefore, the set of irregularity of* (3.1) *is finite.*

$(0:1)$ and $(1:0)$ are regular of (3.1).

PROOF. Assume $i=1$. From the proof above, the necessary and sufficient conditions that $(x_1:y_1)$ is irregular of (3.1) are that equations (3.14), (3.15) and (3.16) hold, which have at most two solutions.

Similarly, we can consider $(x_i:y_i),(x_j:y_j),(x_k:y_k)$ and $(x_l:y_l)$.

From lemma 3.7 and Definition 3.2, there are at most finite irregular pair $((x_m:y_m)$, $(x_n:y_n))$ of (3.1) for each $m,n=i,j,k,l$.

It is clear that $(0:1)$ and $(1:0)$ do not satisfy (3.16). Hence, $(0:1)$ and $(1:0)$ are regular of (3.1). □

LEMMA 3.10. *For each fixed regular pair $((x_k:y_k),(x_l:y_l))$ of* (3.1), *the following* *relation produces an automorphism of* $\boldsymbol{CP}^1$

$$\begin{pmatrix} x_j \\ y_j \end{pmatrix} = K \begin{pmatrix} -(bx_k + Cy_k)x_l - (Bx_k + ey_k)y_l & -(Fx_k + hy_k)x_l - (gx_k)y_l \\ (cy_k)x_l + (dx_k + Ay_k)y_l & (ax_k + Ey_k)x_l + (Dx_k + fy_k)y_l \end{pmatrix} \begin{pmatrix} x_i \\ y_i \end{pmatrix}$$

$$\equiv KM \begin{pmatrix} x_i \\ y_i \end{pmatrix}, \quad K \in \boldsymbol{C}^*, \tag{3.25}$$

which sends regular values to regular values of (3.1)*. In particular, if* $(x_k : y_k) = (x_l : y_l) = (0 : 1)$ (*respectively* $(1 : 0)$)*, then* (3.25) *sends* $(0 : 1)$ (*respectively* $(1 : 0)$) *to* $(0 : 1)$ (*respectively* $(1 : 0)$)*.*

Proof. Since $((x_k : y_k), (x_l : y_l))$ are a regular value, $|M| = |N|$ (c.f. Definition 3.2) is non-zero by Definition 3.3. Hence (3.25) is an automorphism of $\boldsymbol{CP}^1$. Clearly (3.25) is

$$\begin{pmatrix} x_j \\ y_j \end{pmatrix} = K \begin{pmatrix} -\big[(bx_k + Cy_k)x_l + (Bx_k + ey_k)y_l\big]x_i - \big[(Fx_k + hy_k)x_l + (gx_k)y_l\big]y_i \\ \big[(cy_k)x_l + (dx_k + Ay_k)y_l\big]x_i + \big[(ax_k + Ey_k)x_l + (Dx_k + fy_k)y_l\big]y_i \end{pmatrix}$$

which implies

$$\begin{aligned} &\big[(cy_k)x_l + (dx_k + Ay_k)y_l\big]x_i x_j + \big[(ax_k + Ey_k)x_l + (Dx_k + fy_k)y_l\big]y_i x_j \\ &+ \big[(bx_k + Cy_k)x_l + (Bx_k + ey_k)y_l\big]x_i y_j + \big[(Fx_k + hy_k)x_l + (gx_k)y_l\big]y_i y_j = 0. \end{aligned}$$

This is exactly the equation (3.1). By Lemma 3.8, the mapping (3.25) sends regular values of (3.1) to regular values of (3.1). The last statement of the lemma is obvious. □

Remark 3.11. Equation (3.25) is equivalent to equation (3.1). If we write (3.1) as

$$\begin{aligned} &\big\{\big[(cy_k)x_l + (dx_k + Ay_k)y_l\big]x_i + \big[(ax_k + Ey_k)x_l + (Dx_k + fy_k)y_l\big]y_i\big\}x_j \\ &+ \big\{\big[(bx_k + Cy_k)x_l + (Bx_k + ey_k)y_l\big]x_i + \big[(Fx_k + hy_k)x_l + (gx_k)y_l\big]y_i\big\}y_j = 0, \end{aligned}$$

then

$$\begin{aligned} (x_j, y_j) = K\big(&-\big[(bx_k + Cy_k)x_l + (Bx_k + ey_k)y_l\big]x_i - \big[(Fx_k + hy_k)x_l + (gx_k)y_l\big]y_i, \\ &\big[(cy_k)x_l + (dx_k + Ay_k)y_l\big]x_i + \big[(ax_k + Ey_k)x_l + (Dx_k + fy_k)y_l\big]y_i\big) \end{aligned}$$

which is (3.25). Hence, if $((x_k : y_k), (x_l : y_l))$ is a regular pair for (3.1), and $(x_i : y_i)$ is regular for (3.1), then there is a unique $(x_j : y_j)$ solved in terms of $(x_i : y_i)$, $(x_k : y_k)$ and $(x_l : y_l)$. We call such procedure "fixing three variables to solve the another" and call $(x_i : y_i), (x_j : y_j), (x_k : y_k), (x_l : y_l)$ "solved variables".

Lemma 3.12 (Lattice-Isotopy Theorem [**Ra2**]). *If two arrangements are connected by a one-parameter family of arrangements $\{\mathscr{A}(t)\}$ which have the same $L(\mathscr{A})$, then the complements are diffeomorphic, hence of the same homotopy type.*

In order to simplify the proof of our main theorem, the following lemmas are useful.

Lemma 3.13. *Let $\mathscr{A}^*$ be a point arrangement. Then for each two points of $\mathscr{G} = \mathscr{G}(\mathscr{A}^*)$, there are at most two planes passing through both of them.*

Proof. Because $\mathscr{A}^*$ is a point arrangement, $\mathscr{G}$ does not contain any line. For two points in $\mathscr{G}$, if there are more than two planes passing through them, then the intersection line of the planes, say l, has multiplicity $m(l) \geq 3$. It implies that l must be in the pseudo-complex soul $\mathscr{G}$ of $\mathscr{A}^*$. It contradicts that $\mathscr{A}^*$ is a point arrangement. □

Lemma 3.14. *Let $\mathscr{G}$ be a soul, $St(v_1), \ldots, St(v_m)$ be simple joint stars of $\mathscr{G}$ and $\mathscr{G}' = \mathscr{G} - \bigcup_{i=1}^{m} St(v_i)$. If u is a point of $\mathscr{G}'$, then u cannot connect to more than two end points of $St(v_1), \ldots, St(v_m)$ by paths in $\mathscr{G}'$. If u connects two end points of $St(v_1), \ldots, St(v_m)$ by two paths in $\mathscr{G}'$ respectively, then the two paths are unique.*

Proof. Assume u connects to three end points u_1, u_2 and u_3 of $St(v_1), \ldots, St(v_m)$ by paths in $\mathscr{G}'$. Then u_1 connects to other two end points u_2 and u_3 through u. It is a contradiction because $St(v_1), \ldots, St(v_m)$ are simple joint.

If u connects two end points of $St(v_1), \ldots, St(v_m)$ by more than two paths in $\mathscr{G}'$, assume that $\mathscr{P}_1$ and $\mathscr{P}_2$ connect an end point u_1 to u, $\mathscr{P}_3$ connects u to another end point u_2, then there are two paths:

$$(u_1)\mathscr{P}_1,\ (u)\mathscr{P}_3(u_2),$$
$$(u_1)\mathscr{P}_2,\ (u)\mathscr{P}_3,\ (u_2),$$

which connect u_1 and u_2. It is also a contradiction because $St(v_1), \ldots, St(v_m)$ are simple joint. □

Corollary 3.15. *Let $\mathscr{G}$ be a soul, $St(v_1), \ldots, St(v_m)$ be simple joint stars of $\mathscr{G}$. $\mathscr{G}' = \mathscr{G} - \bigcup_{i=1}^{m} St(v_i)$. If u is a point in $\mathscr{G}'$ connecting to $St(v_1), \ldots, St(v_m)$, then only one of the following cases occurs*:

(1) *u connects to only one end point of $St(v_1), \ldots, St(v_m)$ by path in $\mathscr{G}'$.*
(2) *u connects to two end points w_1 and w_2 of $St(v_1), \ldots, St(v_m)$. Moreover, the path in $\mathscr{G}'$ from u to w_i, $i = 1, 2$ is unique.*

Proof. It is obvious from Lemma 3.13. □

Lemma 3.16. *Let $\mathscr{G}$ be a soul, v_1, v_2 and v_3 be three points of $\mathscr{G}$. If v_1, v_2 and v_3 are pairwise connected to each other by paths and each of the paths does not pass all three points, then there is a loop in $\mathscr{G}$.*

Proof. Assume that v_1 and v_2 are connected by the path $\mathscr{P}_1$, v_2 and v_3 are connected by the path $\mathscr{P}_2$, and v_3 and v_1 are connected by the path $\mathscr{P}_3$. Then there is a loop:

$$(v_1)\mathscr{P}_1,\ (v_2)\mathscr{P}_2,\ (v_3)\mathscr{P}_3, (v_1). \qquad \square$$

Corollary 3.17. *Let $\mathscr{G}$ be a soul. If $\mathscr{G}$ has no loop, then any three points in $\mathscr{G}$ can not be pairwise connected each other by the paths, each of which does not pass all three points.*

Proof. It is obvious from Lemma 3.16. $\square$

4. Diffeomorphic types for nice point arrangement in $\boldsymbol{CP}^3$.

Theorem A. *Let $\mathscr{A}_0^*$ and $\mathscr{A}_1^*$ be two nice point arrangements of hyperplanes in $\boldsymbol{CP}^3$. If $L(\mathscr{A}_0^*)$ and $L(\mathscr{A}_1^*)$ are isomorphic, then the complements $M(\mathscr{A}_0^*)$ and $M(\mathscr{A}_1^*)$ in $\boldsymbol{CP}^3$ are diffeomorphic to each other.*

Proof. We represent the two arrangements as $\mathscr{A}_0^* = \{G_1, G_2, \ldots, G_n\}$ and $\mathscr{A}_1^* = \{H_1, H_2, \ldots, H_n\}$ where $G_i = (g_{i1}, g_{i2}, g_{i3}, g_{i4})$ and $H_i = (h_{i1}, h_{i2}, h_{i3}, h_{i4})$ are in $\boldsymbol{CP}^3$. We shall construct a one-parameter family of arrangements $\mathscr{A}^*(t)$ such that $\mathscr{A}^*(0) = \mathscr{A}_0^*$, $\mathscr{A}^*(1) = \mathscr{A}_1^*$ and $L(\mathscr{A}^*(t)) \equiv L(\mathscr{A}_0^*)$ for all $t \in [0,1]$.

Let $\mathscr{A}^* = \{F_1, F_2, \ldots, F_n\}$ where $F_i = x_iG_i + y_iH_i$ for some $x_i, y_i \in \boldsymbol{C}$ such that F_i is in $\boldsymbol{CP}^3$, $i = 1, 2, \ldots, n$. Let $I = \{(i,j,k,l) : 1 \le i < j < k < l \le n\}$. So $|I| = C(n,4)$, where $C(n,4) = \binom{n}{4}$. Consider any quadruple $\{F_i, F_j, F_k, F_l\}$, $(i,j,k,l) \in I$. Denote the matrix

$$\begin{pmatrix} x_ig_{i1}+y_ih_{i1} & x_ig_{i2}+y_ih_{i2} & x_ig_{i3}+y_ih_{i3} & x_ig_{i4}+y_ih_{i4} \\ x_jg_{j1}+y_jh_{j1} & x_jg_{j2}+y_jh_{j2} & x_jg_{j3}+y_jh_{j3} & x_jg_{j4}+y_jh_{j4} \\ x_kg_{k1}+y_kh_{k1} & x_kg_{k2}+y_kh_{k2} & x_kg_{k3}+y_kh_{k3} & x_kg_{k4}+y_kh_{k4} \\ x_lg_{l1}+y_lh_{l1} & x_lg_{l2}+y_lh_{l2} & x_lg_{l3}+y_lh_{l3} & x_lg_{l4}+y_lh_{l4} \end{pmatrix}$$

by $(F_iF_jF_kF_l)$ and its determinant by $|F_iF_jF_kF_l|$. Now we can write

$$\begin{aligned} |F_iF_jF_kF_l| = {} & |G_iG_jG_kG_l|x_ix_jx_kx_l + |H_iG_jG_kG_l|y_ix_jx_kx_l + |G_iH_jG_kG_l|x_iy_jx_kx_l \\ & + |G_iG_jH_kG_l|x_ix_jy_kx_l + |G_iG_jG_kH_l|x_ix_jx_ky_l + |G_iG_jH_kH_l|x_ix_jy_ky_l \\ & + |G_iH_jG_kH_l|x_iy_jx_ky_l + |G_iH_jH_kG_l|x_iy_jy_kx_l + |H_iG_jG_kG_l|y_ix_jx_ky_l \\ & + |H_iG_jH_kG_l|y_ix_jy_kx_l + |H_iH_jG_kG_l|y_iy_jx_kx_l + |G_iH_jH_kH_l|x_iy_jy_ky_l \\ & + |H_iG_jH_kH_l|y_ix_jy_ky_l + |H_iH_jG_kH_l|y_iy_jx_ky_l + |H_iH_jH_kG_l|y_iy_jy_kx_l \\ & + |H_iH_jH_kH_l|y_iy_jy_ky_l. \end{aligned} \tag{4.1}$$

Since it is a point arrangement we only need to consider the case: four planes meet exactly at one point. Replacing $\mathscr{A}^*$ by $\phi(\mathscr{A}^*)$ if necessary where $\phi : \boldsymbol{CP}^3 \to \boldsymbol{CP}^3$ is a complex analytic automorphism, we assume without loss of generality that any one (two, or three) plane(s) in $\mathscr{A}_0^*$ and any three(two or one) plane(s) in $\mathscr{A}_1^*$ do not intersect at a point. Thus, to get $L(\mathscr{A}^*) \equiv L(\mathscr{A}_0^*)$, it is sufficient to have the following: For any $(i,j,k,l) \in I$,

$$rank(F_i, F_j, F_k, F_l) = 3 \text{ if and only if } rank(G_i, G_j, G_k, G_l) = 3.$$

It is equivalent to

$$(1)\quad |F_iF_jF_kF_l| = 0 \text{ if and only if } |G_iG_jG_kG_l| = 0, \tag{4.2}$$

and

(2) there exists one non-zero 3-subdeterminant $D_3(F_iF_jF_kF_l)$ of $|F_iF_jF_kF_l|$ if and only if there exists one non-zero 3-subdeterminant $D_3(G_iG_jG_kG_l)$ of $|G_iG_jG_kG_l|$.

Let $m = \sum_{j\geq 4} C(j,4)p_j(\mathscr{A}_0^*)$. To show (1), we need to consider m equations and $C(n,4) - m$ inequalities

$$P_1 = 0, \ldots, P_m = 0, \tag{4.3}$$

$$Q_1 \neq 0, \ldots, Q_{C(n,4)-m} \neq 0, \tag{4.4}$$

and to show (2), for each $i < j < k < l$ we have to consider a 3-subdeterminant $D_3(F_iF_jF_kF_l)$ in $|F_iF_jF_kF_l|$ such that

$$D_3(F_iF_jF_kF_l) \neq 0 \text{ if and only if } D_3(G_iG_jG_kG_l) \neq 0. \tag{4.5}$$

Here both P_i and Q_j have the forms like (4.1). But for P_i, the first term and last term are zero since $|G_iG_jG_kG_l| = |H_iH_jH_kH_l| = 0$ by (4.2).

To prove the theorem, we need to find a one parameter family of arrangements $\{F_1, F_2, \ldots, F_n\}$ with isomorphic $L(\mathscr{A}_0^*)$. If we can show (1) and show that points $((1:0),(1:0),\ldots,(1:0))$ and $((0:1),(0:1),\ldots,(0:1))$ lie on the same irreducible component of $\{P_1 = 0, \ldots, P_m = 0\}$ but not in varieties $\bigcup_{i=1}^{C(n,4)-m}\{Q_i = 0\}$ and not in the intersection of all $\{D_3(F_iF_jF_kF_l) = 0\}$, then the one parameter family of arrangements with required property can be constructed.

Among $P_1, \ldots, P_m$ at most $c(\mathscr{A}_0^*) = \sum_{j\geq 4}(j-3)p_j(\mathscr{A}_0^*)$ of them are independent. To see this, we consider a j-tuple point v ($j \geq 4$). Let $F_1, \ldots, F_j$ be the planes of $\mathscr{A}^*$ passing though v. We have $C(j,4)$ equations ($|F_iF_jF_kF_l| = 0, \ldots, etc.$). Since $\{F_1, \ldots, F_j\}$ can be linearly generated by three planes, say F_1, F_2 and F_3, the $C(j,4)$ equations are reduced equivalently to $j-3$ equations $|F_1F_2F_3F_k| = 0$ for $i = 4, \ldots, j$. Now consider all j−tuple points ($j \geq 4$). We have a system of $c(\mathscr{A}_0^*)$ equations, say $\{P_1 = 0, \ldots, P_{c(\mathscr{A}_0^*)} = 0\}$ which is equivalent to $\{P_1 = 0, \ldots, P_m = 0\}$.

As we observed before, each P_r can be written as

$$\begin{aligned} P_r = {} & a_r y_{i_r} x_{j_r} x_{k_r} x_{l_r} + b_r x_{i_r} y_{j_r} x_{k_r} x_{l_r} + c_r x_{i_r} x_{j_r} y_{k_r} x_{l_r} + d_r x_{i_r} x_{j_r} x_{k_r} y_{l_r} \\ & + A_r x_{i_r} x_{j_r} y_{k_r} y_{l_r} + B_r x_{i_r} y_{j_r} x_{k_r} y_{l_r} + C_r x_{i_r} y_{j_r} y_{k_r} x_{l_r} + D_r y_{i_r} x_{j_r} x_{k_r} y_{l_r} \\ & + E_r y_{i_r} x_{j_r} y_{k_r} x_{l_r} + F_r y_{i_r} y_{j_r} x_{k_r} x_{l_r} + e_r x_{i_r} y_{j_r} y_{k_r} y_{l_r} + f_r y_{i_r} x_{j_r} y_{k_r} y_{l_r} \\ & + g_r y_{i_r} y_{j_r} x_{k_r} y_{l_r} + h_r y_{i_r} y_{j_r} y_{k_r} x_{l_r} \\ = {} & 0, \end{aligned} \tag{4.6}$$

where $a_r = |H_{i_r}G_{j_r}G_{k_r}G_{l_r}|$, $b_r = |G_{i_r}H_{j_r}G_{k_r}G_{l_r}|$, etc. and $a_rb_rc_rd_rA_rB_rC_rD_rE_rF_re_rf_rg_rh_r \neq 0$ for all $r = 1, \dots, c(\mathscr{A}_0^*)$.

Note that P_r is viewed as a polynomial in $((x_1 : y_1), \dots, (x_n : y_n)) \in (\boldsymbol{CP}^1)^n$. For each r, indices i_r, j_r, k_r, l_r are pairwise distinct and $(i_r, j_r, k_r, l_r) \neq (i_s, j_s, k_s, l_s)$ for $r \neq s$ where $1 \leq i_r, j_r, k_r, l_r, i_s, j_s, k_s, l_s \leq n$ and $1 \leq r, s \leq c(\mathscr{A}_0^*)$.

Since $\mathscr{A}_0^*$ is a nice point arrangement in $\boldsymbol{CP}^3$, then $\mathscr{G}$ has no loop or there are simple joint stars, say $St(v_1), \dots, St(v_s)$ in $\mathscr{G}$ such that they are disjoint and

$$\mathscr{G}' = \mathscr{G} - \bigcup_{i=1}^{s} St(v_i)$$

has no loop, where all $v_i \in \mathscr{G}(0)$.

We shall prove that we can solve all variables in terms of some variables (in the sense of Remark 3.11) without ambiguity. Here we shall use the notation in Definition 2.5.

CASE 0: Assume $\mathscr{G}$ has no loop. We pick a point v_0 with multiplicity k in $\mathscr{G}$. By Definition of $\mathscr{G}$, $k \geq 4$. There are k variables appearing in $k - 3$ equations of (4.6). Without loss of generality we suppose that these variables are $(x_1 : y_1), \dots, (x_k : y_k)$ and $(x_1 : y_1)$, $(x_2 : y_2)$ and $(x_3 : y_3)$ appear in each of these $k - 3$ equations. Thus, we can fix $(x_1 : y_1)$, $(x_2 : y_2)$ and $(x_3 : y_3)$ to solve $(x_4 : y_4), \dots, (x_k : y_k)$. Hence, we can solve all variables at v_0.

From the discussion we know, at each point there are k variables appearing in $k-3$ equations of (4.6). If at most three variables are solved at this point, then we can use these three variables to solve all others. Hence, in the following discussion, we only need to show that at most three variables are solved at each point.

Now, we use induction on the distance from the points to v_0. We consider all 1-points of v_0 which correspond to the end points of $St(v_0)$. Then we shall consider 2-points of v_0, and so on.

Assume we first pick an end point of $St(v_0)$, $u_{1,1}$, it is a 1-point of v_0. By Lemma 3.13 there are at most two planes in the star $St(v_0)$ passing through $u_{1,1}$, which means that at most two variables corresponding to these two planes are solved. Hence we can solve all other variables at $u_{1,1}$. Next we pick another end point of $St(v_0)$, $u_{1,2}$, which does not connect to $u_{1,1}$ by a path that does not pass v_0 from Corollary 3.17. Hence there are at most two planes in the star $St(v_0)$ passing through $u_{1,2}$ by Lemma 3.13. Thus, we can solve all variables at $u_{1,2}$. Continuing this procedure, we can solve all variables at all 1-points of $St(v_0)$.

Assume we can solve all variables at the $(k-1)$-points $u_{k-1,1}, \dots, u_{k-1,m}$. Then consider the k-points. Without loss the generality we assume that k-point $u_{k,1}$ is an end point of $St(u_{k-1,1})$. From induction assumption, all variables at $u_{k-1,1}$ are solved. For $u_{k,1}$, there are at most two planes in $St(u_{k-1,1})$ passing through it by Lemma 3.13 and $u_{k,1}$ can not connect to another j-point $(j < k)$ by the path that does not pass $u_{k-1,1}$ by Corollary 3.17. Hence at most two variables are solved at $u_{k,1}$. Thus, we can solve all variables at $u_{k,1}$. Similarly, we can solve other variables at all k-points. By the induction principle, we can solve all variables at all points of $\mathscr{G}$.

Case 1: Assume $s = 1$ and v_1 is a point of multiplicity k in $\mathscr{A}_0^*$ and $St(v_1)$ is simple joint.

Since $k \geq 4$ by definition of $\mathscr{G}$, there are k variables appearing in $k-3$ equations of (4.6). Similar to Case 0, we can fix three variables and solve all variables at v_1.

The rest of the unsolved variables of equations in (4.6) correspond to the pseudo-complex $\mathscr{G}'$ which has no loop and is a set of the stars.

We also use induction on the distance from the points to v_1.

First, we consider the end points of $St(v_1)$, they are 1-points of v_1.

Case 1.1: If any of the two end points of $St(v_1)$ is not connected by the path in $\mathscr{G}' = \mathscr{G} - St(v_1)$, we can pick each end point of $St(v_1)$ separately. Assume we first pick an end point of $St(v_1)$, $u_{1,1}$. By Lemma 3.13 there are at most two planes in the star $St(v_1)$ passing through $u_{1,1}$, which means that at most two variables corresponding to these two planes are solved. Hence, we can use these variables and solve all other variables at $u_{1,1}$. Next we pick another end point of $St(v_1)$, $u_{1,2}$, which does not connect to $u_{1,1}$ by a path in $\mathscr{G}'$. Hence we can solve all variables at $u_{1,2}$ by the same reason of solving variables at $u_{1,1}$. Continuing this procedure, we can solve all variables at all 1-points of v_1.

Case 1.2: If there are two end points of $St(v_1)$ which are connected by a path in $\mathscr{G}'$, we can choose an end point of $St(v_1)$, say $u_{1,1}$, such that $u_{1,1}$ connects to another end point $u_{1,2}$ of $St(v_1)$. By Lemma 3.13 there are at most two planes, say P_1 and P_2, in the star $St(v_1)$ passing through $u_{1,1}$. We can use the two variables corresponding to P_1 and P_2 and choose another variable, then solve all variables at $u_{1,1}$. Since $St(v_1)$ is simple joint, there is only one path in $\mathscr{G}'$ which connects $u_{1,1}$ and $u_{1,2}$. Assume the plane passing through $u_{1,2}$ in the path is P_3. By Lemma 3.13 there are at most two planes, say P_4 and P_5, in the star $St(v_1)$ passing through $u_{1,2}$. Then we can fix three variables corresponding to P_3, P_4 and P_5, and solve other variables at $u_{1,2}$. Next, consider another end point $u_{1,3}$ of $St(v_1)$. Similarly, since $St(v_1)$ is simple joint, only one of $u_{1,1}$ and $u_{1,2}$ can connect to $u_{1,3}$ by a path in $\mathscr{G}'$. Hence there are at most three variables solved at $u_{1,3}$. Using these three variables we can solve all other variables at $u_{1,3}$. Continuing this procedure, we can solve all variables at all 1-points of v_1.

Assume we can solve all variables at the $(k-1)$-points $u_{k-1,1}, \ldots, u_{k-1,m}$ of v_1. Then consider the k-points of v_1. Without loss of the generality we assume that k-point $u_{k,1}$ is an end point of $St(u_{k-1,1})$ which connects to an end point $u_{1,1}$ of $St(v_1)$. From induction assumption, all variables at $u_{k-1,1}$ are solved. For $u_{k,1}$, there are at most two planes in $St(u_{k-1,1})$ passing through it by Lemma 3.13. $u_{k,1}$ cannot connect to another point that connects to $u_{1,1}$ by Corollary 3.17, and $u_{k,1}$ cannot connect to other two j-point $(j < k)$ by the path in $\mathscr{G}'$ by Corollary 3.14. Hence at most three variables are solved at $u_{k,1}$. Thus, we can solve all variables at $u_{k,1}$. Similarly, using this procedure, we can solve other variables at all k-points.

By induction, we can solve all variables at all points of $\mathscr{G}$.

Case 2: $s = 2$. By the same procedure as above we can solve all variables at v_1 and v_2. If $St(v_1)$ and $St(v_2)$ are not connected by a path in $\mathscr{G}'$, we can solve all variables from them separately. Hence, we only need to consider the case when they are connected.

First, we choose an end point of $St(v_1)$, say $u_{1,1}$. It is a 1-point of v_1. By Lemma 3.13, there are at most two planes in $St(v_1)$ passing through $u_{1,1}$, hence, we can solve the variables at $u_{1,1}$. For other end points of $St(v_1)$, we can solve the variables by the same discussion in Case 1.

Now we consider an end point of $St(v_2)$, say $w_{1,1}$, which connects to an end point of $St(v_1)$, say $u_{1,1}$. We know from Definition 3.6 that $w_{1,1}$ only connects to $u_{1,1}$ by one unique path. Assume the plane passing through $w_{1,1}$ in the path is P_1. Also, by Lemma 3.13, there are at most two planes in $St(v_2)$ which passes through $w_{1,1}$. Assume the planes are P_2 and P_3. Then we use these three solved variables corresponding to P_1, P_2 and P_3 to solve other variables at $w_{1,1}$.

Next, we pick another end point, say $w_{1,2}$. Because $w_{1,2}$ connects to at most one end point of $St(v_1)$ or $St(v_2)$ by Definition 2.6, and at most two planes in $St(v_2)$ pass v_2 and its end point by Lemma 3.13, we know that there are at most three solved variables at $w_{1,2}$. Hence we can use these three solved variables to solve other variables at $w_{1,2}$. Continuing the same procedure, we can solve all variables at the end points of $St(v_1)$ or $St(v_2)$.

Since $\mathscr{G}'$ has no loop, any three points cannot be connected pairwise in $\mathscr{G}'$ by Corollary 3.17 and any point can connect to only one end point or connect to two end points of $St(v_1)$ and $St(v_2)$ by two unique paths in $\mathscr{G}'$ from Corollary 3.15, we can continue this procedure and solve all variables without ambiguity.

Similarly, we can consider the case of $s > 2$.

Thus we can solve all variables in terms of some variables without ambiguity since $\mathscr{G}'$ has no loop.

Now, there are p variables such that all variables are presented as

$$\big((x_1 : y_1), \dots, (x_n : y_n)\big) = f\big((x_1 : y_1), \dots, (x_p : y_p)\big),$$

where each component of f is a composition by some maps as (3.25). So they are homogeneous polynomial of $(x_1 : y_1), \dots, (x_p : y_p)$. Let

$$\begin{aligned} U := (\boldsymbol{CP}^1)^p - \big\{\big((x_1 : y_1), \dots, (x_p : y_p)\big) :\ &\text{for some } 1 \le i \le p, \\ &(x_i : y_i) \text{ is irregular of some equation of (4.6)}\big\}. \end{aligned}$$

By Lemma 3.9, U is an open connected set of $(\boldsymbol{CP}^1)^p$. By Lemma 3.10, f defines an embedding from $U \subset (\boldsymbol{CP}^1)^p$ to $(\boldsymbol{CP}^1)^n$. Since U is irreducible, so is $f(U)$ irreducible. Observe that $(0 : 1)^n = ((0 : 1), \dots, (0 : 1))$ and $(1 : 0)^n = ((1 : 0), \dots, (1 : 0))$ are contained in $f(U)$. We deduce that $(0 : 1)^n$ and $(1 : 0)^n$ are in the same irreducible component of $\{P_1 = 0, \dots, P_{c(\mathscr{A}_0^*)} = 0\}$. In fact, put $(1 : 0)^n$ ($(0 : 1)^n$, respectively) to (4.6), we can see that

$$P_r = 0 \quad \text{for all } r = 1, \dots, c(\mathscr{A}_0^*),$$

and $Q_s = |G_{si}G_{sj}G_{sk}G_{sl}| \neq 0$ ($|H_{si}H_{sj}H_{sk}H_{sl}| \neq 0$, respectively) for all $s = 1, \dots, C(n, 4) - c(\mathscr{A}_0^*)$.

Moreover, let V_r be the variety defined by the zero set of all 3×3 subdeterminants $D_3(F_{ri}F_{rj}F_{rk}F_{rl})$ of $|F_{ri}F_{rj}F_{rk}F_{rl}|$.

To simplify we consider a 3-subdeterminant $D_3(F_iF_jF_kF_l)$ of $|F_iF_jF_kF_l|$ as follows,

$$\begin{aligned} D_3(F_iF_jF_kF_l) &= \begin{vmatrix} x_ig_{i1}+y_ih_{i1} & x_ig_{i2}+y_ih_{i2} & x_ig_{i3}+y_ih_{i3} \\ x_jg_{j1}+y_jh_{j1} & x_jg_{j2}+y_jh_{j2} & x_jg_{j3}+y_jh_{j3} \\ x_kg_{k1}+y_kh_{k1} & x_kg_{k2}+y_kh_{k2} & x_kg_{k3}+y_kh_{k3} \end{vmatrix} \\ &= D_3(G_iG_jG_kG_l)x_ix_jx_k + D_3(H_iG_jG_kG_l)y_ix_jx_k \\ &\quad + D_3(G_iH_jG_kG_l)x_iy_jx_k + D_3(G_iG_jH_kG_l)x_ix_jy_k \\ &\quad + D_3(G_iH_jH_kH_l)x_iy_jy_k + D_3(H_iG_jH_kH_l)y_ix_jy_k \\ &\quad + D_3(H_iH_jG_kH_l)y_iy_jx_k + D_3(H_iH_jH_kH_l)y_iy_jy_k, \end{aligned} \tag{4.7}$$

where $D_3(G_iG_jG_kG_l)$ is the left top 3-subdeterminant of $|G_iG_jG_kG_l|$, and so on.

Put $(1:0)^n$ ($(0:1)^n$, respectively) to (4.7), we can see

$$D_3(F_iF_jF_kF_l) = D_3(G_iG_jG_kG_l) \ \ (\ D_3(H_iH_jH_kH_l), \text{respectively}).$$

Hence, $D_3(F_iF_jF_kF_l) \neq 0$ if and only if $D_3(G_iG_jG_kG_l) \neq 0$ ($D_3(H_iH_jH_kH_l) \neq 0$).

From Definition 2.3 we know that for each point in $\mathscr{G}$ there are three planes passing through it which are in general position. Hence, there exist G_i, G_j and G_k,(H_i, H_j and H_k, respectively) such that

$$D_3(F_iF_jF_kF_l) \neq 0$$

Similarly, we can consider other cases.

Thus, we can see that $(1:0)^n$ and $(0:1)^n$ are not in V_r for $r = 1, \ldots, c(\mathscr{A}_0^*)$.

Now we have shown that (1) and (2) hold in

$$f(U) - \left\{ \left(\bigcup_{s=1}^{C(n,4)-c(\mathscr{A}_0^*)} \{Q_s = 0\} \right) \cup \left(\bigcup_{r=1}^{c(\mathscr{A}_0^*)} V_r \right) \right\}$$

which contains the points $(1:0)^n$ and $(0:1)^n$.

Recall that irreducible variety minus a subvariety is still a connected set. Hence, the irreducible component of $\{P_r = 0$ for $r = 1, \ldots, c(\mathscr{A}_0^*)\}$ minus the subvariety of $\{Q_s = 0$ for all $s = 1, \ldots, C(n,4) - c(\mathscr{A}_0^*)\}$ and the subvarieties V_r for $r = 1, \ldots, c(\mathscr{A}_0^*)$ is still connected. So there is a curve from $((1:0), \ldots, (1:0))$ to $((0:1), \ldots, (0:1))$ such that (4.3), (4.4) and (4.5) are satisfied for any point lying in the curve. This means that we have constructed a one-parameter family of arrangements $\mathscr{A}^*(t)$ such that $\mathscr{A}^*(0) = \mathscr{A}_0^*$, $\mathscr{A}^*(1) = \mathscr{A}_1^*$ and $L(\mathscr{A}^*(t)) \equiv L(\mathscr{A}_0^*)$ for all $t \in [0,1]$.

Now we can apply Lemma 3.12 (Lattice-Isotopy Theorem) and finish the proof of the Theorem. □

In the course of proving Theorem A, we have proved the following Theorem.

Theorem B. *Let $\mathscr{A}^*$ be a nice point arrangement of hyperplanes in $\boldsymbol{CP}^3$. The moduli space of $\mathscr{A}^*$ with fixed combinatorics $L(\mathscr{A}^*)$ is connected.*

Proof. For given two nice point arrangements $\mathscr{A}_0^*$ and $\mathscr{A}_1^*$ of hyperplanes in $\boldsymbol{CP}^3$ with fixed combinatorics $L(\mathscr{A}^*)$, in the proof of Theorem A, we have constructed a one-parameter family $\mathscr{A}^*(t)$ of hyperplanes in $\boldsymbol{CP}^3$ with fixed combinatorics $L(\mathscr{A}^*)$ connecting $\mathscr{A}_0^*$ and $\mathscr{A}_1^*$. Therefore the moduli space of $\mathscr{A}^*$ with fixed combinatorics $L(\mathscr{A}^*)$ is connected. □

Theorem C. *The homotopy groups of the complement $M(\mathscr{A}^*)$ of a nice point arrangement of hyperplanes in $\boldsymbol{CP}^3$ depend only on $L(\mathscr{A}^*)$ (or the lattice $L(\mathscr{A})$).*

Proof. Since the topology of $M(\mathscr{A}^*)$ is determined by $L(\mathscr{A}^*)$, by Theorem A, the homotopy groups of the complement $M(\mathscr{A}^*)$ are determined by $L(\mathscr{A}^*)$. □

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