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Existence and non-existence for strongly coupled quasi-linear cooperative elliptic systems

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Abstract. We study the prototype model of the boundary value problem

$$\begin{aligned} \operatorname{div}(|\nabla u|^{m-2}\nabla u) + u^a v^b = 0 \quad & \text{in } \Omega, \\ \operatorname{div}(|\nabla v|^{m-2}\nabla v) + u^c v^d = 0 \quad & \text{in } \Omega, \\ u = v = 0 \quad & \text{on } \partial\Omega, \end{aligned}$$

where $\Omega \subset \boldsymbol{R}^n$ ($n \geq 2$) is a connected smooth domain, and the exponents $m > 1$ and $a, b, c, d \geq 0$ are non-negative numbers. Under appropriate conditions on the exponents m, a, b, c and d, and on the domain Ω, a variety of results on a priori estimates, existence and non-existence of positive solutions have been established.

1. Introduction.

In the article [**15**], the author studied the existence of positive solutions for strongly coupled systems of semi-linear elliptic differential equations. The purpose of the current paper is to extend the studies in [**15**] to systems with quasi-linear operators as principal parts. More precisely, our main objective is to obtain existence of a positive solution for the system (1.1) below.

Let $\Omega \subset \boldsymbol{R}^n$ ($n \geq 2$) be a connected smooth domain. Consider the following systems of quasi-linear elliptic differential equations

$$\begin{aligned} \Delta_m u + u^a v^b = 0 \quad & \text{in } \Omega, \\ \Delta_m v + u^c v^d = 0 \quad & \text{in } \Omega, \end{aligned} \tag{1.1}$$

where $u, v \geq 0$, $m > 1$ and

$$\Delta_m \cdot = \operatorname{div}(|\nabla \cdot|^{m-2}\nabla \cdot)$$

is the m–Laplace operator, and a, b, c and d are non-negative numbers. We are concerned with the question of existence of a non-negative and non-trivial (vector-valued)[1] function $\boldsymbol{u} = (u, v)$ satisfying (1.1). In the sequel, we use bold face letters to denote vector values.

A function $\boldsymbol{u} \in W^{1,m}_{loc}(\Omega) \cap C(\Omega)$ is said to be a *weak solution* (*weak super-solution*), or simply a *solution* (*super-solution*), of (1.1) if

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[1] In this paper, all relations involving vectors are understood in the component-wise sense.

$$-\int |\nabla u|^{m-2}\nabla u \cdot \nabla\varphi_1 + \int u^a v^b \varphi_1 = (\le)0$$

$$-\int |\nabla v|^{m-2}\nabla v \cdot \nabla\varphi_2 + \int u^c v^d \varphi_2 = (\le)0$$

for all $\Phi = (\varphi_1, \varphi_2) \in C_0^\infty(\Omega)$ ($\Phi = (\varphi_1, \varphi_2) \in C_0^\infty(\Omega)$ non-negative). Whenever Ω is bounded, we associate (1.1) with the homogeneous Dirichlet boundary condition

$$u = v = 0, \qquad x \in \partial\Omega$$

and, naturally, it is understood that solutions of (1.1) will be in the space $W_0^{1,m}(\Omega) \cap C_0(\Omega)$. All solutions considered in this paper are weak solutions in the above sense. For simplicity, we shall also assume that Ω is connected.

The goal of this paper is to develop existence and non-existence and, moreover, a priori estimate results on positive solutions of (1.1). Throughout the entire paper, we shall assume that the non-negative exponents a, b, c and d satisfy

$$\min(b, c) > 0 \quad \text{(i.e., fully-coupled).}$$

Furthermore, we shall only consider the 'super-linear' case in which

$$\beta := bc - \alpha\delta > 0 \quad \text{where} \quad \alpha := m - 1 - a, \quad \delta := m - 1 - d. \tag{1.2}$$

When, in addition,

$$a + d > 0,$$

then (1.1) is called strongly-coupled, a term used in [**15**]. For convenience, for $m < n$, denote

$$m_* := \frac{n(m-1)}{n-m}, \qquad m^* := \frac{n(m-1)+m}{n-m}.$$

($m_* = m^* = \infty$ if $m \ge n$).

We are now ready to state our results. The first result is a Liouville type non-existence theorem on exterior domains. We say that a domain $\Omega \subset \boldsymbol{R}^n$ is exterior if $\Omega \supset \{x \in \boldsymbol{R}^n \mid |x| > R\}$ for some $R > 0$ and $\Omega \ne \boldsymbol{R}^n$.

THEOREM 1.1. *Let $\Omega \subset \boldsymbol{R}^n$ be exterior. Then* (1.1) *does not admit any positive super-solutions, provided that one of the following holds*:

A). $n \le m$.
B). $n > m$, $\min(\alpha, \delta) > 0$ *and*

$$\max\{b + \delta, c + \alpha\} > \frac{n\beta}{mm_*}. \tag{1.3}$$

C). $n > m$, $\delta \le 0 < \alpha$ *and*

$$\max\left\{\frac{\beta}{c-\delta}, c+\alpha\right\} > \frac{n\beta}{mm_*}. \tag{1.4}$$

D). $n > m$, $\alpha \le 0 < \delta$ *and*

$$\max\left\{b+\delta, \frac{\beta}{b-\alpha}\right\} > \frac{n\beta}{mm_*}. \tag{1.5}$$

E). $n > m$, $\max(\alpha, \delta) \le 0$ *and*

$$\min(a+b, c+d) < m_*. \tag{1.6}$$

Remarks.

1. Case B) was proved in [**1**] (see [**1**, Theorem 5.3, p. 41]) for a more general version of (1.1), where different exponents $m_1 = p$ and $m_2 = q$ (associated with the m-Laplace operator) were used in the equations $(1.1)_1$ and $(1.1)_2$ respectively.

2. When $\delta < 0 < \alpha$ and $b+\delta > c+\alpha$, (1.3) is weaker than (1.4) since $\max\{b+\delta, c+\alpha\} = b+\delta > \beta/(c-\delta)$ (they are equivalent if $\delta = 0$ or $b+\delta \le c+\alpha$). Similarly, (1.3) is weaker than (1.5) for $\alpha < 0 < \delta$ and $b+\delta < c+\alpha$. There are similar relations between (1.3) and (1.6).

3. In Case C), (1.4) amounts to (1.6), provided $b+\delta \ge c+\alpha$ since $a+b \ge c+d$ and $\beta/(c-\delta) \ge c+\alpha$, and similarly (1.5) amounts to (1.6) in Case D), provided $b+\delta \le c+\alpha$. Also note, if $\min(\alpha, \delta) \ge 0$, then (1.3) is stronger than (1.6). However, in general (i.e., no restrictions on α and δ), (1.3) does not yield any upper bound for the quantities a, b, c and d. In fact, under (1.3) and the condition $\beta > 0$, all four quantities a, b, c and d could be arbitrarily large simultaneously.

As in [**15**], with the aid of Theorem 1.1, we shall utilize the blow-up method to derive supremum a priori estimates for non-negative monotone solutions of (1.1) on bounded domains. Fixing a positive function $\boldsymbol{h}(x) = (h_1(x), h_2(x)) \in C(\overline{\Omega})$, consider the system of equations

$$\begin{aligned} \Delta_m u + u^a v^b + t h_1(x) &= 0 \quad \text{in } \Omega, \\ \Delta_m v + u^c v^d + t h_2(x) &= 0 \quad \text{in } \Omega, \\ \boldsymbol{u} &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{1.7}$$

where $t \ge 0$ and Ω is a bounded domain in $\boldsymbol{R}^n$. Define[2]

$$\mathscr{C} = \{\boldsymbol{u} \in C_0(\Omega) \mid \boldsymbol{u} \ge 0;\ u, v \text{ monotone in } \Omega\}.$$

Then we have the following supremum a priori estimates.

[2]See Definitions 2.1–2.3 in section 2 for details on the definition of monotone functions.

Theorem 1.2. *Let $\Omega \subset \boldsymbol{R}^n$ be an uniformly normal domain and let $\gamma' \in (0,1)$ be the Holder exponent associated with $\partial\Omega$ (see precise definition in section 2). Assume*

$$\min(a,d) \geq m-1.$$

Suppose that either condition A) or condition E) in Theorem 1.1 holds. If E) holds, we further assume that

$$\max(a,d) < m^*.$$

Then there exists a positive constant $C>0$ depending on the structural constants and $\boldsymbol{h}$ such that

$$t + \max_{x\in\Omega} u(x) + \max_{x\in\Omega} v(x) \leq C$$

for all non-negative solutions $\boldsymbol{u}$ of (1.7) which are monotone in Ω (i.e., $\boldsymbol{u} \in \mathscr{C}$).

Furthermore, there exist $\gamma = \gamma(n,m,a,b,c,d,\Omega) \in (0,\gamma')$ and $C = C(n,m,a,b,c,d,\Omega,t,\boldsymbol{h}) > 0$ such that $\boldsymbol{u} \in C^{1,\gamma}(\overline{\Omega})$ and there holds

$$\|\boldsymbol{u}\|_{1,\gamma,\overline{\Omega}} \leq C$$

for all non-negative solutions $\boldsymbol{u}$ of (1.7) which are monotone in Ω (i.e., $\boldsymbol{u} \in \mathscr{C}$).

Remark. The condition $\min(a,d) \geq m-1$, equivalent to $\max\{\alpha,\delta\} \leq 0$, is removed in [**16**]. As a result, Theorem 1.2, as well as Theorem 1.3 below, remains valid under any one of the five alternatives given in Theorem 1.1.

Theorem 1.2 provides crucial estimates for the subclass of the so-called monotone solutions of (1.1) which turn out to be sufficient for establishing existence of a positive solution (see Theorem 1.3 below). However, the results seem far from optimum, even in the case of $m=2$ (see [**15**], [**16**] and the references therein). In this regard, both a presence of the $m-$Laplace operator ($m \neq 2$) and a strong coupling (i.e., lack of a variational structure) play a key role.

With the aid of the a priori estimates of Theorem 1.2, we are able to employ a fixed point theorem to prove existence of a positive solution for (1.1) on bounded domains.

Below is the main existence result of the paper.

Theorem 1.3. *Let $\Omega \subset \boldsymbol{R}^n$ be an uniformly normal domain and suppose that all conditions of Theorem 1.2 are satisfied. Then (1.1), coupled with the homogeneous Dirichlet boundary conditions, has a positive solution $\boldsymbol{u}$.*

Theorem 1.3 is, to the best knowledge of the author, the first existence result of non-radial solutions for a general (1.1) (i.e., with no variational structure). When $\Omega = B$ is an Euclidean ball, existence of a positive radial solution was obtained in [**2**] for (1.1) under a set of conditions including $\max(a,d) \leq m-1$. Several existence and non-existence results were also obtained in [**4**], [**5**] if (1.1) has a variational structure. In the above mentioned works, two different exponents $m_1 = p$ and $m_2 = q$ (associated with the m-Laplace operator) were used in the equations $(1.1)_1$ and $(1.1)_2$ respectively.

A fixed point theorem is used to prove Theorem 1.3 and the priori estimates of Theorem 1.2 are crucial. In the literature, it has been traditional (in the non-radial case) to derive estimates for *all* positive solutions in this regard. An elementary yet useful observation here is that one only needs estimates for a (certain) *subclass* of positive solutions to obtain existence. Based on this observation, we in section 2 introduce a cone ($\mathscr{C}$) of so-called monotone functions on an uniformly normal domain and show that the m-Laplace operator preserves such a monotonicity. With this new ingredient, we are able to establish the desired a priori estimates for elements in the class of monotone functions and consequently prove Theorem 1.3. This clearly offers a new perspective in this direction, which could be particularly useful for handling systems in the absence of traditional estimates.

The class of uniformly normal domains is rather broad. For instance, it includes bounded convex domains whose boundary has non-negative curvatures (but the convexity is not necessary though). Evidently, Euclidean balls, ellipsoids and rectangular boxes, etc., are included.

It is worth to point out that our approach applies to general quasi-linear cooperative elliptic systems which, in particular, need not have a variational structure (in fact, (1.1) does not have a variational structure for 'almost all' a, b, c, d values). Indeed, the principal terms $\Delta_m u$ in $(1.1)_1$ and $\Delta_m v$ in $(1.1)_2$ can be replaced by $\operatorname{div}(A(x, u, \nabla u))$ and $\operatorname{div}(B(x, v, \nabla v))$ respectively. The pure power terms $u^a v^b$ and $u^c v^d$ may be replaced by general functions $f(x, u, v, \nabla u, \nabla v)$ and $g(x, u, v, \nabla u, \nabla v)$ which are cooperative (see for example [**14**], [**15**] for precise meaning). Moreover, systems consisting of more than two equations can be treated.

The paper is structured as follows: We consider some preliminary results in section 2. The non-existence result (Theorem 1.1) is proved in Section 3. In Section 4, we establish the desired a priori estimates 1.2. Finally, in section 5, we prove the existence result Theorem 1.3.

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2. Preliminaries.

In this section, we discuss some preliminary results which will be used later. We use $C > 0$ throughout to denote generical positive constants depending on structural constants and arguments inside the parentheses, which may vary from one to another.

Let $\Omega \subset \boldsymbol{R}^n$ be a domain. Suppose that $f(x, u) \in C(\Omega \times \boldsymbol{R})$ and $c(x), g(x) \in L^\infty(\Omega)$ are real functions. Consider

$$\Delta_m u + c(x)|u|^{m-2}u + f(x, u) + g(x) = 0 \text{ in } \Omega. \tag{2.1}$$

We first present four lemmas regarding (2.1), being special cases of classical results for quasi-linear elliptic equations. The first one is the $C^{k,\gamma}$-regularity, $k = 0, 1$. For $k \geq 0$

(integer) and $\gamma \in [0,1)$, we use

$$\|u\|_{k,\gamma,\Omega} = \max_{0\le|\eta|\le k} \sup_{x\in\Omega} |D^\eta u(x)| + \max_{|\eta|=k} \sup_{x,y\in\Omega, x\neq y} \frac{|D^\eta u(x) - D^\eta u(y)|}{|x-y|^\gamma}$$

to denote the standard $C^{k,\gamma}$-norms on Ω, where $\eta = (\eta_1, \dots, \eta_n)$ is an n-index of non-negative integers, with $|\eta| = \eta_1 + \cdots + \eta_n$.

Lemma 2.1 ($C^{k,\gamma}$-estimate). *Let $u \in L^\infty(\Omega)$ be a solution of* (2.1) *and suppose $f \equiv 0$. Then the following conclusions hold.*

A). *For any sub-domain $\Omega' \subset\subset \Omega$, there exist*

$$\gamma = \gamma\big(n, m, \|c\|_{L^\infty(\Omega)}, \min\{1, \operatorname{dist}(\Omega', \partial\Omega)\}\big) \in (0,1)$$

and

$$C = C\big(\gamma, n, m, \|c\|_{L^\infty(\Omega)}, \min\{1, \operatorname{dist}(\Omega', \partial\Omega)\}, \Omega'\big) > 0$$

such that $u \in C^{1,\gamma}(\Omega')$ and

$$\|u\|_{1,\gamma,\Omega'} \le C\big(\|g\|_{L^\infty(\Omega)}^{1/(m-1)} + \|u\|_{L^\infty(\Omega)}\big).$$

B). *If, in addition, assume that Ω is a bounded domain with $C^{k,\gamma'}$-boundary for some $\gamma' \in (0,1)$ and $k = 0, 1$ and u vanishes on $\partial\Omega$ and c, $g \in L^\infty(\Omega)$. Then there exist*

$$\gamma = \gamma(n, m, \|c\|_{L^\infty(\Omega)}, \Omega) \in (0, \gamma']$$

and

$$C = C(\gamma, n, m, \|c\|_{L^\infty(\Omega)}, \Omega) > 0$$

such that $u \in C^{k,\gamma}(\overline{\Omega})$, $k = 0, 1$, and

$$\|u\|_{k,\gamma,\overline{\Omega}} \le C\big(\|g\|_{L^\infty(\Omega)}^{1/(m-1)} + \|u\|_{L^\infty(\Omega)}\big).$$

For a proof of Lemma 2.1, we refer the reader to [**6**], [**8**], [**12**] and the references therein. □

The second lemma is the Harnack inequality.

Lemma 2.2 (Harnack Inequality). *Let $u \ge 0$ be a solution of* (2.1) *and suppose $f \equiv 0$. Then for any sub-domain $\Omega' \subset\subset \Omega$, there exists*

$$C = C\big(n, \Omega', \Omega, \min\{1, \operatorname{dist}(\Omega', \partial\Omega)\}, \|c\|_{L^\infty(\Omega)}\big) > 0$$

such that

$$\sup_{\Omega'} u \le C\Big(\inf_{\Omega'} u + \|g\|_{L^\infty(\Omega)}^{1/(m-1)}\Big).$$

PROOF. This follows directly from (a slight variation of) the combination of Theorems 5, 6 and 9 of [**10**]. □

The third lemma is Liouville type non-existence results.

LEMMA 2.3 (Liouville Theorems). *Let $\Omega = \boldsymbol{R}^n$. Suppose*

$$c(x) \equiv 0, \quad g(x) \equiv \kappa(\text{Const.}) \ge 0, \quad f(x,u) \ge 0 \text{ for } u \ge 0.$$

Then there hold

A). *Any non-negative solution u of* (2.1) *must be constant if $\kappa = 0$ and $f \equiv 0$.*
B). *Any non-negative solution u of* (2.1) *must be constant if $\kappa \ge 0$, $f \ge 0$ and $n \le m$.*
C). *If $\kappa > 0$, then* (2.1) *admits no non-negative solution.*
D). *Suppose $\kappa = 0$ and $f(x,u) = cu^p$ for some $c > 0$ and $p \in (0, m^*)$. Then any non-negative solution u of* (2.1) *must be identically zero.*

PROOF. A) and B) are the classical Liouville theorems, see for instance Theorem II(a) [**11**].

Part C) follows directly from Lemma 2.8 [**11**]. Indeed, suppose the contrary and let u be a non-negative solution of (2.1). Then Lemma 2.8 of [**11**] implies $u \equiv 0$ since $\kappa > 0$.[3] This in turn shows that $\kappa = 0$, which is an immediate contradiction.

Part D) was proved in [**1**] when $p \le m_*$ and later extended to the above full range in [**11**], see for example Theorems I and II(c) [**11**]. □

The last lemma concerning (2.1) is a strong maximum principle.

LEMMA 2.4 (Strong Maximum Principle). *Let u be a non-negative solution of* (2.1)*. Suppose $g(x) \ge 0$ and $f(x,u) \ge 0$ for $u \ge 0$. Then either $u \equiv 0$ or $u > 0$ on Ω.*

PROOF. This is due to [**13**], see also Theorem 1 [**9**]. □

We shall also need the following weak comparison principle.

LEMMA 2.5 (Weak Comparison Principle). *Let u and v be continuous functions in $W^{1,m}_{loc}(\Omega)$ and satisfy the distribution inequality*

$$\Delta_m u + h(u) \ge \Delta_m v + h(v), \tag{2.2}$$

where h is a non-increasing function. Suppose that $u \le v$ on $\partial\Omega$. Then $u \le v$ in Ω.

[3] Lemma 2.8 was proved for $\Delta_m u + u^p \le 0$ with $p \in (0, m-1)$ in [**11**]. A slight modification shows the arguments are valid for the inequality $\Delta_m u + \kappa \le 0$ with $\kappa > 0$.

Proof. The result is standard and well-known (cf. [**11**, Lemma 2.2]) and we sketch a proof here for the reader's convenience. Suppose the contrary, that is,

$$w = (u - v)_+ \not\equiv 0.$$

Multiply (2.2) by the test function $w \in W_0^{1,m}(\Omega)$ and integrate over Ω to obtain

$$\int_\Omega \left(|\nabla u|^{m-2}\nabla u - |\nabla v|^{m-2}\nabla v\right)\nabla w \le \int_\Omega (h(u) - h(v))w.$$

Plainly, $(h(u) - h(v))w \le 0$ for all $x \in \Omega$ since h is non-increasing by assumption. It follows that there exists $C > 0$ such that

$$C\int_\Omega (|\nabla u| + |\nabla v|)^{m-2}|\nabla w|^2 \le \int_\Omega (h(u) - h(v))w \le 0.$$

In turn, we have

$$\nabla w \equiv 0 \Longrightarrow w \equiv 0.$$

This is a contradiction and completes the proof. □

We next introduce the notion of uniformly normal domains and the class of monotone functions on an uniformly normal domain. For $\sigma \in \boldsymbol{R}$ and $\nu \in S^{n-1}$ (a direction in $\boldsymbol{R}^n$), denote by $\Gamma_{\sigma,\nu}$ the hyperplane

$$\Gamma_{\sigma,\nu} = \{x \in \boldsymbol{R}^n \,|\, x \cdot \nu = \sigma\}.$$

For a fixed $\nu \in S^{n-1}$, denote

$$\Omega_{\sigma,\nu} = \{x \in \Omega \,|\, x \cdot \nu > \sigma\},$$

the positive cap of Ω with respect to $\Gamma_{\sigma,\nu}$ (in the direction of ν).

Let $x^{\sigma,\nu}$ be the *reflection* in $\Gamma_{\sigma,\nu}$ of a point x in $\boldsymbol{R}^n$, that is,

$$x^{\sigma,\nu} = x + 2(\sigma - x \cdot \nu)\nu,$$

and similarly let $\Omega^{\sigma,\nu}$ be the reflection in $\Gamma_{\sigma,\nu}$ of a set Ω in $\boldsymbol{R}^n$,

$$\Omega^{\sigma,\nu} = \{x^{\sigma,\nu} \,|\, x \in \Omega\}.$$

Now, let Ω be a bounded domain with $C^{1,\gamma'}$-boundary[4] for some $\gamma' \in (0,1)$. For

[4]The regularity requirement that $\partial\Omega$ is of class of $C^{1,\gamma'}$ can be relaxed. For example, rectangular boxes maybe considered.

$z \in \partial\Omega$, denote the (unit) outer-normal $\nu_z = \nu(z)$ at z and put

$$\sigma_z := \sup_{x\in\Omega}\{x \cdot \nu_z\} \geq z_\nu := z \cdot \nu_z.$$

Definition 2.1. Let $z \in \partial\Omega$. We say that Ω is normal at z if there exists $\sigma' < z_\nu$ such that for all $\sigma \in (\sigma', \sigma_z)$

$$(\Omega_{\sigma,\nu_z})^{\sigma,\nu_z} \subset \Omega.$$

Next, we put

$$\beta_z = \beta(z) := \inf\big\{\sigma \in \boldsymbol{R} \mid (\Omega_{t,\nu_z})^{t,\nu_z} \subset \Omega \text{ for all } t \in (\sigma, \sigma_z)\big\} \in (-\infty, z_\nu), \tag{2.3}$$

provided that Ω is normal at z.

For a bounded smooth domain Ω with $C^{1,\gamma}$-boundary (for simplicity), we can introduce the following notion of uniform normality.

Definition 2.2. Let Ω be a bounded domain with $C^{1,\gamma'}$-boundary for some $\gamma' \in (0,1)$. We say that Ω is uniformly normal if Ω is normal at every $z \in \partial\Omega$, and there holds

$$\delta_0 = \delta_0(\Omega, \partial\Omega) := \inf_{z\in\partial\Omega}\{z_\nu - \beta(z)\} > 0, \tag{2.4}$$

where $z_\nu = z \cdot \nu_z$ and ν_z is the (*unit*) outer-normal at z and $\beta(z)$ is given by (2.3).

We shall refer to γ' the Holder exponent associated with $\partial\Omega$.

Let Ω be uniformly normal. We can introduce the class of monotone functions as follows.

Definition 2.3. Let Ω be uniformly normal and let $g(x)$ be a non-negative continuous function on Ω. We say that g is monotone in Ω if for all $z \in \partial\Omega$ and for all $\sigma \in (\beta_z, \sigma_z)$

$$g(x^{\sigma,\nu_z}) \geq g(x), \qquad x \in \Omega_{\sigma,\nu_z}.$$

We denote by $MO(\Omega)$ the set of all monotone functions on Ω.

We conclude this section with two lemmas on properties of the operator $(-\Delta_m)^{-1}$ (with homogeneous Dirichlet boundary data).

Lemma 2.6. *Let Ω be a bounded domain with $C^{1,\gamma'}$-boundary for some $\gamma' \in (0,1)$. Then there exists $\gamma_0 \in (0,\gamma')$ such that the operator*

$$(-\Delta_m)^{-1} : C(\overline{\Omega}) \mapsto C^{1,\gamma_0}(\overline{\Omega}) \cap C_0(\Omega)$$

is well-defined and continuous.

Proof. The proof is standard, but we sketch here for the reader's convenience. For $g(x) \in C(\overline{\Omega})$, clearly the boundary value problem

$$\begin{aligned} \Delta_m u + g(x) = 0 \quad & \text{in } \Omega, \\ u = 0 \quad & \text{on } \partial\Omega \end{aligned}$$

has a unique solution $u \in W_0^{1,m}(\Omega)$. Thus $(-\Delta_m)^{-1}$ is well-defined with $u(x) = (-\Delta_m)^{-1}(g(x))$. By Lemma 2.1, there exist

$$\gamma = \gamma(n, m, \Omega) \in (0, \gamma'], \qquad C = C(\gamma, n, m, \Omega) > 0,$$

where $\gamma' \in (0,1)$ is given in the Lemma, such that $u \in C^{1,\gamma}(\overline{\Omega})$ and there holds

$$\|u\|_{1,\gamma,\overline{\Omega}} \le C \|g\|_{C(\overline{\Omega})}^{1/(m-1)} \tag{2.5}$$

for all $g \in C(\overline{\Omega})$. Take $\gamma_0 = \gamma/2 \in (0,1)$. Suppose for contradiction that

$$(-\Delta_m)^{-1} : C(\overline{\Omega}) \mapsto C^{1,\gamma_0}(\overline{\Omega}) \cap C_0(\Omega)$$

is not continuous. Then there exist $g_0(x) \in C(\overline{\Omega})$, a sequence of functions $\{g_l(x)\} \subset C(\overline{\Omega})$, and $u_l(x) = (-\Delta_m)^{-1}(g_l(x))$ and $u_0(x) = (-\Delta_m)^{-1}(g_0(x))$ such that

$$\lim_{l\to\infty} \|g_l - g_0\|_{C(\overline{\Omega})} = 0, \qquad \liminf_{l\to\infty} \|u_l - u_0\|_{1,\gamma_0,\overline{\Omega}} > 0. \tag{2.6}$$

Without loss of generality, we assume

$$\|g_l\|_{C(\overline{\Omega})} \le \|g_0\|_{C(\overline{\Omega})} + 1.$$

Then, by (2.5), the set $\{u_l(x)\}$ is bounded in $C^{1,\gamma}(\overline{\Omega}) \cap C_0(\Omega)$. By the Ascoli-Arzela theorem, $\{u_l\}$ converges to some $u \in C^{1,\gamma_0}(\overline{\Omega})$ in $C^{1,\gamma_0}(\overline{\Omega})$ (up-to a subsequence). Fix any function $\varphi \in C_0^\infty(\Omega)$. Using φ as a test function in the corresponding equations $\Delta_m u_l(x) + g_l(x) = 0$ and letting $l \to \infty$, one readily deduces that $\Delta_m u(x) + g_0(x) = 0$, namely, $u = (-\Delta_m)^{-1}(g_0(x)) = u_0$. This is an immediate contradiction to (2.6) and completes the proof. □

The next lemma shows that $(-\Delta_m)^{-1}$ preserves the class of monotone functions.

Lemma 2.7. *Suppose that Ω is uniformly normal and $g \in MO(\Omega)$. Let u be a solution of*

$$\begin{aligned} \Delta_m u + g(x) = 0 \quad & \textit{in } \Omega, \\ u = 0 \quad & \textit{on } \partial\Omega. \end{aligned}$$

Then $u \in MO(\Omega)$.

Proof. By Lemmas 2.4 and 2.5, u is either strictly positive or identically zero on Ω since $g(x) \geq 0$. Thus we assume $u > 0$ (nothing left to prove if $u \equiv 0$). It remains to show

$$u(x) \leq u(x^{\sigma,\nu_z}), \qquad x \in \Omega_{\sigma,\nu_z} \tag{2.7}$$

for all $z \in \partial\Omega$ and all $\sigma \in (\beta_z, \sigma_z)$.

Fix $z \in \partial\Omega$ and $\sigma \in (\beta_z, \sigma_z)$, put

$$v(x) = u(x^{\sigma,\nu_z}), \qquad x \in \Omega_{\sigma,\nu_z}.$$

We want to show

$$u(x) - v(x) \leq 0, \qquad x \in \Omega_{\sigma,\nu_z}. \tag{2.8}$$

By the assumption, $g \in MO(\Omega)$, that is,

$$-g(x) \geq -g(x^{\sigma,\nu_z}), \qquad x \in \Omega_{\sigma,\nu_z}.$$

It follows that

$$\begin{aligned} -g(x) = \Delta_m u \geq \Delta_m v = -g(x^{\sigma,\nu_z}), &\quad \text{in } \Omega_{\sigma,\nu_z}, \\ u \leq v, &\quad \text{on } \partial(\Omega_{\sigma,\nu_z}), \end{aligned}$$

since $u > 0$ in Ω. Thus (2.8) follows from Lemma 2.5 at once.

Since $\sigma \in (\beta_z, \sigma_z)$ and $z \in \partial\Omega$ are arbitrary, (2.7) is valid. In particular, $u \geq 0$ belongs to $MO(\Omega)$ and the proof is complete. □

3. Non-existence.

In this section, we prove a Liouville type non-existence result for positive super-solutions of (1.1) on exterior domains. Throughout this section, we assume that Ω is *exterior*, that is, $\Omega \supset \{|x| > R > 0\}$ for some $R > 0$ and $\Omega \neq \boldsymbol{R}^n$. The letter C will be used throughout to denote generic positive constants, which may vary from line to line and only depend on arguments inside the parentheses or are otherwise clear from the context. Also $r = |x|$ will be used throughout this section.

We begin with technical lemmas. Recall that $\beta > 0$ throughout.

Lemma 3.1. *Let u be a continuous positive solution of the inequality*

$$\Delta_m u \leq 0, \qquad x \in \Omega.$$

Then there exists a constant $C = C(m, n, u, R) > 0$ such that for $r > 2R$

$$u(x) \geq Cr^{-(n-m)/(m-1)},$$

provided $n > m$, *while*

$$\liminf_{x \to \infty} u(x) > 0,$$

if $n \leq m$.

PROOF. This is Lemma 2.3 [**11**]. □

By a super-solution $\boldsymbol{u} = (u, v)$ of (1.1), we mean

$$\begin{aligned} -\int |\nabla u|^{m-2} \nabla u \cdot \nabla \varphi_1 + \int u^a v^b \varphi_1 \leq 0 \\ -\int |\nabla v|^{m-2} \nabla v \cdot \nabla \varphi_2 + \int u^c v^d \varphi_2 \leq 0 \end{aligned} \tag{3.1}$$

for all non-negative $\Phi = (\varphi_1, \varphi_2) \in C_0^\infty(\Omega)$. By $B_s = B_s(x_0)$ we mean a ball of radius $s > 0$ and center x_0. In the sequel, we assume that (the closure of) the ball $B_s(x_0)$ of radius s is contained in Ω. By ξ we mean a radially symmetric C^2 cut-off function on $B_1(0)$, see for example [**11**]. For $s > 0$, put

$$\underline{u} = \inf_{B_s} u(x) > 0, \qquad \underline{v} = \inf_{B_s} v(x) > 0.$$

We have the following type of upper-bounds for positive super-solutions of (1.1).

LEMMA 3.2. *Let* $\boldsymbol{u}$ *be a positive super-solution of* (1.1) *in* Ω. *Then there exists a positive constant* $C = C(n, m, a, b, c, d) > 0$ *such that for* $s > 0$

1. *If* $\min(\alpha, \delta) > 0$, *then*

$$\underline{u} \leq Cs^{-m(b+\delta)/\beta}, \qquad \underline{v} \leq Cs^{-m(c+\alpha)/\beta}.$$

2. *If* $\alpha > 0 \geq \delta$, *then*

$$\underline{u}^c \underline{v}^{-\delta} \leq Cs^{-m}, \qquad \underline{v} \leq Cs^{-m(c+\alpha)/\beta}.$$

3. *If* $\delta > 0 \geq \alpha$, *then*

$$\underline{u} \leq Cs^{-m(b+\delta)/\beta}, \qquad \underline{u}^{-\alpha} \underline{v}^b \leq Cs^{-m}.$$

4. *If* $\max(\alpha, \delta) \leq 0$, *then*

$$\underline{u}^c \underline{v}^{-\delta} \leq Cs^{-m}, \qquad \underline{u}^{-\alpha} \underline{v}^b \leq Cs^{-m}.$$

PROOF. Fix $s > 0$ and $x_0 \in \Omega$. For $k, l > m$ and $e, f > 0$, one uses the test function $\Phi = (\xi^k u^{-e}, \xi^l v^{-f}) \geq 0$ ($\xi = \xi(|x - x_0|/s)$) in (3.1) to deduce that (cf. (2.8), [**11**, Lemma 2.4])

$$\frac{e}{2}\int \xi^k u^{-e-1}|\nabla u|^m + \int \xi^k u^{a-e}v^b \leq Cs^{-m}\int \xi^{k-m}u^{m-1-e}, \tag{3.2}$$

and

$$\frac{f}{2}\int \xi^l v^{-f-1}|\nabla v|^m + \int \xi^l u^c v^{d-f} \leq Cs^{-m}\int \xi^{l-m}v^{m-1-f}. \tag{3.3}$$

By (1.2),

$$\max\{c-\alpha, b-\delta\} > 0, \qquad (m-1-b-d)\alpha = (\delta - b)\alpha < b(c-\alpha). \tag{3.4}$$

We next proceed to prove Cases 1-2), with 3-4) being similar and left to the reader.

Case 1). We further divide the proof into two cases.

Case 1-i). $\min\{c-\alpha, b-\delta\} \geq 0$. By (3.4), without loss of generality, assume $c > \alpha$, $b \geq \delta$. Taking $e = a > 0$ and $f = d > 0$ in (3.2) and (3.3), respectively, yields

$$\int \xi^k v^b \leq Cs^{-m}\int \xi^{k-m}u^\alpha, \qquad \int \xi^l u^c \leq Cs^{-m}\int \xi^{l-m}v^\delta. \tag{3.5}$$

By Holder's inequality and (3.5), we have (taking $k, l = k+m$ large and noting $c > \alpha$, $b \geq \delta$)

$$\begin{aligned}\int \xi^k v^b &\leq Cs^{-m}\int \xi^{k-m}u^\alpha \leq Cs^{n(1-\alpha c^{-1})-m}\cdot\Big(\int \xi^{k+m}u^c\Big)^{\alpha c^{-1}}\\ &\leq Cs^{n(1-\alpha c^{-1})-m(1+\alpha c^{-1})}\Big(\int \xi^k v^\delta\Big)^{\alpha c^{-1}}\\ &\leq Cs^{n(1-\alpha c^{-1})-m(1+\alpha c^{-1})+n\alpha c^{-1}(1-\delta b^{-1})}\Big(\int \xi^k v^b\Big)^{\alpha\delta b^{-1}c^{-1}}\\ &= Cs^{nb^{-1}c^{-1}\beta - m(1+\alpha c^{-1})}\Big(\int \xi^k v^b\Big)^{\alpha\delta b^{-1}c^{-1}}.\end{aligned}$$

If $c \geq \alpha$ and $b > \delta$, one derives the above inequality similarly by taking $k = l + m$ large. In turn, there holds for all k large

$$\int \xi^k v^b \leq Cs^{n - mb(c+\alpha)/\beta}, \tag{3.6}$$

since $\beta > 0$. Similarly, one deduces for all l large

$$\int \xi^l u^c \le C s^{n-mc(b+\delta)/\beta}. \tag{3.7}$$

It follows that

$$\underline{v} \le \inf_{x \in B_{s/2}} v \le C\left(s^{-n} \int_{B_{s/2}} v^b\right)^{1/b} \le C\left(s^{-n} \int \xi^k v^b\right)^{1/b} \le C s^{-m(c+\alpha)/\beta},$$

and

$$\underline{u} \le \inf_{x \in B_{s/2}} u \le C\left(s^{-n} \int_{B_{s/2}} u^c\right)^{1/c} \le C\left(s^{-n} \int \xi^l u^c\right)^{1/c} \le C s^{-m(b+\delta)/\beta}.$$

This completes the proof of case 1-i).

Case 1-ii). $\min\{c-\alpha, b-\delta\} < 0$. We first prove (3.6). By (3.4), without loss of generality, assume $c > \alpha$ and $b < \delta$. Fix an $f \in \boldsymbol{R}$ such that

$$\min\{0, m-1-b\} < f < m-1 \quad \text{and} \quad 0 < (f-d)\alpha < b(c-\alpha), \tag{3.8}$$

which is possible since

$$\alpha d > 0, \quad b(c-\alpha) > 0, \quad (m-1-d)\alpha = \delta\alpha > 0 \quad \text{and} \quad (m-1-b-d)\alpha < b(c-\alpha).$$

For $l, k > 0$ to be determined later, with the help of (3.8) and the Holder inequality, we use (3.3) to deduce that

$$\begin{aligned}\int \xi^l u^c v^{d-f} &\le C s^{-m} \int \xi^{l-m} v^{m-1-f} \\ &\le C s^{n-m-n(m-1-f)/b} \left(\int \xi^k v^b\right)^{(m-1-f)b^{-1}},\end{aligned} \tag{3.9}$$

provided

$$bl \ge (m-1-f)k + mb; \tag{3.10}$$

and

$$\begin{aligned}&\int \xi^{[(k-m)c-l\alpha]/(c-\alpha)} v^{(f-d)\alpha/(c-\alpha)} \\ &\quad \le C s^{n[1-(f-d)\alpha/b(c-\alpha)]} \left(\int \xi^k v^b\right)^{(f-d)\alpha/b(c-\alpha)},\end{aligned} \tag{3.11}$$

provided

$$\alpha bl \leq (bc - \alpha(f-d))k - mbc. \tag{3.12}$$

Plainly, for any fixed $f \in \boldsymbol{R}$, there holds

$$(m-1-f)k\alpha + mb\alpha < (bc - \alpha(f-d))k - mbc,$$

provided $k > mb(c+\alpha)/\beta > 0$, since $\alpha, \beta > 0$. It follows that for every $f \in \boldsymbol{R}_+$ satisfying (3.8) and every $k > mb(c+\alpha)/\beta$, we can choose an $l > 0$ so that (3.8), (3.10) and (3.12) hold simultaneously. Consequently (3.9) and (3.11) hold simultaneously for the triple f, k, l. Hence similarly as in 1-i), using $(3.5)_1$, (3.9) and (3.11), direct computations yield (noting $c > \alpha$)

$$\begin{aligned}
\int \xi^k v^b &\leq Cs^{-m} \int \xi^{k-m} u^\alpha = Cs^{-m} \int (\xi^l v^{d-f}) \cdot u^\alpha \cdot \left(\xi^{k-l-m} v^{-(d-f)}\right) \\
&\leq Cs^{-m} \left(\int \xi^l u^c v^{d-f} \right)^{\alpha c^{-1}} \cdot \left(\int \xi^{[(k-m)c - l\alpha]/(c-\alpha)} v^{(f-d)\alpha/(c-\alpha)} \right)^{(c-\alpha)c^{-1}} \\
&\leq Cs^{nb^{-1}c^{-1}\beta - m(1+\alpha c^{-1})} \left(\int \xi^k v^b \right)^{\alpha\delta b^{-1} c^{-1}}.
\end{aligned}$$

Therefore (3.6) holds for all $k > mb(c+\alpha)/\beta$.

Next we prove (3.7). Rewrite (3.3) with $f = d > 0$

$$\int |\nabla w|^m + \int \xi^l u^c \leq Cl^m s^{-m} \int \xi^{-m} w^m, \tag{3.13}$$

where $w = \xi^{l/m} v^{\delta/m}$ and $C = C(m,n,d)$. For simplicity, assume $m < n$, with the case $m \geq n$ being similar. For $\varepsilon \in (0,m)$ and

$$p = \frac{nm}{(n-m)(m-\varepsilon)} > \frac{n}{n-m} > 1, \qquad p' = \frac{p}{p-1} \in (1, n/m),$$

we apply the Holder, Sobolev and Young inequalities to deduce

$$\begin{aligned}
\int \xi^{-m} w^m &= \int (\xi^{-m} w^\varepsilon) \cdot w^{m-\varepsilon} \leq \left(\int (\xi^{-m} w^\varepsilon)^{p'} \right)^{1/p'} \cdot \left(\int w^{nm/(n-m)} \right)^{1/p} \\
&\leq C(n,m) \left(\int (\xi^{-m} w^\varepsilon)^{p'} \right)^{1/p'} \cdot \left(\int |\nabla w|^m \right)^{(m-\varepsilon)/m} \\
&\leq K(\varepsilon) s^{m - m^2/\varepsilon} \left(\int \xi^{l\varepsilon p'/m - mp'} v^{\delta\varepsilon p'/m} \right)^{m/p'\varepsilon} + \varepsilon^{2m} s^m \int |\nabla w|^m. \qquad (3.14)
\end{aligned}$$

Fix any $k > 0$ so that (3.6) holds. Since $\delta > 0$, we may choose $l, \varepsilon > 0$ so that

$$l^{1/2} \geq \max\left\{\delta p' C^{-1/2m}/mb, C^{1/2m}(m^2 + \delta k/b), C^{-1/2m}\right\}$$
$$\varepsilon = l^{-1/2} C^{-1/2m} (\in (0,1]),$$

where $C > 0$ was given in (3.13). It follows that

$$\varepsilon \delta p' \leq mb, \qquad \varepsilon^{2m} C l^m \leq 1, \qquad (l\varepsilon p'/m - mp') \geq \delta \varepsilon p' k/mb. \tag{3.15}$$

Substituting (3.14) into (3.13), with the help of (3.15) and the Holder inequality if necessary, we deduce that for $l > 0$ large

$$\int \xi^l u^c \leq K(\varepsilon) s^{-m^2/\varepsilon} \left(\int \xi^{l\varepsilon p'/m - mp'} v^{\delta\varepsilon p'/m} \right)^{m/p'\varepsilon}$$
$$\leq K(\varepsilon) s^{-m^2/\varepsilon} \cdot s^{n[1-\delta\varepsilon p'/bm]m/p'\varepsilon} \cdot \left(\int \xi^k v^b \right)^{\delta/b} \leq K(\varepsilon) s^{n - mc(b+\delta)/\beta},$$

where we have used (3.6). This is (3.7) and the rest of the proof follows similarly as in 1-i) and is left to the reader, which completes the proof of case 1).

Case 2). By $(3.5)_2$, we immediately have

$$\underline{u}^c \leq \inf_{x \in B_{s/2}} u^c \leq C s^{-n} \int_{B_{s/2}} u^c \leq C s^{-n} \int \xi^l u^c$$
$$\leq C s^{-n-m} \int \xi^{l-m} v^\delta \leq C s^{-m} \sup_{x \in B_s} v^\delta = \frac{C s^{-m}}{\underline{v}^{-\delta}},$$

since $\delta \leq 0$. It follows that

$$\underline{u}^c \underline{v}^{-\delta} \leq C s^{-m}.$$

To prove the other inequality, we again consider two cases.

2-i). $c \geq \alpha > 0$. Then one proceeds exactly as in 1-i) to derive

$$\underline{v}^b \leq C s^{-n} \int \xi^k v^b \leq C s^{-n-m} \int \xi^{k-m} u^\alpha \leq C s^{-n\alpha c^{-1} - m} \left(\int \xi^l u^c \right)^{\alpha c^{-1}}$$
$$\leq C s^{-m(1+\alpha c^{-1})} \cdot \left(s^{-n} \int \xi^{l-m} v^\delta \right)^{\alpha c^{-1}} \leq C s^{-m(1+\alpha c^{-1})} \cdot \underline{v}^{\alpha\delta c^{-1}},$$

since $\delta \leq 0$. This implies the second inequality.

2-ii). $0 < c < \alpha$. Then one proceeds as in 1-ii) to rewrite (3.2) with $e = a > 0$

$$\int |\nabla w|^m + \int \xi^k v^b \le C k^m s^{-m} \int \xi^{-m} w^m,$$

where $w = \xi^{k/m} u^{\alpha/m}$ and $C = C(m, n, a)$. Arguing similarly as in 1-ii), one derives

$$\int \xi^k v^b \le K(\varepsilon) s^{n(1-\alpha/c)-m} \cdot \left(\int \xi^l u^c \right)^{\alpha/c} \le K(\varepsilon) s^{n(1-\alpha/c)-m(1+\alpha/c)} \left(\int \xi^{l-m} v^\delta \right)^{\alpha/c},$$

and the rest follows. The proof is complete. □

Now we are ready to prove the non-existence result Theorem 1.1.

Proof of Theorem 1.1. Since $\beta > 0$, Lemmas 3.1 and 3.2 apply. We argue by contradiction. Suppose that (1.1) has a positive super-solution $\boldsymbol{u}$. Take a sequence of points $\{x^i\} \subset \boldsymbol{R}^n$ such that $r_i = |x^i| > 3R$ and $r_i \to \infty$. Put $B_i = B_{r_i/2}(x^i)$ and

$$u_i = \inf_{B_i} u(x), \qquad v_i = \inf_{B_i} v(x).$$

Case A). $n \le m$. By Lemma 3.2, one infers that there exist $\varepsilon > 0$ and $C > 0$ such that

$$\min\{u_i, v_i\} \le C r_i^{-\varepsilon}, \qquad i = 1, 2, \ldots.$$

Since $n \le m$, Lemma 3.1 implies that there exists $C > 0$ such that

$$\min\{u_i, v_i\} \ge C, \qquad i = 1, 2, \ldots.$$

This is an immediate contradiction since $r_i \to \infty$, as required.

Case B). $n > m$, $\min\{\alpha, \delta\} > 0$ and (1.3) holds. By Lemma 3.1, noting $n > m$, there exist $y^i, z^i \in \overline{B_i}$ and $C > 0$ such that

$$r_i \le 2|y^i| \le 4r_i, \qquad r_i \le 2|z^i| \le 4r_i,$$
$$u(y^i) = u_i \ge C|y^i|^{-(n-m)/(m-1)} \ge C r_i^{-(n-m)/(m-1)} \tag{3.16}$$

and

$$v(z^i) = v_i \ge C|z^i|^{-(n-m)/(m-1)} \ge C r_i^{-(n-m)/(m-1)}. \tag{3.17}$$

By Lemma 3.2, Case 1), there holds

$$u_i \le C r_i^{-m(b+\delta)/\beta}, \qquad v_i \le C r_i^{-m(c+\alpha)/\beta} \tag{3.18}$$

for some $C > 0$ independent of i. It follows that, by combining (3.16)–(3.18),

$$r_i^{-(n-m)/(m-1)} \le C r_i^{-m(b+\delta)/\beta}, \qquad r_i^{-(n-m)/(m-1)} \le C r_i^{-m(c+\alpha)/\beta}.$$

In turn,

$$r_i^{\max(b+\delta,c+\alpha)-\beta(n-m)/m(m-1)} \le C.$$

But this is impossible since $\max(b+\delta, c+\alpha) - \beta(n-m)/m(m-1) > 0$ by (1.3) and $r_i \to \infty$.

Case C). $n > m$, $\alpha > 0 \ge \delta$ and (1.4) holds. Using the lower bounds Lemma 3.1, one has (noting $-\delta \ge 0$)

$$u_i^c v_i^{-\delta} \ge C r_i^{-(c-\delta)(n-m)/(m-1)}, \qquad v_i \ge C r_i^{-(n-m)/(m-1)}.$$

On the other hand, using the upper bounds Lemma 3.2-2), we have

$$u_i^c v_i^{-\delta} \le C r_i^{-m}, \qquad v_i \le C r_i^{-m(c+\alpha)/\beta}.$$

Combining the above inequalities yields (noting $c - \delta > 0$),

$$r_i^{-(n-m)/(m-1)} \le C r_i^{-m/(c-\delta)}, \qquad r_i^{-(n-m)/(m-1)} \le C r_i^{-m(c+\alpha)/\beta}.$$

That is,

$$r_i^{\max(\beta/(c-\delta),c+\alpha)-\beta(n-m)/m(m-1)} \le C.$$

This, again, yields an immediate contradiction since $\max(\beta/(c-\delta), c+\alpha) - \beta(n-m)/m(m-1) > 0$ by (1.4) and $r_i \to \infty$.

Case D). $n > m$, $\alpha \le 0 < \delta$ and (1.5) holds. The proof is an asymmetric analogue to Case C) and left to the reader.

Case E). $n > m$, $\alpha \le 0$, $\delta \le 0$ and (1.6) holds. The proof is a combination of Cases C) and D) and again left to the reader, noting that in this case (1.6) is equivalent to the inequality

$$\max\left\{\frac{1}{c-\delta}, \frac{1}{b-\alpha}\right\} > \frac{n-m}{m(m-1)}. \qquad \square$$

4. A priori estimates.

We shall assume that Ω is an uniformly normal domain throughout this section and denote by $\gamma' \in (0,1)$ the Holder exponent associated with $\partial\Omega$. For $\boldsymbol{p} = (p_1, p_2) \ge 0$ and $\boldsymbol{u} = (u_1, u_2) \ge 0$, put

$$\boldsymbol{u}^{\boldsymbol{p}} = u_1^{p_1} u_2^{p_2}.$$

Fixing a positive function $\boldsymbol{h}(x) = (h_1(x), h_2(x)) \in C(\overline{\Omega})$, consider the system of equations

$$\begin{aligned} \Delta_m u_i + \boldsymbol{u}^{\boldsymbol{p}_i} + t h_i(x) = 0 \quad &\text{in } \Omega, \\ \boldsymbol{u} = 0 \quad &\text{on } \partial\Omega, \end{aligned} \tag{4.1}$$

where $t \geq 0$, $\boldsymbol{p}_i = (p_{i1}, p_{i2}) \geq 0$, $i = 1, 2$, and $\boldsymbol{u} = (u_1, u_2)$. We shall derive supremum a priori estimates for the class of non-negative monotone solutions $\boldsymbol{u}$ of (4.1), and continue to use the notations from section 3.

As in the introduction, define

$$\mathscr{C} = \{\boldsymbol{u} \in C_0(\Omega) \mid \boldsymbol{u} \geq 0;\, u_1, u_2 \text{ monotone in } \Omega\}.$$

Let $\boldsymbol{u} \in \mathscr{C}$ be a positive solution of (4.1). For $i = 1, 2$, write

$$n_i(x) := \boldsymbol{u}^{\boldsymbol{p}_i}(x) u_i^{-(m-1)}(x),$$

and

$$U_i := \max_{x \in \Omega} u_i(x) > 0, \qquad N_i := \sup_{x \in \Omega} n_i(x) > 0.$$

We have the following supremum a priori estimates.

Theorem 4.1. *Let $\boldsymbol{u} \in \mathscr{C}$ be a solution of* (4.1). *Suppose that the exponents $\boldsymbol{p}_i = (p_{i1}, p_{i2}) \geq 0$ satisfy the conditions*

$$\min(p_{11}, p_{22}) \geq m - 1,$$

and

$$\beta := p_{12} \cdot p_{21} - ((m-1) - p_{11})((m-1) - p_{22}) > 0.$$

Moreover, assume that one of the following holds:

A). $n \leq m$.
B). $n > m$ *and*

$$\min(p_{11} + p_{12}, p_{21} + p_{22}) < m_*, \qquad \max(p_{22}, p_{11}) < m^*.$$

Then there exists $C > 0$ depending on the structural constants and $\boldsymbol{h}$ (independent of $\boldsymbol{u}$ or t) such that

$$t + U_1 + U_2 \leq C. \tag{4.2}$$

Furthermore, there exist positive constants $\gamma = \gamma(n, m, \Omega) \in (0, \gamma')$ *and* $C = C(n, m, \boldsymbol{p}_i, \Omega) > 0$ *such that* $\boldsymbol{u} \in C^{1,\gamma}(\overline{\Omega})$ *and there holds*

$$\|\boldsymbol{u}\|_{1,\gamma,\overline{\Omega}} \leq C.$$

Remark. This is Theorem 1.2 given in the introduction.

Proof. This is an extension of Theorem 4.1 of semi-linear case in [**15**] to the quasi-linear case and the proof is based on a blow-up argument (cf. [**7**], [**14**]). Suppose for contradiction that (4.2) is false. Then there exist a sequence of non-negative solutions $\boldsymbol{u}_l = (u_{1,l}(x), u_{2,l}(x)) \in \mathscr{C}$ of (4.1) and a corresponding sequence of numbers $t_l \geq 0$ such that

$$0 < t_l + \max_{x \in \Omega} u_{1,l}(x) + \max_{x \in \Omega} u_{2,l}(x) \to \infty.$$

As in [**15**], without loss of generality, we assume in the sequel that $\boldsymbol{u}_l > 0$ is strictly positive and

$$\lim_{l \to \infty} U_{1,l} = \lim_{l \to \infty} \max\{U_{1,l}, U_{2,l}\} = \infty, \tag{4.3}$$

where we use $n_{i,l}(x)$, $U_{i,l}$ and $N_{i,l}$, corresponding to $\boldsymbol{u}_l$, to denote the various quantities given at the beginning of this section.

Since $p_{11}, p_{22} \geq m-1$ and $p_{12} \cdot p_{21} > ((m-1) - p_{11})((m-1) - p_{22}) \geq 0$, we have

$$\min\{p_{12}, p_{21}\} > 0, \qquad \min_{i=1,2}\{|\boldsymbol{p}_i|, |\boldsymbol{q}_i|\} > m-1, \tag{4.4}$$

where $\boldsymbol{q}_i := (p_{1i}, p_{2i})$, $i = 1, 2$.

By our assumption, Ω is uniformly normal and $\{\boldsymbol{u}_l\}$ are monotone in Ω. That is, for all $z \in \partial\Omega$ and for all $\gamma \in (\beta_z, \gamma_z)$

$$\boldsymbol{u}_l(x^{\gamma,\nu_z}) \geq \boldsymbol{u}_l(x), \qquad x \in \Omega_{\gamma,\nu_z}.$$

In particular, there exists $\delta_0 > 0$ (cf. (2.4)) such that

$$\max_{x \in \Omega} u_{i,l}(x) = \max_{x \in \Omega_0} u_{i,l}(x), \qquad i = 1, 2;\ l = 1, 2, \ldots,$$

where

$$\Omega_0 = (\Omega \backslash \Omega_{\delta_0}) := \{x \in \Omega \mid \operatorname{dist}(x, \partial\Omega) \geq \delta_0\}.$$

It follows, with the help of (4.4), the assumption $p_{ii} \geq m-1$ and the monotonicity of $\boldsymbol{u}_l$, that there exist $\xi_{i,l} \in \Omega_0$ and $\zeta_{i,l} \in \Omega_0$ such that

$$U_{i,l} = u_{i,l}(\xi_{i,l}) = \max_{x\in\Omega} u_{i,l}(x), \qquad N_{i,l} = n_{i,l}(\zeta_{i,l}) = \max_{x\in\Omega} n_{i,l}(x)$$

for $i = 1, 2;\ l = 1, 2, \ldots$. In particular,

$$\min\{\mathrm{dist}(\xi_{i,l}, \partial\Omega), \mathrm{dist}(\zeta_{i,l}, \partial\Omega)\} \geq \delta_0. \tag{4.5}$$

In the sequel, passing to a subsequence if necessary, we assume that all sequences (of numerals) converge to a non-negative quantity including infinity. Moreover, we shall assume $\boldsymbol{h} \equiv (1, 1)$ for convenience.

For $z^l \in \Omega$ and $Q_l \geq 1$ to be determined later, make the change of variables

$$v_{i,l}(y) = \frac{u_{i,l}(x)}{u_{i,l}(z^l)}, \quad y = (x - z^l)Q_l; \qquad l = 1, 2, \ldots \tag{4.6}$$

and

$$\Omega_l := \{y \in \boldsymbol{R}^n \mid y = (x - z^l)Q_l, x \in \Omega\}; \qquad \tau_l := \mathrm{dist}(z^l, \partial\Omega)Q_l = \mathrm{dist}(0, \partial\Omega_l). \tag{4.7}$$

Clearly

$$\boldsymbol{v}_l(0) = (v_{1,l}(0), v_{2,l}(0)) \equiv (1, 1), \qquad l = 1, 2, \ldots. \tag{4.8}$$

By direct calculations, $\boldsymbol{v}_l$ satisfies

$$\begin{aligned} \Delta_m v_{i,l} + Q_l^{-m} n_{i,l}(y) v_{i,l}^{m-1} + Q_l^{-m} t_l u_{i,l}^{-(m-1)}(z^l) &= 0 \quad \text{in } \Omega_l, \\ \boldsymbol{v}_l &= 0 \quad \text{on } \partial\Omega_l, \end{aligned} \tag{4.9$_{i,l}$}$$

for $i = 1, 2$ and $l = 1, 2, \ldots$, where (abusing notation)

$$n_{i,l}(y) = n_{i,l}(x) = \boldsymbol{u}_l^{\boldsymbol{p}_i}(x) u_{i,l}^{-(m-1)}(x) = \boldsymbol{u}_l^{\boldsymbol{p}_i}\big(z^l + yQ_l^{-1}\big) u_{i,l}^{-(m-1)}\big(z^l + yQ_l^{-1}\big).$$

Denote[5]

$$\bar{N}_i = \lim_{l\to\infty} N_{i,l} \in [0, \infty], \qquad i = 1, 2.$$

We divide the proof into three cases.

Case I). $\bar{N}_1 = 0$. We claim that either

$$t_l U_{1,l}^{-(m-1)} \to 0 \quad \text{or} \quad \max\big\{N_{2,l}, t_l u_{2,l}^{-(m-1)}(\xi_{1,l})\big\} \to \infty. \tag{4.10}$$

[5]It is understood that the convergence (here and in the sequel) is up-to a subsequence.

Suppose $t_l U_{1,l}^{-(m-1)} \to c > 0$ and $N_{2,l} \to \bar{N}_2 < \infty$. Then we have

$$t_l^{-1} = O\big(U_{1,l}^{-(m-1)}\big) \quad \text{and} \quad U_{1,l}^{|\boldsymbol{q}_1|-(m-1)} u_{2,l}^{|\boldsymbol{q}_2|-(m-1)}(\xi_{1,l}) \le N_{1,l} N_{2,l} = o(1).$$

It follows that

$$t_l^{-1} = O\big(U_{1,l}^{-(m-1)}\big) \quad \text{and} \quad u_{2,l}(\xi_{1,l}) = o\big(U_{1,l}^{((m-1)-|\boldsymbol{q}_1|)/(|\boldsymbol{q}_2|-(m-1))}\big).$$

Therefore

$$\begin{aligned} t_l^{-1} u_{2,l}^{m-1}(\xi_{1,l}) &= o\big([U_{1,l}^{((m-1)-|\boldsymbol{q}_1|)/(|\boldsymbol{q}_2|-(m-1))}]^{m-1}\big) \cdot O(U_{1,l}^{-(m-1)}) \\ &= o\big([U_{1,l}^{(2(m-1)-|\boldsymbol{q}_1|-|\boldsymbol{q}_2|)/(|\boldsymbol{q}_2|-(m-1))}]^{m-1}\big) = o(1), \end{aligned}$$

since $2(m-1) - |\boldsymbol{q}_1| - |\boldsymbol{q}_2| < 0$, $|\boldsymbol{q}_2| - (m-1) > 0$ by (4.4) and $U_{1,l} \to \infty$. This is (4.10). Next we further consider two sub-cases in accordance with (4.10).

Sub-case I-1). $t_l U_{1,l}^{-(m-1)} \to 0$. In (4.6), take

$$z^l = \xi_{1,l}, \qquad Q_l \equiv 1.$$

Plainly

$$\begin{gathered} 0 \le v_{1,l}(y) \le 1, \quad \big|Q_l^{-m} n_{1,l}(y) v_{1,l}^{m-1}\big| \le N_{1,l} = o(1), \\ \big|Q_l^{-m} t_l U_{1,l}^{-(m-1)}\big| = t_l U_{1,l}^{-(m-1)} = o(1) \end{gathered}$$

uniformly for all $y \in \Omega_l$.

Without loss of generality, assume $\Omega_l = \Omega$ for all $l \ge 1$ (they possibly differ by a translation). Applying Lemma 2.1-B) to the first equations $(4.9)_{1,l}$ on $\Omega_l = \Omega$, we see that there exists $\gamma \in (0,1)$ such that the sequence $\{v_{1,l}\}$ are bounded in the Banach space $C^{1,\gamma}(\overline{\Omega}) \cap C_0(\Omega)$. It follows, by the Ascoli-Arzela theorem, that there exists $v \in C^{1,\gamma/2}(\overline{\Omega}) \cap C_0(\Omega)$ such that

$$\lim_{l\to\infty} v_{1,l}(y) = v(y), \qquad v(0) = \lim_{l\to\infty} v_{1,l}(0) = 1$$

in $C^{1,\gamma/2}(\overline{\Omega}) \cap C_0(\Omega)$. Fix any function $\varphi \in C_0^\infty(\Omega)$. Taking φ as a test function in the first equations $(4.9)_{1,l}$ and letting $l \to \infty$, one immediately deduces that v satisfies

$$\begin{aligned} \Delta_m v = 0, &\quad \text{in } \Omega, \\ v = 0, &\quad \text{on } \partial\Omega. \end{aligned}$$

It follows at once that $v \equiv 0$ in Ω, which contradicts the fact $v(0) = 1$.

Sub-case I-2). $\max\{N_{2,l}, t_l u_{2,l}^{-(m-1)}(\xi_{1,l})\} \to \infty$. Now take

$$z^l = \begin{cases} \zeta_{2,l} & \text{if } N_{2,l} \geq t_l u_{2,l}^{-(m-1)}(\xi_{1,l}), \\ \xi_{1,l} & \text{if } N_{2,l} < t_l u_{2,l}^{-(m-1)}(\xi_{1,l}), \end{cases}$$

and

$$Q_l^m = \max\left\{N_{2,l}, t_l u_{1,l}^{-(m-1)}(z^l), t_l u_{2,l}^{-(m-1)}(z^l)\right\} \to \infty.$$

By (4.5) and (4.7),

$$\tau_l = Q_l \operatorname{dist}(z^l, \partial\Omega) \geq Q_l \delta_0 \to \infty, \qquad B_{\tau_l}(0) \subset \Omega_l \to \boldsymbol{R}^n. \tag{4.11}$$

By the choices of Q_l, we have

$$\left|Q_l^{-m} n_{i,l}(y)\right| \leq 1, \qquad \left|Q_l^{-m} t_l u_{i,l}^{-(m-1)}(z^l)\right| \leq 1, \tag{4.12}$$

uniformly for $i = 1, 2$ and for all $y \in \Omega_l$.

Fix any $R > 0$ and any compact subset $\Gamma \subset B_R(0) \subset \boldsymbol{R}^n$. Applying the Harnack inequality Lemma 2.2 to the equations $(4.9)_{i,l}$ with $\Omega = B_{R+1}(0)$ and $\Omega' = \Gamma$, with the aid of (4.12), we obtain that $\boldsymbol{v}_l$ are uniformly bounded on Γ for l sufficiently large (so $B_{R+1}(0) \subset \Omega_l$) since $\boldsymbol{v}_l(0) = (1, 1)$. Similarly as in I-1), applying Lemma 2.1-A) to the equations $(4.9)_{i,l}$ with $\Omega = B_{R+1}(0)$ and $\Omega' = \Gamma$, with the aid of (4.12), we deduce that there exists an $\gamma \in (0, 1)$ such that $\{\boldsymbol{v}_l\}$ are bounded in the Banach space $C^{1,\gamma}(\overline{\Gamma})$. Since Γ is arbitrary, one then uses the Ascoli-Arzela theorem to infer that there exists $\boldsymbol{v} \in C_{loc}^{1,\gamma/2}(\boldsymbol{R}^n)$ such that

$$\lim_{l\to\infty} \boldsymbol{v}_l(y) = \boldsymbol{v}(y) \geq 0, \qquad \boldsymbol{v}(0) = (1, 1)$$

uniformly on Γ (in $C^{1,\gamma/2}$-topology), noting this time $\Omega_l \to \boldsymbol{R}^n$ by (4.11). Moreover, by the choices of Q_l, there exist constants $\delta_1, \delta_2, \kappa_1, \kappa_2 \geq 0$ such that for $i = 1, 2$

$$\lim_{l\to\infty} Q_l^{-m} t_l u_{i,l}^{-(m-1)}(z^l) = \delta_i, \tag{4.13}$$

and

$$\lim_{l\to\infty} Q_l^{-m} n_{i,l}(y) v_{i,l}^{m-1}(y) = \lim_{l\to\infty} Q_l^{-m} n_{i,l}(z^l) \boldsymbol{v}_l^{\boldsymbol{p}_i}(y) = \kappa_i \boldsymbol{v}^{\boldsymbol{p}_i}(y) \tag{4.14}$$

uniformly on Γ, with $\delta_1 + \delta_2 + \kappa_1 + \kappa_2 \geq 1 > 0$. For any function $\varphi \in C_0^\infty(\boldsymbol{R}^n)$, fix Γ so that $\operatorname{supp}(\varphi) \subset \Gamma$, where $\operatorname{supp}(\varphi)$ is the support of φ and thus compact. Using φ as a test function in the equations $(4.9)_{i,l}$ for l sufficiently large (so $\Gamma \subset\subset \Omega_l$) and letting $l \to \infty$, one readily verifies that the limiting function $\boldsymbol{v}$ satisfies the following limiting equations

$$\Delta_m v_i + \kappa_i \boldsymbol{v}^{\boldsymbol{p}_i}(y) + \delta_i = 0 \text{ in } \boldsymbol{R}^n, \qquad i = 1, 2,$$

where we have used the limits (4.13) and (4.14). Applying Lemma 2.3 to each equation above respectively, one sees that $\delta_1 = \delta_2 = 0$ since $\kappa_1, \kappa_2, \delta_1, \delta_2 \geq 0$ and $\boldsymbol{v}(y) \geq 0$. It follows that

$$\Delta_m v_i + \kappa_i \boldsymbol{v}^{\boldsymbol{p}_i}(y) = 0 \text{ in } \boldsymbol{R}^n$$

with $\kappa_1 + \kappa_2 \geq 1 > 0$. If both κ_1 and κ_2 are positive, then Theorem 1.1 applies to the above system by our assumptions and consequently $\boldsymbol{v}(y) \equiv 0$, a contradiction to $\boldsymbol{v}(0) = (1, 1)$. Therefore either κ_1 or κ_2 must be zero. If, say, $\kappa_1 = 0$, then $\kappa_2 = 1$ and $v_1(y) \equiv 1$ by Lemma 2.3 since $\Delta_m v_1 = 0$ and $v_1(0) = 1$. Thus the second equation (noting $v_1 \equiv 1$) reduces to

$$\Delta_m v_2 + v_2^{p_{22}}(y) = 0 \text{ in } \boldsymbol{R}^n.$$

If $m \geq n$, then $v_2 \equiv 1$ (noting $v_2(0) = 1$) by Lemma 2.3, which implies $\kappa_2 = 0$, a contradiction. If $m < n$, then $0 < p_{22} < m^*$ by our assumption. Thus $v_2(y) \equiv 0$ by Lemma 2.3. This is again impossible since $v_2(0) = 1$.

This completes the proof of Case I).

Case II). $\bar{N}_1 = \infty$. In (4.6), take

$$N_l = \max\{N_{1,l}, N_{2,l}\}, \qquad z^l = \zeta_{i,l} \quad \text{if } N_l = N_{i,l},$$

and

$$Q_l^m = \max\left\{N_l, t_l u_{1,l}^{-(m-1)}(z^l), t_l u_{2,l}^{-(m-1)}(z^l)\right\} \to \infty.$$

We omit the rest of the proof which is essentially the same as in Case I-2).

Case III). $\bar{N}_1 \in (0, \infty)$. We claim that

$$\max\left\{N_{2,l}, t_l U_{1,l}^{-(m-1)}, t_l U_{2,l}^{-(m-1)}\right\} \to \infty. \tag{4.15}$$

Suppose for contradiction that (4.15) is false. Then there exists $C > 0$ such that

$$\max\left\{N_{1,l}, N_{2,l}, t_l U_{1,l}^{-(m-1)}, t_l U_{2,l}^{-(m-1)}\right\} \leq C.$$

Namely, there holds

$$|n_{i,l}(x)| \leq C, \qquad \left|t_l U_{i,l}^{-(m-1)}\right| \leq C \tag{4.16}$$

uniformly for $x \in \Omega$, $i = 1, 2$ and $l = 1, 2, \ldots$. For $i = 1, 2$ and $l = 1, 2, \ldots$, rewrite (4.1) into

$$\begin{aligned} \Delta_m w_{i,l} + n_{i,l}(x) w_{i,l}^{m-1} + t_l U_{i,l}^{-(m-1)} &= 0 \quad \text{in } \Omega, \\ \boldsymbol{w}_l &= 0 \quad \text{on } \partial\Omega, \end{aligned}$$

where $w_{i,l}(x) = u_{i,l}(x)/U_{i,l}$. Applying the Harnack inequality Lemma 2.2 to the above system on Ω, with the help of (4.16), we derive that there exists $C > 0$ (independent of l) such that

$$\max_{x\in\Omega_0} u_{i,l}(x) \le C \min_{x\in\Omega_0} u_{i,l}(x), \qquad i = 1, 2;\ l = 1, 2, \ldots. \tag{4.17}$$

By the definition of $N_{1,l}$ and (4.17), there exist $C_l, D_l \in [1/(2C), 2C]$ such that

$$\begin{aligned} u_{2,l}(\xi_{1,l}) = C_l u_{2,l}(\zeta_{1,l}) &= C_l \big[u_{1,l}^{(m-1)-p_{11}}(\zeta_{1,l}) N_{1,l}\big]^{1/p_{12}} \\ &= C_l [D_l U_{1,l}]^{((m-1)-p_{11})/p_{12}} \big(\bar{N}_1^{1/p_{12}} + o(1)\big), \end{aligned}$$

since $p_{12} > 0$ by (4.4) and since $\xi_{1,l}, \zeta_{1,l} \in \Omega_0$. It follows that

$$\begin{aligned} N_{2,l} &\ge n_{2,l}(\xi_{1,l}) = U_{1,l}^{p_{21}} u_{2,l}^{p_{22}-(m-1)}(\xi_{1,l}) \\ &= C_l^{\sigma_1} D_l^{\sigma_2} U_{1,l}^{p_{21}-((m-1)-p_{11})((m-1)-p_{22})/p_{12}} \big(\bar{N}_1^{(p_{22}-(m-1))/p_{12}} + o(1)\big) \to \infty \end{aligned}$$

since $p_{21}p_{12} > ((m-1) - p_{11})((m-1) - p_{22})$ and $U_{1,l} \to \infty$, where σ_1 and σ_2 are real numbers. This contradicts (4.16) and so (4.15) holds.

With (4.15) available, now take

$$z^l = \begin{cases} \zeta_{2,l} & \text{if } \max\big\{N_{2,l}, t_l U_{1,l}^{-(m-1)}, t_l U_{2,l}^{-(m-1)}\big\} = N_{2,l}, \\ \xi_{1,l} & \text{if } \max\big\{N_{2,l}, t_l U_{1,l}^{-(m-1)}, t_l U_{2,l}^{-(m-1)}\big\} = t_l U_{1,l}^{-(m-1)} \\ \xi_{2,l} & \text{if } \max\big\{N_{2,l}, t_l U_{1,l}^{-(m-1)}, t_l U_{2,l}^{-(m-1)}\big\} = t_l U_{2,l}^{-(m-1)}. \end{cases}$$

$$Q_l^m = \max\big\{N_{2,l}, t_l u_{1,l}^{-(m-1)}(z^l), t_l u_{2,l}^{-(m-1)}(z^l)\big\} \to \infty.$$

Then the rest of the proof proceeds again similarly as in Case I-2) and is left to the reader.

In conclusion, the contradictions we just derived imply that the hypothesis (4.3) is false and therefore the a priori estimate (4.2) must be valid.

The $C^{1,\gamma}$-estimate follows directly from Lemma 2.1 and the proof is complete. □

Proof of Theorem 1.2. It is precisely Theorem 4.1 by taking $p_{11} = a$, $p_{12} = b$, $p_{21} = c$ and $p_{22} = d$. □

5. Existence.

In this section, we prove the existence result – Theorem 1.3. As in [**14**], we shall apply a fixed point theorem and the a priori estimates obtained in section 4 are crucial.

Throughout this section, we assume that $\Omega \subset \boldsymbol{R}^n$ is an uniformly normal domain. By Lemma 2.1, there exists $\gamma = \gamma(n, m, \Omega) \in (0, \gamma')$, where $\gamma' \in (0,1)$ is the Holder exponent associated with $\partial\Omega$, such that all non-negative continuous solutions of (1.1) are in the space $C^{1,\gamma}(\overline{\Omega}) \cap C_0(\Omega)$. For $\gamma_0 = \gamma/2$, put

$$X := C(\overline{\Omega}), \qquad Y := C^{1,\gamma_0}(\overline{\Omega}) \cap C_0(\Omega).$$

Both X and Y are Banach spaces equipped with the standard norm. Define

$$\mathscr{C} = \{\boldsymbol{u} \in C_0(\Omega) \mid \boldsymbol{u} \geq 0;\ u, v \in MO(\Omega)\}. \tag{5.1}$$

One readily verifies that $\mathscr{C}$ is a cone in X. Clearly, for $\boldsymbol{u} \in \mathscr{C}$ one has

$$\boldsymbol{u} \geq 0; \qquad U = \max_{x\in\Omega} u(x) = \max_{x\in\Omega_0} u(x), \qquad V = \max_{x\in\Omega} v(x) = \max_{x\in\Omega_0} v(x) \tag{5.2}$$

and

$$N_{i,l} = \max_{x\in\Omega} n_{i,l}(x) = \max_{x\in\Omega_0} n_{i,l}(x), \tag{5.3}$$

where Ω_0 is given in section 4. Denote

$$G := (-\Delta_m)^{-1}\mathbf{I} :\ X \mapsto Y,$$

with the homogeneous Dirichlet data, where $\mathbf{I}$ is the 2×2 identity matrix. By Lemmas 2.1 and 2.6, $G : X \mapsto Y$ is well-defined, continuous and bounded. Set

$$T(\boldsymbol{u}) := (|u|^a|v|^b, |u|^c|v|^d) :\ X \mapsto X.$$

One readily verifies that T is continuous and bounded.

Now consider the operator

$$F = G \circ T :\ X \to Y \hookrightarrow X.$$

the operator F is *continuous and compact* since $G \circ T : X \to Y$ is continuous and bounded and the embedding $Y \hookrightarrow X$ is compact (and continuous). In the sequel, $\|\cdot\|$ denotes the standard norm on X, i.e., the supremum norm on $\overline{\Omega}$.

Now we are ready to prove Theorem 1.3.

Proof of Theorem 1.3. Proceeding exactly as in [**14**], we apply Proposition 4.1 (due to [**3**]) from [**14**] to the compact operator F on the cone $\mathscr{C}$ and divide the proof into five steps.

STEP 1. $F(\mathbf{0}) = \mathbf{0}$ and the map F is $\mathscr{C}$-preserving. It is obvious $F(\mathbf{0}) = \mathbf{0}$ since $T(\mathbf{0}) = \mathbf{0}$ and $G(\mathbf{0}) = \mathbf{0}$. It remains to show that F is $\mathscr{C}$-preserving.

Let $\boldsymbol{u} \in \mathscr{C}$ and put $\boldsymbol{v} = (\bar{u}, \bar{v}) = F(\boldsymbol{u})$. Therefore

$$\begin{aligned} \Delta_m \bar{u} + g_1(x) = 0, &\quad \text{in } \Omega, \\ \Delta_m \bar{v} + g_2(x) = 0, &\quad \text{in } \Omega, \\ \bar{u} = \bar{v} = 0, &\quad \text{on } \partial\Omega. \end{aligned}$$

where

$$g_1(x) = u^a v^b, \qquad g_2(x) = u^c v^d.$$

Plainly, $u(x) \in MO(\Omega)$ and $v(x) \in MO(\Omega)$ since $\boldsymbol{u} \in \mathscr{C}$. Thus one readily deduces that both g_1 and g_2 belong to the class $MO(\Omega)$ since $a, b, c, d \geq 0$. Therefore Lemma 2.7 applies and one concludes that both $\bar{u}$ and $\bar{v}$ belong to the class $MO(\Omega)$. Consequently, $\boldsymbol{v} \in \mathscr{C}$ and $F : \mathscr{C} \mapsto \mathscr{C}$ is a $\mathscr{C}$-cone preserving map.

STEP 2. For $t \in [0, 1]$, there exists a positive number r such that $\boldsymbol{u} \neq tF(\boldsymbol{u})$ for $\|\boldsymbol{u}\| = r$. Consider $\boldsymbol{u} = tF(\boldsymbol{u})$, that is,

$$\begin{aligned} \Delta_m u + t u^a v^b = 0, &\quad \text{in } \Omega, \\ \Delta_m v + t u^c v^d = 0, &\quad \text{in } \Omega, \\ u = v = 0, &\quad \text{on } \partial\Omega. \end{aligned}$$

Multiply by the first equation by u, second by v and integrate over Ω and add to obtain

$$\int_\Omega (|\nabla u|^m + |\nabla v|^m) = t \int_\Omega (u^{a+1} v^b + u^c v^{d+1}) = \int_\Omega o(|u|^m + |v|^m)$$

as $\|\boldsymbol{u}\| \to 0$, since $\min(a+b, c+d) > m-1$. It follows that there exists $r_0 > 0$ such that the equation $\boldsymbol{u} = tF(\boldsymbol{u})$ has no solution in $B_{r_0}(0) - \{0\}$ for all $t \in [0, 1]$.

STEP 3. There exist positive numbers t_0 and R and a vector $\boldsymbol{u}_0 \in \mathscr{C} - \{0\}$ such that

$$\boldsymbol{u} \neq F(\boldsymbol{u}) + t\boldsymbol{u}_0 \tag{5.4}$$

for $t \geq t_0$ and $\boldsymbol{u} \in \mathscr{C}$ with $\|\boldsymbol{u}\| \leq R$.

Let φ be a (fixed) positive function in the class $MO(\Omega)$ (i.e., the distance function to the boundary). Take

$$\phi = (-\Delta_m)^{-1} \varphi, \qquad \boldsymbol{u}_0 = (\phi, \phi) \in \mathscr{C} - \{0\}.$$

Obviously $\phi \in MO(\Omega)$ by Lemma 2.7. Thus $\boldsymbol{u}_0 \in \mathscr{C}$. Consider the equation

$$\boldsymbol{u} = F(\boldsymbol{u}) + t\boldsymbol{u}_0, \tag{5.5}$$

that is,

$$\Delta_m u + u^a v^b + t\varphi = 0, \quad \text{in } \Omega,$$
$$\Delta_m v + u^c v^d + t\varphi = 0, \quad \text{in } \Omega.$$

By Theorem 1.2 (taking $h_1 = h_2 = \varphi$), there exists $C > 0$ such that

$$t + \|\boldsymbol{u}\| \leq C.$$

Take $t_0 = C + 1$. Then (5.4) holds as long as $t \geq t_0$. Note particularly that the choice of $R > 0$ can be arbitrary.

Step 4. There exists a positive number R such that (5.4) holds for all $t \geq 0$ and $\boldsymbol{u} \in \mathscr{C}$ with $\|\boldsymbol{u}\| = R$. Just take $R = C + 1$ in Step 3 and the result follows.

Step 5. Now we can finish the proof by applying Proposition 4.1 in [**14**]. Plainly, taking $X = X$, $\mathscr{C} = \mathscr{C}$ and $T = F$, one readily verifies that all conditions of Proposition 4.1 are satisfied by Steps 1–4 above. Therefore the mapping F has a fixed point $\boldsymbol{w} \in \mathscr{C}$ with $\|\boldsymbol{w}\| \in [r, R]$, which is a non-negative solution of (1.1), with at least one non-vanishing component ($\|\boldsymbol{w}\| \geq r > 0$).

It remains to show $\boldsymbol{w} = (w_1, w_2) > 0$. By the strong maximum principle Lemma 2.4, both w_1 and w_2 must be either strictly positive or identically zero. On the other hand, since $b > 0$ and $c > 0$, none of $\boldsymbol{w}$'s components can vanish identically unless $\boldsymbol{w} \equiv 0$. Thus $\boldsymbol{w} > 0$ is a positive solution of (1.1) and the proof is complete. $\square$

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