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Mapping tori with first Betti number at least two

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Abstract. We show that given a finitely presented group G with $\beta_1(G) \geq 2$ which is a mapping torus Γ_θ for Γ a finitely generated group and θ an automorphism of Γ then if the Alexander polynomial of G is non-constant, we can take $\beta_1(\Gamma)$ to be arbitrarily large. We give a range of applications and examples, such as any group G with $\beta_1(G) \geq 2$ that is F_n-by-$\boldsymbol{Z}$ for F_n the non-abelian free group of rank n is also F_m-by-$\boldsymbol{Z}$ for infinitely many m. We also examine 3-manifold groups where we show that a finitely generated subgroup cannot be conjugate to a proper subgroup of itself.

1. Introduction.

Given a group Γ and an endomorphism θ of Γ, the (algebraic) mapping torus $G = \Gamma_\theta$ is the quotient group of the free product of Γ and $\boldsymbol{Z} = \langle t \rangle$ formed by adding the relations $t\gamma t^{-1} = \theta(\gamma)$ for all $\gamma \in \Gamma$, or equivalently for a generating set of Γ, so that if Γ is finitely generated (respectively finitely presented) then Γ_θ will be too, but in general such properties possessed by Γ_θ will not be inherited by Γ.

The name comes from the mapping torus construction in topology where we have a continuous map f from a path-connected topological space to itself. The mapping torus is then the space $Y = (X \times [0,1])/\sim$ for the equivalence relation $(x,0) \sim (f(x),1)$, with $\pi_1 Y = (\pi_1 X)_{f_*}$ for f_* the induced endomorphism of $\pi_1 X$ obtained from f. We can also go the other way: for any group Γ there is a CW-complex X with $\pi_1 X = \Gamma$ that is aspherical, which implies that any endomorphism of Γ is induced by a continuous map of X to itself.

In this paper we generally restrict to the situation where our endomorphism θ is injective, in which case Γ_θ naturally contains a copy of Γ, and if so then θ is an automorphism if and only if this copy of Γ is normal in Γ_θ. Then we have $\Gamma_\theta/\Gamma \cong \boldsymbol{Z}$, so that if Γ has a property $\mathscr{P}$ then Γ_θ is $\mathscr{P}$-by-cyclic. Conversely if a group G has a homomorphism onto $\boldsymbol{Z}$ with kernel K then we can lift the copy of $\boldsymbol{Z}$ up to G, so that we can set $G = K_\theta$ for some automorphism θ.

Clearly any mapping torus G has its first Betti number $\beta_1(G)$ at least 1. If $\beta_1(G) = 1$ then there are just two homomorphisms of G onto $\boldsymbol{Z}$, both with the same kernel, but we are interested in the case where $\beta_1(G) \geq 2$ so that there are a whole range of surjective homomorphisms to examine. A question that can be asked is: if we have one homomorphism χ giving rise to a decomposition of G as a mapping torus K_θ where $K = \ker\chi$ possesses an appropriate property, can we express G as a mapping torus using other homomorphisms whilst still retaining our nice property? A fundamental example

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of such a result is in the paper [**6**] in 1987 introducing the Bieri-Neumann-Strebel (BNS) invariant of a finitely generated group. This is used to generalise a theorem [**28**] of W. D. Neumann in 1979 which shows that, on putting a natural topology on the set of kernels of homomorphisms from the finitely generated group G to $\boldsymbol{Z}$, the subset of finitely generated kernels is open.

In this paper we combine the study of the BNS invariant of a finitely presented group G along with its Alexander polynomial. Although both are well established objects, the utility of this approach is that the BNS invariant carries important qualitative information about the kernels of homomorphisms from G onto $\boldsymbol{Z}$ (such as finite generation) but is hard to compute in practice, whereas the Alexander polynomial can be calculated directly from any finite presentation of a group and gives us quantitative information, in particular the first Betti numbers of the kernels.

We introduce these two invariants and properties required in Section 2, which gives us our main general result in Corollary 2.4: if G is a finitely presented group with $\beta_1(G) \geq 2$ and G can be expressed as a mapping torus Γ_θ for Γ finitely generated (respectively finitely presented) and θ an automorphism of Γ, then G has similar decompositions H_ϕ where H is finitely generated (respectively finitely presented) and ϕ is an automorphism of H with $\beta_1(H)$ being arbitrarily large, provided only that the Alexander polynomial of G is non-constant.

This result would be of little use if we did not have a ready supply of such mapping tori with non-constant Alexander polynomials to apply it to, but we give conditions in Section 3 that ensure this, such as G having a deficiency 1 presentation and $\beta_1(G) \geq 3$. We obtain Theorem 3.4 with the striking statement that if a group G has $\beta_1(G) \geq 2$ and is F_n-by-$\boldsymbol{Z}$ for F_n the non-abelian free group of rank n then G is F_m-by-$\boldsymbol{Z}$ for infinitely many m. In Example 1 we look at the group G in [**24**] which was shown there to be both a mapping torus of F_2 with respect to an injective but non-surjective endomorphism and also to be F_3-by-$\boldsymbol{Z}$. We prove G is in fact F_k-by-$\boldsymbol{Z}$ for all $k \geq 3$ and we also display, given any $n \geq 2$, a specific example of a free-by-cyclic group which is F_k-by-$\boldsymbol{Z}$ for exactly $k \geq n$.

Some time ago it was asked by G. P. Scott if groups of the form F_k-by-$\boldsymbol{Z}$ are subgroup separable, with [**11**] giving the first example of one that is not. The group G in [**24**] is shown using this non-surjective endomorphism to contain a non-free but locally free group and to have a finitely generated free subgroup which is conjugate in G to a proper subgroup of itself, thus G is F_k-by-$\boldsymbol{Z}$ but not subgroup separable. In Section 4 we turn our attention to the fundamental groups of 3-manifolds (or 3-manifold groups as we call them). The finitely generated 3-manifold groups are known to possess symmetric BNS invariants and Alexander polynomials and on exploiting this symmetry we show in Theorem 4.1 that no finitely generated subgroup of a 3-manifold group G can be conjugate in G to a proper subgroup of itself, thus this method can never be used to prove that a finitely generated 3-manifold group is not subgroup separable. We also obtain very quickly the (known) fact that a 3-manifold group cannot contain non-trivial Baumslag-Solitar groups.

If M is a compact orientable irreducible 3-manifold then any expression of $\pi_1 M$ as a mapping torus Γ_θ for Γ finitely generated and θ an automorphism implies that M is fibred over the circle by a surface with fundamental group Γ, using a well known and important theorem of Stallings (with an extension to the non-orientable case in [**18**]). Thus if $\beta_1(M) \geq 2$, the openness property of finitely generated kernels means that if M is fibred, it is fibred in infinitely many different ways. We can ask about the possible topological types of the various fibres, with a result in [**27**] stating that for closed fibred 3-manifolds M with $\beta_1(M) \geq 2$, there are infinitely many non-homeomorphic fibres (provided the original fibre is not one of the obvious small surfaces). The proof involves differential geometry but we present an alternative proof in Theorem 4.2 which applies without change for 3-manifolds fibred over the circle by a compact surface with boundary, thus we obtain a generalisation of this result.

We then provide examples to illustrate the constructive nature of our approach, in that if the BNS invariant of the fibred closed orientable 3-manifold is known, we can determine the exact list of fibres. We give in Example 4 a specific hyperbolic 3-manifold which is fibred by all closed orientable surfaces of genus 2 or more. Moreover we show that we can determine all fibres in the case of 3-manifolds with boundary, even though the topological type of a surface is not determined by its first Betti number, by Dehn filling of the cusps. This easily proves that the list of fibres is all n-punctured tori for the Whitehead link (where $n \geq 2$) and for every once punctured torus bundle with first Betti number at least two (here $n \geq 1$). We can also apply this technique to trivial fibre bundles where the fundamental group is a direct product, recovering old results for closed fibre bundles and giving the full list of fibres where the trivial fibre has any genus and one boundary component.

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2. The BNS invariant and the Alexander polynomial.

Let G be a finitely generated group and suppose that $b = \beta_1(G) > 0$. The Bieri-Neumann-Strebel (BNS) invariant gives us information on when the kernels of homomorphisms from G onto $\boldsymbol{Z}$ are finitely generated. This is done in [**6**] by identifying non-zero homomorphisms of G into $\boldsymbol{R}$, up to multiplication by a positive constant, with the sphere S^{b-1}. The BNS invariant of G is a subset Σ of S^{b-1}, with a homomorphism χ of G onto $\boldsymbol{Z}$ having finitely generated kernel if and only if χ (which we think of as a rationally defined point in S^{b-1}) is in both Σ and $-\Sigma$. The original definition of Σ was in terms of finite generation over submonoids but an equivalent definition in [**10**] or in [**6**, Chapter 4] allows us to relate its use to mapping tori, as we now describe.

Given a finitely generated group G that has a decomposition of the form $G = \Gamma_\theta$ for Γ any group and θ an endomorphism of Γ, we define the associated homomorphism χ of the mapping torus $\Gamma_\theta = \langle \Gamma, t \rangle$ by $\chi(t) = 1$ and $\chi(\Gamma) = 0$. Now given any surjective homomorphism $\chi : G \to \boldsymbol{Z}$, we say as in Proposition 3.1 of [**10**] that χ is in the BNS invariant Σ of G if χ is the associated homomorphism of a decomposition of G into the form Γ_θ where Γ is finitely generated and θ is an injective endomorphism of Γ. Although there are in general many decompositions of this form with χ the associated homomorphism, [**10**, Corollary 3.2] implies that χ has finitely generated kernel if and only if it is associated to Γ_θ where θ is an automorphism (whereupon the decomposition with $\Gamma = \ker\chi$ is the only one having χ as associated homomorphism), as then $-\chi$ is associated to $\Gamma_{\theta^{-1}}$, thus if and only if χ is in both Σ and $-\Sigma$.

If we are given a non-injective endomorphism θ of a group Γ, it is still possible to form the mapping torus Γ_θ in the same way, but we do not need to concentrate on this case. This is because under the natural map from $\Gamma * \boldsymbol{Z}$ to the quotient group Γ_θ we have, on setting the image of Γ to be Γ',

$$\Gamma' = \Gamma / \bigcup_{i=1}^{\infty} \text{Ker } \theta^i.$$

Thus we can always think of Γ_θ as a mapping torus $\Gamma'_{\theta'}$ with θ' injective after all, and the associated homomorphisms will be the same.

One of the main results of [**6**] is that the BNS invariant Σ of G is an open subset of S^{b-1}, so that once we have found a homomorphism χ onto $\boldsymbol{Z}$ with finitely generated kernel then all "nearby" homomorphisms in S^{b-1} (assuming $b \geq 2$) have finitely generated kernel too (this was also proved in [**28**]). Indeed this also works on replacing finitely generated by finitely presented by [**16**]. The one drawback is that on being given a particular group G, say by a finite presentation, it can be difficult to determine the exact subset Σ. We will see some methods to do this in the examples later on, but let us now look at the Alexander polynomial which although not able to give us so much information, has the advantage that it is straightforward to work out from a finite presentation of a group using Fox's free differential calculus. Therefore we give a brief description adopting the approach of Crowell and Fox in [**12**].

Let the finitely presented group G be $\langle x_1, \ldots, x_n | r_1, \ldots, r_m \rangle$ in terms of generators and relators, and let its free abelianisation be $\text{ab}(G)$, which will be isomorphic to $\boldsymbol{Z}^{\beta_1(G)}$. If F_n is the free group of rank n with free basis $x_1, \ldots, x_n$ then a derivation of the integral group ring $\boldsymbol{Z}[F_n]$ is a map from $\boldsymbol{Z}[F_n]$ to itself satisfying

$$\begin{aligned} D(v_1 + v_2) &= Dv_1 + Dv_2, \\ D(v_1 v_2) &= (Dv_1)\tau(v_2) + v_1 Dv_2 \end{aligned}$$

where τ is the trivialiser: namely the ring homomorphism from $\boldsymbol{Z}[F_n]$ to $\boldsymbol{Z}$ with $\tau(x) = 1$ for all $x \in F_n$. It is a fact that for each free generator x_j there exists a unique derivation D_j, also written $\partial/\partial x_j$, such that $\partial x_i / \partial x_j = \delta_{ij}$ and this gives rise to the fundamental formula for any element v of $\boldsymbol{Z}[F_n]$:

$$v - \tau(v) = \sum_{j=1}^{n} \frac{\partial v}{\partial x_j}(x_j - 1).$$

To calculate the "partial derivative" $\partial w / \partial x_j$ for any $w \in F_n$ we can use the formal rules

$$\frac{\partial x_i}{\partial x_j} = \delta_{ij}, \qquad \frac{\partial x_i^{-1}}{\partial x_j} = -\delta_{ij} x_i^{-1}, \qquad \frac{\partial(w_1 w_2)}{\partial x_j} = \frac{\partial w_1}{\partial x_j} + w_1 \frac{\partial w_2}{\partial x_j}$$

where generally w_2 will be the last letter in the word $w = w_1w_2$. Let γ be the natural map from $\boldsymbol{Z}[F_n]$ to $\boldsymbol{Z}[G]$ and let α be the same from $\boldsymbol{Z}[G]$ to $\boldsymbol{Z}[\mathrm{ab}(G)]$. Then the Alexander matrix A of the presentation is the $m \times n$ matrix with entries

$$a_{ij} = \alpha\gamma\left(\frac{\partial r_i}{\partial x_j}\right).$$

We define the kth elementary ideal $E_k(A)$ to be the ideal of $\boldsymbol{Z}[\mathrm{ab}(G)]$ generated by the $(n-k)\times(n-k)$ minors of A if $0 < n-k \le m$, thus under this notation k is the number of columns that are deleted in forming the minors. If $k \ge n$ then we define $E_k(A) = \boldsymbol{Z}[\mathrm{ab}(G)]$ and if $k < n-m$ then we let $E_k(A) = 0$. Finally we define the Alexander polynomial Δ_G to be the generator (up to units) of the smallest principal ideal containing $E_1(A)$. To calculate it we can choose a basis $(t_1, \ldots, t_b)$ for $\mathrm{ab}(G)$, apply the free differential calculus as above and then form our matrix by evaluating. From here we can determine the minors and their highest common factor. Of course this would be of little use if it depended on the presentation of G, but that it is invariant can be seen directly, as shown in [**12**, VII 4.5], by observing that applying a Tietze transformation to a presentation does not change the elementary ideals. Alternatively we have a topological definition of the Alexander polynomial, as described in [**26**, Section 2] or [**13**, Section 3]: if X is a finite CW-complex with $\pi_1 X = G$ and $f : \tilde{X} \to X$ is the regular cover corresponding to the homomorphism α from G to $\mathrm{ab}(G) = \boldsymbol{Z}^{\beta_1(G)}$ then, taking $p \in X$, the Alexander module of X over the group ring $\boldsymbol{Z}[\mathrm{ab}(G)]$ is $H_1(\tilde{X}, f^{-1}(p); \boldsymbol{Z})$. The connection between the two approaches is that by taking a free resolution of this module, we obtain the Alexander matrix as above (or rather under our notation it is the transpose of A). The Alexander polynomial Δ_G is only defined up to units, thus we can think of Δ_G as a Laurent polynomial in $\boldsymbol{Z}[t_1^{\pm 1}, \ldots, t_b^{\pm 1}]$ up to multiplication by $\pm t_1^{k_1} \cdots t_b^{k_b}$. Of course the actual coefficients depend on this choice of basis, but we can make a change of basis if necessary by putting $t_i = s_1^{k_{i1}} \cdots s_b^{k_{ib}}$ with the vectors $(k_{i1}, \ldots, k_{ib})$ making up an element of $GL(b, \boldsymbol{Z})$.

A variant on the Alexander polynomial is obtained if, rather than taking the abelianisation map α from $\boldsymbol{Z}[G]$ to $\boldsymbol{Z}[\mathrm{ab}(G)]$ as above, we replace α with any group homomorphism β of G onto a free abelian group freely generated by say $e_1, \ldots, e_k$ (where $k \le b$). We can define the Alexander polynomial $\Delta_{G,\beta}$ relative to β by exactly the same process. As any β will factor through α, writing $\beta = \tilde{\beta}\alpha$ we have

$$\Delta_G\big(\tilde{\beta}(t_1), \ldots, \tilde{\beta}(t_b)\big) \text{ divides } \Delta_{G,\beta}(e_1, \ldots, e_k) \in \boldsymbol{Z}\big[e_1^{\pm 1}, \ldots, e_k^{\pm 1}\big]$$

because the left hand side is evaluation of Δ_G (the highest common factor of the minors) under $\tilde{\beta}$, whereas the right hand side is obtained by first evaluating the matrix A under $\tilde{\beta}$ and then taking the highest common factor of these resulting minors. In particular we can take any surjective homomorphism $\chi : G \to \boldsymbol{Z}$ and calculate $\Delta_{G,\chi}$.

We need to see what form this Alexander polynomial takes when G is a mapping torus of a finitely presented group. The proposition below can be thought of as the appropriate generalisation in algebraic terms that a fibred knot in S^3 has a monic Alexander polynomial.

PROPOSITION 2.1. *Suppose G is a finitely presented group and $\chi : G \to \boldsymbol{Z}$ is a surjective homomorphism with $\ker\chi = K$. Then the relative Alexander polynomial $\Delta_{G,\chi} \in \boldsymbol{Z}[t^{\pm 1}]$ has degree $\beta_1(K;\boldsymbol{Q})$. Furthermore $\Delta_{G,\chi}$ is monic if K is finitely presented.*

PROOF. The first part is a well known fact, found for instance as Theorem 6.17 in [**25**]. The proof there is only for a knot complement but it is applicable to the general case without any change. Note that this result also includes $\Delta_{G,\chi} = 0$ if and only if $\beta_1(K;\boldsymbol{Q})$ is infinite.

Now suppose K has a presentation of the form $\langle x_1, \ldots, x_k | r_1, \ldots, r_l\rangle$ then we can take for G the presentation

$$\langle x_1, \ldots, x_k, t | tx_1t^{-1}w_1^{-1}, \ldots, tx_kt^{-1}w_k^{-1}, r_1, \ldots, r_l\rangle$$

for w_i some word in $x_1, \ldots, x_k$. Forming the $(k+l) \times (k+1)$ matrix A from this ordered presentation using the free differential calculus and evaluating with respect to the homomorphism χ, we consider the top left $k \times k$ submatrix $S = (s_{ij})$. As these entries are formed by taking the derivative of the relator $tx_it^{-1}w_i^{-1}$ with respect to x_j, we obtain $s_{ij} = t\delta_{ij} - m_{ij}$, where $m_{ij} \in \boldsymbol{Z}$ is just the exponent sum of x_j in w_i. Now $\Delta_{G,\chi}$ is the highest common factor of all $k \times k$ minors of A so it must divide $\det S$, which is just the characteristic equation of (m_{ij}) and so is monic. □

This can also be adapted to any finitely presented mapping torus Γ_θ, regardless of θ, despite the fact that if θ is not an automorphism then Γ is not unique.

COROLLARY 2.2. *If G is a finitely presented group that is a mapping torus $\Gamma_\theta = \langle \Gamma, t\rangle$ for θ any endomorphism and Γ any group, then $\beta_1(\Gamma)$ is at least $\beta_1(\ker\chi)$ where χ is the associated homomorphism of Γ_θ. Moreover $\Delta_{G,\chi}$ is monic if Γ is finitely presented.*

PROOF. Although $\Gamma \neq \ker\chi$ in general, let Γ' be the natural image of Γ in G as before in the case where θ is not injective (and equal to Γ otherwise), then

$$\ker\chi = \bigcup_{n=0}^{\infty} t^{-n}\Gamma' t^n$$

and so $\beta_1(\ker\chi) \leq \beta_1(\Gamma') \leq \beta_1(\Gamma)$, where the first inequality follows because if ker χ surjects onto $\boldsymbol{Z}^k$, we can take k elements in ker χ which are mapped onto a generating set, and these will all lie in $t^{-N}\Gamma' t^N$ for some N. Moreover the second part goes through exactly as in Proposition 2.1 for any θ when Γ is finitely presented because we can take the same presentation for G with $w_i = \theta(x_i)$, and so $\Delta_{G,\chi}$ is monic. □

We now wish to show how the Alexander polynomial allows us to find kernels with arbitrarily high first Betti number.

THEOREM 2.3. *If G is a finitely presented group with $\beta_1(G) = b \geq 2$ and the multivariable Alexander polynomial Δ_G is not constant then in any non-empty open set $U \subseteq S^{b-1}$ there are homomorphisms χ from G onto $\boldsymbol{Z}$ with $\beta_1(\ker\chi;\boldsymbol{Q})$ arbitrarily high.*

PROOF. Given such a χ in U, we can take a presentation $\langle g_1,\dots,g_n|r_1,\dots,r_m\rangle$ for G such that $\chi(g_1)=1,\chi(g_i)=0$ for $i>1$ and such that each of $g_1,\dots,g_b$ has zero exponential sum in each relation r_i. For instance this can be achieved by using Nielsen transformations whose effect on the abelianised relation matrix of this presentation for G converts it into standard form. Thus the images of $g_1,\dots,g_b$ form a basis for $\mathrm{ab}(G)$ which we denote as $t_1,\dots,t_b$.

We take the multivariable Alexander polynomial $\Delta_G(t_1,\dots,t_b)$ which is non-constant and consider the two variable polynomial $\Delta_G(t_1,t_2,1,\dots,1)$. We require that this has more than one non-trivial term. We can assume it is non-zero because $\Delta_G(t,1,\dots,1)$ divides the polynomial $\Delta_{G,\chi}(t)$ so if it is zero then we already have $\beta_1(\ker\chi;\boldsymbol{Q})=\infty$. However it could be constant, in which case we make a change of basis just amongst $(t_2,\dots,t_b)$ so that the effect of χ is unchanged. We describe how to do this for $b=3$ and then the general result follows by reducing the dimension by one each time.

Think of the monomial $t_1^{n_1}t_2^{n_2}t_3^{n_3}$ as the lattice point $(n_1,n_2,n_3)\in\boldsymbol{Z}^3$ and consider the finite subset L of $\boldsymbol{Z}^3$ which represents the non-trivial terms of Δ_G. As we can multiply by units, we can assume that L contains the origin and that all points in L have $n_2\ge 0$. We then choose another point $m=(m_1,m_2,m_3)\in L$ with m_2 maximal and m_3 maximal subject to this. For $k>0$ consider the change of basis where t_2 goes to $s_2^{-k}s_3$ and t_1,t_3 go to s_1,s_2 respectively. Then on setting $s_3=1$ we have (n_1,n_2,n_3) becoming (n_1,n_3-kn_2) so that if $k>\max(n_3/n_2)$ for those points in L with $n_2\neq 0$ then no other point in $\boldsymbol{Z}^2$ can cancel out with the origin. Also if a point n were to cancel with m under their projections in $\boldsymbol{Z}^2$ then we have $m_3-n_3=k(m_2-n_2)$ so also taking $k>\max(m_3-n_3)/(m_2-n_2)$ where defined, we would have $m=n$. Thus $\Delta_G(s_1,s_2,1)$ is non-constant.

Having made our change of basis so that $\Delta_G(t_1,t_2,1,\dots,1)$ has at least two terms, we adapt our notation and refer to this as $\delta(t_1,t_2)$, as well as setting $u=g_1$ and $v=g_2$ with images $x=t_1,y=t_2$ respectively in $\mathrm{ab}(G)$. We know that up to units $\delta(x,y)=p_r(y)x^r+\dots+p_0(y)$ with $p_i\in\boldsymbol{Z}[y^{\pm1}]$, $r\ge 1$ and $p_0(y),p_r(y)$ not the zero polynomial (or if $r=0$ then $\delta(x,y)$ is purely a polynomial in y that is at least linear). We multiply by an appropriate unit y^k to make p_0, and hence also $\delta(x,y)$, have a non-zero constant term. We further assume that p_r has a non-zero term in y^l for $l\ge 0$ which can be achieved by replacing y with y^{-1} if necessary.

We are now going to construct homomorphisms χ_m for $m\ge 1$ whose Alexander polynomial Δ_{G,χ_m} has arbitrarily high degree. If F_2 is the free group on u,v then there exists a free basis (α,β) of F_2 such that u is the same as $\alpha^{m^2+1}\beta^m$ in homology and v is equivalent to $\alpha^m\beta$. Let $\chi_m:G\to\boldsymbol{Z}$ be defined by $\chi_m(\alpha)=1$, $\chi_m(\beta)=0$, $\chi_m(g_i)=0$ for $i>2$, so that $\chi_m(u)=m^2+1,\chi_m(v)=m$. We have $\chi_m\to\chi$ in the natural topology of S^{b-1} so χ_m is eventually in U.

We now estimate $\beta_1(\ker\chi_m;\boldsymbol{Q})$ using the Alexander polynomial; it is the degree of Δ_{G,χ_m} which is at least that of Δ_G when evaluated at χ_m. But this is $\Delta_G(t^{m^2+1},t^m,1,\dots,1)$ which is $\delta(t^{m^2+1},t^m)$, thus a term $x^{k_1}y^{k_2}$ of $\delta(x,y)$ is now $t^{k_1(m^2+1)+k_2m}$. Our non-zero constant term of δ remains constant and can only be cancelled out by another term on evaluation if (m^2+1) divides the power of y, which will not happen for m large. Considering the y^lx^r term of $\delta(x,y)$, where we choose l

maximal with non-zero coefficient, this is now $t^{r(m^2+1)+lm}$ and for large enough m no other term can result in this high a power of t, because $r > 0$ was chosen as maximal amongst the coefficients of x in the terms of $\delta(x,y)$ and $l \geq 0$ chosen as maximal subject to this (or $l > 0$ if only y terms appear). Thus no other term in $\delta(x,y)$ can cancel with this one, so $\beta_1(\ker \chi_m; \boldsymbol{Q})$ is at least the degree of δ as a polynomial in t, which is in turn at least $r(m^2+1)+lm$ and this tends to infinity with m. □

We now have our main result. By combining Theorem 2.3 with the results on openness of finitely generated and finitely presented kernels mentioned at the beginning of this section, we immediately obtain:

Corollary 2.4. *If the finitely presented group $G = \Gamma_\theta$ is the mapping torus of the finitely generated* (*respectively finitely presented*) *group Γ using the automorphism θ such that $\beta_1(G) \geq 2$ and the multivariable Alexander polynomial Δ_G is not constant then $G = H_\phi$ for infinitely many finitely generated* (*respectively finitely presented*) *groups H and corresponding automorphisms ϕ, with $\beta_1(H)$ being arbitrarily large.*

Thus we now need criteria to tell us when the multivariable Alexander polynomial of a finitely presented mapping torus is non-constant and this is the subject of the next section.

3. Deficiency.

Given a finitely presented group G, recall that the deficiency of a finite presentation of G is the number of generators minus the number of relators. We can clearly add relators so we are interested in the maximum deficiency over all presentations of G, which is bounded above by $\beta_1(G)$, and we call this the deficiency $d(G)$ of the group G. In applying the results of the last section it is handy to divide up the cases according to $d(G)$.

A. Deficiency at least two.

If $d(G) \geq 2$ then the results only apply in a negative sense. We have that the multivariable Alexander polynomial Δ_G is identically zero, so the kernel K of any surjective homomorphism $\chi : G \to \boldsymbol{Z}$ has $\beta_1(K; \boldsymbol{Q})$ infinite and thus is certainly infinitely generated. Indeed we see by Corollary 2.2 that G is not even a mapping torus with respect to any finitely generated group Γ and any endomorphism θ.

B. Deficiency one.

It is here that the theory is most complete. We assume that $G = \langle x_1, \ldots, x_n | r_1, \ldots, r_{n-1} \rangle$ and work out when Δ_G is non-constant by using a theorem of McMullen in [**26**] where it was applied to 3-manifolds with boundary. The proof appeared there in outline; here we reproduce it more fully in order to emphasise that the result applies to any deficiency 1 presentation.

Theorem 3.1. *If G has a deficiency* 1 *presentation with $\beta_1(G) = b > 1$ then, taking $(t_1, \ldots, t_b)$ as any basis for* $\mathrm{ab}(G)$ *so that G has multivariable Alexander polynomial $\Delta_G(t_1, \ldots, t_b)$, for any surjective homomorphism $\chi : G \to \boldsymbol{Z}$* (*which factors as $\tilde{\chi}\alpha$ for α*

the natural map from G to $\mathrm{ab}(G)$) *we have*

$$\Delta_{G,\chi}(t) = (t-1)\Delta_G\big(t^{\tilde{\chi}(t_1)}, \dots, t^{\tilde{\chi}(t_b)}\big).$$

Proof. Given our presentation $\langle x_1, \dots, x_n | \, r_1, \dots, r_{n-1}\rangle$ we can change it as in Theorem 2.3 so that the image of the generators $(x_1, \dots, x_b)$ under the natural ring homomorphism $\alpha\gamma$ is our basis $(t_1, \dots, t_b)$ and that for $j > b$ we have $\alpha\gamma(x_j) = 1 \in \boldsymbol{Z}[\mathrm{ab}(G)]$. We form the Alexander matrix A of the presentation which has size $(n-1)\times n$, so let m_k be the minor of A obtained by deleting the kth column. We find an alternative expression for m_k by choosing another column, say the jth for $1 \le j \le b$ and $j \ne k$, and multiplying each term in the jth column by $1 - t_j$. Thus the determinant of this new matrix is $\pm(1-t_j)m_k$. But now we can replace the entries in the jth column by using the fundamental formula in Section 2 which when applied here with $v = r_i$ for any of the relators r_i tells us that for any i we have $(1-x_j)\partial r_i/\partial x_j$ equal in $\boldsymbol{Z}[F_n]$ to

$$\tau(r_i) - r_i + (x_1 - 1)\frac{\partial r_i}{\partial x_1} + \dots + (x_{j-1} - 1)\frac{\partial r_i}{\partial x_{j-1}} + (x_{j+1} - 1)\frac{\partial r_i}{\partial x_{j+1}} + \dots + (x_n - 1)\frac{\partial r_i}{\partial x_n}$$

where $\tau(r_i) = 1$. But on mapping this expression to $\boldsymbol{Z}[\mathrm{ab}(G)]$ by $\alpha\gamma$, we have that r_i goes to 1 too, with $x_1, \dots, x_b$ going to $t_1, \dots, t_b$ and $x_{b+1}, \dots, x_n$ going to 1. Thus if we replace $(1-x_j)\partial r_i/\partial x_j$ by the above expression in each entry of the jth column, we can now use linear multiples of each column to remove all terms appearing in the entries of the jth column except the image of $(x_k - 1)\partial r_i/\partial x_k$. But the determinant of the resulting matrix is just zero for $k > b$ and $\pm(1-t_k)m_j$ for $1 \le k \le b$ thus we have $m_k = 0$ for $k > b$ and

$$m_k(1-t_j) = \pm m_j(1-t_k) \qquad \text{for } 1 \le k \le b.$$

In the latter case $m_j = \delta_j(1-t_j)$ for some $\delta_j \in \boldsymbol{Z}[\mathrm{ab}(G)]$ because this is a unique factorisation domain. Thus $\delta_k = \pm\delta_j = \Delta$ (say) for $1 \le j, k \le b$, meaning that the Alexander polynomial $\Delta_G = \mathrm{hcf}(m_1, \dots, m_b)$ is Δ.

Now in calculating the Alexander polynomial with respect to χ, let $(\chi(x_1), \dots, \chi(x_b)) = (k_1, \dots, k_b)$ so we evaluate the minors at $\tilde{\chi}$ and then take the highest common factor of

$$\Delta_G(t^{k_1}, \dots, t^{k_b})(t^{k_1} - 1), \dots, \Delta_G(t^{k_1}, \dots, t^{k_b})(t^{k_b} - 1)$$

so we need $\mathrm{hcf}(t^{k_1} - 1, \dots, t^{k_b} - 1)$ which is $t - 1$ because the irreducible factors of the ith term are the cyclotomic polynomials dividing k_i, but $\mathrm{hcf}(k_1, \dots, k_b)$ is 1 as χ is onto. □

We can now make some general applications.

Corollary 3.2. *Suppose G has deficiency* 1. *If $\beta_1(G) \ge 3$ then $\beta_1(\ker\chi; \boldsymbol{Q})$ is unbounded over homomorphisms χ from G onto $\boldsymbol{Z}$. If $\beta_1(G) = 2$ then either*

$\beta_1(\ker\chi;\boldsymbol{Q})=1$ for all χ or $\beta_1(\ker\chi;\boldsymbol{Q})$ is unbounded, as determined by whether Δ_G is non-zero and constant or not.

PROOF. We see that $\Delta_G = k \neq 0$ implies that $\beta_1(\ker\chi;\boldsymbol{Q}) = 1$ for all χ by Theorem 3.1 and Proposition 2.1, and if Δ_G is non-constant then Theorem 2.3 applies. If $\beta_1(\ker\chi;\boldsymbol{Q}) = 1$ then $\beta_1(G) \leq 2$ so for $\beta_1(G) \geq 3$ the former case cannot hold. $\square$

COROLLARY 3.3. *If $G = \Gamma_\theta$ is a group of deficiency 1 with $\beta_1(G) \geq 3$ and is a mapping torus of the finitely generated group Γ with respect to the automorphism θ then we can take $\beta_1(\Gamma)$ to be arbitrarily large.*

Now we look at free-by-cyclic groups.

THEOREM 3.4. *If F_n is the free group of rank n and G is F_n-by-$\boldsymbol{Z}$ where $\beta_1(G) \geq 2$ and $n \geq 2$ then G is F_m-by-$\boldsymbol{Z}$ for infinitely many m.*

PROOF. We know G certainly has a deficiency 1 presentation as in Proposition 2.1 with $l = 0$ (which gives the deficiency of G). By Corollary 3.2 Δ_G is non-constant for $n \geq 2$ so we can now apply Corollary 2.4 to get that G has homomorphisms onto $\boldsymbol{Z}$ with finitely presented kernels K having $\beta_1(K)$ arbitrarily large. To show that K is always free, observe that G has cohomological dimension $\mathrm{cd}(G) = 2$ because we can build a 2 dimensional CW-complex X with fundamental group G whose universal cover $\tilde{X} = T \times \boldsymbol{R}$ where T is a tree, thus $\tilde{X}$ is contractible. By Shapiro's Lemma, K a subgroup of G means that $\mathrm{cd}(K) \leq 2$. Then by [**4**, Corollary 8.6] as $K \trianglelefteq G$ and both are finitely presented, we conclude that either K is free or it must have finite index in G. $\square$

Note that [**15**] proves that mapping tori of free group injective endomorphisms are coherent, that is all finitely generated subgroups are finitely presented.

EXAMPLE 1. In [**24**] the group

$$G = \langle a, b, t | Tat = b, Tbt = abA\rangle$$

(where we use A, B, T as inverses) is discussed. This is Γ_θ for θ the injective (but not surjective) endomorphism of $F_2 = \langle a, b\rangle$ given by $\theta(a) = b, \theta(b) = abA$ (with t and T swapped compared to the notation in our paper). It is pointed out that $\ker\chi$, where $\chi : G \to \boldsymbol{Z}$ is given by $\chi(t) = 1, \chi(a) = \chi(b) = 0$ is infinitely generated, non-free and locally free but that G is F_3-by-$\boldsymbol{Z}$. We therefore have from Theorem 3.4 that G is F_m-by-$\boldsymbol{Z}$ for other m, and here we can give an exact description.

PROPOSITION 3.5. *This group G is F_m-by-$\boldsymbol{Z}$ for exactly $m \geq 3$.*

PROOF. We can eliminate b so that $G = \langle a, t\rangle$ is a 2 generator group with 1 relator. This then allows us to use the algorithm of K. S. Brown which is Theorem 4.4 in [**10**] to determine exactly which surjective homomorphisms onto $\boldsymbol{Z}$ have finitely generated kernel. The relator is $r = T^2at^2aTAtA$ so we draw it out on a grid and discover that Σ contains all homomorphisms except the original homomorphism $\chi_1(a) = 0, \chi_1(t) = 1$; the homomorphism $\chi_2(a) = -1, \chi_2(t) = 0$ and the homomorphism $\chi_3(a) = 1, \chi_3(t) = -1$.

Thus all kernels except those of the three homomorphisms above are finitely generated and free. We now calculate the Alexander polynomial Δ_G; using a, t also to stand for the obvious basis of $\mathrm{ab}(G)$, we have

$$\frac{\partial r}{\partial a} = t^{-2}(1-t)(1+t-at)$$
$$\frac{\partial r}{\partial t} = t^{-2}(a-1)(1+t-at)$$

so $\Delta_G = 1 + t - at$. Now suppose we have a homomorphism $\chi(a) = p, \chi(t) = q$ for p, q coprime and (without loss of generality) $p \neq 0, q > 0$ then to find $\Delta_{G,\chi}$ we can replace $a = s^p$ and $t = s^q$ in the above (or use Theorem 3.1) to conclude that $\Delta_{G,\chi}(s) = (s-1)(1+s^q-s^{p+q})$ which has degree $1+q+p$ for $p > 0$, $1+q$ for $q > -p > 0$ and $1-p$ for $-p > q$, thus any integer at least 3 can be obtained. □

We have many variations on this example. For instance we can proceed in exactly the same way as in Example 1 but with the relation $r_n = T^n a t^n a T A t A$ for $n \geq 2$. This allows us to conclude that the group $G_n = \langle a, t | r_n \rangle$ (which again has three exceptional homomorphisms) is F_k-by-$\boldsymbol{Z}$ for exactly $k > n$ and contains a non-free locally free subgroup (see Examples 6 and 7 for groups that are F_k-by-$\boldsymbol{Z}$ for exactly $k > 1$).

It was also shown in [**24**] that Example 1 is not subgroup separable. Of the many equivalent definitions (and alternative names as it is also called LERF) we say that a finitely generated group G is subgroup separable if for all $g \in G$ and for all finitely generated subgroups H of G with $g \notin H$, we can find N normal and of finite index in G such that $gN \notin HN/N$. It was pointed out by Blass and P. M. Neumann in [**8**] that if G has a finitely generated subgroup H which is conjugate to a proper subgroup of itself, so in particular if $\Sigma \neq -\Sigma$ (when restricted to the rationally defined points), then G is not subgroup separable.

Example 2. We can answer Problem 5 in [**28**] (see the concluding remarks for statements of these problems) by taking the family of examples $G_n = \langle x, y | (xyXY)^n \rangle$. An easy calculation gives $\Delta_{G_n} = n$ so that $\Delta_{G,\chi}(t) = n(t-1)$ and $\beta_1(\ker \chi) = 1$ for all χ, with Brown's algorithm again showing there are no finitely generated kernels for $n \geq 2$. Note that G_n is a Fuchsian group with a closed surface subgroup of finite index, but this subgroup has deficiency more than two so all the kernels of the subgroup have infinitely generated abelianisation.

C. Deficiency at most zero.

The groups here that most interest us are the fundamental groups of closed 3-manifolds (which have deficiency at least zero, and exactly zero if they have no prime factors that are S^2 bundles over S^1 by [**14**, Theorem 2.5]). In this case we have:

Theorem 3.6. *Let M be a closed orientable 3-manifold with $\beta_1(M) = b > 1$ and $G = \pi_1 M$ with $(t_1, \ldots, t_b)$ as any basis for $\mathrm{ab}(G)$. Then for any surjective homomorphism $\chi : G \to \boldsymbol{Z}$ (which factors as $\tilde{\chi}\alpha$ for α the natural map from G to $\mathrm{ab}(G)$) we have*

$$\Delta_{G,\chi}(t) = (t-1)^2 \Delta_G\big(t^{\tilde{\chi}(t_1)}, \ldots, t^{\tilde{\chi}(t_b)}\big).$$

PROOF. This is essentially Theorem 5.1 of [**26**] which shows that for such an M there exists a balanced presentation $\langle g_1, \ldots, g_n | r_1, \ldots, r_n \rangle$ for G (with the images $t_1, \ldots, t_b$ of $g_1, \ldots, g_b$ a basis for $\mathrm{ab}(G)$ and those of the other generators trivial) such that on calculating the minors m_{ij} obtained from the Alexander matrix of the presentation, there is Δ with

$$m_{ij} = \pm(1 - t_i)(1 - t_j)\Delta \text{ for } 1 \leq i, j \leq n.$$

Even though this does not hold for every balanced presentation of G (for instance the one given in Proposition 4.3), we can argue exactly as in Theorem 3.1 to get $\Delta_G(t_1, \ldots, t_b) = \Delta$ and

$$\Delta_{G,\chi}(t) = (t-1)^2 \Delta_G\big(t^{\tilde{\chi}(t_1)}, \ldots, t^{\tilde{\chi}(t_b)}\big). \qquad \square$$

We now look at a more general example just to show that no equivalent will hold for all groups of deficiency zero.

EXAMPLE 3. In the hypothesis of Theorem 2.3 we assumed that the full Alexander polynomial Δ_G of G was not constant, but all that is required for the proof to work is a non-constant relative Alexander polynomial $\Delta_{G,\beta}$ where β is a homomorphism of G onto $\boldsymbol{Z}^k$ for $k \geq 2$. The example of $G = F_2 \times F_2$ shows that this can occur even when $\Delta_G = 1$. Moreover this is an important example because it is well known that the homomorphism $\chi : G \to \boldsymbol{Z}$ that sends each generator to 1 has finitely generated but not finitely presented kernel. The obvious presentation is

$$G = \langle a, b, c, d | acAC, bcBC, adAD, bdBD \rangle$$

from which we obtain the Alexander matrix, which has entries consisting of zeros and terms of the form $a - 1$ and the like. On taking a few 3×3 minors, we readily see that $\Delta_G = 1$. However if we now set $\beta : G \to \boldsymbol{Z}^2$ given by $\beta(a) = \beta(b) = (1, 0)$ and $\beta(c) = \beta(d) = (0, 1)$, which we write as s and t respectively in the group ring of $\boldsymbol{Z}^2$, we find that all minors of the relative Alexander matrix with respect to β are (ignoring signs) $(s-1)^2(t-1)$ or $(s-1)(t-1)^2$ so that $\Delta_{G,\beta} = (s-1)(t-1)$. Thus by putting $s = t$ we have $\beta_1(\ker \chi) = 3$ (and G must have deficiency zero by Theorem 3.1), showing that Theorem 3.6 does not hold for this group.

We can also answer Problem 2 in [**28**] as stated at the end of this paper. Let G_k be the direct product of k copies of F_n for $n \geq 2$.

PROPOSITION 3.7. *For $k \geq 2$ there are finitely generated kernels of homomorphisms of $\boldsymbol{Z} \times G_k$ onto $\boldsymbol{Z}$ with cohomological dimension k and $k+1$.*

PROOF. We know $\boldsymbol{Z} \times G_k$ has cohomological dimension $k+1$ so this is an upper bound for all subgroups. The kernel of any homomorphism that sends every element in a free generating set for each direct factor of G_k to 1 but the $\boldsymbol{Z}$ factor to 0 must have cohomological dimension $k+1$, because by taking a non-trivial element in the kernel from each factor we see that it contains $\boldsymbol{Z}^{k+1}$. But this kernel is finitely generated for

$k \geq 2$; see [**9**, Exercise 5.2.2]. But a homomorphism that sends a generator of the $\boldsymbol{Z}$ factor to 1 has its kernel isomorphic to G_k. □

Many other examples of mapping tori with deficiency at most zero can be created by starting with a finitely presented group Γ and producing an automorphism θ of Γ. If $\beta_1(\Gamma_\theta) \geq 2$ (for which we will need $\beta_1(\Gamma) \geq 1$) and if the Alexander polynomial of Γ_θ is non-constant then Corollary 2.4 can be applied. Moreover if the conclusion of Theorem 3.1 fails then Γ_θ has non-positive deficiency.

4. Applications to 3-manifold groups.

Any 3-manifold M (possibly with boundary but always assumed connected) has a countable fundamental group (we refer to $\pi_1 M$ as a 3-manifold group) and if M is compact then $\pi_1 M$ is finitely presented. Although no complete characterisation of 3-manifold groups is known, they have important properties: two of these which are often used to provide examples of finitely presented groups that are not 3-manifold groups are:

(1) Every finitely generated subgroup of $\pi_1 M$ is finitely presented.

(2) If M is compact then the deficiency of $\pi_1 M$ is at least zero.

Property (1) follows from the Scott compact core theorem: if M is a non-compact 3-manifold with $\pi_1 M$ finitely generated then there exists a compact submanifold M_C in M with $\pi_1 M_C = \pi_1 M$, thus a finitely generated subgroup of a 3-manifold group is the fundamental group of a compact 3-manifold.

Another established but less well known property that is most useful to us here is that if M is compact then $G = \pi_1 M$ has to be in some sense a "symmetric" group: both the Alexander polynomial Δ_G and the BNS invariant Σ of G are symmetric. This means that $\Delta_G(t_1, \dots, t_b) = \Delta_G(t_1^{-1}, \dots, t_b^{-1})$ up to units, see [**32**], and that $\Sigma = -\Sigma$ which is Corollary F in [**6**]. (Incidentally it is readily seen that in general symmetry of one does not imply symmetry of the other by playing with Brown's algorithm.) This allows us to give many examples of finitely presented non 3-manifold groups, and often it is easier to determine the non-symmetry of Δ_G than the failure of (1) or (2). Moreover we can use these symmetry properties to establish quickly two general facts about any 3-manifold group G. The Baumslag-Solitar group $B_{k,l}$ is given by the presentation $\langle a, t | ta^k t^{-1} = a^l \rangle$ and we say that it is a non-trivial Baumslag-Solitar group if $k, l \neq 0$ and $|k| \neq |l|$. It is known that the trivial Baumslag-Solitar groups do occur as 3-manifold groups whereas the non-trivial ones cannot ([**20**], [**22**]). One can also show this by computing $\Delta_{B_{k,l}}(t) = kt - l$ which is not symmetric.

The next property we establish is to do with subgroup separability of 3-manifold groups and does not appear to be known. If G is a finitely generated 3-manifold group then we cannot use the non-symmetry of the BNS invariant Σ to show that G is not subgroup separable as we did in Section 3 for other groups. In fact we cannot even use the Blass-Neumann argument which needs a finitely generated subgroup that is conjugate to a proper subgroup of itself.

Theorem 4.1. *If $G = \pi_1 M$ for M any 3-manifold and H is a finitely generated subgroup of G with $t \in G$ such that $tHt^{-1} \subseteq H$ then $tHt^{-1} = H$.*

Proof. Taking $K = \langle t, H \rangle$, we have that conjugation by t is an injective endomorphism θ of H and also there is a surjective homomorphism ϕ from the mapping torus H_θ to K. If ϕ is injective then the decomposition of K as H_θ has its associated homomorphism χ in the BNS invariant Σ of K, but K is a finitely generated 3-manifold group thus $-\chi$ is in Σ and hence θ is an automorphism. Otherwise we must have $t^n \in H$ for $n > 0$ by [**17**, Lemma 3.1], but if $tHt^{-1} \subset H$ we would have $t^nHt^{-n} \subsetneqq H$. $\square$

If M is a compact 3-manifold then we can ask the same question of $\pi_1 M$ as we have of other groups in the previous sections: does it have a homomorphism onto $\boldsymbol{Z}$ with finitely generated kernel? This is not just of independent interest but is of major geometric importance because of a famous theorem [**29**] of Stallings: if M is compact, orientable, irreducible and possesses a surjective homomorphism to $\boldsymbol{Z}$ with finitely generated kernel K then M is fibred over the circle with the fibre having fundamental group K. Indeed even without the conditions of orientability and irreducibility, the conclusion still holds provided we remove any fake 3-cells from M and cap off any 2-spheres in its boundary, as shown in [**18**, Theorem 11.6(i)], with the sole exception that M might only be homotopy equivalent to $P^2 \times S^1$ if $\pi_1 M = \boldsymbol{Z}_2 \times \boldsymbol{Z}$.

Conversely a 3-manifold M which is the topological mapping torus of a compact surface S constructed from a self-homeomorphism f of S is compact and (unless $S = S^2$) irreducible, with M orientable if and only if S is orientable and f is orientation preserving, so that $\pi_1 M$ is equal to the algebraic mapping torus Γ_θ for $\Gamma = \pi_1 S$ and θ the automorphism of Γ induced by f. Thus we can see how the machinery built up previously applies in this case: knowing the BNS invariant Σ of $\pi_1 M$ means we know all the homomorphisms χ onto $\boldsymbol{Z}$ that represent fibres. Then for such a χ the degree of the Alexander polynomial $\Delta_{\pi_1 M,\chi}$ gives us the first Betti number of the surface that M is fibred by with respect to χ; indeed $\Delta_{\pi_1 M,\chi}(t)$ is the characteristic polynomial of the monodromy matrix.

If M is a compact oriented 3-manifold then alongside the BNS invariant we have another important concept introduced by Thurston in [**30**], that of the Thurston norm on the second homology group $H_2(M;\boldsymbol{R})$ or on $H_2(M,\partial M;\boldsymbol{R})$ if M has boundary. A (semi-)norm is put on this space by defining for a given $a \in H_2(M;\boldsymbol{Z})$ or $H_2(M,\partial M;\boldsymbol{Z})$ the value $x(a)$ which is the minimum of $\chi_-(S)$ for S an embedded surface representing the class a (there will exist such a surface, which can be taken as oriented although it might be disconnected). Here $\chi_-(S)$ is the sum of $\max(0,-\chi(S_j))$ where the S_j's run over the connected components of S. The function x is then extended to $H_2(M;\boldsymbol{R})$ or $H_2(M,\partial M;\boldsymbol{R})$ thus obtaining a unit ball B for the norm, which is shown to be a polyhedron with vertices that are lattice points. By duality this space can be thought of as $H^1(M;\boldsymbol{R})$ and some of the top dimensional faces of ∂B are defined as fibred faces, meaning that they have the property that any ray defined in $H^1(M;\boldsymbol{Z})$ passing through the interior of a fibred face of ∂B has a point p on it in $H^1(M;\boldsymbol{Z})$ represented by a fibre S of M. This gives us by [**6**, Theorem E] that the BNS invariant Σ of $\pi_1 M$ is exactly the projection to the unit sphere in $H^1(M;\boldsymbol{R})$ of the fibred faces of ∂B. Moreover in this case we have $x(p) = -\chi(S)$, so that if $\chi(S) < 0$ then by taking rays which pass through a fibred face and whose intersection with $H^1(M;\boldsymbol{Z})\backslash\{0\}$ is non-empty but arbitrarily far from the origin, we obtain the following theorem in the oriented case. This theorem generalises [**27**] which shows it for a 3-manifold M fibred by a closed surface.

THEOREM 4.2. *If M is a 3-manifold fibred over the circle by a compact surface that is not the sphere, torus, projective plane, Klein bottle, disc, annulus or Möbius strip (the small surfaces) with $\beta_1(M) \geq 2$ then M is fibred by surfaces which have arbitrarily high first Betti number.*

PROOF. If M is non-orientable then let $\hat{M} \to M$ be the oriented double cover, with $G = \pi_1(M)$ and $\hat{G} = \pi_1(\hat{M})$. Then the inclusion $i : \hat{G} \to G$ induces an injective homomorphism $i^* : \mathrm{Hom}(G; \boldsymbol{Z}) \to \mathrm{Hom}(\hat{G}; \boldsymbol{Z})$. Let $\chi \in \mathrm{Hom}(G; \boldsymbol{Z})$ be the homomorphism given by the fibration in the theorem. Then $i^*\chi$ corresponds to a point in the Thurston fibred face. If $\chi_n \in \mathrm{Hom}(G; \boldsymbol{Z})$ are distinct and $[\chi_n] \to [\chi]$ in the space $\mathrm{Hom}(G; \boldsymbol{R})\backslash\{0\}/\boldsymbol{R}_+$ then $[i^*\chi_n]$ is in the same Thurston fibred face as in $[i^*\chi]$ and the Thurston norm of $i^*\chi_n$ tends to ∞, i.e. $\beta_1(\ker(\chi_n \circ i)) \to \infty$. This shows $\beta_1(\ker \chi_n) \to \infty$ since the surface which corresponds to $\ker(\chi_n \circ i)$ is at most a double cover of the surface corresponding to $\ker \chi_n$. □

However the proof of Theorem 4.2 does not answer the question of exactly which surfaces a given 3-manifold is fibred by. For this purpose Theorems 2.3 and 4.2, together with accurate knowledge of Σ are useful.

EXAMPLE 4. We would like a closed hyperbolic 3-manifold that has every orientable surface of genus 2 or more as a fibre. A place to look for closed hyperbolic 3-manifolds which have manageable group presentations is in the census. However there is only one 3-manifold M in this list which has $\beta_1(M) > 1$. Fortunately this 3-manifold v1539(5,1) will do.

PROPOSITION 4.3. *The closed orientable hyperbolic* 3*-manifold* v1539(5,1) *is fibred by every closed orientable surface of genus* 2 *or more.*

PROOF. We have for $G = \pi_1 \mathrm{v}1539(5,1)$ the 2 generator 2 relator presentation

$$\langle a, b | aBab^3aBaBAbAbABaBaBAbAbAB, aBab^4aBaBA^4B \rangle$$

so this has Alexander polynomial $\Delta_G = a^2b + ab^2 + ab + a + b$. Although we cannot directly apply Brown's algorithm to G, we can use it for the two groups G_1, G_2 defined by only one of these relations. Then a homomorphism of G_i onto $\boldsymbol{Z}$ with finitely generated kernel will continue to have this property as a homomorphism from G, provided only that it factors through G. In fact in the second case we see that all homomorphisms are in Σ except possibly the two homomorphisms (up to sign) $\chi(a) = \pm\chi(b) = \pm 1$. Moreover these have infinitely generated kernel which we can conclude by Proposition 2.1 because $\Delta_{G,\chi}(t)$, which we know is $(t-1)^2$ times Δ_G evaluated at χ, is not monic and G is a coherent group. Hence in this case Σ consists of all but these four points of S^1. Thus taking any other surjective homomorphism χ and setting $\chi(a) = p, \chi(b) = q$ for p, q coprime we have, for say $p \geq q \geq 0$, that our relative Alexander polynomial

$$\Delta_{G,\chi}(t) = (t-1)^2\Delta_G(t^p, t^q) = (t-1)^2(t^{2p+q} + t^{p+2q} + t^{p+q} + t^p + t^q)$$

which has degree $2p + 2$ so our fibre is of genus $p + 1$. □

The manifold M in Proposition 4.3 has a lot of fibrations for which the relative Alexander polynomials are different but have the same degree. Thus the fibres have the same genus (and we can get a whole range of genera in this way) but the fibre bundles cannot be weakly equivalent because the respective Alexander polynomials are the characteristic polynomials of the induced monodromy maps, and these would have to be conjugate matrices.

Now let us look at 3-manifolds that have cusps, namely torus boundary components. Suppose that the orientable 3-manifold M has cusps and is fibred with respect to a particular homomorphism then the fibre will have free fundamental group. It may seem that although the Alexander polynomial can tell us the rank r of this fibre subgroup, it cannot tell us the topological type of the fibre, namely the genus g and the number of punctures p where $r = 2g + p - 1$. However provided we know the elements of the fundamental group of each cusp, we can determine g by Dehn filling. Given a cusp T and a homomorphism χ of $\pi_1 M$ onto $\boldsymbol{Z}$, we define a χ-longitude to be a non-identity primitive element d of $\pi_1 T = \boldsymbol{Z} \times \boldsymbol{Z}$ (which will inject into $\pi_1 M$ unless M is a solid torus because M is irreducible) with $\chi(d) = 0$. So in Dehn filling along the χ-longitude of each cusp, we obtain a 3-manifold N which will be fibred by a closed surface of the same genus g. To determine this genus, we work out the Alexander polynomial of $\pi_1 N$ with respect to χ, just as we did with $\pi_1 M$, and this will have degree $2g$.

EXAMPLE 5. Let us take the cusped hyperbolic 3-manifold which is the exterior of the Whitehead link. This is well known to have the twice punctured torus as a fibre, so let us see it fibred in other ways (compare with [**30**, Example 1]). Indeed here we are able to obtain all of its fibres.

PROPOSITION 4.4. *The fibres of the Whitehead link exterior are precisely the n-punctured torus for $n \geq 2$.*

PROOF. We have a presentation for its fundamental group of the form

$$G = \langle a, b | abaBABabABAbabAB \rangle$$

which can be obtained by considering the Whitehead link as a 2-bridge link. Using Brown's algorithm to draw out the relator, we find that the homomorphism $\chi(a) = p, \chi(b) = q$ for $\mathrm{hcf}(p, q) = 1$ (and without loss of generality $q \geq 0$) has finitely generated kernel except for $(p, q) = (1, 0)$ or $(0, 1)$. We calculate the full Alexander polynomial $\Delta_G(a, b) = (a - 1)(b - 1)$ so that

$$\Delta_{G,\chi}(t) = (t - 1)\Delta_G(t^p, t^q) = (t - 1)(t^p - 1)(t^q - 1).$$

This has degree $1 + |p| + q$ which is at least 3 if $p, q \neq 0$. Otherwise we have the zero polynomial but these are exactly the homomorphisms which do not represent fibres, so the rank of the fibre can be anything from 3 upwards.

We can determine its topological type for a given χ by Dehn filling along both χ-longitudes to get N and then finding $\Delta_{\pi_1 N, \chi}$. It is the case that a represents a meridian of the knot and $w = baBABab$ commutes with a thus this is also in the cusp because M is a hyperbolic 3-manifold. Hence we see that $AbaBABab$ is trivial in homology and so must be one longitude. Moreover by spotting from the relation that the other meridian b commutes with $abABAba$, we see that $BabABAba$ is the other longitude. But as we already know we have the 2-punctured torus as a fibre, on Dehn filling these χ-longitudes we obtain a torus bundle which can only have a torus as a fibre (for instance the fundamental group must be solvable). But we have just seen that the longitudes do not depend on χ so we always obtain the same 3-manifold by this process. This argument works for any cusped 3-manifold fibred by an n-punctured torus where all the cusps have rank 1 in homology. □

EXAMPLE 6. One never hears about a once-punctured torus bundle M being fibred by anything other than a once-punctured torus. Usually we have $\beta_1(M) = 1$ anyway so this is the only fibre (as happens for all cases when M is hyperbolic) but an important non-hyperbolic example where $\beta_1(M) = 2$ is given by the presentation

$$G = \langle a, b, t | taT = ab, tbT = b \rangle$$

and indeed all once-punctured torus bundles with $\beta_1(M) = 2$ have a monodromy matrix which is conjugate to a power of this one and hence are obtained as cyclic covers of M. Moreover M is a Siefert fibred space and was the first known example of a compact 3-manifold with a fundamental group which is not subgroup separable, see [**11**]. We can eliminate the generator b from the presentation, allowing us to conclude that $\Delta_G(a, t) = t - 1$ and via Brown's algorithm that all homomorphisms except $\chi(a) = \pm 1, \chi(t) = 0$ represent fibres. Thus the trick above in Example 6 tells us that the fibres are exactly tori with any (strictly positive) number of punctures. This also applies to the cyclic covers of M as the Alexander polynomial and BNS invariant will be the same. This leaves only the trivial once-punctured torus bundle which is dealt with next.

EXAMPLE 7. As our last example, we return to trivial fibre bundles. If the 3-manifold M is the trivial fibre bundle with fibre the compact surface S (which is not a small surface) then we have $\pi_1 M = F \times \boldsymbol{Z}$ where $F = \pi_1 S$ is free or a closed surface group. Taking a standard generating set $a_1, \ldots, a_r, t$ for $\pi_1 M$, we have an obvious presentation. Now on being given a homomorphism χ from $\pi_1 M$ onto $\boldsymbol{Z}$ that sends $(a_1, \ldots, a_r, t)$ to $(k_1, \ldots, k_r, m)$, it is not hard to see by projecting ker χ to F that ker χ is finitely generated if and only if $m \neq 0$, in which case ker χ is isomorphic to a subgroup of index $|m|$ in F.

This gives us the result in [**31**], [**21**] and [**19**] that the closed trivial fibre bundle with fibre a surface S_g of genus $g \geq 1$ is also fibred by precisely the surfaces of genus $m(g-1)+1$ for $m \geq 1$, and it also says that if F were free of rank $r \geq 1$ then the other fibres would have fundamental group free of rank $m(r-1)+1$. To obtain their topological type under each homomorphism χ we can Dehn fill as above which we now illustrate: suppose that our surface S has genus g and 1 boundary component so that $r = 2g$. We can choose generators for $\pi_1 S$ so that the cusp has basis $p = a_1 a_2 a_1^{-1} a_2^{-1} \cdots a_{2g-1} a_{2g} a_{2g-1}^{-1} a_{2g}^{-1}$ and t. We have multivariable Alexander polynomial $\Delta_{\pi_1 M} = (t-1)^{r-1}$ and if $\chi(t) = m \neq 0$ then we Dehn fill the curve p to obtain the closed 3-manifold N because this is always the χ-longitude. On doing this for the original homomorphism we obviously get $N = S_g \times S^1$ so that $\Delta_{\pi_1 N}(a_1, \ldots, a_r, t) = (t-1)^{r-2}$. But we always fill the same curve, giving

$$\Delta_{\pi_1 N,\chi}(s) = (s-1)^2 \Delta_{\pi_1 N}\big(s^{\chi(a_1)}, \dots, s^{\chi(a_r)}, s^{\chi(t)}\big) = (s-1)^2 (s^m - 1)^{r-2}$$

so that the degree is $|m|(r-2)+2$. This means that under χ the fibre has genus $|m|(g-1)+1$ and $|m|$ boundary components, so that unless S is a once-punctured torus, both the number of boundary components and the genus go to infinity with $|m|$.

5. Concluding remarks.

The questions we have been considering can be thought of as originating in [**28**] which finishes by posing six problems. To end this paper and to show progress has been made since then, we reproduce these problems along with answers where known. Here G is a finitely generated group with $\beta_1(G) \geq 2$ and $fgK(G)$, $fgfK(G)$, $fpK(G)$ are the subspaces of finitely generated, finitely generated free and finitely presented kernels in the space of all kernels of homomorphisms from G onto $\boldsymbol{Z}$.

Problem 1. The subset $fgfK(G)$ is open in $K(G)$. Find a group theoretic proof of this (the proof in [**28**] involved 3-manifold topology).

We can invoke the original group theoretic proof in [**28**] that $fgK(G)$ is open and combine it with the results quoted in Theorem 3.4 that $fgfK(G) \neq \varnothing$ implies both $fgfK(G) = fpK(G)$ (by [**4**, Corollary 8.6]) and that $fpK(G) = fgK(G)$ by [**15**]. But the latter result could be regarded as proved by geometric considerations, as could the proof in [**16**] that $fpK(G)$ is open.

Problem 2. Do all elements of $fgK(G)$ have the same cohomological dimension?

No. Although [**4**] tells us that only n or $n-1$ can occur, where n is the cohomological dimension of G, and moreover if 1 occurs then this is the only value (see Problem 1), we showed in Proposition 3.7 that both n and $n-1$ can occur for any $n \geq 3$.

Problem 3. If $fgfK(G) \neq \varnothing$ then is $fgfK(G) = fgK(G)$?

Yes, see Problem 1.

Problem 4. Describe the type of subsets that can occur as $fgK(G)$ when G is a 3-manifold group and in general.

The 3-manifold case is dealt with in [**30**] and see [**6**] after Corollary F for the general case.

Problem 5. Defining $fgbK(G)$ as the set of kernels with finite first Betti number, is $fgbK(G)$ contained in the closure of $fgK(G)$?

No (as expected); see Example 2.

Problem 6. Is $fpK(G)$ open and when does it equal $fgK(G)$?

As mentioned in Problem 1, [**16**] proves the former. The latter is not true in general, for instance the direct product of free groups $F_2 \times F_2$ in Example 3 has $fpK(G) = \varnothing$ but the closure of $fgK(G)$ is all of $K(G)$, and $F_2 \times F_2 \times F_2$ has $fpK(G) \neq \varnothing$ but $fpK(G) \neq fgK(G)$.

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