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Decay properties of the Stokes semigroup in exterior domains with Neumann boundary condition

By Yoshihiro Shibata* and Senjo Shimizu†

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Abstract. In this paper, we obtain local energy decay estimates and L_p-L_q estimates of the solutions to the Stokes equations with Neumann boundary condition which is obtained as a linearized equation of the free boundary problem for the Navier-Stokes equations. Comparing with the non-slip boundary condition case, we have a better decay estimate for the gradient of the semigroup because of the null force at the boundary.

1. Introduction.

Let Ω be an exterior domain in $\boldsymbol{R}^n$ ($n \geqq 3$) with boundary Γ which is a $C^{2,1}$ compact hypersurface. ν is the unit outward normal to Γ. This paper is concerned with the decay properties of solutions to the Stokes equation with Neumann boundary condition:

$$\begin{aligned} \partial_t u - \operatorname{Div} S(u,\pi) &= 0 \quad \text{in} \quad \Omega, \qquad t>0 \\ \operatorname{div} u &= 0 \quad \text{in} \quad \Omega, \qquad t>0 \\ S(u,\pi)\nu &= 0 \quad \text{on} \quad \Gamma, \qquad t>0 \\ u|_{t=0} &= u_0 \quad \text{in} \quad \Omega \end{aligned} \tag{1.1}$$

where $u = {}^t(u_1,\dots,u_n)$ and π are unknown velocity vector and pressure, respectively. u_0 is an initial velocity vector. $S(u,\pi)$ is the stress tensor given by

$$S(u,\pi) = D(u) - \pi I$$
$$D(u) = (D_{jk}(u))_{j,k=1}^n, \quad D_{jk}(u) = \partial u_j/\partial x_k + \partial u_k/\partial x_j$$

(1.1) is a linearized problem of the free boundary problem (cf. [**22**]):

$$\begin{aligned} &\partial_t v + (v\cdot\nabla)v - \Delta v + \nabla q = f(x,t) && \text{in } \Omega(t), \qquad t>0 \\ &\nabla\cdot v = 0 && \text{in } \Omega(t), \qquad t>0 \end{aligned}$$

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$$S(v,q)\nu(t) + q_0(x,t)\nu(t) = 0 \qquad \text{on } \partial\Omega(t), \quad t > 0$$
$$v|_{t=0} = v_0 \qquad \text{in } \Omega(0) \tag{1.2}$$

where v_0 is an initial velocity vector, $f(x,t)$ is a prescribed external mass force and $q_0(x,t)$ is a pressure. $\Omega(t)$ is occupied by the fluid which is given only on the initial time $t = 0$, while $\Omega(t)$ for $t > 0$ is to be determined. $\nu(t)$ is the unit outer normal to $\partial\Omega(t)$, and $v(x,t)$ and $q(x,t)$ are unknown velocity and pressure, respectively. In this model we do not take the surface tension into account.

In order to solve (1.2) global in time at least with small initial data, it is important to investigate the decay properties of solutions to (1.1), which is one of the motivations of this paper. Another motivation is due to Kozono [**13**]. In fact, according to Kozono [**13**], when we consider the nonstationary Stokes equation with nonslip boundary condition in an exterior domain $\Omega \subset \boldsymbol{R}^n$ $(n \geqq 3)$, to obtain the optimal decay rate $(n/2)(1-(1/r))$ of the L_r norm of solutions $(1 < r \leqq \infty)$ it is necessary and sufficient that the net force exerted by the fluid on the boundary is zero (the related results are cited therein). In (1.1) the force on the boundary itself vanishes, and therefore we can expect to get better decay properties of solutions compared with the nonslip boundary condition case. And such better decay rate really appears in the estimate of the gradient of solutions to (1.1). Namely, for any solution u to (1.1) there holds the gradient estimate:

$$\|\nabla u(t,\cdot)\|_{L_p(\Omega)} \leqq C_p\, t^{-1/2} \|u_0\|_{L_p(\Omega)}, \ t \to \infty \tag{1.3}$$

for any p with $1 < p < \infty$, while this estimate holds only for p with $1 < p \leqq n$ in the nonslip boundary condition case. Moreover, there holds the L_∞ estimate of the gradient of u as follows:

$$\|\nabla u(t,\cdot)\|_{L_\infty(\Omega)} \leqq C_p\, t^{-n/(2p)-1/2} \|u_0\|_{L_p(\Omega)}, \ t \to \infty \tag{1.4}$$

for any p with $1 \leqq p < \infty$, which can not be obtained in the nonslip boundary condition case.

Now, we shall state our results precisely. To do this we shall formulate (1.1) in the analytic semigroup theoretical framework, following Grubb and Solonnikov [**11**] and Grubb [**9**], [**10**]. For $1 < p < \infty$ there holds the second Helmholtz decomposition:

$$L_p(\Omega)^n = J_p(\Omega) \oplus G_p(\Omega), \quad \oplus : \text{direct sum}$$

corresponding to (1.1) with the following notation:

$$J_p(\Omega) = \{u \in L_p(\Omega)^n \mid \nabla \cdot u = 0 \quad \text{in } \Omega\}$$
$$G_p(\Omega) = \{\nabla\pi \mid \pi \in \dot{X}_p(\Omega)\}$$
$$\dot{X}_p(\Omega) = \{\pi \in X_p(\Omega) \mid \pi|_\Gamma = 0\}$$
$$X_p(\Omega) = \{\pi \in \hat{W}^1_p(\Omega) \mid \|\pi\|_{X_p(\Omega)} < \infty\}$$

$$\hat{W}_p^1(\Omega) = \{\pi \in L_{p,\mathrm{loc}}(\overline{\Omega}) \mid \nabla\pi \in L_p(\Omega)^n\}$$

$$\|\pi\|_{X_p(\Omega)} = \begin{cases} \|\nabla\pi\|_{L_p(\Omega)} + \|\pi/d\|_{L_p(\Omega)} & n \leqq p < \infty \\ \|\nabla\pi\|_{L_p(\Omega)} + \|\pi/d\|_{L_p(\Omega)} + \|\pi\|_{L_{\frac{np}{n-p}}(\Omega)} & 1 < p < n \end{cases}$$

$$d(x) = \begin{cases} 1 + |x| & p \neq n \\ (1 + |x|)\log(2 + |x|) & p = n \end{cases}$$

Let P_p be the solenoidal projection: $L_p(\Omega)^n \to J_p(\Omega)$ along $G_p(\Omega)$. To introduce the Stokes operator associated with (1.1), we consider the resolvent problem corresponding to (1.1):

$$\lambda v - \mathrm{Div}\, S(v,\theta) = P_p f, \quad \mathrm{div}\, v = 0 \ \text{ in } \Omega, \quad S(v,\theta)\nu|_\Gamma = 0 \tag{1.5}$$

If we take the divergence of (1.5) and multiply the boundary condition by ν, we have

$$\Delta\theta = 0 \ \text{ in } \Omega, \quad \theta|_\Gamma = \nu \cdot (D(v)\nu) - \mathrm{div}\, v|_\Gamma \tag{1.6}$$

because $\nu \cdot \nu = 1$ on Γ. We know that given $v \in W_p^2(\Omega)^n$ there exists a unique $\theta \in X_p(\Omega)$ which solves (1.6) and enjoys the estimate: $\|\theta\|_{X_p(\Omega)} \leqq C_p\|v\|_{W_p^2(\Omega)}$. From this point of view, let us define the map $K : W_p^2(\Omega)^n \to X_p(\Omega)$ by $\theta = K(v)$. By using this symbol, we know that (1.5) is equivalent to the reduced Stokes equation:

$$\lambda v - \mathrm{Div}\, S(v, K(v)) = P_p f \ \text{ in } \Omega, \quad S(v, K(v))\nu|_\Gamma = 0 \tag{1.7}$$

(cf. Grubb and Solonnikov [**11**]). Therefore we define the Stokes operator A_p corresponding to (1.1) by the following formulas:

$$A_p u = -\Delta u + \nabla K(u) \quad \text{for } u \in \mathscr{D}(A_p)$$

$$\mathscr{D}(A_p) = \{u \in J_p(\Omega) \cap W_p^2(\Omega)^n \mid S(u, K(u))\nu|_\Gamma = 0\}$$

From Grubb and Solonnikov [**11**] and Shibata and Shimizu [**21**], we know that A_p generates an analytic semigroup $\{T(t)\}_{t\geqq 0}$ on $J_p(\Omega)$ for $1 < p < \infty$, the details of which will be explained in Section 2, below.

The first result is concerning the local energy decay estimate. Let R_0 be a fixed number such that $\boldsymbol{R}^n \setminus \Omega \subset B_{R_0}$, where $B_L = \{x \in \boldsymbol{R}^n \mid |x| < L\}$ for given $L > 0$. Set $\Omega_R = \Omega \cap B_R$ and

$$L_{p,R}(\Omega)^n = \{f \in L_p(\Omega)^n \mid f(x) = 0 \ \text{ for } x \notin B_R\}$$

THEOREM 1.1. *Let $1 < p < \infty$ and $R \geqq R_0$. Then for every $f \in L_{p,R}(\Omega)^n$ and $t \geqq 1$ there holds the estimate*:

$$\|T(t)P_p f\|_{W_p^2(\Omega_R)} \leqq C_{p,R}\, t^{-\frac{n}{2}} \|f\|_{L_p(\Omega)} \tag{1.8}$$

The second results are concerned with the L_p-L_q decay estimate. We define the solenoidal space $J_1(\Omega)$ by the completion of the space $C^\infty_{0,\sigma}(\boldsymbol{R}^n) = \{u \in C^\infty(\boldsymbol{R}^n)^n \mid \operatorname{div} u = 0 \text{ in } \boldsymbol{R}^n$ and u vanishes outside of some large ball$\}$ in $L_1(\Omega)^1$. Then, combining Theorem 1.1 and the L_p-L_q estimate for the whole space problem by cut-off technique, we can show the following theorem along the standard argument (cf. [**12**], [**14**]).

THEOREM 1.2. *For every $f \in J_p(\Omega)$ and $t > 0$ there hold the estimates*:

$$\|T(t)f\|_{L_q(\Omega)} \leqq C_{p,q}\, t^{-\frac{n}{2}\left(\frac{1}{p}-\frac{1}{q}\right)} \|f\|_{L_p(\Omega)} \quad \text{for } 1 \leqq p \leqq q \leqq \infty \ (p \neq \infty,\ q \neq 1) \tag{1.9}$$

$$\|\nabla T(t)f\|_{L_q(\Omega)} \leqq C_{p,q}\, t^{-\frac{n}{2}\left(\frac{1}{p}-\frac{1}{q}\right)-\frac{1}{2}} \|f\|_{L_p(\Omega)} \quad \text{for } 1 \leqq p \leqq q \leqq n \ (q \neq 1) \tag{1.10}$$

Moreover, thanks to the null force at the boundary, we obtain the following theorem.

THEOREM 1.3. *Let $n < q \leqq \infty$ and $1 \leqq p \leqq q \leqq \infty$ $(p \neq \infty)$. For every $f \in J_p(\Omega)$ and $t > 0$ we have*

$$\|\nabla T(t)f\|_{L_q(\Omega)} \leqq C_{p,q}\, t^{-\frac{n}{2}\left(\frac{1}{p}-\frac{1}{q}\right)-\frac{1}{2}} \|f\|_{L_p(\Omega)} \tag{1.11}$$

Theorem 1.3 shows a significant difference of asymptotic behavior of solutions between the Neumann boundary condition and nonslip boundary condition. In fact, as already mentioned in (1.3) and (1.4), if we consider the nonslip boundary condition $u|_\Gamma = 0$ instead of the Neumann boundary condition, then we only have (1.9) and (1.10) (cf. [**3**], [**4**], [**5**], [**12**], [**15**] and [**19**]). Moreover, the condition $1 \leqq p \leqq q \leqq n$ $(q \neq 1)$ is unavoidable to get (1.10), which was proved by Maremonti and Solonnikov [**15**].

To end this section, we explain the notation which we shall use throughout the paper. Given vector or matrix M, ${}^tM$ denotes the transposed M. Given Banach space X with norm $\|\cdot\|_X$, we set

$$X^n = \left\{v = {}^t(v_1, \ldots, v_n) \mid v_j \in X\right\}, \quad \|v\|_X = \sum_{j=1}^n \|v_j\|_X$$

The dot $\cdot$ denotes the inner-product of $\boldsymbol{R}^n$. $F = (F_{ij})$ means the $n \times n$ matrix whose i-th column and j-th row component is F_{ij}. For the differentiation of the $n \times n$ matrix of functions $F = (F_{ij})$, the n-vector of functions $u = {}^t(u_1, \ldots, u_n)$ and the scalar function π, we use the following symbols: $\partial_j \pi = \partial \pi / \partial x_j$,

$$\nabla \pi = {}^t(\partial_1 \pi, \ldots, \partial_n \pi), \quad \operatorname{div} u = \sum_{j=1}^n \partial_j u_j, \quad \operatorname{Div} F = {}^t\left(\sum_{j=1}^n \partial_j F_{1j}, \ldots, \sum_{j=1}^n \partial_j F_{nj}\right)$$

$$\nabla u = (\partial_i u_j), \quad D(u) = (\partial_i u_j + \partial_j u_i), \quad I = (\delta_{ij}), \quad S(u, \pi) = D(u) - \pi I$$

where δ_{ij} is the Kronecker's delta symbol, namely $\delta_{ij} = 1$ $(i = j)$ and $= 0$ $(i \neq j)$. The

[1] In fact, for $1 < p < \infty$ we see that $C^\infty_{0,\sigma}(\boldsymbol{R}^n)$ is dense in $J_p(\Omega)$, which will be proved in the appendix, below.

inner product $(\cdot,\cdot)_\Omega$ is defined by

$$(u,v)_\Omega = \int_\Omega u(x)\cdot v(x)\,dx$$

For Banach spaces X and Y, $\mathscr{L}(X,Y)$ denotes the set of all bounded linear operators from X into Y. We write $\mathscr{L}(X)=\mathscr{L}(X,X)$. By C we denote a generic constant and $C_{a,b,\dots}$ denotes the constant depending on the quantities a, b, $\dots$. The constants C and $C_{a,b,\dots}$ may change from line to line.

2. An analytic semigroup associated with reduced Stokes equation.

In this section, we shall give an analytic semigroup theoretical formulation of (1.1) and we shall show the generation of an analytic semigroup associated with reduced Stokes equation corresponding to (1.1). Our argument here is based on the theory concerning the corresponding resolvent problem:

$$\lambda u - \operatorname{Div} S(u,\pi) = f, \quad \operatorname{div} u = 0 \quad \text{in } \Omega, \quad S(u,\pi)\nu|_\Gamma = 0 \tag{2.1}$$

We use the following theorem which was proved by Grubb and Solonnikov [**11**] and Shibata and Shimizu [**21**].

THEOREM 2.1. *Let $1<p<\infty$, $0<\epsilon<\pi$ and $\delta>0$. Set*

$$\Sigma_\epsilon = \{\lambda \in \boldsymbol{C}\setminus\{0\} \mid |\arg \lambda| \leqq \pi-\epsilon\}$$

For every $f\in L_p(\Omega)^n$ and $\lambda\in \boldsymbol{C}\setminus(-\infty,0]$, (2.1) admits a unique solution $(u,\pi)\in W_p^2(\Omega)^n\times X_p(\Omega)$, which enjoys the estimates:

$$|\lambda|\|u\|_{L_p(\Omega)} + |\lambda|^{\frac12}\|\nabla u\|_{L_p(\Omega)} + \|u\|_{W_p^2(\Omega)} + \|\pi\|_{X_p(\Omega)} \leqq C_{p,\epsilon,\delta}\|f\|_{L_p(\Omega)}$$

provided that $\lambda\in\Sigma_\epsilon$ with $|\lambda|\geqq\delta$.

Letting $\lambda\to\infty$ in (2.1) and using Theorem 2.1 we have the following lemma.

LEMMA 2.2. *Let $1<p<\infty$. Then, for any $f\in L_p(\Omega)^n$, there exist $g\in J_p(\Omega)$ and $\pi\in\dot{X}_p(\Omega)$ such that*

$$f = g + \nabla\pi \quad \textit{in } \Omega \tag{2.2}$$

$$\|g\|_{L_p(\Omega)} + \|\pi\|_{X_p(\Omega)} \leqq C\|f\|_{L_p(\Omega)} \tag{2.3}$$

PROOF. By Theorem 2.1 we see that for any integer $m\geqq 1$, there exists a sequence $\{(u_m,\pi_m)\}_{m=1}^\infty \subset W_p^2(\Omega)^n\times X_p(\Omega)$ such that (u_m,π_m) satisfies the equation:

$$m u_m - \operatorname{Div} S(u_m,\pi_m) = f, \quad \operatorname{div} u_m = 0 \quad \text{in } \Omega, \quad S(u_m,\pi_m)\nu|_\Gamma = 0 \tag{2.4}$$

and the estimate:

$$m\|u_m\|_{L_p(\Omega)} + \|u_m\|_{W_p^2(\Omega)} + \|\pi_m\|_{X_p(\Omega)} \leqq C\|f\|_{L_p(\Omega)} \tag{2.5}$$

where C is independent of m. Set

$$\begin{aligned} W_p^{\ell-1/p}(\Gamma) &= \{u \in W_p^{\ell-1}(\Gamma) \mid \exists\, v \in W_p^\ell(\Omega), v = u \text{ on } \Gamma\} \quad \ell = 1, 2 \\ \|u\|_{W_p^{\ell-1/p}(\Gamma)} &= \inf\{\|v\|_{W_p^\ell(\Omega)} \mid v \in W_p^\ell(\Omega), v = u \text{ on } \Gamma\} \end{aligned}$$

By the definition of the trace to the boundary we have

$$\|u_m\|_{W_p^{2-1/p}(\Gamma)} + \|\pi_m\|_{W_p^{1-1/p}(\Gamma)} \leqq C\|f\|_{L_p(\Omega)}$$

for any integer $m \geqq 1$. In view of the compactness theorem due to Rellich we see that there exist a subsequence $\{(u_{m_j}, \pi_{m_j})\}$ of $\{(u_m, \pi_m)\}$, $g \in L_p(\Omega)^n$, $u \in W_p^2(\Omega)^n$ and $\pi \in X_p(\Omega)$ such that

$$\begin{aligned} m_j u_{m_j} &\to g && \text{weakly in } L_p(\Omega)^n \\ \partial_x^\alpha u_{m_j} &\to \partial_x^\alpha u && \text{weakly in } L_p(\Omega)^n, \quad |\alpha| \leqq 2 \\ \partial_x^\alpha \pi_{m_j} &\to \partial_x^\alpha \pi && \text{weakly in } L_p(\Omega), \quad |\alpha| \leqq 1 \\ u_{m_j} &\to u && \text{strongly in } W_p^1(\Gamma)^n \\ \pi_{m_j} &\to \pi && \text{strongly in } L_p(\Gamma) \end{aligned} \tag{2.6}$$

as $m_j \to \infty$. By (2.5) we have $\|u_m\|_{L_p(\Omega)} \leqq Cm^{-1}\|f\|_{L_p(\Omega)}$, which implies that $u = 0$. Therefore, letting $m_j \to \infty$ in (2.4) and using (2.6), we see that g and π are required functions, which completes the proof of the lemma. □

By using the uniqueness of solutions to the Laplace equation with zero Dirichlet condition we see the uniqueness of the decomposition in (2.2), and therefore we have

$$L_p(\Omega)^n = J_p(\Omega) \oplus G_p(\Omega) \tag{2.7}$$

We call this the second Helmholtz decomposition corresponding to the Neumann boundary condition case.

We can show the following theorem by standard argument (cf. Fujiwara and Morimoto [**8**]).

Theorem 2.3. *Let $1 < p < \infty$ and $1/p + 1/p' = 1$. Then, $J_p(\Omega)^* = J_{p'}(\Omega)$.*

Now, we shall eliminate π in (2.1). To do this, we need the following lemma.

Lemma 2.4. *Let $1 < p < \infty$. Then, for any $h \in W_p^{1-1/p}(\Gamma)$ there exists a unique $\pi \in X_p(\Omega)$ which solves the Laplace equation*:

$$\Delta \pi = 0 \quad in\ \Omega, \quad \pi|_\Gamma = h$$

and satisfies the estimate:

$$\|\pi\|_{X_p(\Omega)} \leqq C\|h\|_{W_p^{1-1/p}(\Gamma)}$$

Let $(u, \pi) \in W_p^2(\Omega)^n \times X_p(\Omega)$ solve (2.1). Set P_p be a continuous projection from $L_p(\Omega)^n$ into $J_p(\Omega)$ along $G_p(\Omega)$. We take the second Helmholtz decomposition: $f = P_p f + \nabla\theta$ with $\theta \in \dot{X}_p(\Omega)$. Inserting this formula into (2.1), we have

$$\lambda u - \operatorname{Div} S(u, \pi - \theta) = P_p f, \quad \operatorname{div} u = 0 \quad \text{in } \Omega, \quad S(u, \pi - \theta)\nu|_\Gamma = 0 \tag{2.8}$$

Set $\pi - \theta = \rho$. Taking the divergence of (2.8), we have

$$\Delta\rho = 0 \quad \text{in} \ \ \Omega \tag{2.9}$$

because $\operatorname{div} u = 0$ and $\operatorname{div} g = 0$ in Ω. Since $|\nu|^2 = 1$, multiplying the boundary condition by ν, we have $\nu\cdot(D(u)\nu) - \rho|_\Gamma = 0$. Since $\operatorname{div} u = 0$ on Γ, we have the boundary condition:

$$\rho|_\Gamma = \nu \cdot (D(u)\nu) - \operatorname{div} u|_\Gamma \tag{2.10}$$

In view of Lemma 2.4, let $\rho \in X_p(\Omega)$ be a solution to the Laplace equation (2.9) with side condition (2.10). Let K be a bounded linear operator from $W_p^2(\Omega)^n$ into $X_p(\Omega)$ defined by $K(u) = \rho$. Set $\pi = \theta + K(u)$, then we finally arrive at the equation:

$$\lambda u - \operatorname{Div} S(u, K(u)) = P_p f \quad \text{in } \Omega, \quad S(u, K(u))\nu|_\Gamma = 0 \tag{2.11}$$

On the other hand, if $u \in W_p^2(\Omega)^n$ satisfies (2.11), then $\operatorname{div} u = 0$. In fact, $\operatorname{div} u$ enjoys the equation: $(\lambda - \Delta)(\operatorname{div} u) = 0$ in Ω. By (2.11), we have $0 = \nu \cdot (D(u)\nu) - K(u)|_\Gamma$, which combined with (2.10) implies that $\operatorname{div} u|_\Gamma = 0$. Therefore, by Lemma 2.4 we have $\operatorname{div} u = 0$.

From these observations, we see that the problem (2.1) is equivalent to the problem (2.11). Therefore, let us define the reduced Stokes operator A_p by

$$A_p u = -\operatorname{Div} S(u, K(u)), \quad u \in \mathscr{D}(A_p) \tag{2.12}$$

$$\mathscr{D}(A_p) = \big\{u \in W_p^2(\Omega)^n \cap J_p(\Omega) \mid S(u, K(u))\nu|_\Gamma = 0\big\} \tag{2.13}$$

By Theorem 2.1 we have the following theorem.

THEOREM 2.5. *Let $1 < p < \infty$. Then, $\boldsymbol{C} \setminus (-\infty, 0]$ is contained in the resolvent set of A_p. Moreover, for any $\epsilon \in (0, \pi)$ and $\delta > 0$ there exists a constant $C = C_{p,\epsilon,\delta}$ such that*

$$|\lambda|\|(\lambda+A_p)^{-1}f\|_{L_p(\Omega)}+|\lambda|^{\frac{1}{2}}\|\nabla(\lambda+A_p)^{-1}f\|_{L_p(\Omega)}+\|(\lambda+A_p)^{-1}f\|_{W_p^2(\Omega)} \leqq C_{p,\epsilon,\delta}\|f\|_{L_p(\Omega)} \tag{2.14}$$

for any $f \in J_p(\Omega)$ *provided that* $\lambda \in \Sigma_\epsilon$ *with* $|\lambda| \geqq \delta$.

To show the generation of analytic semigroup associated with A_p we have to show the following lemma.

Lemma 2.6. *Let* $1 < p < \infty$. *Then,* A_p *is a densely defined closed operator on* $J_p(\Omega)$.

Proof. First we shall show that $\mathscr{D}(A_p)$ is dense in $J_p(\Omega)$. Let $f \in J_p(\Omega)$, and then by Theorem 2.1 there exists a sequence $\{(u_m,\pi_m)\}_{m=1}^\infty \subset W_p^2(\Omega)^n \times X_p(\Omega)$ such that

$$mu_m - \operatorname{Div} S(u_m,\pi_m) = f, \quad \operatorname{div} u_m = 0 \quad \text{in } \Omega \tag{2.15}$$

$$S(u_m,\pi_m)\nu|_\Gamma = 0 \tag{2.16}$$

$$m\|u_m\|_{L_p(\Omega)} + \|u_m\|_{W_p^2(\Omega)} + \|\pi_m\|_{X_p(\Omega)} \leqq C\|f\|_{L_p(\Omega)} \tag{2.17}$$

Since $\pi_m = K(u_m)$, (2.15) and (2.16) imply that $u_m \in \mathscr{D}(A_p)$. Employing the same argument as in the proof of Lemma 2.2, passing to the subsequence if necessary, we see that

$$\begin{aligned} &mu_m \to g && \text{weakly in } L_p(\Omega)^n \\ &\partial_x^\alpha u_m \to 0 && \text{weakly in } L_p(\Omega)^n, \quad |\alpha| \leqq 2 \\ &\partial_x^\alpha \pi_m \to \partial_x^\alpha \pi && \text{weakly in } L_p(\Omega), \quad |\alpha| \leqq 1 \end{aligned}$$

Letting $m \to \infty$ in (2.15) we have

$$g + \nabla\pi = f, \quad \operatorname{div} g = 0 \quad \text{in } \Omega, \quad \pi|_\Gamma = 0$$

with some $g \in L_p(\Omega)^n$ and $\pi \in X_p(\Omega)$. Since $f \in J_p(\Omega)$, we have $f = g$ and $\pi = 0$. In particular, setting $v_m = mu_m$, we see that v_m converges to g weakly in $L_p(\Omega)^n$ and $v_m \in \mathscr{D}(A_p)$. By Mazur's theorem, we can choose a convex combination of sequence $\{v_m\}$, which is in $\mathscr{D}(A_p)$ and converges to g strongly in $L_p(\Omega)^n$. This shows that $\mathscr{D}(A_p)$ is dense in $J_p(\Omega)$. Now we shall show that A_p is closed operator. Let $\{u_j\}_{j=1}^\infty \subset \mathscr{D}(A_p)$ be a sequence such that

$$u_j \to u \quad \text{in } L_p(\Omega)^n, \quad A_pu_j \to v \quad \text{in } L_p(\Omega)^n \tag{2.18}$$

for some u, $v \in L_p(\Omega)^n$. Since $\overline{\mathscr{D}(A_p)} = J_p(\Omega)$, $u \in J_p(\Omega)$. If we set $f_j = u_j + A_pu_j$, then $f_j \to u + v$ in $L_p(\Omega)^n$ as $j \to \infty$. By Theorem 2.1 with $\lambda = 1$, we have

$$\|u_j - u_k\|_{W_p^2(\Omega)} \leqq C\|f_j - f_k\|_{L_p(\Omega)}$$

as j, $k \to \infty$, and therefore there exists a $w \in \mathscr{D}(A_p)$ such that $u_j \to w$ in $W_p^2(\Omega)^n$ as $j \to \infty$, which combined with (2.18) implies that $u = w \in \mathscr{D}(A_p)$ and $A_p u = v$, which completes the proof of the lemma. □

Combining Theorem 2.5 with Lemma 2.6, we have the following theorem.

THEOREM 2.7. *Let $1 < p < \infty$. Then A_p generates an analytic semigroup $\{T(t)\}_{t \geqq 0}$ on $J_p(\Omega)$.*

REMARK 2.8. We can show by the standard argument that $A_p^* = A_{p'}$ provided that $1 < p < \infty$ and $1/p + 1/p' = 1$ (cf. Fujiwara and Morimoto [**8**], Miyakawa [**16**]).

3. Analysis of the whole space problem.

In this section, we consider the resolvent problem for the Stokes equation in the whole space:

$$\lambda u - \Delta u + \nabla \pi = f, \quad \operatorname{div} u = 0 \qquad \text{in } \boldsymbol{R}^n \tag{3.1}$$

For $f \in L_p(\boldsymbol{R}^n)^n$, $1 < p < \infty$ and $\lambda \in \boldsymbol{C} \setminus (-\infty, 0]$, let us define the solution operators to (3.1) by

$$R_0(\lambda) f(x) = \mathscr{F}_\xi^{-1}\left[\frac{P(\xi)\hat{f}(\xi)}{\lambda + |\xi|^2}\right](x), \qquad \Pi f(x) = \mathscr{F}_\xi^{-1}\left[\frac{-i\xi \cdot \hat{f}(\xi)}{|\xi|^2}\right](x) \tag{3.2}$$

where $(P(\xi))_{jk} = \delta_{jk} - \xi_j \xi_k / |\xi|^2$. Given $R > 0$, we set

$$L_{p,R}(\boldsymbol{R}^n)^n = \left\{ f \in L_p(\boldsymbol{R}^n)^n \mid f(x) = 0 \text{ for } x \notin B_R \right\}$$
$$\mathscr{L}_{p,R}(\boldsymbol{R}^n) = \mathscr{L}\left(L_{p,R}(\boldsymbol{R}^n)^n, W_p^2(B_R)^n\right)$$

The following theorem is the main result in this section.

THEOREM 3.1. *Let $1 < p < \infty$ and $0 < \epsilon < \pi/2$. Then there exist $G_j(\lambda) \in Anal(U_{1/2}, \mathscr{L}_{p,R}(\boldsymbol{R}^n))$, $j = 1, 2$, such that $R_0(\lambda)$ has the following expansion*:

$$R_0(\lambda) = \lambda^{\frac{n}{2}-1} (\log \lambda)^{\sigma(n)} G_1(\lambda) + G_2(\lambda) \tag{3.3}$$

for any $\lambda \in \boldsymbol{C} \setminus (-\infty, 0]$ with $|\lambda| \leqq 1/2$, where $\sigma(n) = 1$ $(n \geqq 4, \text{ even})$ and $\sigma(n) = 0$ $(n \geqq 3, \text{ odd})$. Moreover $G_j(\lambda)$ satisfies the relation:

$$\nabla \cdot (G_j(\lambda) f) = 0 \quad \text{in } \boldsymbol{R}^n \tag{3.4}$$

for any $f \in L_{p,R}(\boldsymbol{R}^n)^n$ and $\lambda \in \boldsymbol{C} \setminus (-\infty, 0]$ with $|\lambda| \leqq 1/2$. Here $U_r = \{\lambda \in \boldsymbol{C} \mid |\lambda| \leqq r\}$

and $Anal(U_r, X)$ *denotes the set of all* X*-valued analytic function on* U_r.

REMARK 3.2. Iwashita [**12**] gave an expansion formula corresponding to (3.3) by using the result due to Murata [**17**], and therefore he had to use some weighted spaces, which required more complicated and unessential arguments to obtain several estimates in $W^2_p(\Omega_R)^n$. To prove Theorem 1.1 without using such weighted spaces unlike [**12**], we shall show Theorem 3.1 by our own method, below. Varnhorn [**23**] also gave an expansion formula like (3.3) by using the Stokes potential and the expansion formula for the Bessel functions, but we use the Fourier transform to represent the solution formula of the Stokes resolvent problem, and therefore our proof below is also essentially different from Varnhorn's one.

PROOF. Let $\psi(r) \in C^\infty_0(\boldsymbol{R})$ such that $\psi(r) = 1$ ($|r| \leqq 1$) and $\psi(r) = 0$ ($|r| \geqq 2$), and set $\phi_0(\xi) = \psi(|\xi|)$ and $\phi_\infty(\xi) = 1 - \psi(|\xi|)$. Given $\lambda \in \boldsymbol{C} \setminus (-\infty, 0]$ with $|\lambda| \leqq 1/2$, we set

$$R^N_0(\lambda)f = \mathscr{F}^{-1}_\xi\left[\frac{\phi_N(\xi)P(\xi)\hat{f}(\xi)}{\lambda + |\xi|^2}\right](x), \quad N = 0, \infty$$

First we shall show the analyticity of $R^\infty_0(\lambda)f$. Since $(\lambda + |\xi|^2)^{-1}$ is an analytic function of λ when $|\xi| \geqq 1$ and $|\lambda| \leqq 3/4$, we have

$$\frac{1}{\lambda + |\xi|^2} = \frac{1}{2\pi i}\int_{|t|=\frac{3}{4}} \frac{dt}{(t-\lambda)(t+|\xi|^2)} = \sum_{m=0}^\infty \frac{1}{2\pi i}\int_{|t|=\frac{3}{4}} \frac{dt}{(t+|\xi|^2)t^{m+1}}\lambda^m$$

and therefore $R^\infty_0(\lambda)f$ is formally given by

$$R^\infty_0(\lambda)f = \sum_{m=0}^\infty \frac{1}{2\pi i}\int_{|t|=\frac{3}{4}} \mathscr{F}^{-1}_\xi\left[\frac{\phi_\infty(\xi)P(\xi)\hat{f}(\xi)}{t + |\xi|^2}\right](x)\frac{dt}{t^{m+1}}\lambda^m \tag{3.5}$$

Since $|t + |\xi|^2| \geqq (1/8)(1 + |\xi|^2)$ when $|t| = 3/4$ and $|\xi| \geqq 1$, we have

$$\left|\partial^\beta_\xi\left[\xi^\alpha\phi_\infty(\xi)P(\xi)(t + |\xi|^2)^{-1}\right]\right| \leqq C_\beta|\xi|^{-|\beta|}$$

for any $\beta \in \boldsymbol{N}^n_0$, $|\alpha| \leqq 2$ and $|t| = 3/4$, where $\boldsymbol{N}_0 = \boldsymbol{N} \cup \{0\}$. By the Fourier multiplier theorem,

$$\begin{aligned}
&\sum_{m=0}^\infty \left\|\frac{1}{2\pi i}\int_{|t|=3/4} \mathscr{F}^{-1}_\xi\left[\frac{\phi_\infty(\xi)P(\xi)\hat{f}(\xi)}{t+|\xi|^2}\right]\frac{dt}{t^{m+1}}\lambda^m\right\|_{L_p(\boldsymbol{R}^n)} \\
&\quad \leqq \sum_{m=0}^\infty C_p\|f\|_{L_p(\boldsymbol{R}^n)}\frac{1}{2\pi}\int_{|t|=3/4}\frac{|dt|}{|t|^{m+1}}|\lambda|^m \leqq C_p\|f\|_{L_p(\boldsymbol{R}^n)}\sum_{m=0}^\infty\left(\frac{4}{3}|\lambda|\right)^m
\end{aligned} \tag{3.6}$$

The right hand side of (3.6) converges uniformly when $|\lambda| < 3/4$. Thus

$$R_0^\infty(\lambda) \in Anal\big(U_{1/2}, \mathscr{L}\big(L_p(\boldsymbol{R}^n)^n, W_p^2(\boldsymbol{R}^n)^n\big)\big) \subset Anal\big(U_{1/2}, \mathscr{L}_{p,R}(\boldsymbol{R}^n)\big)$$

Moreover, we obviously have $\nabla \cdot R_0^\infty(\lambda)f = 0$.

Next we consider $R_0^0(\lambda)f$. Let $f = {}^t(f_1, \dots, f_n) \in L_{p,R}(\boldsymbol{R}^n)^n$. Changing the variables $\xi = r\omega$, $\omega \in S^{n-1}$ and using $e^{i(x-y)\cdot r\omega} = \sum_{l=0}^\infty [i(x-y)\cdot r\omega]^l / l!$, we have

$$\begin{aligned}
\big(R_0^0(\lambda)f\big)_j &= \left(\mathscr{F}_\xi^{-1}\left[\frac{\phi_0(\xi)P(\xi)}{\lambda+|\xi|^2}\right] * f(x)\right)_j \\
&= \sum_{k=1}^n \frac{1}{(2\pi)^n}\int_{\boldsymbol{R}^n}\int_{|\omega|=1}\int_0^\infty \frac{e^{i(x-y)\cdot r\omega}\psi(r)(\delta_{jk}-\omega_j\omega_k)}{\lambda+r^2} r^{n-1} f_k(y)\,dr\,d\omega\,dy \\
&= \sum_{k=1}^n \sum_{l=0}^\infty \frac{1}{(2\pi)^n}\int_{\boldsymbol{R}^n}\int_{|\omega|=1} \frac{(i(x-y)\cdot\omega)^l}{l!}(\delta_{jk}-\omega_j\omega_k) f_k(y)\,d\omega\,dy \\
&\quad \times \int_0^\infty \frac{\psi(r) r^{n-1+l}}{\lambda+r^2}\,dr
\end{aligned}$$

where $(\cdots)_j$ denotes the j th component of $\cdots$. We prepare the following lemma.

LEMMA 3.3. *Let $m \in \boldsymbol{N}_0$, $\lambda \in \boldsymbol{C}\setminus(-\infty, 0]$ and $|\lambda| \leqq 1/2$. Then we have*

$$\int_0^\infty \frac{\psi(r)r^{2m}}{\lambda+r^2}\,dr = \frac{(-1)^m\pi}{2}\frac{\lambda^m}{\sqrt{\lambda}} + h_{2m}(\lambda) \tag{3.7}$$

$$\int_0^\infty \frac{\psi(r)r^{2m+1}}{\lambda+r^2}\,dr = \frac{(-1)^{m+1}}{2}\lambda^m \log\lambda + h_{2m+1}(\lambda) \tag{3.8}$$

where $\sqrt{\lambda}$ takes the branch $Re\sqrt{\lambda} > 0$, and $h_{2m}(\lambda)$ and $h_{2m+1}(\lambda)$ are analytic functions of λ when $|\lambda| \leqq 1/2$ which satisfy the estimates: $|h_{2m}(\lambda)| \leqq C\,2^{2m}$ and $|h_{2m+1}(\lambda)| \leqq C\,2^{2m+1}$, respectively, where C is a constant independent of m.

PROOF. Let us write

$$\int_0^\infty \frac{\psi(r)r^k}{\lambda+r^2}\,dr = \int_1^\infty \frac{\psi(r)r^k}{\lambda+r^2}\,dr + \int_0^1 \frac{\psi(r)r^k}{\lambda+r^2}\,dr = I_k(\lambda) + II_k(\lambda)$$

Since $|\lambda + r^2| \geqq r^2 - |\lambda| \geqq 1/2$ when $r \geqq 1$ and $\lambda \leqq 1/2$, $I_k(\lambda)$ is an analytic function when $|\lambda| \leqq 1/2$ and

$$|I_k(\lambda)| \leqq 2\int_1^2 r^k\,dr \leqq 4\cdot 2^k \quad \text{for all } \ k \geqq 0$$

Now, we shall analyze $II_k(\lambda)$. First, we consider the case that k is even. Set $k = 2m$. Then

$$
\begin{aligned}
II_{2m}(\lambda) &= \int_0^1 \frac{(r^2+\lambda-\lambda)^m}{r^2+\lambda}\,dr \\
&= \sum_{l=1}^m \binom{m}{l} (-\lambda)^{m-l} \int_0^1 (r^2+\lambda)^{l-1}\,dr + (-\lambda)^m \int_0^1 \frac{dr}{\lambda+r^2}
\end{aligned}
$$

By the residue theorem,

$$
\int_0^1 \frac{dr}{\lambda+r^2} = \frac{1}{2}\int_{-\infty}^{\infty} \frac{dr}{r^2+\lambda} - \int_1^{\infty} \frac{dr}{r^2+\lambda} = \frac{\pi}{2}\frac{1}{\sqrt{\lambda}} - \int_1^{\infty} \frac{dr}{r^2+\lambda}
$$

If we set

$$
II_{2m,2}(\lambda) = \sum_{l=1}^m \binom{m}{l} (-\lambda)^{m-l} \int_0^1 (r^2+\lambda)^{l-1}\,dr - (-\lambda)^m \int_1^{\infty} \frac{dr}{r^2+\lambda}
$$

then $II_{2m,2}(\lambda)$ is an analytic function in $|\lambda| < 1$, and $|II_{2m,2}(\lambda)| \leqq 2\cdot 2^m$ when $|\lambda| \leqq 1/2$. Therefore setting $h_{2m}(\lambda) = I_{2m}(\lambda) + II_{2m,2}(\lambda)$, we obtain (3.7).

Next, we consider the case that k is odd. Set $k = 2m+1$. Changing the variable $r^2 = s$, we obtain

$$
\begin{aligned}
\int_0^1 \frac{\psi(r) r^{2m+1}}{\lambda+r^2}\,dr &= \frac{1}{2}\int_0^1 \frac{(s+\lambda-\lambda)^m}{s+\lambda}\,ds \\
&= \frac{1}{2}\sum_{l=1}^m \binom{m}{l} (-\lambda)^{m-l} \int_0^1 (s+\lambda)^{l-1}\,ds + \frac{1}{2}(-\lambda)^m \int_0^1 \frac{ds}{s+\lambda} \\
&= \frac{1}{2}\sum_{l=1}^m \binom{m}{l} (-\lambda)^{m-l} \frac{1}{l}\{(1+\lambda)^l - \lambda^l\} + \frac{1}{2}(-\lambda)^m (\log(1+\lambda) - \log\lambda)
\end{aligned}
$$

If we set

$$
II_{2m+1,2}(\lambda) = \frac{1}{2}\sum_{l=1}^m \binom{m}{l} (-\lambda)^{m-l} \frac{1}{l}\{(1+\lambda)^l - \lambda^l\} + \frac{1}{2}(-\lambda)^m \log(1+\lambda)
$$

then $II_{2m+1,2}(\lambda)$ is an analytic function in $|\lambda| < 1$, and $|II_{2m+1,2}(\lambda)| \leqq C\,2^m$ when $|\lambda| \leqq 1/2$. Therefore setting $h_{2m+1}(\lambda) = I_{2m+1}(\lambda) + II_{2m+1,2}(\lambda)$, we obtain (3.8). □

Now we continue the proof of Theorem 3.1. In order to consider the analyticity of $R_0^0(\lambda)f$, we set

$$
(S_l f)_j = \frac{1}{(2\pi)^n} \sum_{k=1}^n \int_{\boldsymbol{R}^n} \int_{|\omega|=1} (i(x-y)\cdot\omega)^l (\delta_{jk} - \omega_j\omega_k) f_k(y)\,d\omega dy \tag{3.9}
$$

First we consider the case that n is odd. By using the fact that

$$\int_{|\omega|=1} (i(x-y)\cdot\omega)^{2m+1}\, d\omega = 0 \tag{3.10}$$

for any $m \in \boldsymbol{N}_0$, we have $S_{2m+1}f = 0$ for $m \in \boldsymbol{N}_0$. Since $n-1+2l$ is even, by (3.7) we have

$$R_0^0(\lambda)f = \sum_{l=0}^{\infty} \frac{S_{2l}f}{(2l)!} \int_0^\infty \frac{\psi(r)r^{n-1+2l}}{\lambda+r^2}\, dr = \sum_{l=0}^{\infty} \frac{S_{2l}f}{(2l)!} \left\{ \frac{(-1)^{\frac{n-1}{2}+l}}{2} \pi\lambda^{\frac{n}{2}-1+l} + h_{n-1+2l}(\lambda) \right\}$$

for $\lambda \in \boldsymbol{C} \setminus (-\infty, 0]$ with $|\lambda| \leqq 1/2$. If we set

$$G_1^0(\lambda)f = \sum_{l=0}^{\infty} \frac{S_{2l}f}{(2l)!} \frac{(-1)^{\frac{n-1}{2}+l}}{2} \pi\lambda^l, \quad G_2^0(\lambda)f = \sum_{l=0}^{\infty} \frac{S_{2l}f}{(2l)!} h_{n-1+2l}(\lambda)$$

then

$$R_0^0(\lambda)f = \lambda^{\frac{n}{2}-1} G_1^0(\lambda)f + G_2^0(\lambda)f$$

for $\lambda \in \boldsymbol{C} \setminus (-\infty, 0]$ with $|\lambda| \leqq 1/2$. For every $f \in L_{p,R}(\boldsymbol{R}^n)^n$, we obtain

$$\left\|\partial_x^\alpha S_l f(x)\right\|_{L_p(B_R)} \leqq C(l+1)^2 \int_{B_R} |x-y|^l |f(y)|\, dy \leqq C_p R^{\frac{n}{p'}} (l+1)^2 (2R)^l \|f\|_{L_p(\boldsymbol{R}^n)}$$

when $|x| \leqq R$ and $|\alpha| \leqq 2$. By this inequality and Lemma 3.3 we have

$$\sum_{l=0}^{\infty} \left\| \frac{S_{2l}f}{(2l)!} h_{n-1+2l}(\lambda) \right\|_{W_p^2(B_R)} \leqq C_p\, R^{\frac{n}{p'}} 2^{n-1} \|f\|_{L_p(\boldsymbol{R}^n)} \sum_{l=0}^{\infty} \frac{(2l+1)^2 (4R)^{2l}}{(2l)!}$$

which implies that $G_2^0(\lambda) \in Anal(U_{1/2}, \mathscr{L}_{p,R}(\boldsymbol{R}^n))$. Moreover we obtain $\nabla \cdot S_l f = 0$, since $\sum_{j,k=1}^n i\omega_j(\delta_{jk} - i\omega_j\omega_k) f_k(y) = 0$ when $l \geqq 1$ and since $S_0 f$ is independent of x when $l = 0$. Thus we have $\nabla \cdot G_2^0(\lambda)f = 0$. In the same manner, we see that $G_1^0(\lambda) \in Anal(U_{1/2}, \mathscr{L}_{p,R}(\boldsymbol{R}^n))$ and that $\nabla \cdot G_1^0(\lambda)f = 0$.

Next we consider the case that $n(\geqq 4)$ is even. Since $n-1+2l$ is odd, by (3.8), (3.9) and (3.10) we have

$$R_0^0(\lambda)f = \sum_{l=0}^{\infty} \frac{S_{2l}f}{(2l)!} \int_0^\infty \frac{\psi(r)r^{n-1+2l}}{\lambda+r^2}\, dr = \sum_{l=0}^{\infty} \frac{S_{2l}f}{(2l)!} \left\{ \frac{(-1)^{\frac{n}{2}+l}}{2} \lambda^{\frac{n}{2}-1+l} \log\lambda + h_{n-1+2l}(\lambda) \right\}$$

If we set

$$G_1^0(\lambda)f = \sum_{l=0}^{\infty} \frac{S_{2l}f}{(2l)!} \frac{(-1)^{\frac{n}{2}+l}}{2} \lambda^l, \quad G_2^0(\lambda)f = \sum_{l=0}^{\infty} \frac{S_{2l}f}{(2l)!} h_{n-1+2l}(\lambda)$$

then

$$R_0^0(\lambda)f = \lambda^{\frac{n}{2}-1}\log\lambda\, G_1^0(\lambda)f + G_2^0(\lambda)f$$

for $\lambda \in \boldsymbol{C}\setminus(-\infty,0]$ with $|\lambda| \leqq 1/2$. Employing the same argument as in the case that n is odd, we have $G_j^0(\lambda) \in Anal(U_{1/2}, \mathscr{L}_{p,R}(\boldsymbol{R}^n))$ and $\nabla\cdot G_j^0(\lambda)f = 0$, $j = 1, 2$. Therefore if we set $G_1(\lambda) = G_1^0(\lambda)$, $G_2(\lambda) = G_2^0(\lambda) + R_0^\infty(\lambda)$, then $G_j(\lambda)$, $j = 1, 2$, satisfy the desired properties, which completes the proof of the theorem. □

In the next theorem, we show some properties of the operator $R_0(\lambda)$ when $\lambda = 0$.

Theorem 3.4. *Let $1 < p < \infty$ and $0 < \epsilon < \pi/2$.*
(1) *For every $f \in L_p(\boldsymbol{R}^n)^n$ and $\lambda \in \Sigma_\epsilon$, there holds the estimate*:

$$|\lambda|\|R_0(\lambda)f\|_{L_p(\boldsymbol{R}^n)} + |\lambda|^{\frac{1}{2}}\|\nabla R_0(\lambda)f\|_{L_p(\boldsymbol{R}^n)} + \left\|\nabla^2 R_0(\lambda)f\right\|_{L_p(\boldsymbol{R}^n)} \leqq C_{p,\epsilon}\|f\|_{L_p(\boldsymbol{R}^n)} \tag{3.11}$$

(2) *If we define*

$$R_0(0)f = \mathscr{F}_\xi^{-1}\left[(P(\xi)/|\xi|^2)\hat{f}(\xi)\right](x)$$

then for any $f \in L_{p,R}(\boldsymbol{R}^n)^n$ there holds the estimate:

$$\begin{aligned}\sup_{|x|\geqq R+1}|R_0(0)f(x)||x|^{n-2} + \sup_{|x|\geqq R+1}|\nabla R_0(0)f(x)||x|^{n-1} + \|R_0(0)f\|_{W_p^1(B_{R+1})}\\ + \left\|\nabla^2 R_0(0)f\right\|_{L_p(\boldsymbol{R}^n)} + \sup_{|x|\geqq R+1}|\Pi f(x)||x|^{n-1} + \|\Pi f\|_{L_p(B_{R+1})} + \|\nabla\Pi f\|_{L_p(\boldsymbol{R}^n)}\\ \leqq C_{p,R}\,\|f\|_{L_p(\boldsymbol{R}^n)}\end{aligned} \tag{3.12}$$

(3) *For every $\lambda \in \boldsymbol{C}\setminus(-\infty,0]$ with $|\lambda| \leqq 1/2$ and $f \in L_{p,R}(\boldsymbol{R}^n)^n$, there holds the estimate*

$$\|R_0(\lambda)f - R_0(0)f\|_{W_p^2(B_R)} \leqq C\,p_n(|\lambda|)\|f\|_{L_p(\boldsymbol{R}^n)} \tag{3.13}$$

with a positive constant $C = C_{p,\epsilon,R}$, where

$$p_n(|\lambda|) = \max\left(|\lambda|,\ |\lambda|^{\frac{n}{2}-1}|\log\lambda|^{\sigma(n)}\right) \tag{3.14}$$

Proof. (1) Since

$$|\lambda + |\xi|^2| \geqq \sin(\epsilon/2)(|\lambda| + |\xi|^2) \tag{3.15}$$

for every $\lambda \in \Sigma_\epsilon$ and $\xi \in \boldsymbol{R}^n$, we obtain (3.11) by using the Fourier multiplier theorem.

(2) By using the formula:

$$R_0(0)f = \frac{1}{2\omega_n}\left(\frac{x_j x_k}{|x|^n} + \frac{\delta_{jk}}{n-2}|x|^{2-n}\right) * f$$

where ω_n denotes the surface area of unit sphere in $\boldsymbol{R}^n$ (cf. [**7**], [**23**]), and by [**21**, Theorem 3.5], we obtain (3.12).

(3) Since $R_0(\lambda)f \to R_0(0)f$ in $W_p^2(B_R)^n$ as $\lambda \to 0$ for $f \in L_{p,R}(\boldsymbol{R}^n)^n$, we see that

$$R_0(0) = G_2(0) \tag{3.16}$$

in Theorem 3.1. Therefore we have (3.13) by Theorem 3.1. □

4. Preliminaries.

Let D be a bounded domain in $\boldsymbol{R}^n$ ($n \geqq 2$) and the boundary ∂D be a $C^{2,1}$ hypersurface. In this section we consider the unique solvability of the problem:

$$-\mathrm{Div}\, S(u,\pi) = f, \quad \mathrm{div}\, u = 0 \quad \text{in} \quad D, \qquad S(u,\pi)\nu|_{\partial D} = g \tag{4.1}$$

where ν is the unit outward normal to ∂D. In order to consider the uniqueness of (4.1), we introduce the rigid space $\mathscr{R}$:

$$\mathscr{R} = \{Ax + b \mid A \text{ is an anti-symmetric matrix and } b \in \boldsymbol{R}^n\}$$

Let $\{p_l\}_{l=1}^M$ ($M = n(n-1)/2 + n$) be an orthogonal basis in $\mathscr{R}$ such as $(p_j, p_k)_D = \delta_{jk}$. We know that

$$D(u) = 0 \underset{\text{iff}}{\iff} u \in \mathscr{R} \tag{4.2}$$

(cf. Duvaut and Lions [**6**]) and that if $u \in \mathscr{R}$, then $\mathrm{div}\, u = 0$. Set

$$\dot{L}_p(D)^n = \big\{u \in L_p(D)^n \mid (u, p_l)_D = 0, \quad l = 1, \ldots, M\big\}$$

For the existence of the solution to (4.1), f and g should satisfy the compatibility condition:

$$(f, p_l)_D + (g, p_l)_{\partial D} = 0 \quad \text{for} \quad l = 1, \ldots, M \tag{4.3}$$

In fact, if (u, π) is a solution to (4.1), then for any $p_l \in \mathscr{R}$ we have

$$\begin{aligned}(f, p_l)_D &= (-\mathrm{Div}\, S(u,\pi), p_l)_D \\ &= -(g, p_l)_{\partial D} + (1/2)(D(u), D(p_l))_D - (\pi, \mathrm{div}\, p_l)_D = -(g, p_l)_{\partial D}\end{aligned}$$

because $D(p_l) = 0$ and $\mathrm{div}\, p_l = 0$.

The theorem which follows is the main result in this section.

Theorem 4.1. *Let* $1 < p < \infty$. *For every* $f \in L_p(D)^n$ *and* $g \in W_p^{1-1/p}(\partial D)^n$ *which satisfy* (4.3), (4.1) *admits a unique solution* $(u,\pi) \in (W_p^2(D)^n \cap \dot{L}_p(D)^n) \times W_p^1(D)$ *having the estimate*:

$$\|u\|_{W_p^2(D)} + \|\pi\|_{W_p^1(D)} \leqq C_{p,D}\big(\|f\|_{L_p(D)} + \|g\|_{W_p^{1-1/p}(\partial D)}\big) \tag{4.4}$$

To show the uniqueness of the solution to (4.1), we prepare the following lemma.

Lemma 4.2. *Let* $1 < p < \infty$. $(u,\pi) \in W_p^2(D)^n \times W_p^1(D)$ *satisfies the homogeneous equation*:

$$-\mathrm{Div}\, S(u,\pi) = 0, \quad \mathrm{div}\, u = 0 \quad \text{in} \quad D, \qquad S(u,\pi)\nu|_{\partial D} = 0 \tag{4.5}$$

if and only if $u \in \mathscr{R}$ *and* $\pi = 0$.

Proof. Let $(u,\pi) \in W_p^2(D)^n \times W_p^1(D)$ satisfy (4.5). Then (u,π) satisfies

$$u - \mathrm{Div}\, S(u,\pi) = u, \quad \mathrm{div}\, u = 0 \quad \text{in} \quad D \qquad S(u,\pi)\nu|_{\partial D} = 0$$

By the boot-strap argument we know that $(u,\pi) \in W_q^2(D)^n \times W_q^1(D)$ for any $q \in [p,\infty)$. The boundedness of D implies that $(u,\pi) \in W_q^2(D)^n \times W_q^1(D)$ for any $q \in (1,p]$. Therefore $(u,\pi) \in W_2^2(D)^n \times W_2^1(D)$. By integration by parts, we have $D(u) = 0$, and by (4.2) $u \in \mathscr{R}$. Moreover since $\nabla\pi = 0$ in D and $\pi|_{\partial D} = 0$, $\pi = 0$. The necessity is obvious, which completes the proof of the lemma. □

To show the existence of the solution to (4.1), we consider the auxiliary problem:

$$u - \mathrm{Div}\, S(u,\pi) = f, \quad \mathrm{div}\, u = 0 \quad \text{in} \quad D, \qquad S(u,\pi)\nu|_{\partial D} = g \tag{4.6}$$

Concerning (4.6), we know the following lemma which was proved in [**21**, Theorem 1.1].

Lemma 4.3. *Let* $1 < p < \infty$. *For every* $f \in L_p(D)^n$ *and* $g \in W_p^{1-1/p}(\partial D)^n$, (4.6) *admits a unique solution* $(u,\pi) \in W_p^2(D)^n \times W_p^1(D)$.

Proof of Theorem 4.1. If $(u,\pi) \in (W_p^2(D)^n \cap \dot{L}_p(D)^n) \times W_p^1(D)$ satisfies (4.5), then by Lemma 4.2, $u \in \mathscr{R}$ and $\pi = 0$. Since $u \in \dot{L}_p(D)^n$, $u = 0$, which completes the proof of the uniqueness.

Now we shall show the existence. If $(u,\pi) \in W_p^2(D)^n \times W_p^1(D)$ solves (4.6) and if f and g satisfy (4.3), then $u \in \dot{L}_p(D)^n$. In fact, for $p_l \in \mathscr{R}$ we have

$$\begin{aligned}(u,p_l)_D &= (\mathrm{Div}\, S(u,\pi), p_l)_D + (f,p_l)_D \\ &= (S(u,\pi)\nu, p_l)_{\partial D} - (1/2)(D(u), D(p_l))_D + (\pi, \mathrm{div}\, p_l)_D + (f,p_l)_D \\ &= (g,p_l)_{\partial D} + (f,p_l)_D = 0 \end{aligned} \tag{4.7}$$

where we have used the facts: $D(p_l) = 0$ and $\operatorname{div} p_l = 0$. Therefore, by using the solution of (4.6) we can reduce (4.1) to the case where $g = 0$. From this observation it is sufficient to consider the equation:

$$-\operatorname{Div} S(u, \pi) = f, \quad \operatorname{div} u = 0 \quad \text{in} \quad D, \qquad S(u, \pi)\nu|_{\partial D} = 0 \tag{4.8}$$

for $f \in \dot{L}_p(D)^n$. By (4.7) and Lemma 4.3 we see that for every $f \in \dot{L}_p(D)^n$, there exists a unique solution $(v, \theta) \in (W_p^2(D)^n \cap \dot{L}_p(D)^n) \times W_p^1(D)$ to the problem:

$$v - \operatorname{Div} S(v, \theta) = f, \quad \operatorname{div}\, v = 0 \quad \text{in} \quad D, \qquad S(v, \theta)\nu|_{\partial D} = 0 \tag{4.9}$$

which enjoys $\|v\|_{W_p^2(D)} + \|\theta\|_{W_p^1(D)} \leqq C_{p,D}\|f\|_{L_p(D)}$. Now, let us define the maps K, K_1 and K_2 by the formulas

$$K_1 f = v, \quad K_2 f = \theta, \quad K f = (K_1 f, K_2 f)$$

We know that $K_1 : \dot{L}_p(D)^n \to W_p^2(D)^n \cap \dot{L}_p(D)^n$ and $K_2 : \dot{L}_p(D)^n \to W_p^1(D)$ are bounded linear operators, respectively. Since

$$\begin{aligned} -\operatorname{Div} S(K_1 h, K_2 h) &= K_1 h - \operatorname{Div} S(K_1 h, K_2 h) - K_1 h \\ &= h - K_1 h = (I - K_1)h \quad \text{in} \quad D \end{aligned} \tag{4.10}$$

if we show the existence of the inverse operator $(I - K_1)^{-1} : \dot{L}_p(D)^n \to \dot{L}_p(D)^n$, then $(u, \pi) = K(I - K_1)^{-1} f$ is a solution to (4.8). Since $K_1 \in \mathscr{L}(\dot{L}_p(D)^n)$ is a compact operator, in order to show the existence of $(I - K_1)^{-1}$, it is sufficient to show that $I - K_1$ is injective in view of the Fredholm alternative theorem. Let $h \in \dot{L}_p(D)^n$ such that $(I - K_1)h = 0$. If we set $(v, \theta) = Kh$, then by (4.10) $v \in W_p^2(D)^n \cap \dot{L}_p(D)^n$ and $\theta \in W_p^1(D)$ enjoy the homogeneous equation (4.5), which implies that $v = 0$ and $\theta = 0$, namely $h = v - \operatorname{Div} S(v, \theta) = 0$. Therefore, we have the injectivity of $I - K_1$, which completes the proof of the theorem. $\square$

Finally we shall state some technical lemmas which will be used to keep the divergence free condition in what follows. First we shall state so-called the Bogovskiĭ-Pileckas lemma. To do this we introduce the following function spaces:

$$\dot{W}_p^m(D) = \overline{C_0^\infty(D)}^{W_p^m(D)}, \quad \dot{W}_p^0(D) = L_p(D)$$
$$\dot{W}_{p,a}^m(D) = \left\{ f \in \dot{W}_p^m(D) \mid \int_D f\, dx = 0 \right\}$$

Lemma 4.4 (cf. [**1**], [**2**] and [**18**]). *Let $1 < p < \infty$ and $m \in \boldsymbol{N}_0$. There exists a linear operator $\boldsymbol{B} : \dot{W}_{p,a}^m(D) \to W_p^{m+1}(\boldsymbol{R}^n)^n$ such that*

$$\nabla \cdot \boldsymbol{B}[f] = f_0 \quad \text{in} \quad \boldsymbol{R}^n, \qquad \operatorname{supp} \boldsymbol{B}[f] \subset D$$

$$\|\boldsymbol{B}[f]\|_{W_p^{m+1}(\boldsymbol{R}^n)} \leqq C_{m,p}\|f\|_{W_p^m(D)}$$

where $f_0 = f$ $(x \in D)$ *and* $f_0 = 0$ $(x \notin D)$.

The next lemma was proved in [**21**, Lemma 8.3].

Lemma 4.5. *Let* $k \in \boldsymbol{N}_0$, $r_j \in \boldsymbol{R}$, $j = 1, 2, 3, 4$, *such that* $0 < r_1 < r_3 < r_4 < r_2$ *and* $\chi \in C^\infty(\boldsymbol{R}^n)$ *such that* $\operatorname{supp}\nabla\chi \subset D_{r_3,r_4}$. *If* $u \in W_p^k(D_{r_1,r_2})$ *satisfies the condition*: $\operatorname{div} u = 0$ *in* D_{r_1,r_2}, *then there exists* $v \in W_p^k(\boldsymbol{R}^n)^n$ *which possesses the properties*: $\operatorname{supp} v \subset D_{r_1,r_2}$, $\operatorname{div} v = 0$ *in* $\boldsymbol{R}^n$, $(\nabla\chi)\cdot v = (\nabla\chi)\cdot u$ *in* $\boldsymbol{R}^n$ *and* $\|v\|_{W_p^k(\boldsymbol{R}^n)} \leqq C\|u\|_{W_p^k(D_{r_1,r_2})}$.

Combining Lemmas 4.4 and 4.5, we obtain the following lemma.

Lemma 4.6. *Let* $k \geqq 1$, $0 < r_1 < r_2$ *and* χ *be the same function in Lemma* 4.5. *If* $u \in W_p^k(D_{r_1,r_2})$ *satisfies the condition*: $\operatorname{div} u = 0$ *in* D_{r_1,r_2}, *then* $(\nabla\chi)\cdot u \in \dot{W}_{p,a}^k(D_{r_1,r_2})$ *and therefore*

$$\boldsymbol{B}[(\nabla\chi)\cdot u] \in W_p^{k+1}(\boldsymbol{R}^n), \quad \operatorname{supp}\boldsymbol{B}[(\nabla\chi)\cdot u] \subset D_{r_1,r_2}$$
$$\nabla\cdot\boldsymbol{B}[(\nabla\chi)\cdot u] = (\nabla\chi)\cdot u \quad \text{in} \quad \boldsymbol{R}^n$$
$$\|\boldsymbol{B}[(\nabla\chi)\cdot u]\|_{W_p^{k+1}(\boldsymbol{R}^n)} \leqq C_{p,k,r_1,r_2}\|(\nabla\chi)\cdot u\|_{W_p^k(D_{r_1,r_2})}$$

Proof. By Lemma 4.5 and the divergence theorem

$$\int_{D_{r_1,r_2}} (\nabla\chi)\cdot u\,dx = \int_{\boldsymbol{R}^n} (\nabla\chi)\cdot v\,dx = \int_{\boldsymbol{R}^n} \operatorname{div}(\chi v)\,dx = 0$$

which implies that $(\nabla\chi)\cdot u \in \dot{W}_{p,a}^k(D_{r_1,r_2})$. Therefore by Lemma 4.4, we obtain the lemma. □

5. An expansion formula of the resolvent around the origin in Ω.

In this section, we investigate the behavior of solutions to the resolvent problem (2.1) at $\lambda = 0$. Set

$$\mathscr{L}_{p,R}(\Omega) = \mathscr{L}\big(L_{p,R}(\Omega)^n, W_p^2(\Omega_R)^n\big)$$

The theorem which follows is the main result in this section.

Theorem 5.1. *Let* $1 < p < \infty$, $0 < \epsilon < \pi/2$ *and* $R > R_0+3$. *Then there exist* $\lambda_0 = \lambda_{p,R} > 0$, $H_0 \in \mathscr{L}_{p,R}(\Omega)$, $H_1(\lambda) \in B(\dot{U}_{\lambda_0}, \mathscr{L}_{p,R}(\Omega))$ *and* $H_2(\lambda) \in Anal(U_{\lambda_0}, \mathscr{L}_{p,R}(\Omega))$ *such that*

$$(\lambda + A_p)^{-1}P_p f = \lambda^{\frac{n}{2}-1}(\log\lambda)^{\sigma(n)}\,H_0 f + \lambda^{\frac{n}{2}-1}H_1(\lambda)f + H_2(\lambda)f \tag{5.1}$$

in Ω_R *for any* $f \in L_{p,R}(\Omega)^n$ *and* $\lambda \in \Sigma_\epsilon$ *with* $|\lambda| \leqq \lambda_0$. *Moreover* H_0 *and* $H_j(\lambda)$ $(j = 1, 2)$ *satisfy the relation*:

$$\nabla \cdot (H_0 f) = 0, \quad \nabla \cdot (H_j(\lambda) f) = 0 \text{ in } \Omega, \quad j = 1, 2 \tag{5.2}$$

for any $f \in L_{p,R}(\Omega)^n$ *and* $\lambda \in \Sigma_\epsilon$ *with* $|\lambda| \leqq \lambda_0$. *Here* $B(\dot{U}_{\lambda_0}, \mathscr{L}_{p,R}(\Omega))$ *denotes the set of all* $\mathscr{L}_{p,R}(\Omega)$*-valued bounded analytic functions on* $\dot{U}_{\lambda_0} = U_{\lambda_0} \setminus (-\infty, 0]$.

Proof. For $f \in L_{p,R}(\Omega)^n$, we set $f_0(x) = f(x)$ $(x \in \Omega)$ and $f_0(x) = 0$ $(x \notin \Omega)$, and set $\gamma f = f|_{\Omega_{R+1}}$. Let $(R_0(\lambda) f_0, \Pi f_0)$ be given by (3.2). Let (u, π) be a solution to the problem:

$$\begin{aligned} &-\operatorname{Div} S(u, \pi) = \gamma f + M(\lambda) f, \quad \operatorname{div} u = 0 \quad \text{in } \Omega_{R+1} \\ &S(u, \pi)\nu|_\Gamma = 0, \quad S(u, \pi)\nu_0|_{S_{R+1}} = S(R_0(\lambda) f_0, \Pi f_0)\nu_0|_{S_{R+1}} \end{aligned} \tag{5.3}$$

where ν_0 is the unit outward normal to $S_{R+1} = \{x \in \boldsymbol{R}^n \mid |x| = R + 1\}$ and $M(\lambda) f$ is defined by the formula:

$$M(\lambda) f = -\sum_{k=1}^{M} (S(R_0(\lambda) f_0, \Pi f_0)\nu_0, p_k)_{S_{R+1}} p_k - \sum_{k=1}^{M} (\gamma f, p_k)_{\Omega_{R+1}} p_k \tag{5.4}$$

From the definition of $M(\lambda)$, we have

$$(\gamma f + M(\lambda) f, p_l)_{\Omega_{R+1}} + (S(R_0(\lambda) f_0, \Pi f_0)\nu_0, p_l)_{S_{R+1}} = 0 \tag{5.5}$$

In view of (5.5), by Theorem 4.1 we know that (5.3) admits a unique solution

$$(u, \pi) \in \big(W_p^2(\Omega_{R+1})^n \cap \dot{L}_p(\Omega_{R+1})^n\big) \times W_p^1(\Omega_{R+1})$$

We define the operator $(A'(\lambda), B(\lambda))$ by the formula: $u = A'(\lambda) f$ and $\pi = B(\lambda) f$. If (u, π) solve (5.3), then $(u + \sum_{k=1}^{M} a_k p_k, \pi)$ also solve (5.3). Therefore for the later use, we define the solution operator $A(\lambda)$ by

$$A(\lambda) f = A'(\lambda) f + \sum_{k=1}^{M} \big(R_0(\lambda) f_0 - A'(\lambda) f, p_k\big)_{\Omega_{R+1}} p_k \tag{5.6}$$

In particular, $(A(\lambda) f, B(\lambda) f)$ solves (5.3) and satisfies the condition:

$$(A(\lambda) f - R_0(\lambda) f_0, p_l)_{\Omega_{R+1}} = 0, \quad l = 1, \ldots, M \tag{5.7}$$

Now we discuss the expansion of $(A(\lambda), B(\lambda))$ at $\lambda = 0$. First we shall give expansion formulas of $S(R_0(\lambda) f_0, \Pi f_0)$ and $M(\lambda) f$. Let $G_j(\lambda)$ $(j = 1, 2)$ be the operators defined in Theorem 3.1. We see that

$$G_j(\lambda) = \sum_{m=0}^{\infty} G_{jm}\lambda^m \quad (|\lambda| \leqq 1/2), \quad G_{jm} = \frac{1}{2\pi i}\int_{|z|=r} G_j(z)\frac{dz}{z^{m+1}} \ (0 < r < 1/2)$$

$$\|G_{jm}\|_{\mathscr{L}_{p,R}(\boldsymbol{R}^n)} \leqq L_j r^{-m} \ (0 < r < 1/2), \quad \nabla \cdot G_{jm} f = 0 \ \text{ in } \boldsymbol{R}^n \ \text{ for } f \in L_{p,R}(\boldsymbol{R}^n)^n \tag{5.8}$$

By Theorem 3.1 and (5.7)

$$\begin{aligned} &S(R_0(\lambda)f_0, \Pi f_0) \\ &\quad = S(R_0(0)f_0, \Pi f_0) + \lambda^{\frac{n}{2}-1}(\log\lambda)^{\sigma(n)} \sum_{m=0}^{\infty} D(G_{1m}f_0)\lambda^m + \sum_{m=1}^{\infty} D(G_{2m}f_0)\lambda^m \end{aligned}$$

where we have used the fact that $G_2(0) = G_{20} = R_0(0)$ (cf. (3.16)). By the divergence theorem

$$(\gamma f, p_l)_{\Omega_{R+1}} + (S(R_0(0)f_0, \Pi f_0)\nu_0, p_l)_{S_{R+1}} = 0, \quad l = 1, \dots, M \tag{5.9}$$

and therefore by (5.4) formally we have

$$\begin{aligned} M(\lambda)f = &-\lambda^{\frac{n}{2}-1}(\log\lambda)^{\sigma(n)} \sum_{m=0}^{\infty} \left[\sum_{k=1}^{M} (D(G_{1m}f_0)\nu_0, p_k)_{S_{R+1}} p_k\right]\lambda^m \\ &- \sum_{m=1}^{\infty}\left[\sum_{k=1}^{M} (D(G_{2m}f_0)\nu_0, p_k)_{S_{R+1}} p_k\right]\lambda^m \end{aligned}$$

Since $(\operatorname{Div} D(G_{jm}f_0), p_l)_{B_{R+1}} = (D(G_{jm}f_0)\nu_0, p_l)_{S_{R+1}}$, if we set

$$M_{jm}f = -\sum_{k=1}^{M} (\operatorname{Div} D(G_{jm}f_0), p_k)_{B_{R+1}} p_k \tag{5.10}$$

then formally we have

$$M(\lambda)f = \lambda^{\frac{n}{2}-1}(\log\lambda)^{\sigma(n)} \sum_{m=0}^{\infty} M_{1m}f\,\lambda^m + \sum_{m=1}^{\infty} M_{2m}f\,\lambda^m$$

Since $\|M_{jm}f\|_{L_p(\Omega_{R+1})} \leqq Cr^{-m}\|f\|_{L_p(\Omega)}$ for $f \in L_{p,R}(\boldsymbol{R}^n)^n$ and $0 < r < 1/2$ as follows from (5.8), if we set

$$M_1(\lambda)f = \sum_{m=0}^{\infty} (M_{1m}f)\lambda^m, \quad M_2(\lambda)f = \sum_{m=1}^{\infty} (M_{2m}f)\lambda^m$$

then we have

$$M_j(\lambda) \in Anal\big(U_{1/2}, \mathscr{L}(L_{p,R}(\Omega)^n, L_p(\Omega_{R+1})^n)\big), \quad j = 1, 2$$
$$M(\lambda) = \lambda^{\frac{n}{2}-1}(\log \lambda)^{\sigma(n)} M_1(\lambda) + M_2(\lambda) \tag{5.11}$$

Now, we decompose (5.3) into the following problems:

$$- \operatorname{Div} S(u_{20}, \pi_{20}) = \gamma f, \quad \operatorname{div} u_{20} = 0 \quad \text{in} \quad \Omega_{R+1}$$
$$S(u_{20}, \pi_{20})\nu|_\Gamma = 0, \quad S(u_{20}, \pi_{20})\nu_0|_{S_{R+1}} = S(R_0(0)f_0, \Pi f_0)\nu_0|_{S_{R+1}} \tag{5.12}$$

$$- \operatorname{Div} S(u_{jm}, \pi_{jm}) = M_{jm} f, \quad \operatorname{div} u_{jm} = 0 \quad \text{in} \quad \Omega_{R+1}$$
$$S(u_{jm}, \pi_{jm})\nu|_\Gamma = 0, \quad S(u_{jm}, \pi_{jm})\nu_0|_{S_{R+1}} = D(G_{jm} f_0)\nu_0|_{S_{R+1}} \tag{5.13}$$

for $m = 0, 1, 2, \dots$ when $j = 1$ and for $m = 1, 2, \dots$ when $j = 2$. By (5.9), the right members of (5.12) satisfy the compatibility condition (4.3). Noting that

$$(M_{jm} f, p_\ell)_{\Omega_{R+1}} + (D(G_{jm} f_0)\nu_0, p_l)_{S_{R+1}} = 0, \quad l = 1, \dots, M$$

as follows from (5.10) and the divergence theorem, we see that the right members of (5.13) also satisfy the compatibility condition (4.3). By Theorem 4.1 we know the existence of the solution $(u_{jm}, \pi_{jm}) \in (W_p^2(\Omega_{R+1})^n \cap \dot{L}_p(\Omega_{R+1})^n) \times W_p^1(\Omega_{R+1})$. Therefore we define the solution operator (A'_{jm}, B_{jm}) of (5.12) and (5.13) by $u_{jm} = A'_{jm} f$ and $\pi_{jm} = B_{jm} f$ for $m = 0, 1, 2, \dots$ and $j = 1, 2$. Obviously

$$A'_{jm} \in \mathscr{L}\big(L_{p,R}(\Omega)^n, W_p^2(\Omega_{R+1})^n \cap \dot{L}_p(\Omega_{R+1})^n\big), \quad B_{jm} \in \mathscr{L}\big(L_{p,R}(\Omega)^n, W_p^1(\Omega_{R+1})\big)$$

By (5.8), (5.10), Theorems 3.4 and 4.1 we have

$$\begin{aligned}
&\big\|A'_{20} f\big\|_{W_p^2(\Omega_{R+1})} + \|B_{20} f\|_{W_p^1(\Omega_{R+1})} \\
&\quad \leqq C\Big(\|\gamma f\|_{L_p(\Omega_{R+1})} + \|S(R_0(0)f_0, \Pi f_0)\nu_0\|_{W_p^{1-1/p}(S_{R+1})}\Big) \leqq C\|f\|_{L_p(\Omega)} \\
&\big\|A'_{jm} f\big\|_{W_p^2(\Omega_{R+1})} + \|B_{jm} f\|_{W_p^1(\Omega_{R+1})} \\
&\quad \leqq C\Big(\|M_{jm} f\|_{L_p(\Omega_{R+1})} + \|D(G_{jm} f_0)\nu_0\|_{W_p^{1-1/p}(S_{R+1})}\Big) \leqq C L_j r^{-m} \|f\|_{L_p(\Omega)}
\end{aligned} \tag{5.14}$$

for $f \in L_{p,R}(\Omega)^n$ and $0 < r < 1/2$, where C is independent of m. Therefore if we set

$$A'_j(\lambda) f = \sum_{m=0}^{\infty} A'_{jm} f \lambda^m, \quad B_j(\lambda) f = \sum_{m=0}^{\infty} B_{jm} f \lambda^m$$

then we have

$$
\begin{aligned}
&A'_j(\lambda) \in Anal\big(U_{1/2}, \mathscr{L}(L_{p,R}(\Omega)^n, W_p^2(\Omega_{R+1})^n \cap \dot{L}_p(\Omega_{R+1})^n)\big), \quad j=1,2 \\
&\nabla \cdot A'_j(\lambda)f = 0 \ \text{ in } \Omega_{R+1} \ \text{ for } f \in L_{p,R}(\Omega)^n, \quad j=1,2 \\
&B_j(\lambda) \in Anal\big(U_{1/2}, \mathscr{L}(L_{p,R}(\Omega)^n, W_p^1(\Omega_{R+1}))\big), \quad j=1,2 \qquad (5.15)
\end{aligned}
$$

and we see that $A'(\lambda)$ and $B(\lambda)$ have the following expansion:

$$
A'(\lambda) = \lambda^{\frac{n}{2}-1}(\log\lambda)^{\sigma(n)}A'_1(\lambda) + A'_2(\lambda), \quad B(\lambda) = \lambda^{\frac{n}{2}-1}(\log\lambda)^{\sigma(n)}B_1(\lambda) + B_2(\lambda) \quad (5.16)
$$

In view of (5.6), (5.8), (5.15) and Theorem 3.1, setting

$$
A_j(\lambda)f = A'_j(\lambda)f + \sum_{k=1}^{M}\big(G_j(\lambda)f_0 - A'_j(\lambda)f, p_k\big)_{\Omega_{R+1}}p_k, \quad j=1,2
$$

we have

$$
\begin{aligned}
&A(\lambda) = \lambda^{\frac{n}{2}-1}(\log\lambda)^{\sigma(n)}A_1(\lambda) + A_2(\lambda) \\
&A_j(\lambda) \in Anal\big(U_{1/2}, \mathscr{L}(L_{p,R}(\Omega)^n, W_p^2(\Omega_{R+1})^n \cap \dot{L}_p(\Omega_{R+1})^n)\big) \quad j=1,2 \\
&\nabla \cdot A_j(\lambda)f = 0 \ \text{ in } \Omega_{R+1} \ \text{ for } f \in L_{p,R}(\Omega)^n, \quad j=1,2 \qquad (5.17)
\end{aligned}
$$

Now we shall construct the parametrix of the problem (2.1). Let $\phi \in C_0^\infty(\boldsymbol{R}^n)$ such that $\phi(x) = 1$ ($|x| \leqq R-3/2$) and $\phi(x) = 0$ ($|x| \geqq R-5/4$). We define

$$
\begin{aligned}
&\Phi(\lambda)f = (1-\phi)R_0(\lambda)f_0 + \phi A(\lambda)f + \boldsymbol{B}[(\nabla\phi)\cdot(R_0(\lambda)f_0 - A(\lambda)f)] \\
&\Psi(\lambda)f = (1-\phi)\Pi f_0 + \phi B(\lambda)f \qquad (5.18)
\end{aligned}
$$

Since $\operatorname{supp}\nabla\phi \subset D_{R-3/2,R-5/4}$, $\operatorname{div}R_0(\lambda)f_0 = 0$ in $\boldsymbol{R}^n$ and $\operatorname{div}A(\lambda)f = 0$ in Ω_{R+1}, by Lemma 4.6 we have $(\nabla\phi)\cdot(R_0(\lambda)f_0 - A(\lambda)f) \in \dot{W}_{p,a}^2(D_{R-2,R-1})$ and therefore

$$
\begin{aligned}
&\boldsymbol{B}[(\nabla\phi)\cdot(R_0(\lambda)f_0 - A(\lambda)f)] \in W_p^3(\boldsymbol{R}^n) \\
&\operatorname{supp}\boldsymbol{B}[(\nabla\phi)\cdot(R_0(\lambda)f_0 - A(\lambda)f)] \subset D_{R-2,R-1} \\
&\nabla\cdot\boldsymbol{B}[(\nabla\phi)\cdot(R_0(\lambda)f_0 - A(\lambda)f)] = (\nabla\phi)\cdot(R_0(\lambda)f_0 - A(\lambda)f) \quad \text{in} \quad \boldsymbol{R}^n \\
&\big\|\boldsymbol{B}[(\nabla\phi)\cdot(R_0(\lambda)f_0 - A(\lambda)f)]\big\|_{W_p^3(\boldsymbol{R}^n)} \leqq C_{p,R}\big\|(\nabla\phi)\cdot(R_0(\lambda)f_0 - A(\lambda)f)]\big\|_{W_p^2(\boldsymbol{R}^n)} \\
&\qquad (5.19)
\end{aligned}
$$

By Theorem 3.1, (5.17), (5.18) and (5.19), $\Phi(\lambda)$ has the following expansion:

$$
\begin{aligned}
&\Phi(\lambda) = \lambda^{\frac{n}{2}-1}(\log\lambda)^{\sigma(n)}\Phi_1(\lambda) + \Phi_2(\lambda), \quad \Phi_1(\lambda),\ \Phi_2(\lambda) \in Anal(U_{1/2}, \mathscr{L}_{p,R}(\Omega)) \\
&\nabla\cdot\Phi_j(\lambda)f = 0 \ \text{ in } \Omega_R \quad \text{for } f \in L_{p,R}(\Omega)^n, \quad j=1,2 \qquad (5.20)
\end{aligned}
$$

By (3.1), (5.3) and (5.19), $(\Phi(\lambda)f, \Psi(\lambda)f)$ satisfies

$$\begin{aligned}&\lambda\Phi(\lambda)f - \operatorname{Div} S(\Phi(\lambda)f, \Psi(\lambda)f) = f + Q(\lambda)f, \quad \operatorname{div}\Phi(\lambda)f = 0 \quad \text{in } \Omega\\ &S(\Phi(\lambda)f, \Psi(\lambda)f)\nu|_\Gamma = 0\end{aligned} \tag{5.21}$$

where

$$\begin{aligned}Q(\lambda)f =& 2(\nabla\phi)\nabla(R_0(\lambda)f_0 - A(\lambda)f) + (\Delta\phi)(R_0(\lambda)f_0 - A(\lambda)f) + \lambda\phi A(\lambda)f + \phi M(\lambda)f\\ &+ (\lambda - \Delta)\boldsymbol{B}[(\nabla\phi)\cdot(R_0(\lambda)f_0 - A(\lambda)f)] - (\nabla\phi)\Pi f_0 + (\nabla\phi)B(\lambda)f\end{aligned} \tag{5.22}$$

If we show the existence of $(I + Q(\lambda))^{-1} \in \mathscr{L}(L_{p,R}(\Omega)^n)$ for $\lambda \in \dot{U}_{\lambda_0}$ with some λ_0, then by Theorem 3.4, (5.18), (5.19) and (5.21) we see that

$$\big(\Phi(\lambda)(I + Q(\lambda))^{-1}f, \Psi(\lambda)(I + Q(\lambda))^{-1}f\big) \in W_p^2(\Omega)^n \times X_p(\Omega) \tag{5.23}$$

and it satisfies (2.1). Thus the uniqueness assertion in Theorem 2.5 implies that

$$(\lambda + A_p)^{-1}P_p f = \Phi(\lambda)(I + Q(\lambda))^{-1}f \tag{5.24}$$

for any $f \in L_{p,R}(\Omega)^n$ and $\lambda \in \dot{U}_{\lambda_0}$. By Theorem 3.1, (5.11), (5.17) and (5.19) we can write

$$Q(\lambda) - Q(0) = \lambda^{\frac{n}{2}-1}(\log\lambda)^{\sigma(n)}Q_1(\lambda) + \lambda Q_2(\lambda) \tag{5.25}$$

with some $Q_1(\lambda)$, $Q_2(\lambda) \in Anal(U_{1/2}, \mathscr{L}(L_{p,R}(\Omega)^n, W_p^1(\Omega_R)^n))$. In particular, we have

$$\|Q(\lambda)f - Q(0)f\|_{W_p^1(\Omega)} \leqq Cp_n(|\lambda|)\|f\|_{L_p(\Omega)} \tag{5.26}$$

for any $f \in L_{p,R}(\Omega)^n$ and $\lambda \in \dot{U}_{1/2}$, where $p_n(|\lambda|)$ is given by (3.14). Therefore if we show the existence of $(I + Q(0))^{-1} \in \mathscr{L}(L_{p,R}(\Omega)^n)$, then there exists a $\lambda_0 > 0$ such that for $\lambda \in \dot{U}_{\lambda_0}$

$$(I + Q(\lambda))^{-1} = (I + Q(0))^{-1}\sum_{j=0}^{\infty}\left[(Q(0) - Q(\lambda))(I + Q(0))^{-1}\right]^j$$

which combined with (5.20), (5.24), (5.25) and (5.26) implies (5.1) and (5.2). From these observations, to complete the proof of Theorem 5.1 it suffices to show the following lemma.

LEMMA 5.2. *$(I + Q(0))^{-1} \in \mathscr{L}(L_{p,R}(\Omega)^n)$.*

PROOF. Since $Q(0) \in \mathscr{L}(L_{p,R}(\Omega)^n)$ is a compact operator, in view of the Fredholm alternative theorem to show the lemma it suffices to show that $I + Q(0)$ is injective. Let

$f \in L_{p,R}(\Omega)^n$ satisfy $(I + Q(0))f = 0$ in Ω. If we set $u = \Phi(0)f$ and $\pi = \Psi(0)f$, by (5.21) we see that

$$-\mathrm{Div}\, S(u,\pi) = 0 \quad \mathrm{div}\, u = 0 \quad \text{in} \quad \Omega, \quad S(u,\pi)\nu|_\Gamma = 0 \tag{5.27}$$

By Theorems 3.4 and 4.1 and Lemma 4.6, we have

$$\begin{aligned}
&u = (1-\phi)R_0(0)f_0 + \phi A(0)f + \boldsymbol{B}[(\nabla\phi)\cdot[R_0(0)f_0 - A(0)f]] \in W^2_{p,loc}(\bar\Omega)\\
&\pi = (1-\phi)\Pi f_0 + \phi B(0)f \in W^1_{p,loc}(\bar\Omega)\\
&|u(x)| \leqq C_{p,R}\,|x|^{-(n-2)}\|f\|_{L_p(\Omega)}, \quad |\nabla u(x)| \leqq C_{p,R}\,|x|^{-(n-1)}\|f\|_{L_p(\Omega)} \quad \text{for} \quad |x| \geqq R+1\\
&|\pi(x)| \leqq C_{p,R}\,|x|^{-(n-1)}\|f\|_{L_p(\Omega)} \quad \text{for} \quad |x| \geqq R+1
\end{aligned} \tag{5.28}$$

By the boot-strap argument, we see that $u \in W^2_{2,loc}(\bar\Omega)^n$ and $\pi \in W^1_{2,loc}(\bar\Omega)$. Let $\rho(x) \in C^\infty_0(\boldsymbol{R}^n)$ such that $\rho(x) = 1$ ($|x| \leqq 1$) and $\rho(x) = 0$ ($|x| \geqq 2$), and set $\rho_L(x) = \rho(x/L)$. By the divergence theorem

$$\begin{aligned}
0 &= (-\mathrm{Div}\, S(u,\pi), \rho_L u)_\Omega = (D(u), \nabla(\rho_L u))_\Omega - (\pi, \mathrm{div}\,(\rho_L u))_\Omega\\
&= (1/2)(D(u), D(u)\rho_L)_\Omega + (D(u), (\nabla\rho_L)u)_\Omega - (\pi, (\nabla\rho_L)\cdot u)_\Omega
\end{aligned} \tag{5.29}$$

By (5.28), for $L > R+2$ we have

$$\begin{aligned}
&\big|(D(u), (\nabla\rho_L)u)_\Omega\big|, \ \big|(\pi, (\nabla\rho_L)\cdot u)_\Omega\big|\\
&\quad\leqq C_{p,R}L^{-1}\int_{L\leqq|x|\leqq 2L} |x|^{-(n-2)}|x|^{-(n-1)}\,dx\|f\|^2_{L_p(\Omega)}\\
&\quad\leqq C_{p,R}L^{-(n-2)}\|f\|^2_{L_p(\Omega)} \quad \text{as} \quad L \to \infty
\end{aligned}$$

because $n \geqq 3$. So we obtain $\|D(u)\|^2_{L_2(\Omega)} = 0$ by $L \to \infty$ in (5.29). Thus $D(u) = 0$, namely $u \in \mathscr{R}$. Since u is represented by the formula: $u = Ax + b$ with some anti-symmetric matrix A and $b \in \boldsymbol{R}^n$, by (5.28) $u = 0$. Since $\nabla\pi = 0$, by (5.28) $\pi = 0$. Thus we have

$$\Phi(0)f = 0 \quad \Psi(0)f = 0 \quad \text{in} \quad \Omega \tag{5.30}$$

By the definition of $\phi(x)$, we have

$$A(0)f = B(0)f = 0 \quad |x| \leqq R-2, \quad R_0(0)f = \Pi f_0 = 0 \quad |x| \geqq R-1 \tag{5.31}$$

If we set

$$w = \begin{cases} A(0)f & x \in \Omega_{R+1}\\ 0 & x \notin \Omega \end{cases} \qquad \theta = \begin{cases} B(0)f & x \in \Omega_{R+1}\\ 0 & x \notin \Omega \end{cases}$$

then from (5.3) and (5.31) it follows that $(w,\theta) \in W_p^2(B_{R+1})^n \times W_p^1(B_{R+1})$ and

$$-\operatorname{Div} S(w,\theta) = f_0, \quad \operatorname{div} w = 0 \quad \text{in } B_{R+1}$$
$$S(w,\theta)\nu_0|_{S_{R+1}} = S(R_0(0)f_0, \Pi f_0)\nu_0|_{S_{R+1}} \tag{5.32}$$

On the other hand, by (5.31) $(R_0(0)f_0, \Pi f_0)$ also satisfies (5.32). Therefore $(w - R_0(0)f_0, \theta - \Pi f_0)$ satisfies (4.5) with $D = B_{R+1}$. By Theorem 4.1 and (5.7) with $\lambda = 0$ we have

$$A(0)f - R_0(0)f_0 = 0, \quad B(0)f - \Pi f_0 = 0 \quad \text{in} \quad \Omega_{R+1} \tag{5.33}$$

By (5.28), (5.30), (5.33) and $\operatorname{supp}\phi \subset B_{R-1}$,

$$0 = R_0(0)f_0 + \phi(A(0)f - R_0(0)f_0) = R_0(0)f_0 \quad \text{in} \ \ \Omega_{R+1}$$
$$0 = \Pi f_0 + \phi(B(0)f - \Pi f_0) = \Pi f_0 \quad \text{in} \ \ \Omega_{R+1}$$

Thus we obtain

$$f_0 = -\operatorname{Div} S(R_0(0)f_0, \Pi f_0) = 0 \quad \text{in} \ \ B_{R+1}$$

namely $f = 0$, which completes the proof of the lemma. □

6. Proofs of main theorems.

Applying Theorems 2.5 and 5.1 to the representation formula of the analytic semigroup $\{T(t)\}_{t \geqq 0}$ in terms of $(\lambda + A_p)^{-1}P_p$, we can prove Theorem 1.1 in the same manner as in Iwashita [**12**] and Kubo and Shibata [**14**], and therefore we may omit the detailed proof of Theorem 1.1. And also, replacing Lemma 2.5 in [**14**] by Lemma 4.6 and combining the L_p-L_q estimates of the Stokes semigroup in $\boldsymbol{R}^n$ and the local energy decay in Theorem 1.1 by cut-off technique, we can prove Theorem 1.2 in the same manner as in Iwashita [**12**] and Kubo and Shibata [**14**]. Therefore, we may also omit the detailed proof of Theorem 1.2.

Now, we shall prove Theorem 1.3. Once we obtain the next lemma, we immediately prove Theorem 1.3.

LEMMA 6.1. *Let $n < p \leqq q \leqq \infty$ $(p \neq \infty)$. For every $f \in J_p(\Omega)$ and $t > 0$ we have*

$$\|\nabla T(t)f\|_{L_q(\Omega)} \leqq C_{p,q}\, t^{-\frac{n}{2}\left(\frac{1}{p}-\frac{1}{q}\right)-\frac{1}{2}} \|f\|_{L_p(\Omega)} \tag{6.1}$$

In fact, if $1 < p \leqq n < q \leqq \infty$, we choose r in such a way that $n < r \leqq q \leqq \infty$ and $r \neq \infty$. Then, for $f \in J_p(\Omega)$ by Lemma 6.1, (1.9) and the semigroup property: $\nabla T(t)f = \nabla T(t/2)[T(t/2)f]$, we have

$$\begin{aligned}\|\nabla T(t)f\|_{L_q(\Omega)} &\leqq C_{r,q}(t/2)^{-\frac{n}{2}\left(\frac{1}{r}-\frac{1}{q}\right)-\frac{1}{2}}\|T(t/2)f\|_{L_r(\Omega)} \\ &\leqq C_{r,q}C_{p,r}(t/2)^{-\frac{n}{2}\left(\frac{1}{r}-\frac{1}{q}\right)-\frac{1}{2}}(t/2)^{-\frac{n}{2}\left(\frac{1}{p}-\frac{1}{r}\right)}\|f\|_{L_p(\Omega)} \\ &= C_{p,q}t^{-\frac{n}{2}\left(\frac{1}{p}-\frac{1}{q}\right)-\frac{1}{2}}\|f\|_{L_p(\Omega)}\end{aligned}$$

which shows Theorem 1.3 in the case that $1 < p \leqq n < q \leqq \infty$. Therefore, we shall prove Lemma 6.1, below.

The lemma which follows is a key to prove Lemma 6.1.

Lemma 6.2. *Let $0 < \epsilon < \pi/2$ and $n < p \leqq q \leqq \infty$ $(p \neq \infty)$. Then, there exist positive constants λ_0 and $C_{p,q,\epsilon}$ such that*

$$\left\|\nabla(\lambda + A_p)^{-1}f\right\|_{L_q(\Omega)} \leqq C_{p,q,\epsilon}\,|\lambda|^{\frac{n}{2}\left(\frac{1}{p}-\frac{1}{q}\right)-\frac{1}{2}}\|f\|_{L_p(\Omega)} \tag{6.2}$$

for every $\lambda \in \Sigma_\epsilon$ with $|\lambda| \leqq \lambda_0$ and $f \in J_p(\Omega)$.

In order to prove Lemma 6.2, we prepare an auxiliary lemma for the solution operator $R_0(\lambda)$.

Lemma 6.3. *Let $0 < \epsilon < \pi/2$ and $n < p \leqq q \leqq \infty$ $(p \neq \infty)$.*
(1) *For any $f_0 \in L_p(\boldsymbol{R}^n)^n$ and $\lambda \in \Sigma_\epsilon$ with $|\lambda| \leqq 1$ there holds the estimate*:

$$\|\nabla R_0(\lambda)f_0\|_{L_q(\boldsymbol{R}^n)} \leqq C_{p,q,\epsilon}\,|\lambda|^{\frac{n}{2}\left(\frac{1}{p}-\frac{1}{q}\right)-\frac{1}{2}}\|f_0\|_{L_p(\boldsymbol{R}^n)} \tag{6.3}$$

(2) *For any $f_0 \in L_p(\boldsymbol{R}^n)^n \cap L_1(\boldsymbol{R}^n)^n$ and $\lambda \in \Sigma_\epsilon$ with $|\lambda| \leqq 1$ there holds the estimate*:

$$\|\nabla R_0(\lambda)f_0\|_{L_q(\boldsymbol{R}^n)} \leqq C_{p,q,\epsilon}\big(\|f_0\|_{L_p(\boldsymbol{R}^n)} + \|f_0\|_{L_1(\boldsymbol{R}^n)}\big) \tag{6.4}$$

Proof. In the course of the proof below, we always assume that $\lambda \in \Sigma_\epsilon$ and $|\lambda| \leqq 1$. First we shall show the assertion (1) when $q = \infty$. Let $\psi_0(\xi) \in C_0^\infty(\boldsymbol{R}^n)$ such that $\psi_0(\xi) = 1$ $(|\xi| \leqq 2)$ and $\psi_0(\xi) = 0$ $(|\xi| \geqq 3)$, and set $\psi_\infty(\xi) = 1 - \psi_0(\xi)$ and

$$R_0^N(\lambda)f_0 = \mathscr{F}^{-1}\big[\psi_N(\xi)P(\xi)\hat{f}_0(\xi)/(\lambda + |\xi|^2)\big](x), \quad N = 0, \infty.$$

To estimate $R_0^\infty(\lambda)f_0$, we observe that

$$\big|\partial_\xi^\alpha\big[\psi_\infty(\xi)(\lambda + |\xi|^2)^{-1}\big]\big| \leqq C_\alpha(1 + |\xi|^2)^{-1}|\xi|^{-|\alpha|}$$

for any $\alpha \in \boldsymbol{N}_0^n$ and $\xi \in \boldsymbol{R}^n$, because $|\lambda + |\xi|^2| \geqq (1/2)(|\xi|^2 + 1)$ when $|\xi| \geqq 2$ and $|\lambda| \leqq 1$. By the Fourier multiplier theorem we have

$$\big\|R_0^\infty(\lambda)f_0\big\|_{W_p^2(\boldsymbol{R}^n)} \leqq C_p\|f_0\|_{L_p(\boldsymbol{R}^n)} \tag{6.5}$$

Since $n < p \leqq q \leqq \infty$ and $p \neq \infty$, by the Sobolev imbedding theorem, we know that

$$W_p^1(D) \subset L_q(D), \quad \|u\|_{L_q(D)} \leqq C_{p,q}\|u\|_{W_p^1(D)} \tag{6.6}$$

for $D = \boldsymbol{R}^n$, Ω and Ω_{R+1}. By (6.6) and (6.5) we have

$$\big\|\nabla R_0^\infty(\lambda)f_0\big\|_{L_\infty(\boldsymbol{R}^n)} \leqq C_p\big\|\nabla R_0^\infty(\lambda)f_0\big\|_{W_p^1(\boldsymbol{R}^n)} \leqq C_p\|f_0\|_{L_p(\boldsymbol{R}^n)} \tag{6.7}$$

Next we consider $R_0^0(\lambda)f_0$. By (3.15) we have

$$\big|\partial_\xi^\alpha\big[\psi_0(\xi)(\lambda+|\xi|^2)^{-1}\big]\big| \leqq C_{\alpha,\epsilon}(|\lambda|+|\xi|^2)^{-1}|\xi|^{-|\alpha|}$$

for $|\xi| \leqq 3$ and $\alpha \in \boldsymbol{N}_0^n$. Therefore

$$\big|\partial_\xi^\alpha\big[\psi_0(\xi)\xi_j P(\xi)(\lambda+|\xi|^2)^{-1}\big]\big| \leqq C_{\alpha,\epsilon}|\xi|^{-1-|\alpha|} \quad \text{or} \quad \leqq C_{\alpha,\epsilon}|\lambda|^{-\frac{1}{2}}|\xi|^{-|\alpha|} \tag{6.8}$$

for $\alpha \in \boldsymbol{N}_0^n$. If we set

$$K_\lambda^j(x) = \mathscr{F}_\xi^{-1}\big[\psi_0(\xi)i\xi_j P(\xi)(\lambda+|\xi|^2)^{-1}\big](x)$$

then $\partial_j R_0^0(\lambda)f_0 = K_\lambda^j * f_0(x)$. To estimate $K_\lambda^j(x)$, we use the following theorem (cf. [**20**, Theorem 2.3]):

THEOREM 6.4. *Let B be a Banach space and $|\cdot|_B$ its corresponding norm. Let α be a number $> -n$ and set $\alpha = N + \sigma - n$ where $N \geqq 0$ is an integer and $0 < \sigma \leqq 1$. Let $f(\xi)$ be a function in $C^\infty(\boldsymbol{R}^n \setminus \{0\}; B)$ such that*

$$\partial_\xi^\gamma f(\xi) \in L_1(\boldsymbol{R}^n; B) \quad \text{for } |\gamma| \leqq N$$
$$\big|\partial_\xi^\gamma f(\xi)\big|_B \leqq C_\gamma|\xi|^{\alpha-|\gamma|} \quad \forall\xi \neq 0, \quad \forall\gamma$$

Let

$$g(x) = \int_{\boldsymbol{R}^n} e^{-ix\cdot\xi} f(\xi)\,d\xi$$

Then, we have

$$|g(x)|_B \leqq C_{n,\alpha}\Big(\max_{|\gamma|\leqq N+2} C_\gamma\Big)\,|x|^{-(n+\alpha)}, \quad \forall x \neq 0$$

where $C_{n,\alpha}$ is a constant depending only on n and α.

By Theorem 6.4 and (6.8) we have

$$\left|K_\lambda^j(x)\right| \leqq C_\epsilon |x|^{-(n-1)} \quad \text{for all } x \neq 0 \tag{6.9}$$

$$\left|K_\lambda^j(x)\right| \leqq C_\epsilon |\lambda|^{-\frac{1}{2}} |x|^{-n} \quad \text{for all } x \neq 0 \tag{6.10}$$

By (6.9) and (6.10) we have

$$\begin{aligned}\int_{\boldsymbol{R}^n} \left|K_\lambda^j(x)\right|^{p'} dx &\leqq C_{p,\epsilon}\Big(\int_{|x|\leqq|\lambda|^{-\frac{1}{2}}} |x|^{-p'(n-1)} dx + |\lambda|^{-\frac{p'}{2}} \int_{|x|\geqq|\lambda|^{-\frac{1}{2}}} |x|^{-p'n} dx\Big) \\ &\leqq C_{p,\epsilon}\big(|\lambda|^{-\frac{1}{2}\{n-p'(n-1)\}} + |\lambda|^{-\frac{p'}{2}} |\lambda|^{-\frac{1}{2}(-p'n+n)}\big) \leqq C_{p,\epsilon}\, |\lambda|^{\left(\frac{n}{2p}-\frac{1}{2}\right)p'}\end{aligned}$$

Therefore by the Young inequality we have

$$\left\|\partial_j R_0^0(\lambda) f_0\right\|_{L_\infty(\boldsymbol{R}^n)} \leqq \left\|K_\lambda^j\right\|_{L_{p'}(\boldsymbol{R}^n)} \|f_0\|_{L_p(\boldsymbol{R}^n)} \leqq C_{p,\epsilon}\, |\lambda|^{\frac{n}{2p}-\frac{1}{2}} \|f_0\|_{L_p(\boldsymbol{R}^n)} \tag{6.11}$$

Since $n/(2p) - 1/2 < 0$ and $|\lambda| \leqq 1$, combining (6.5) with (6.11), we obtain

$$\|\nabla R_0(\lambda) f_0\|_{L_\infty(\boldsymbol{R}^n)} \leqq C_{p,\epsilon}\, |\lambda|^{\frac{n}{2p}-\frac{1}{2}} \|f_0\|_{L_p(\boldsymbol{R}^n)} \tag{6.12}$$

which shows (6.3) when $q = \infty$ and $n < p < \infty$. When $q = p < \infty$, by (3.11) we obtain

$$\|\nabla R_0(\lambda) f_0\|_{L_p(\boldsymbol{R}^n)} \leqq C_{p,\epsilon}\, |\lambda|^{-\frac{1}{2}} \|f_0\|_{L_p(\boldsymbol{R}^n)} \tag{6.13}$$

for every $f_0 \in L_p(\boldsymbol{R}^n)^n$ and $\lambda \in \Sigma_\epsilon$, which shows (6.3) when $q = p < \infty$. When $n < p < q < \infty$, using the interpolation inequality:

$$\|\nabla R_0(\lambda) f_0\|_{L_q(\boldsymbol{R}^n)} \leqq C_{p,q} \|\nabla R_0(\lambda) f_0\|_{L_p(\boldsymbol{R}^n)}^{\frac{p}{q}} \|\nabla R_0(\lambda) f_0\|_{L_\infty(\boldsymbol{R}^n)}^{1-\frac{p}{q}}$$

and (6.12) and (6.13), we obtain (6.3), which completes the proof of (6.3).

In order to prove the assertion (2), it suffices to prove that

$$\|\nabla R_0(\lambda) f_0\|_{L_p(\boldsymbol{R}^n)} \leqq C_{p,\epsilon}\big(\|f_0\|_{L_p(\boldsymbol{R}^n)} + \|f_0\|_{L_1(\boldsymbol{R}^n)}\big) \tag{6.14}$$

for any $f_0 \in L_p(\boldsymbol{R}^n)^n \cap L_1(\boldsymbol{R}^n)^n$ and $\lambda \in \Sigma_\epsilon$ with $|\lambda| \leqq 1$. In fact, since

$$\|\nabla R_0(\lambda) f_0\|_{L_q(\boldsymbol{R}^n)} \leqq C_{p,q}\big(\|\nabla^2 R_0(\lambda) f\|_{L_p(\boldsymbol{R}^n)} + \|\nabla R_0(\lambda) f\|_{L_p(\boldsymbol{R}^n)}\big)$$

as follows from (6.6), by (6.14) and (3.11) we have (6.4).

By (6.9) and the fact: $(n-1)p > (n-1)n > n$, we have

$$\int_{|x|\geqq 1} \left|K_\lambda^j(x)\right|^p dx \leqq C_\epsilon \int_{|x|\geqq 1} |x|^{-(n-1)p} dx \leqq C_\epsilon$$

Since

$$\left|K_\lambda^j(x)\right| \leqq C_\epsilon \int_{|\xi|\leqq 3} |\xi|^{-1} d\xi = C_\epsilon$$

as follows from (6.8) with $\alpha = 0$, we have

$$\int_{|x|\leqq 1} \left|K_\lambda^j(x)\right|^p dx \leqq C_{p,\epsilon}$$

Therefore $\|K_\lambda^j\|_{L_p(\boldsymbol{R}^n)} = C_{p,\epsilon} < \infty$. By the Young inequality we obtain

$$\left\|\partial_j R_0^0(\lambda) f_0\right\|_{L_p(\boldsymbol{R}^n)} \leqq \left\|K_\lambda^j\right\|_{L_p(\boldsymbol{R}^n)} \|f_0\|_{L_1(\boldsymbol{R}^n)} \leqq C_{p,\epsilon} \|f_0\|_{L_1(\boldsymbol{R}^n)} \tag{6.15}$$

Combining (6.5) with (6.15), we obtain (6.14), which completes the proof of the lemma. □

Proof of Lemma 6.2. Let $R_0(\lambda)$ and Π be the operators defined in (3.2). Since $(R_0(\lambda)f_0, \Pi f_0 + c)$ solves (3.1) for any constant c, we may assume that

$$\int_{\Omega_{R+1}} \Pi f_0 dx = 0$$

and therefore by Poincaré's inequality and (3.12) we have

$$\|\Pi f_0\|_{W_p^1(\Omega_{R+1})} \leqq C \|\nabla \Pi f_0\|_{L_p(\boldsymbol{R}^n)} \leqq C \|f\|_{L_p(\Omega)} \tag{6.16}$$

Let (u, π) be a solution of (2.1) for $\lambda \in \Sigma_\epsilon$ with $|\lambda| \leqq 1$ and set

$$u = R_0(\lambda) f_0|_\Omega + v, \quad \pi = \Pi f_0|_\Omega + \theta \tag{6.17}$$

Then, (v, θ) enjoys the equation:

$$\begin{aligned} &\lambda v - \operatorname{Div} S(v, \theta) = 0, \quad \operatorname{div} v = 0 \ \text{ in } \Omega \\ &S(v, \theta)\nu|_\Gamma = -S(R_0(\lambda) f_0, \Pi f_0)\nu|_\Gamma \end{aligned} \tag{6.18}$$

To represent (v, θ), we shall introduce (w, τ) which is a solution to the equation:

$$\begin{aligned} &-\operatorname{Div} S(w, \tau) = g(\lambda), \quad \operatorname{div} w = 0 \ \text{ in } \Omega_{R+1} \\ &S(w, \tau)\nu|_\Gamma = -S(R_0(\lambda) f_0, \Pi f_0)\nu|_\Gamma, \quad S(w, \tau)\nu_0|_{S_{R+1}} = 0 \end{aligned} \tag{6.19}$$

where we have set

$$g(\lambda) = -\sum_{k=1}^{M} (\lambda R_0(\lambda) f_0, p_k)_{\mathscr{O}}\, p_k$$

with $\mathscr{O} = \boldsymbol{R}^n \setminus \Omega$. Since

$$(g(\lambda), p_l)_{\Omega_{R+1}} - (S(R_0(\lambda) f_0, \Pi f_0)\nu, p_l)_\Gamma = 0, \quad l = 1, \dots, M$$

by Theorem 4.1 we know the unique existence of (w, τ). Let $\phi \in C_0^\infty(\boldsymbol{R}^n)$ such that $\phi(x) = 1$ ($|x| \leqq R-2$) and $\phi(x) = 0$ ($|x| \geqq R-1$). By Lemma 4.6, we can define $\boldsymbol{B}[(\nabla\phi)\cdot w]$. Thus we set

$$v = \phi w - \boldsymbol{B}[(\nabla\phi)\cdot w] + U, \quad \theta = \phi\tau + \Psi \tag{6.20}$$

where (U, Ψ) is a solution to

$$\lambda U - \operatorname{Div} S(U, \Psi) = G(\lambda), \quad \operatorname{div} U = 0 \ \text{ in } \Omega, \quad S(U, \Psi)\nu|_\Gamma = 0 \tag{6.21}$$

with

$$G(\lambda) = -\phi\lambda w + (\lambda - \Delta)\boldsymbol{B}[(\nabla\phi)\cdot w] + 2(\nabla\phi)(\nabla w) + (\Delta\phi)w - (\nabla\phi)\tau - \phi g(\lambda)$$

Since $n/(2p) - 1/2 < 0$ and $|\lambda| \leqq 1$, by (3.11) we have

$$\begin{aligned}\|g(\lambda)\|_{L_p(\Omega_{R+1})} &\leqq C_{p,R}\|\lambda R_0(\lambda) f_0\|_{L_p(B_{R+1})} \\ &\leqq C_{p,\epsilon,R}\|f\|_{L_p(\Omega)} \leqq C_{p,\epsilon,R}|\lambda|^{\frac{n}{2p}-\frac{1}{2}}\|f\|_{L_p(\Omega)}\end{aligned} \tag{6.22}$$

By (3.11), (3.12), (6.3) with $q = \infty$ and (6.16), we have

$$\begin{aligned}&\|S(R_0(\lambda) f_0, \Pi f_0)\|_{W_p^1(\Omega_{R+1})} \leqq C_p\big(\|\nabla R_0(\lambda) f_0\|_{W_p^1(\Omega_{R+1})} + \|\nabla \Pi f_0\|_{L_p(\Omega_{R+1})}\big) \\ &\quad\leqq C\big(\|\nabla^2 R_0(\lambda) f_0\|_{L_p(\boldsymbol{R}^n)} + \|\nabla R_0(\lambda) f_0\|_{L_\infty(\boldsymbol{R}^n)} + \|\nabla \Pi f_0\|_{L_p(\Omega_{R+1})}\big) \\ &\quad\leqq C_{p,\epsilon,R}\,|\lambda|^{\frac{n}{2p}-\frac{1}{2}}\|f\|_{L_p(\Omega)}\end{aligned} \tag{6.23}$$

By (4.4), (6.22) and (6.23), we have

$$\begin{aligned}\|w\|_{W_p^2(\Omega_{R+1})} + \|\tau\|_{W_p^1(\Omega_{R+1})} &\leqq C_{p,R}\big(\|g(\lambda)\|_{L_p(\Omega_{R+1})} + \|S(R_0(\lambda) f_0, \Pi f_0)\|_{W_p^1(\Omega_{R+1})}\big) \\ &\leqq C_{p,\epsilon,R}\,|\lambda|^{\frac{n}{2p}-\frac{1}{2}}\,\|f\|_{L_p(\Omega)}\end{aligned} \tag{6.24}$$

By (6.22), (6.24) and Lemma 4.6, we have

$$\|G(\lambda)\|_{L_p(\Omega)} \leqq C_{p,\epsilon}\,|\lambda|^{\frac{n}{2p}-\frac{1}{2}}\|f\|_{L_p(\Omega)} \tag{6.25}$$

From the proof of Theorem 5.1 we know that there exists a constant $\lambda_0 > 0$ such that for $\lambda \in \dot{U}_{\lambda_0}$ we can write

$$\begin{aligned} U &= (1-\phi)R_0(\lambda)\big[(I+Q(\lambda))^{-1}G(\lambda)\big]_0 + \phi A(\lambda)(I+Q(\lambda))^{-1}G(\lambda) \\ &\quad + \boldsymbol{B}\big[(\nabla\phi)\cdot(R_0(\lambda)\big[(I+Q(\lambda))^{-1}G(\lambda)\big]_0 - A(\lambda)(I+Q(\lambda))^{-1}G(\lambda))\big] \\ \Psi &= (1-\phi)\Pi\big[(I+Q(\lambda))^{-1}G(\lambda)\big]_0 + \phi B(\lambda)(I+Q(\lambda))^{-1}G(\lambda) \end{aligned} \tag{6.26}$$

where $(A(\lambda), B(\lambda))$ is the solution operator of (5.3) which satisfies (5.7). By (5.15), (5.17), Lemma 5.2, (6.25) and (6.6), we have

$$\begin{aligned} \big\|A(\lambda)(I+Q(\lambda))^{-1}G(\lambda)\big\|_{W^1_q(\Omega_{R+1})} &\leqq C_{p,q}\big\|A(\lambda)(I+Q(\lambda))^{-1}G(\lambda)\big\|_{W^2_p(\Omega_{R+1})} \\ &\leqq C_{p,q}\big\|(I+Q(\lambda))^{-1}G(\lambda)\big\|_{L_p(\Omega)} \\ &\leqq C_{p,q}\|G(\lambda)\|_{L_p(\Omega)} \leqq C_{p,q,\epsilon}\,|\lambda|^{\frac{n}{2p}-\frac{1}{2}}\|f\|_{L_p(\Omega)} \end{aligned} \tag{6.27}$$

for every $\lambda \in \Sigma_\epsilon$ with $|\lambda| \leqq \lambda_0$. By Theorem 3.1, Lemma 5.2, (6.4), (6.25), the fact that $\operatorname{supp}(I+Q(\lambda))^{-1}G(\lambda) \subset B_R$ and (6.6), we have

$$\begin{aligned} &\big\|\nabla\big[R_0(\lambda)[(I+Q(\lambda))^{-1}G(\lambda)]_0\big]\big\|_{L_q(\boldsymbol{R}^n)} + \big\|R_0(\lambda)[(1+Q(\lambda))^{-1}G(\lambda)]_0\big\|_{W^1_q(B_{R+1})} \\ &\leqq C_{p,q,\epsilon}\big(\|(I+Q(\lambda))^{-1}G(\lambda)\|_{L_p(\Omega)} + \|(I+Q(\lambda))^{-1}G(\lambda)\|_{L_1(\Omega)}\big) \\ &\leqq C_{p,q,\epsilon}|\lambda|^{\frac{n}{2p}-\frac{1}{2}}\|f\|_{L_p(\Omega)} \end{aligned} \tag{6.28}$$

for every $\lambda \in \Sigma_\epsilon$ with $|\lambda| \leqq \lambda_0$. Since

$$\nabla(\lambda + A_p)^{-1}f = \nabla\big(R_0(\lambda)f_0|_\Omega + \phi w - \boldsymbol{B}[(\nabla\phi)\cdot w] + U\big)$$

as follows from (6.17) and (6.20), it suffices to estimate the $L_q(\Omega)$–norm of the right hand side. By (6.3), we have

$$\|\nabla R_0(\lambda)f_0\|_{L_q(\Omega)} \leqq C_{p,q,\epsilon}|\lambda|^{\frac{n}{2}\left(\frac{1}{p}-\frac{1}{q}\right)-\frac{1}{2}}\,\|f\|_{L_p(\Omega)}$$

By (6.24) and Lemma 4.6, we have

$$\|\nabla(\phi w - \boldsymbol{B}[(\nabla\phi)\cdot w])\|_{L_q(\Omega)} \leqq C_{p,q}\|w\|_{W^2_p(\Omega_{R+1})} \leqq C_{p,q,\epsilon}|\lambda|^{\frac{n}{2p}-\frac{1}{2}}\|f\|_{L_p(\Omega)}$$

By (6.27), (6.28) and Lemma 4.6, we have

$$\|\nabla U\|_{L_q(\Omega)} \leqq C_{p,q,\epsilon}|\lambda|^{\frac{n}{2p}-\frac{1}{2}}\|f\|_{L_p(\Omega)}$$

Combining these estimates and noting that $|\lambda|^{\frac{n}{2p}-\frac{1}{2}} \leqq |\lambda|^{\frac{n}{2}(\frac{1}{p}-\frac{1}{q})-\frac{1}{2}}$ for $|\lambda| \leqq 1$, we have Lemma 6.2. □

Proof of Lemma 6.1. Set $\gamma = \{se^{\pm i(\pi-\epsilon)} \mid s > 0\}$ for $0 < \epsilon < \pi/2$ and

$$\nabla T(t)f = \frac{1}{2\pi i}\left(\int_{\gamma,|\lambda|\leqq\lambda_0} + \int_{\gamma,|\lambda|\geqq\lambda_0}\right)e^{\lambda t}\nabla(\lambda + A_p)^{-1}f\,d\lambda = I(t) + II(t)$$

for $f \in J_p(\Omega)$. By (6.2) we have

$$\|I(t)\|_{L_q(\Omega)} \leqq C_{p,q,\epsilon}\, t^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{q})-\frac{1}{2}}\|f\|_{L_p(\Omega)} \tag{6.29}$$

Since $n < p < \infty$ and $p \leqq q \leqq \infty$, by (6.6) and (2.14) we obtain

$$\|II(t)\|_{L_q(\Omega)} \leqq C_{p,q}\|II(t)\|_{W_p^1(\Omega)} \leqq C_{p,q,\epsilon}\, t^{-1}e^{-(\cos\epsilon)\lambda_0 t}\|f\|_{L_p(\Omega)}$$

which combined with (6.29) implies (6.1) for $t \geqq 1$. When $0 < t < 1$, by using (2.14) we obtain

$$\|\nabla^2 T(t)f\|_{L_p(\Omega)} \leqq C_{p,\epsilon}\, t^{-1}\|f\|_{L_p(\Omega)}, \quad \|\nabla T(t)f\|_{L_p(\Omega)} \leqq C_{p,\epsilon}\, t^{-\frac{1}{2}}\|f\|_{L_p(\Omega)} \tag{6.30}$$

By the interpolation inequality we have

$$\|\nabla T(t)f\|_{L_q(\Omega)} \leqq C_{p,q}\|\nabla^2 T(t)f\|^a_{L_p(\Omega)}\|\nabla T(t)f\|^{1-a}_{L_p(\Omega)}$$

with $a = n(1/p - 1/q)$, because $n < p \leqq q \leqq \infty$ and $p \neq \infty$, and therefore by (6.30) we obtain

$$\|II(t)\|_{L_q(\Omega)} \leqq C_{p,q,\epsilon}\, t^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{q})-\frac{1}{2}}\|f\|_{L_p(\Omega)} \tag{6.31}$$

for $0 < t < 1$. This completes the proof of Lemma 6.1. □

A. The denseness of $C^\infty_{0,\sigma}(\boldsymbol{R}^n)$ in $J_p(\Omega)$.

In the appendix, we shall show the following proposition.

Proposition A.1. *Let $1 < p < \infty$. Then, $C^\infty_{0,\sigma}(\boldsymbol{R}^n)$ is dense in $J_p(\Omega)$.*

Proof. By Lemma 2.6, $\mathscr{D}(A_p)$ is dense in $J_p(\Omega)$, and therefore for any $u \in J_p(\Omega)$ and $\epsilon > 0$ there exists a $v \in W^2_p(\Omega)^n$ such that $\operatorname{div} v = 0$ in Ω and $\|u - v\|_{L_p(\Omega)} < \epsilon/3$. Let φ be a function in $C^\infty_0(\boldsymbol{R}^n)$ such that $\varphi(x) = 1$ for $|x| \leqq 1$ and $\varphi(x) = 0$ for $|x| \geqq 2$, and set $\varphi_R(x) = \varphi(x/R)$. In view of Lemma 4.6, if we set $w_R = \varphi_R v - \boldsymbol{B}[(\nabla\varphi_R)\cdot v]$, then we have

$$w_R \in W^2_p(\Omega)^n, \quad \operatorname{div} w_R = 0 \text{ in } \Omega \quad \text{and} \quad w_R = 0 \text{ for } |x| \geqq 2R \tag{A.1}$$

Since

$$\|\boldsymbol{B}[(\nabla\varphi_R)\cdot v]\|_{W_p^3(\Omega)} \leqq C\|(\nabla\varphi_R)\cdot v\|_{W_p^2(\Omega)} \leqq C\|\nabla\varphi\|_{W_\infty^2(\Omega)}R^{-1}\|v\|_{W_p^2(\Omega)}$$

for $R>1$ as follows from Lemma 4.6, we have $\|w_R - v\|_{W_p^2(\Omega)} \to 0$ as $R\to\infty$, which shows that there exists an $R>1$ such that $\|w_R - v\|_{L_p(\Omega)} < \epsilon/3$. By the Lions extension method we know that there exists a $y \in W_p^2(\boldsymbol{R}^n)$ such that $y = w_R$ on Ω and $\|y\|_{W_p^2(\Omega)} \leqq C\|w_R\|_{W_p^2(\Omega)}$. Since $y = w_R$ on Ω and $\operatorname{div} w_R = 0$ in Ω, we have $\operatorname{div} y = 0$ on Ω, which implies that $\operatorname{div} y \in \dot W_p^1(\boldsymbol{R}^n\setminus\overline{\Omega})$. To use Lemma 4.4 we observe that

$$\int_{\Omega^c}\operatorname{div} y\,dx = -\int_\Gamma \nu\cdot y\,d\sigma = -\int_\Gamma \nu\cdot w_R\,d\sigma = -\int_\Omega \operatorname{div} w_R\,dx = 0$$

where $d\sigma$ denotes the surface element of Γ and we have used (A.1), which implies that $\operatorname{div} y \in \dot W_{p,a}^1(\boldsymbol{R}^n\setminus\overline{\Omega})$. By Lemma 4.4, we see that $\boldsymbol{B}[\operatorname{div} y] \in W_p^2(\boldsymbol{R}^n)^n$, $\operatorname{div}\boldsymbol{B}[\operatorname{div} y] = \operatorname{div} y$ in $\boldsymbol{R}^n$ and $\boldsymbol{B}[\operatorname{div} y]$ vanishes on Ω. Therefore, if we set $z = y - \boldsymbol{B}[\operatorname{div} y]$, then $z \in W_p^2(\boldsymbol{R}^n)^n$, $\operatorname{div} z = 0$ in $\boldsymbol{R}^n$ and $z = w_R$ on Ω. Let $\psi(x)$ be a function in $C_0^\infty(\boldsymbol{R}^n)$ such that $\int_{\boldsymbol{R}^n}\psi\,dx = 1$ and set $\psi_\tau(x) = \tau^{-n}\psi(x/\tau)$. Then, $z_\tau = \psi_\tau * z$ has the properties that

$$z_\tau \in C_{0,\sigma}^\infty(\boldsymbol{R}^n), \quad \lim_{\tau\to 0}\|z_\tau - z\|_{W_p^2(\boldsymbol{R}^n)} = 0$$

where $*$ denotes the convolution operator. Since $z = w_R$ on Ω, we have

$$\lim_{\tau\to 0}\|z_\tau - w_R\|_{L_p(\Omega)} = \lim_{\tau\to 0}\|z_\tau - z\|_{L_p(\Omega)} \leqq \lim_{\tau\to 0}\|z_\tau - z\|_{L_p(\boldsymbol{R}^n)} = 0$$

Therefore, there exists a $\tau>0$ such that $\|z_\tau - w_R\|_{L_p(\Omega)} < \epsilon/3$. Combining these results implies that $\|u - z_\tau\|_{L_p(\Omega)} < \epsilon$, which completes the proof of the proposition. □

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Yoshihiro Shibata
Department of Mathematical Sciences
School of Science and Engineering
Waseda University
Ohkubo 3-4-1, Shinjuku-ku
Tokyo 169-8555, Japan
E-mail: yshibata@waseda.jp

Senjo Shimizu
Faculty of Engineering
Shizuoka University
Hamamatsu
Shizuoka 432-8561, Japan
E-mail: tssshim@ipc.shizuoka.ac.jp