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# Hypoellipticity of a second order operator with a principal symbol changing sign across a smooth hypersurface

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**Abstract.** We give sufficient conditions for hypoellipticity of a second order operator with real-valued infinitely differentiable coefficients whose principal part is the product of a real-valued infinitely differentiable function $\phi(x)$ and the sum of squares of first order operators $X_1, \dots, X_r$. These conditions are related to the way in which $\phi(x)$ changes its sign, and the rank of the Lie algebra generated by $\phi X_1, \dots, \phi X_r$ and $X_0$ where $X_0$ is the first order term of the operator. Our result is an extension of that of [**4**], and it includes some cases not treated in [**1**], [**5**] and [**8**].

## 1. Introduction.

Let $X_0, X_1, \dots, X_r$ be first order linear partial differential operators with real-valued $C^\infty$ coefficients defined in an open subset $\Omega$ of $R^n$, $n \geq 2$, and let $\phi(x), c(x)$ be real-valued $C^\infty$ functions defined in $\Omega$. We consider the second order operator of the form:

$$L = \phi(x) \sum_{i=1}^{r} X_i^2 + X_0 + c(x).$$

We say that $L$ is hypoelliptic in $\Omega$ if for any open subset $\omega$ of $\Omega$ and any $u \in \mathscr{D}'(\omega)$, $Lu \in C^\infty(\omega)$ implies $u \in C^\infty(\omega)$. We also define rank Lie $(X_0, \phi X_1, \dots, \phi X_r)(x)$ as the maximal number of linearly independent operators in the Lie algebra generated by $X_0, \phi X_1, \dots, \phi X_r$ considered at a point $x$. When $\phi(x) = 1$ in $\Omega$, it is well-known by [**6**] that $L$ is hypoelliptic in $\Omega$ if rank Lie $(X_0, X_1, \dots, X_r)(x) = n$ for all $x \in \Omega$.

In this paper we shall prove the following theorem by using fundamental theorems on commutators, adjoints, and boundedness of pseudo-differential operators, and establishing estimates of the subelliptic kind in localized Sobolev spaces.

Theorem 1.1. *Assume that*

(H1) $X_i \phi(x) = 0$ $(1 \leq i \leq r)$ *for all* $x \in \Omega$ *such that* $\phi(x) = 0$,
(H2) rank Lie $(X_0, \phi X_1, \dots, \phi X_r)(x) = n$ *for all* $x \in \Omega$,
(H3) *there exist nonempty open sets* $\Omega_+$, $\Omega_-$ *and a relatively closed set* $\Gamma \subset \Omega$ *such that*
  (H3-1) $\Omega = \Omega_+ \cup \Omega_- \cup \Gamma$,
  (H3-2) $\Omega_+ \cap \Omega_- = \varnothing$,

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(H3-3) $\Gamma = \partial\Omega_+ = \partial\Omega_-$ *and* $\Gamma$ *is a* $C^\infty$ *hypersurface, that is, for any* $p \in \Gamma$ *there exist an open neighborhood* $\omega_p$ *of* $p$ *and a function* $g(x) \in C^\infty(\omega_p)$ *such that* $\Gamma \cap \omega_p = \{x \mid g(x) = 0\}$ *where* $\operatorname{grad} g(x) \neq 0$ $(x \in \omega_p)$,

(H3-4) $\phi(x) \leq 0$ *in* $\Omega_-$, $\phi(x) \geq 0$ *in* $\Omega_+$,

(H3-5) $(X_0(p), n(p)) > 0$ *for all* $p \in \Gamma$, *where* $X_0(p)$ *is the vector defined at* $p$ *by the vector field* $X_0$, *and* $n(p)$ *is the unit normal of* $\Gamma$ *at* $p$ *towards* $\Omega_+$.

*Then* $L$ *is hypoelliptic in* $\Omega$.

Let us consider the case where the coefficients of $X_i$ $(0 \leq i \leq r)$, $\phi(x)$ and $c(x)$ are analytic in $\Omega$, and $X_0, \phi X_1, \ldots, \phi X_r$ do not vanish simultaneously at any point of $\Omega$. It was proved in Theorem II.I(iii) and (ii) of [**10**] that (H1) and (H2) are necessary for $L$ to be hypoelliptic in $\Omega$. There are preceding results about the necessity of (H2) for $L$ to be hypoelliptic in $\Omega$. When $\phi(x) = 1$ in $\Omega$, it was proved in Theorem 2.2 of [**3**]. When $\phi(x) \geq 0$ in $\Omega$, it was proved in Theorem 2.8.2 of [**9**]. It was also proved in Theorem II.I(iv) of [**10**] that the condition:

$$X_0\phi(x) \geq 0 \quad \text{for all} \;\; x \in \phi^{-1}(0)$$

is necessary for hypoellipticity of $L$. (H3) is a special case of this condition.

Since hypoellipticity is a local property, to prove Theorem 1.1 it is sufficient to show that $L$ is hypoelliptic in an open neighborhood of any $p \in \Omega$. Let us write the operator $L$ in the divergent form:

$$L = \sum_{k=1}^{n} \frac{\partial}{\partial x_k} Y_k + Y_0 + c(x),$$

where $Y_0, Y_1, \ldots, Y_n$ are first order operators with coefficients belonging to $C^\infty(\Omega)$. We know from Theorem 1.1 of [**2**] that (H1) and (H2) imply that

$$\operatorname{rank} \operatorname{Lie}(Y_0, Y_1, \ldots, Y_n)(x) = n \text{ for all } x \in \Omega.$$

Then according to Theorem 2.6.4 of [**9**], $L$ is hypoelliptic in an open neighborhood of $p$ if $p \in \Omega_+ \cup \Omega_-$. Hence, it only remains to show the hypoellipticity of $L$ in an open neighborhood of any $p \in \Gamma$.

Fix any $p \in \Gamma$. Then, by (H3) there exists a diffeomorphism $\Psi$ from an open neighborhood $V_p$ of $p$ to an open set $\{(y', y_n) \mid |y'| < T,\, |y_n| < T\}$, $y' = (y_1, \ldots, y_{n-1})$, $T > 0$, such that

$$\begin{aligned} \Psi(p) &= (0,0), \\ \Psi(V_p \cap \Omega_-) &= \bigl\{(y', y_n) \mid |y'| < T,\ -T < y_n < 0\bigr\}, \\ \Psi(V_p \cap \Omega_+) &= \bigl\{(y', y_n) \mid |y'| < T,\ 0 < y_n < T\bigr\}, \\ \Psi_*(X_0) &= \frac{\partial}{\partial y_n}. \end{aligned}$$

Hence we may assume from the beginning that

$$\Omega = \{(x', x_n) \mid |x'| < T,\ |x_n| < T\}, \tag{1.1}$$

$$L = x_n a(x) \sum_{i=1}^{r} X_i^2 + \frac{\partial}{\partial x_n} + c(x), \quad x \in \Omega, \tag{1.2}$$

and the following three conditions are satisfied.

$$X_i(x_n a(x)) = 0 \ (1 \le i \le r) \text{ for all } x \in \Omega \text{ such that } x_n a(x) = 0, \tag{1.3}$$

$$\operatorname{rank} \operatorname{Lie}\left(\frac{\partial}{\partial x_n}, x_n a X_1, \dots, x_n a X_r\right)(x) = n \text{ for all } x \in \Omega, \tag{1.4}$$

$$a(x) \ge 0 \text{ in } \Omega. \tag{1.5}$$

In the later sections we shall show that under the hypotheses (1.3)–(1.5) the operator $L$ defined by (1.2) is hypoelliptic in $\Omega$ defined by (1.1).

Remark. In case of $a(x) > 0$ ($x \in \Omega$), the hypoellipticity of $L$ was proved in [**4**] which extended the previous result Theorem III.1 of [**10**]. In case of $a(x) = x_n^2 A(x)$ where $A(x)$ is a non-negative infinitely differentiable function defined in $\Omega$, the hypoellipticity of $L$ was proved in Theorem A of [**8**].

Example. Let $\alpha(x_1, x_2, x_3)$ be a non-negative infinitely differentiable function which is defined in $R^3$, and not representable in the form of a finite sum of squares of infinitely differentiable functions in any neighborhood of the origin. The existence of such a function was proved in Lemma 2.6.5 of [**9**]. Let $n \ge 4$ and put

$$L = x_n\big(\alpha(x_1, x_2, x_3) + x_n^2\big)\left(\frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_{n-1}^2} + \left(x_n \frac{\partial}{\partial x_n}\right)^2\right) + \frac{\partial}{\partial x_n}.$$

Then $L$ does not belong to the classes of operators treated in [**1**], [**4**], [**5**] and [**8**]. However the conditions (1.3)–(1.5) are satisfied for $L$, and so $L$ is hypoelliptic in $\Omega$.

## 2. Notations and elementary lemmas.

We introduce notations and state elementary lemmas which will be used in the later sections.

Notation 2.1. Let $m$ be a real number. We say that a $C^\infty$ function $p(x, \xi)$ defined on $R^n \times R^n$ is a symbol of class $S^m$ if for any multi-indices $\alpha$, $\beta$ there exists a constant $C_{\alpha,\beta}$ such that $\left|\frac{\partial^\alpha}{\partial \xi^\alpha} \frac{\partial^\beta}{\partial x^\beta} p(x, \xi)\right| \le C_{\alpha,\beta}(1 + |\xi|^2)^{(m-|\alpha|)/2}$, $(x, \xi) \in R^n \times R^n$. We denote by $\operatorname{Op}(S^m)$ the set of all pseudo-differential operators with symbols belonging to $S^m$.

Notation 2.2. Let $s$ be a real number and $\varepsilon$ be a positive number. We denote by $E_s$ the pseudo-differential operator with the symbol $(1+|\xi|^2)^{s/2}$, and by $E_s^{(\varepsilon)}$ the pseudo-differential operator with the symbol $(1+|\xi'|^2+|\varepsilon\xi_n|^2)^{s/2}$ where $\xi'=(\xi_1,\dots,\xi_{n-1})$. We denote by $(\ ,\ )$ the inner product in $L^2(R^n)$, and by $(\ ,\ )_s$ and $\|\ \|_s$ the inner product and the norm respectively in the Sobolev space $H_s$. We also use the notations:

$$(u,v)_{s,\varepsilon}=\big(E_s^{(\varepsilon)}u,E_s^{(\varepsilon)}v\big),\quad \|u\|_{s,\varepsilon}=(u,u)_{s,\varepsilon}^{1/2},\quad u,v\in H_s.$$

Here we note that the norms $\|u\|_s$ and $\|u\|_{s,\varepsilon}$ are equivalent for any fixed $\varepsilon>0$.

Notation 2.3. For $p\in R^n$ and $\rho>0$ we define $B(p,\rho)=\{x\mid |x-p|<\rho\}$.

Lemma 2.1. *Let $s$ be a real number and $A_{2s}\in \mathrm{Op}(S^{2s})$. There exists a constant $C>0$ such that for any $\delta>0$*

$$\big|(v,A_{2s}w)\big|\le \delta\|v\|_s^2+\frac{C}{\delta}\|w\|_s^2,\quad v,\,w\in C_0^\infty(R^n).$$

Lemma 2.2. *Let $m_1,m_2$ and $s$ be real numbers such that $m_1+m_2=2s$. If $A_i\in\mathrm{Op}(S^{m_i})\,(i=1,2)$, then there exists a constant $C>0$ such that*

$$\big|(A_1u,A_2u)\big|\le C\|u\|_s^2,\quad u\in C_0^\infty(R^n).$$

Lemma 2.3. *Let $m_j$ $(j=1,\dots,p)$ and $s$ be real numbers such that $m_1+\cdots+m_p=2s+1$. Let $A_j\in\mathrm{Op}(S^{m_j})$ $(j=1,\dots,p)$, and suppose that $A_j-A_j^*\in\mathrm{Op}(S^{m_j-1})$ or $A_j+A_j^*\in\mathrm{Op}(S^{m_j-1})$ $(j=1,\dots,p)$. Then for any permutation $(j_1,\dots,j_p)$ of $(1,\dots,p)$ there exists a constant $C>0$ such that*

$$\big|(A_1\cdots A_lu,A_{l+1}\cdots A_pu)\big|\le\big|(A_{j_1}\cdots A_{j_m}u,A_{j_{m+1}}\cdots A_{j_p}u)\big|+C\|u\|_s^2,\quad u\in C_0^\infty(R^n).$$

The following lemma is a generalization of Proposition 3.1 of [**8**].

Lemma 2.4. *Let $Y_1\in\mathrm{Op}(S^1)$ and $A_{2s}\in\mathrm{Op}(S^{2s})$. Suppose that $Y_1^*+Y_1\in\mathrm{Op}(S^0)$ and $A_{2s}^*-A_{2s}\in\mathrm{Op}(S^{2s-1})$. Then there exists a constant $C>0$ such that*

$$\big|\mathrm{Re}\,(Y_1u,A_{2s}u)\big|\le C\|u\|_s^2,\quad u\in C_0^\infty(R^n).$$

## 3. Localization.

Let $A_0,\dots,A_r$ be first order operators with real-valued $C^\infty$ coefficients defined in $\Omega$. For every multi-index $J=(j_1,\dots,j_l)$ $(0\le j_k\le r,\ 1\le k\le l)$ we introduce the notations:

$$A_J=\big[A_{j_1},[A_{j_2},\cdots,[A_{j_{l-1}},A_{j_l}]\cdots]\big],\tag{3.1}$$

$$l_0(J)=\text{the number of } k \text{ such that } j_k=0,\tag{3.2}$$

$$\tau(J)=2^{-(l+l_0(J))+1}.\tag{3.3}$$

We set

$$Y_0 = \frac{\partial}{\partial x_n}, \qquad Y_1 = x_n a X_1, \ \ldots, \ Y_r = x_n a X_r, \tag{3.4}$$

$$Z_0 = x_n \frac{\partial}{\partial x_n}, \quad Z_1 = x_n a X_1, \ \ldots, \ Z_r = x_n a X_r. \tag{3.5}$$

Then we have the following

LEMMA 3.1. *For any multi-index $J = (j_1, \ldots, j_l)$ $(0 \leq j_k \leq r, \ 1 \leq k \leq l)$ we have the expression*

$$x_n^{l_0(J)} Y_J = Z_J + \sum_{I \subset J} g_{J,I} Z_I \ \text{in } \Omega, \tag{3.6}$$

*where $g_{J,I} \in C^\infty(\Omega)$ $(I \subset J)$ and $I \subset J$ means that $I = (i_1, \ldots, i_m)$ is a multi-index such that $m < l$ and $i_k = j_{l_k}$ $(1 \leq k \leq m, \ l_1 < \cdots < l_m)$.*

PROOF. We shall show by induction on $l$ that there exist $h_{J,I} \in C^\infty(\Omega)$ $(I \subset J)$ such that

$$Z_J = x_n^{l_0(J)} Y_J + \sum_{I \subset J} h_{J,I} x_n^{l_0(I)} Y_I. \tag{3.7}$$

Then (3.6) follows from (3.7) by induction on $l$. It is obvious that (3.7) holds when $l = 1$. Assume that (3.7) holds when $l = k$. Let $J = (j_1, j_2, \ldots, j_{k+1})$ and set $J' = (j_2, \ldots, j_{k+1})$. Then we have

$$Z_J = \big[Z_{j_1}, Z_{J'}\big] = \Big[Z_{j_1}, x_n^{l_0(J')} Y_{J'} + \sum_{I \subset J'} h_{J',I} x_n^{l_0(I)} Y_I\Big]. \tag{3.8}$$

First we consider the case of $j_1 = 0$. Then $Z_{j_1} = x_n \partial/\partial x_n$, $l_0(J') + 1 = l_0(J)$ and we have by (3.8)

$$\begin{aligned} Z_J &= x_n^{l_0(J)} Y_J - x_n^{l_0(J')} Y_{J'}(x_n) \frac{\partial}{\partial x_n} + l_0(J') x_n^{l_0(J')} Y_{J'} \\ &\quad + \sum_{I \subset J'} \Big\{ h_{J',I} x_n^{l_0((0,I))} Y_{(0,I)} - h_{J',I} x_n^{l_0(I)} Y_I(x_n) \frac{\partial}{\partial x_n} \Big\} \\ &\quad + \sum_{I \subset J'} \Big( x_n \frac{\partial h_{J',I}}{\partial x_n} + l_0(I) h_{J',I} \Big) x_n^{l_0(I)} Y_I. \end{aligned} \tag{3.9}$$

If $l_0(J') \geq 1$, then we can write with $f_{J'} = x_n^{l_0(J')-1} Y_{J'}(x_n)$

$$x_n^{l_0(J')} Y_{J'}(x_n) \frac{\partial}{\partial x_n} = f_{J'} x_n \frac{\partial}{\partial x_n} = f_{J'} x_n^{l_0((0))} Y_{(0)}. \tag{3.10}$$

If $l_0(J')=0$, then $j_s\geq 1$ $(2\leq s\leq k+1)$ and we can write with some $f_{J'}\in C^\infty(\Omega)$

$$\begin{aligned}x_n^{l_0(J')}Y_{J'}(x_n)\frac{\partial}{\partial x_n}&=\big[x_naX_{j_2},[x_naX_{j_3},\dots,[x_naX_{j_k},x_naX_{j_{k+1}}]\cdots]\big](x_n)\frac{\partial}{\partial x_n}\\&=f_{J'}x_n^{l_0((0))}Y_{(0)}.\end{aligned}\tag{3.11}$$

In the same way we can write with some $f_I\in C^\infty(\Omega)$

$$x_n^{l_0(I)}Y_I(x_n)\frac{\partial}{\partial x_n}=f_Ix_n^{l_0((0))}Y_{(0)},\quad I\subset J'.\tag{3.12}$$

Since $(0)\subset J$ and $(0,I)\subset(0,J')=J$, we obtain (3.7) by (3.9)–(3.12).

Next we consider the case of $j_1\geq 1$. Then $Z_{j_1}=Y_{j_1}=x_naX_{j_1}$, $l_0(J')=l_0(J)$, and $l_0(I)=l_0((j_1,I))$. Hence we have by (3.8)

$$\begin{aligned}Z_J=x_n^{l_0(J)}Y_J+l_0(J')aX_{j_1}(x_n)x_n^{l_0(J')}Y_{J'}+\sum_{I\subset J'}h_{J',I}x_n^{l_0((j_1,I))}Y_{(j_1,I)}\\+\sum_{I\subset J'}\big\{x_naX_{j_1}(h_{J',I})x_n^{l_0(I)}Y_I+l_0(I)ah_{J',I}X_{j_1}(x_n)x_n^{l_0(I)}Y_I\big\}.\end{aligned}$$

Since $(j_1,I)\subset(j_1,J')=J$, we obtain (3.7) by the above equation. □

Let $p$ be an arbitrary point in $\Omega$. It follows from hypothesis (1.4) that there exist an open neighborhood $\Omega_p\subset\Omega$ of $p$, multi-indices $J_1,\dots,J_n$ and functions $f_{j,k}\in C^\infty(\Omega_p)$ $(j,k=1,\dots,n)$ such that

$$\frac{\partial}{\partial x_j}=\sum_{k=1}^n f_{j,k}Y_{J_k}\text{ in }\Omega_p,\ 1\leq j\leq n.\tag{3.13}$$

We define

$$l_0=\max_{1\leq k\leq n}l_0(J_k).\tag{3.14}$$

We note that $l_0$ is a non-negative integer depending on $p$. Let $p\in\{x\mid |x|<T,\,x_n=0\}$ and suppose that $l_0=0$. Then $l_0(J_k)=0$ $(1\leq k\leq n)$, and so $Y_{J_k}$ $(1\leq k\leq n)$ are obtained by taking commutators successively starting from $x_naX_1,\dots,x_naX_r$. Hence $Y_{J_k}=0$ $(1\leq k\leq n)$ at $p$, which contradicts (3.13). Thus we see that

$$l_0\geq 1\text{ if }p\in\{x\mid |x|<T,\ x_n=0\}.\tag{3.15}$$

It follows from (3.13) and Lemma 3.1 that

$$x_n^{l_0}\frac{\partial}{\partial x_j}=\sum_{k=1}^n\sum_{I\subseteq J_k}g_{j,k,I}Z_I\text{ in }\Omega_p,\ 1\leq j\leq n,\tag{3.16}$$

where $g_{j,k,I} \in C^\infty(\Omega_p)$ and $I \subseteq J_k$ means that $I \subset J_k$ or $I = J_k$.

We choose an open set $\omega_p$ and a function $\chi(x)$ so that

$$p \in \omega_p \subset \bar{\omega}_p \subset \Omega_p, \tag{3.17}$$

and

$$\chi(x) \in C_0^\infty(\Omega_p), \quad \chi(x) = 1 \text{ in } \omega_p. \tag{3.18}$$

Now we put

$$\tilde{a} = \chi a, \quad \tilde{X}_i = \chi X_i \ (1 \le i \le r), \quad \tilde{c} = \chi c, \tag{3.19}$$

and

$$\tilde{L} = x_n \tilde{a} \sum_{i=1}^r \tilde{X}_i^2 + \chi \frac{\partial}{\partial x_n} + \tilde{c}. \tag{3.20}$$

## 4. Energy estimates.

In this section we shall fix $p \in \{(x', x_n) \mid |x'| < T,\ x_n = 0\}$ and establish energy estimates for the operator $\tilde{L}$ defined by (3.20).

PROPOSITION 4.1. *Let $s$ be a real number and let $A_s$ be a pseudo-differential operator belonging to* $\mathrm{Op}(S^s)$. *Assume that*

$$A_s^* - A_s \in \mathrm{Op}(S^{s-1}) \quad \textit{or} \quad A_s^* + A_s \in \mathrm{Op}(S^{s-1}).$$

*Then there exists a constant $C_s > 0$ depending on $s$ such that*

$$\sum_{i=1}^r \big(x_n^2 \tilde{a} A_s \tilde{X}_i u, A_s \tilde{X}_i u\big) \le -\mathrm{Re}\,\big(x_n \tilde{L} u, A_s^* A_s u\big) + C_s \|u\|_s^2, \quad u \in C_0^\infty(R^n). \tag{4.1}$$

PROOF. We shall denote by $C_1$ and $C_2$ positive constants depending on $s$ and independent of $u \in C_0^\infty(R^n)$. We have by (3.20)

$$\begin{aligned}
&-\mathrm{Re}\,\big(x_n \tilde{L} u, A_s^* A_s u\big) \\
&\quad = -\sum_{i=1}^r \mathrm{Re}\,\big(x_n^2 \tilde{a} \tilde{X}_i^2 u, A_s^* A_s u\big) - \mathrm{Re}\left(x_n \chi \frac{\partial}{\partial x_n} u, A_s^* A_s u\right) - \mathrm{Re}\,\big(x_n \tilde{c} u, A_s^* A_s u\big).
\end{aligned}$$

We can apply Lemma 2.4 to the second term in the right-hand side of the above equation, and Lemma 2.2 to the third one. Hence

$$-\mathrm{Re}\,\big(x_n \tilde{L} u, A_s^* A_s u\big) \ge -\sum_{i=1}^r \mathrm{Re}\,\big(x_n^2 \tilde{a} \tilde{X}_i^2 u, A_s^* A_s u\big) - C_1 \|u\|_s^2. \tag{4.2}$$

We estimate the first term in the right-hand side of (4.2). We write $\tilde{X}_i^* = -\tilde{X}_i + f_i$, $f_i \in C_0^\infty(R^n)$. Then

$$\begin{aligned}&- \mathrm{Re}\,\big(x_n^2\tilde{a}\tilde{X}_i^2u, A_s^*A_su\big)\\ &\quad = \big(x_n^2\tilde{a}A_s\tilde{X}_iu, A_s\tilde{X}_iu\big) + \mathrm{Re}\,\big(\tilde{X}_iu, [A_s, x_n^2\tilde{a}]^*A_s\tilde{X}_iu\big) + \mathrm{Re}\,\big(x_n^2\tilde{a}\tilde{X}_iu, [\tilde{X}_i, A_s^*A_s]u\big)\\ &\qquad - \mathrm{Re}\,\big(x_n^2\tilde{a}\tilde{X}_iu, f_iA_s^*A_su\big) - \mathrm{Re}\,\big([x_n^2\tilde{a}, \tilde{X}_i]\tilde{X}_iu, A_s^*A_su\big).\end{aligned}$$

By hypothesis we can apply Lemma 2.4 to the last four terms in the right-hand side of the above equation. Hence we have

$$-\mathrm{Re}\,\big(x_n^2\tilde{a}\tilde{X}_i^2u, A_s^*A_su\big) \ge \big(x_n^2\tilde{a}A_s\tilde{X}_iu, A_s\tilde{X}_iu\big) - C_2\|u\|_s^2, \quad 1 \le i \le r. \tag{4.3}$$

Thus we obtain (4.1) by (4.2) and (4.3). □

Corollary 4.1. *Let $A_s$ be a pseudo-differential operator as in Proposition* 4.1. *Let $p_i(x) \in C_0^\infty(R^n)$, $i = 1, \dots, r$, and suppose that*

$$|p_i(x)|^2 \le Cx_n^2\tilde{a}(x), \quad x \in R^n,\ i = 1, \dots, r,$$

*where $C > 0$ is a constant. Then there exists a constant $C_s > 0$ such that*

$$\sum_{i=1}^r \big\|A_sp_i\tilde{X}_iu\big\|_0^2 \le -2C\mathrm{Re}\,\big(x_n\tilde{L}u, A_s^*A_su\big) + C_s\|u\|_s^2, \quad u \in C_0^\infty(R^n), \tag{4.4}$$

$$\sum_{i=1}^r \big\|p_i\tilde{X}_iu\big\|_s^2 \le C_s\big(\|\tilde{L}u\|_s^2 + \|u\|_s^2\big), \quad u \in C_0^\infty(R^n). \tag{4.5}$$

Lemma 4.1. *There exists a constant $M > 0$ such that*

$$|\tilde{a}(x)|^2 + \sum_{k=1}^n \big|\tilde{a}_{x_k}(x)\big|^2 \le M\tilde{a}(x), \quad x \in R^n, \tag{4.6}$$

$$\sum_{i=1}^r \big|\tilde{X}_i(x_n\tilde{a}(x))\big|^2 \le Mx_n^2\tilde{a}(x), \quad x \in R^n. \tag{4.7}$$

Proof. According to Lemma 1.7.1 of [**9**] there exists a constant $C_1 > 0$ such that

$$|\tilde{a}_{x_k}(x)|^2 \le C_1\tilde{a}(x), \quad x \in R^n,\ 1 \le k \le n. \tag{4.8}$$

On the other hand, it follows from (3.19) and hypothesis (1.3) that

$$\tilde{X}_i(x_n)\tilde{a}(x)|_{x_n=0} = \chi(x)^2X_i(x_na(x))|_{x_n=0} = 0.$$

Then there exists $\alpha_i(x) \in C_0^\infty(R^n)$ such that $\tilde{X}_i(x_n)^2\tilde{a}(x) = x_n^2\alpha_i(x)$, because

$\tilde{X}_i(x_n)^2\tilde{a}(x) \ge 0$ in $R^n$. Hence, taking (4.8) into account we have

$$|\tilde{X}_i(x_n\tilde{a}(x))|^2 \le 2\tilde{X}_i(x_n)^2\tilde{a}(x)^2 + 2x_n^2\tilde{X}_i(\tilde{a}(x))^2 \le C_2x_n^2\tilde{a}(x), \quad x \in R^n,\ 1 \le i \le r. \tag{4.9}$$

The lemma holds by (4.8) and (4.9). □

Let $A_s = E_s$ in Corollary 4.1. Then Corollary 4.1 and Lemma 4.1 yield immediately the following corollary.

COROLLARY 4.2. *Let $M$ be the positive constant determined in Lemma 4.1. For any real number $s$ there exists a constant $C_s > 0$ depending on $s$ such that for all $k = 1, \ldots, n$ and $u \in C_0^\infty(R^n)$*

$$\sum_{i=1}^r \left\|x_n\tilde{a}\tilde{X}_iu\right\|_s^2 \le -2M\mathrm{Re}\left(x_n\tilde{L}u, u\right)_s + C_s\|u\|_s^2,$$

$$\sum_{i=1}^r \left\|x_n\tilde{a}_{x_k}\tilde{X}_iu\right\|_s^2 \le -2M\mathrm{Re}\left(x_n\tilde{L}u, u\right)_s + C_s\|u\|_s^2,$$

$$\sum_{i=1}^r \left\|\tilde{X}_i(x_n\tilde{a})\tilde{X}_iu\right\|_s^2 \le -2M\mathrm{Re}\left(x_n\tilde{L}u, u\right)_s + C_s\|u\|_s^2,$$

$$\sum_{i=1}^r \left\|x_n\tilde{a}\tilde{X}_iu\right\|_s^2 + \sum_{i=1}^r \left\|x_n\tilde{a}_{x_k}\tilde{X}_iu\right\|_s^2 + \sum_{i=1}^r \left\|\tilde{X}_i(x_n\tilde{a})\tilde{X}_iu\right\|_s^2 \le C_s\left(\|\tilde{L}u\|_s^2 + \|u\|_s^2\right).$$

PROPOSITION 4.2. *For any real number $s$, there exist $\varepsilon_s > 0$, $\rho_s > 0$ and $C_s > 0$ depending on $s$ such that $B(p, \rho_s) \subset \omega_p$ and*

$$\sum_{i=1}^r \left(\tilde{a}E^{(\varepsilon_s)}_{s-\frac{1}{2}}\tilde{X}_iu, E^{(\varepsilon_s)}_{s-\frac{1}{2}}\tilde{X}_iu\right) + \left\|\frac{\partial}{\partial x_n}u\right\|^2_{s-\frac{1}{2},\varepsilon_s} \le -2\mathrm{Re}\left(x_n\tilde{L}u, u\right)_s + 2\mathrm{Re}\left(\tilde{L}u, \frac{\partial}{\partial x_n}u\right)_{s-\frac{1}{2},\varepsilon_s} + C_s\|u\|_s^2, \quad u \in C_0^\infty(B(p, \rho_s)). \tag{4.10}$$

PROOF. In the proof we shall use positive parameters $\varepsilon$, $\rho$ and $\delta$. We shall denote by $C_{s,\varepsilon,\delta}$ a positive constant depending on $s$, $\varepsilon$ and $\delta$, by $C^{(1)}_{s,\varepsilon}$ and $C^{(2)}_{s,\varepsilon}$ positive constants depending on $s$ and $\varepsilon$, and by $C^{(1)}_{s,\varepsilon,\rho}$, $C^{(2)}_{s,\varepsilon,\rho}$ and $C^{(3)}_{s,\varepsilon,\rho}$ positive constants depending on $s$, $\varepsilon$ and $\rho$.

Since by (3.18) $\chi\frac{\partial}{\partial x_n}u = \frac{\partial}{\partial x_n}u$ for $u \in C_0^\infty(\omega_p)$, it follows that

$$\left(\tilde{L}u, \frac{\partial}{\partial x_n}u\right)_{s-\frac{1}{2},\varepsilon} = \sum_{i=1}^r \left(x_n\tilde{a}\tilde{X}_i^2u, E^{(\varepsilon)}_{2s-1}\frac{\partial}{\partial x_n}u\right) + \left\|\frac{\partial}{\partial x_n}u\right\|^2_{s-\frac{1}{2},\varepsilon} + \left(\tilde{c}u, E^{(\varepsilon)}_{2s-1}\frac{\partial}{\partial x_n}u\right), \quad u \in C_0^\infty(\omega_p). \tag{4.11}$$

We write for $1 \le i \le r$

$$\tilde{X}_i^* = -\tilde{X}_i + f_i, \quad f_i \in C_0^\infty(R^n). \tag{4.12}$$

Then we have for $1 \le i \le r$

$$\big(x_n\tilde{a}\tilde{X}_i^2\big)^* = x_n\tilde{a}\tilde{X}_i^2 + 2\big\{\tilde{X}_i(x_n\tilde{a}) - f_i x_n\tilde{a}\big\}\tilde{X}_i + g_i, \quad g_i \in C_0^\infty(R^n). \tag{4.13}$$

Hence

$$\begin{aligned}
\operatorname{Re}\left(x_n\tilde{a}\tilde{X}_i^2 u, E_{2s-1}^{(\varepsilon)}\frac{\partial}{\partial x_n}u\right) &= \frac{1}{2}\left(x_n\tilde{a}\tilde{X}_i^2 u, E_{2s-1}^{(\varepsilon)}\frac{\partial}{\partial x_n}u\right) + \frac{1}{2}\left(E_{2s-1}^{(\varepsilon)}\frac{\partial}{\partial x_n}u, x_n\tilde{a}\tilde{X}_i^2 u\right)\\
&= \frac{1}{2}\left(\left[x_n\tilde{a}\tilde{X}_i^2, E_{2s-1}^{(\varepsilon)}\frac{\partial}{\partial x_n}\right]u, u\right) - \left(E_{2s-1}^{(\varepsilon)}\frac{\partial}{\partial x_n}u, \big\{\tilde{X}_i(x_n\tilde{a}) - f_i x_n\tilde{a}\big\}\tilde{X}_i u\right)\\
&\quad - \frac{1}{2}\left(E_{2s-1}^{(\varepsilon)}\frac{\partial}{\partial x_n}u, g_i u\right), \quad u \in C_0^\infty(R^n).
\end{aligned} \tag{4.14}$$

Let $\tilde{X}_i(x,\xi)$, $1 \le i \le r$, be the symbol of $\tilde{X}_i$. Then the principal symbol of $\big[x_n\tilde{a}\tilde{X}_i^2, E_{2s-1}^{(\varepsilon)}\frac{\partial}{\partial x_n}\big]$ is expressed in the form:

$$\begin{aligned}
&A_{2s}^{(i,\varepsilon)}(x,\xi)x_n\tilde{a}(x)\tilde{X}_i(x,\xi) + \sum_{k=1}^n B_{2s}^{(i,k,\varepsilon)}(x,\xi)x_n\tilde{a}_{x_k}(x)\tilde{X}_i(x,\xi)\\
&\quad - (2s-1)\varepsilon^2 E_{2s-3}^{(\varepsilon)}(\xi)\xi_n^2\tilde{a}(x)\tilde{X}_i(x,\xi)^2 - \tilde{a}(x)E_{2s-1}^{(\varepsilon)}(\xi)\tilde{X}_i(x,\xi)^2,
\end{aligned}$$

where $A_{2s}^{(i,\varepsilon)}(x,\xi)$ and $B_{2s}^{(i,k,\varepsilon)}(x,\xi)$ belong to $S^{2s}$, $1 \le i \le r$, $1 \le k \le n$, $\varepsilon > 0$. Hence there exists $C_{2s}^{(i,\varepsilon)} \in \mathrm{Op}(S^{2s})$, $1 \le i \le r$, $\varepsilon > 0$, such that

$$\begin{aligned}
\left[x_n\tilde{a}\tilde{X}_i^2, E_{2s-1}^{(\varepsilon)}\frac{\partial}{\partial x_n}\right] &= A_{2s}^{(i,\varepsilon)}x_n\tilde{a}\tilde{X}_i + \sum_{k=1}^n B_{2s}^{(i,k,\varepsilon)}x_n\tilde{a}_{x_k}\tilde{X}_i\\
&\quad + (2s-1)\varepsilon^2\frac{\partial}{\partial x_n}E_{s-\frac{1}{2}}^{(\varepsilon)}\tilde{a}\tilde{X}_i^2 E_{s-\frac{5}{2}}^{(\varepsilon)}\frac{\partial}{\partial x_n} + \tilde{X}_i^* E_{s-\frac{1}{2}}^{(\varepsilon)}\tilde{a}E_{s-\frac{1}{2}}^{(\varepsilon)}\tilde{X}_i + C_{2s}^{(i,\varepsilon)}.
\end{aligned} \tag{4.15}$$

Let $\delta > 0$. It follows from (4.11), (4.14), (4.15) and Lemma 2.1 that

$$\begin{aligned}
\operatorname{Re}\left(\tilde{L}u, \frac{\partial}{\partial x_n}u\right)_{s-\frac{1}{2},\varepsilon} &\ge \frac{1}{2}\sum_{i=1}^r\left(\tilde{a}E_{s-\frac{1}{2}}^{(\varepsilon)}\tilde{X}_i u, E_{s-\frac{1}{2}}^{(\varepsilon)}\tilde{X}_i u\right) + \left\|\frac{\partial}{\partial x_n}u\right\|_{s-\frac{1}{2},\varepsilon}^2\\
&\quad - \left|s - \frac{1}{2}\right|\varepsilon^2\sum_{i=1}^r\left|\left(\tilde{a}\tilde{X}_i^2 E_{s-\frac{5}{2}}^{(\varepsilon)}\frac{\partial}{\partial x_n}u, E_{s-\frac{1}{2}}^{(\varepsilon)}\frac{\partial}{\partial x_n}u\right)\right|\\
&\quad - \delta\sum_{i=1}^r\left(\big\|\tilde{X}_i(x_n\tilde{a})\tilde{X}_i u\big\|_s^2 + \big\|x_n\tilde{a}\tilde{X}_i u\big\|_s^2 + \sum_{k=1}^n\big\|x_n\tilde{a}_{x_k}\tilde{X}_i u\big\|_s^2\right)\\
&\quad - C_{s,\varepsilon,\delta}\|u\|_s^2, \quad u \in C_0^\infty(\omega_p).
\end{aligned}$$

Now let $M$ be the positive constant determined in Lemma 4.1, and put $\delta = 1/2M(n+2)$. Then we have by the above inequality and Corollary 4.2

$$\begin{aligned}\operatorname{Re}\left(\tilde{L}u, \frac{\partial}{\partial x_n}u\right)_{s-\frac{1}{2},\varepsilon} \geq{}& \frac{1}{2}\sum_{i=1}^{r}\left(\tilde{a}E^{(\varepsilon)}_{s-\frac{1}{2}}\tilde{X}_i u, E^{(\varepsilon)}_{s-\frac{1}{2}}\tilde{X}_i u\right) + \left\|\frac{\partial}{\partial x_n}u\right\|^2_{s-\frac{1}{2},\varepsilon} \\ &- \left|s-\frac{1}{2}\right|\varepsilon^2\sum_{i=1}^{r}\left|\left(\tilde{a}\tilde{X}_i^2 E^{(\varepsilon)}_{s-\frac{5}{2}}\frac{\partial}{\partial x_n}u, E^{(\varepsilon)}_{s-\frac{1}{2}}\frac{\partial}{\partial x_n}u\right)\right| \\ &+ \operatorname{Re}\left(x_n\tilde{L}u, u\right)_s - C^{(1)}_{s,\varepsilon}\|u\|_s^2, \quad u \in C_0^\infty(\omega_p). \qquad (4.16)\end{aligned}$$

We estimate the third term in the right-hand side of (4.16). We write

$$\tilde{X}_i = \sum_{k=1}^{n}\tilde{c}_{ik}(x)\frac{\partial}{\partial x_k}, \quad 1 \leq i \leq r. \qquad (4.17)$$

Then we have

$$\begin{aligned}&\left|s-\frac{1}{2}\right|\varepsilon^2\left|\left(\tilde{a}\tilde{X}_i^2 E^{(\varepsilon)}_{s-\frac{5}{2}}\frac{\partial}{\partial x_n}u, E^{(\varepsilon)}_{s-\frac{1}{2}}\frac{\partial}{\partial x_n}u\right)\right| \\ &\quad\leq 4r\left|s-\frac{1}{2}\right|^2\varepsilon^4 n^2\sum_{k,l=1}^{n}\left\|\tilde{a}\tilde{c}_{ik}\tilde{c}_{il}\frac{\partial^2}{\partial x_k\partial x_l}E^{(\varepsilon)}_{s-\frac{5}{2}}\frac{\partial}{\partial x_n}u\right\|_0^2 + \frac{1}{4r}\left\|\frac{\partial}{\partial x_n}u\right\|^2_{s-\frac{1}{2},\varepsilon} + C^{(2)}_{s,\varepsilon}\|u\|_s^2, \\ &\qquad\qquad u \in C_0^\infty(R^n). \quad (4.18)\end{aligned}$$

Now we choose $\rho > 0$ so small that $B(p,\rho) \subset \omega_p$. Let $\psi_\rho(x)$ be a function belonging to $C_0^\infty(B(p,3\rho))$ such that $0 \leq \psi_\rho(x) \leq 1$ in $R^n$ and $\psi_\rho(x) = 1$ in $B(p,2\rho)$. Then it follows that

$$\begin{aligned}&\varepsilon^4\sum_{k,l=1}^{n}\left\|\tilde{a}\tilde{c}_{ik}\tilde{c}_{il}\frac{\partial^2}{\partial x_k\partial x_l}E^{(\varepsilon)}_{s-\frac{5}{2}}\frac{\partial}{\partial x_n}u\right\|_0^2 \\ &\quad\leq \varepsilon^4\sum_{k,l=1}^{n}\left\|\psi_\rho\tilde{a}\tilde{c}_{ik}\tilde{c}_{il}\frac{\partial^2}{\partial x_k\partial x_l}E^{(\varepsilon)}_{s-\frac{5}{2}}\frac{\partial}{\partial x_n}u\right\|_0^2 + C^{(1)}_{s,\varepsilon,\rho}\|u\|_s^2 \\ &\quad\leq \varepsilon^4\sum_{k,l=1}^{n}\left\{\sup_{x\in R^n}\left|\psi_\rho(x)\tilde{a}(x)\tilde{c}_{ik}(x)\tilde{c}_{il}(x)\right|^2\right\}(2\pi)^{-n}\int\left|\xi_k\xi_l E^{(\varepsilon)}_{s-\frac{5}{2}}(\xi)\xi_n\hat{u}(\xi)\right|^2 d\xi \\ &\qquad + C^{(1)}_{s,\varepsilon,\rho}\|u\|_s^2, \quad u \in C_0^\infty(B(p,\rho)). \qquad (4.19)\end{aligned}$$

First we consider the case of $k = n$ or $l = n$. Since $\tilde{c}_{in}(x) = \tilde{X}_i(x_n)$ by (4.17), it follows from (4.7) that $\tilde{a}(x)\tilde{c}_{in}(x)|_{x_n=0} = 0$, $1 \leq i \leq r$. Hence there exists a constant $M' > 0$ such that

$$\left|\tilde{a}(x)\tilde{c}_{ik}(x)\tilde{c}_{il}(x)\right|^2 \le M'|x_n|^2, \quad x \in R^n,\ 1 \le i \le r,\ k = n \text{ or } l = n. \tag{4.20}$$

Then we have

$$\begin{aligned}
&\varepsilon^4\left\{\sup_{x\in R^n}\left|\psi_\rho(x)\tilde{a}(x)\tilde{c}_{ik}(x)\tilde{c}_{il}(x)\right|^2\right\}\left|\xi_k\xi_l E^{(\varepsilon)}_{-2}(\xi)\right|^2 \\
&\le M'(3\rho)^2\left\{(\varepsilon^2|\xi'|^2 + \varepsilon^2|\xi_n|^2)E^{(\varepsilon)}_{-2}(\xi)\right\}^2 \\
&\le M'(3\rho)^2, \quad \xi \in R^n,\ 0 < \varepsilon \le 1,\ 1 \le i \le r,\ k = n \text{ or } l = n.
\end{aligned} \tag{4.21}$$

Next we consider the case of $1 \le k,\ l \le n-1$. We put

$$M'' = \max_{1\le i\le r, 1\le k,l\le n-1}\ \sup_{x\in R^n}\left|\tilde{a}(x)\tilde{c}_{ik}(x)\tilde{c}_{il}(x)\right|^2. \tag{4.22}$$

Then we have

$$\begin{aligned}
&\varepsilon^4\left\{\sup_{x\in R^n}\left|\psi_\rho(x)\tilde{a}(x)\tilde{c}_{ik}(x)\tilde{c}_{il}(x)\right|^2\right\}\left|\xi_k\xi_l E^{(\varepsilon)}_{-2}(\xi)\right|^2 \le M''\varepsilon^4, \\
&\qquad\qquad \xi \in R^n,\ \varepsilon > 0,\ 1 \le i \le r,\ 1 \le k,\ l \le n-1.
\end{aligned} \tag{4.23}$$

It follows from (4.21) and (4.23) that

$$\begin{aligned}
&\varepsilon^4\sum_{k,l=1}^n\left\{\sup_{x\in R^n}\left|\psi_\rho(x)\tilde{a}(x)\tilde{c}_{ik}(x)\tilde{c}_{il}(x)\right|^2\right\}(2\pi)^{-n}\int\left|\xi_k\xi_l E^{(\varepsilon)}_{s-\frac{5}{2}}(\xi)\xi_n\hat{u}(\xi)\right|^2 d\xi \\
&\le C(\rho^2+\varepsilon^4)\left\|\frac{\partial}{\partial x_n}u\right\|^2_{s-\frac{1}{2},\varepsilon}, \quad u \in C_0^\infty(R^n),\ 0 < \varepsilon \le 1,\ 1 \le i \le r,
\end{aligned} \tag{4.24}$$

where $C = 9n^2(M' + M'')$ and $C$ is independent of $s$, $\varepsilon$ and $\rho$. We have by (4.18), (4.19) and (4.24)

$$\begin{aligned}
&\left|s-\frac{1}{2}\right|\varepsilon^2\left|\left(\tilde{a}\tilde{X}_i^2 E^{(\varepsilon)}_{s-\frac{5}{2}}\frac{\partial}{\partial x_n}u, E^{(\varepsilon)}_{s-\frac{1}{2}}\frac{\partial}{\partial x_n}u\right)\right| \\
&\le 4r\left|s-\frac{1}{2}\right|^2 n^2 C(\rho^2+\varepsilon^4)\left\|\frac{\partial}{\partial x_n}u\right\|^2_{s-\frac{1}{2},\varepsilon} + \frac{1}{4r}\left\|\frac{\partial}{\partial x_n}u\right\|^2_{s-\frac{1}{2},\varepsilon} + C^{(2)}_{s,\varepsilon,\rho}\|u\|_s^2, \\
&\qquad\qquad u \in C_0^\infty(B(p,\rho)),\ 0 < \varepsilon \le 1,\ 1 \le i \le r.
\end{aligned} \tag{4.25}$$

It follows from (4.16) and (4.25) that

$$\begin{aligned}
&\operatorname{Re}\left(\tilde{L}u, \frac{\partial}{\partial x_n}u\right)_{s-\frac{1}{2},\varepsilon} \\
&\quad \geq \frac{1}{2}\sum_{i=1}^{r}\left(\tilde{a}E^{(\varepsilon)}_{s-\frac{1}{2}}\tilde{X}_i u, E^{(\varepsilon)}_{s-\frac{1}{2}}\tilde{X}_i u\right) + \left\{\frac{3}{4} - 4r^2\left|s-\frac{1}{2}\right|^2 n^2 C(\rho^2+\varepsilon^4)\right\}\left\|\frac{\partial}{\partial x_n}u\right\|^2_{s-\frac{1}{2},\varepsilon} \\
&\qquad + \operatorname{Re}\left(x_n\tilde{L}u, u\right)_s - C^{(3)}_{s,\varepsilon,\rho}\|u\|_s^2, \quad u \in C_0^\infty(B(p,\rho)),\ 0<\varepsilon\leq 1.
\end{aligned}$$

Now we choose $\varepsilon_s,\ \rho_s > 0$ so small that $0 < \varepsilon_s \leq 1$, $B(p,\rho_s) \subset \omega_p$ and $3/4 - 4r^2|s - 1/2|^2 n^2 C(\rho^2+\varepsilon^4) \geq 1/2$. Then we obtain (4.10). □

Corollary 4.3. *For any real number $s$, let $\rho_s > 0$ be the radius determined in Proposition 4.2. There exists $C_s > 0$ depending on $s$ such that*

$$\left\|\frac{\partial}{\partial x_n}u\right\|^2_{s-\frac{1}{2}} \leq C_s\big(\|\tilde{L}u\|_s^2 + \|u\|_s^2\big), \quad u \in C_0^\infty(B(p,\rho_s)), \tag{4.26}$$

$$\left\|x_n\frac{\partial}{\partial x_n}u\right\|^2_{s-\frac{1}{2}} \leq C_s\big(\|\tilde{L}u\|_s^2 + \|u\|_s^2\big), \quad u \in C_0^\infty(B(p,\rho_s)). \tag{4.27}$$

## 5. An estimate of the subelliptic kind.

In this section we shall fix $p \in \{(x', x_n) \mid |x'| < T,\ x_n = 0\}$ and establish an estimate of the subelliptic kind for the operator $\tilde{L}$ defined by (3.20).

Lemma 5.1. *Let $s$ be a real number such that $s \leq 1/2$. Let $R$ be a first order operator with real-valued coefficients belonging to $C_0^\infty(R^n)$. Then there exists a constant $C_s > 0$ depending on $s$ such that for all $i = 1, \dots, r$*

$$\big\|[x_n\tilde{a}\tilde{X}_i, R]u\big\|^2_{s-1} \leq C_s\big(\|\tilde{L}u\|_0^2 + \|u\|_0^2 + \|Ru\|^2_{2s-1}\big), \quad u \in C_0^\infty(R^n). \tag{5.1}$$

Proof. We put $Q_{2s-1} = E_{2s-2}[x_n\tilde{a}\tilde{X}_i, R]$. Then $Q_{2s-1} \in \operatorname{Op}(S^{2s-1})$ and we have

$$\begin{aligned}
\big\|[x_n\tilde{a}\tilde{X}_i, R]u\big\|^2_{s-1} &= \big(x_n\tilde{a}\tilde{X}_i Ru, Q_{2s-1}u\big) - \big(Rx_n\tilde{a}\tilde{X}_i u, Q_{2s-1}u\big) \\
&= \big(Q^*_{2s-1}Ru, (x_n\tilde{a}\tilde{X}_i)^* u\big) + \big([Q^*_{2s-1}, x_n\tilde{a}\tilde{X}_i]Ru, u\big) \\
&\quad - \big(x_n\tilde{a}\tilde{X}_i u, Q_{2s-1}R^* u\big) - \big(x_n\tilde{a}\tilde{X}_i u, [R^*, Q_{2s-1}]u\big) \\
&\leq C\big(\|x_n\tilde{a}\tilde{X}_i u\|_0^2 + \|Ru\|^2_{2s-1} + \|u\|_0^2 + \|u\|^2_{2s-1}\big), \quad u \in C_0^\infty(R^n),
\end{aligned}$$

where $C$ is a positive constant depending on $s$. We note that $2s - 1 \leq 0$ by hypothesis, and so $\|u\|_{2s-1} \leq \|u\|_0$. Then we obtain (5.1) by Corollary 4.2. □

Lemma 5.2. *Let $t$ be a real number and let $A_t \in \operatorname{Op}(S^t)$. We have the expression*

$$\left[x_n^2\tilde{a}\tilde{X}_i^2, A_t\right] = B_t^{(i)} x_n \tilde{a}\tilde{X}_i + \sum_{k=1}^{n} C_t^{(i,k)} x_n \tilde{a}_{x_k}\tilde{X}_i + D_t^{(i)}, \quad 1 \le i \le r, \tag{5.2}$$

*where* $B_t^{(i)}$, $C_t^{(i,k)}$, $D_t^{(i)} \in \mathrm{Op}(S^t)$ $(1 \le i \le r,\ 1 \le k \le n)$.

Proof. We denote by $\tilde{X}_i(x,\xi)$ the symbol of $\tilde{X}_i$ and by $A_t(x,\xi)$ the symbol of $A_t$. Then the principal symbol of $[x_n^2\tilde{a}\tilde{X}_i^2, A_t]$ is equal to $B_t^{(i)}(x,\xi)x_n\tilde{a}(x)\tilde{X}_i(x,\xi) + \sum_{k=1}^n C_t^{(i,k)}(x,\xi)x_n\tilde{a}_{x_k}(x)\tilde{X}_i(x,\xi)$ where

$$\begin{aligned} B_t^{(i)}(x,\xi) &= \sum_{k=1}^{n} -2\sqrt{-1}x_n\Big\{\big(\tilde{X}_i(x,\xi)\big)_{\xi_k}\big(A_t(x,\xi)\big)_{x_k} - \big(A_t(x,\xi)\big)_{\xi_k}\big(\tilde{X}_i(x,\xi)\big)_{x_k}\Big\} \\ &\quad + 2\sqrt{-1}\big(A_t(x,\xi)\big)_{\xi_n}\tilde{X}_i(x,\xi), \\ C_t^{(i,k)}(x,\xi) &= \sqrt{-1}x_n\big(A_t(x,\xi)\big)_{\xi_k}\tilde{X}_i(x,\xi). \end{aligned}$$

Let $B_t^{(i)}$ and $C_t^{(i,k)}$ $(1 \le i \le r,\ 1 \le k \le n)$ be the pseudo-differential operators with the symbols $B_t^{(i)}(x,\xi)$ and $C_t^{(i,k)}(x,\xi)$ $(1 \le i \le r,\ 1 \le k \le n)$ respectively. Then $B_t^{(i)}$, $C_t^{(i,k)} \in \mathrm{Op}(S^t)$ $(1 \le i \le r,\ 1 \le k \le n)$ and

$$\left[x_n\tilde{a}\tilde{X}_i^2, A_t\right] \equiv B_t^{(i)} x_n \tilde{a}\tilde{X}_i + \sum_{k=1}^{n} C_t^{(i,k)} x_n \tilde{a}_{x_k}\tilde{X}_i \pmod{\mathrm{Op}(S^t)}.$$

Thus the lemma has been proved. □

Lemma 5.3. *Let $t$ be a real number and let $A_t \in \mathrm{Op}(S^t)$. There exists a constant $C > 0$ such that for all $i = 1, \dots, r$*

$$\left|(A_t u, x_n^2\tilde{a}\tilde{X}_i^2 u)\right| \le C\big(\|\tilde{L}u\|_0^2 + \|u\|_0^2 + \|A_t u\|_0^2 + \|A_t u\|_t^2\big), \quad u \in C_0^\infty(R^n). \tag{5.3}$$

Proof. We shall denote by $C_1, C_2, \dots$ positive constants independent of $u \in C_0^\infty(R^n)$. We write $\tilde{X}_i^* = -\tilde{X}_i + f_i$, $f_i \in C_0^\infty(R^n)$. Then

$$\big(x_n^2\tilde{a}\tilde{X}_i\big)^* = -x_n^2\tilde{a}\tilde{X}_i + h_i, \tag{5.4}$$

where

$$h_i(x) = -\tilde{X}_i\big(x_n^2\tilde{a}(x)\big) + f_i(x)x_n^2\tilde{a}(x). \tag{5.5}$$

We note that (4.7) and Corollary 4.1 with $s = 0$ imply that

$$\big\|h_i\tilde{X}_i u\big\|_0^2 \le C_1\big(\|\tilde{L}u\|_0^2 + \|u\|_0^2\big). \tag{5.6}$$

Then we have

$$\begin{aligned}\big|(A_t u, x_n^2\tilde a \tilde X_i^2 u)\big| &= \big| -(x_n^2 \tilde a \tilde X_i A_t u, \tilde X_i u) + (A_t u, h_i \tilde X_i u)\big| \\ &\le \big|(x_n^2\tilde a \tilde X_i A_t u, \tilde X_i u)\big| + \|A_t u\|_0^2 + C_1\big(\|\tilde L u\|_0^2 + \|u\|_0^2\big). \qquad (5.7)\end{aligned}$$

We choose $\psi(x) \in C_0^\infty(R^n)$ such that $0 \le \psi(x) \le 1$ $(x \in R^n)$ and $\psi(x) = 1$ $(x \in \Omega_p)$. Then we see from (3.18)–(3.20) that $\tilde X_i v = \tilde X_i(\psi v)$ and $\tilde L v = \tilde L(\psi v)$ $(v \in C^\infty(R^n))$. Hence it follows from Proposition 4.1 with $A_s = 1$ that

$$\begin{aligned}\big|(x_n^2\tilde a \tilde X_i A_t u, \tilde X_i u)\big| &= \big|(x_n^2 \tilde a \tilde X_i \psi A_t u, \tilde X_i u)\big| \\ &\le \big(x_n^2 \tilde a \tilde X_i \psi A_t u, \tilde X_i \psi A_t u\big) + \big(x_n^2\tilde a \tilde X_i u, \tilde X_i u\big) \\ &\le -\operatorname{Re}\big(x_n \tilde L \psi A_t u, \psi A_t u\big) + C_2\|\psi A_t u\|_0^2 - \operatorname{Re}\big(x_n \tilde L u, u\big) + C_2\|u\|_0^2 \\ &\le -\operatorname{Re}\big(x_n \tilde L A_t u, \psi A_t u\big) + C_3\big(\|A_t u\|_0^2 + \|\tilde L u\|_0^2 + \|u\|_0^2\big). \qquad (5.8)\end{aligned}$$

By definition (3.20) of $\tilde L$ and Lemma 5.2 we have

$$\begin{aligned}&\big(x_n \tilde L A_t u, \psi A_t u\big) \\ &= \big(A_t x_n \tilde L u, \psi A_t u\big) + \bigg(\Big[x_n \chi \frac{\partial}{\partial x_n} + x_n \tilde c, A_t\Big] u, \psi A_t u\bigg) + \sum_{i=1}^r \big([x_n^2 \tilde a \tilde X_i^2, A_t] u, \psi A_t u\big) \\ &= \big(x_n \tilde L u, A_t^* \psi A_t u\big) + \bigg(u, \Big[x_n\chi\frac{\partial}{\partial x_n} + x_n\tilde c, A_t\Big]^* \psi A_t u\bigg) + \sum_{i=1}^r \big(x_n \tilde a \tilde X_i u, B_t^{(i)*}\psi A_t u\big) \\ &\quad + \sum_{i=1}^r\sum_{k=1}^n \big(x_n \tilde a_{x_k} \tilde X_i u, C_t^{(i,\,k)*}\psi A_t u\big) + \sum_{i=1}^r \big(u, D_t^{(i)*}\psi A_t u\big).\end{aligned}$$

Hence it follows from Corollary 4.2 with $s = 0$ that

$$\big|(x_n \tilde L A_t u, \psi A_t u)\big| \le C_4\big(\|\tilde L u\|_0^2 + \|A_t u\|_t^2 + \|u\|_0^2\big). \qquad (5.9)$$

In virtue of (5.7), (5.8) and (5.9) we obtain (5.3). □

LEMMA 5.4. *Let $s$ be a real number such that $0 \le s \le 1/2$. Let $R$ be a first order operator with real-valued coefficients belonging to $C_0^\infty(R^n)$. There exists a constant $C_s > 0$ depending on $s$ such that*

$$\bigg\|\Big[x_n \frac{\partial}{\partial x_n}, R\Big] u\bigg\|_{s-1}^2 \le C_s\big(\|\tilde L u\|_0^2 + \|u\|_0^2 + \|Ru\|_{4s-1}^2\big), \quad u \in C_0^\infty(\omega_p). \qquad (5.10)$$

PROOF. We shall denote by $C_1, C_2, \dots$ positive constants depending on $s$ and independent of $u \in C_0^\infty(\omega_p)$. We remember that the function $\chi(x)$ was determined by (3.18) so that $\chi(x) \in C_0^\infty(\Omega_p)$ and $\chi(x) = 1$ in $\omega_p$. Hence

$$\left[x_n\frac{\partial}{\partial x_n},R\right]u=\left[x_n\chi\frac{\partial}{\partial x_n},R\right]u,\quad u\in C_0^\infty(\omega_p). \tag{5.11}$$

We put

$$P_{2s-1}=E_{2s-2}\left[x_n\chi\frac{\partial}{\partial x_n},R\right]. \tag{5.12}$$

Then $P_{2s-1}\in \mathrm{Op}(S^{2s-1})$ and it follows from definition (3.20) of $\tilde{L}$ that

$$\begin{aligned}
&\left\|\left[x_n\frac{\partial}{\partial x_n},R\right]u\right\|_{s-1}^2\\
&\quad=\left(\left[x_n\chi\frac{\partial}{\partial x_n}+x_n\tilde{c},P_{2s-1}^*R\right]u,u\right)+\left(\left[P_{2s-1}^*,x_n\chi\frac{\partial}{\partial x_n}+x_n\tilde{c}\right]Ru,u\right)\\
&\qquad-\big([x_n\tilde{c},R]u,P_{2s-1}u\big)\\
&\quad=\big(x_n\tilde{L}P_{2s-1}^*Ru,u\big)-\big(x_n\tilde{L}u,R^*P_{2s-1}u\big)\\
&\qquad+\sum_{i=1}^r\big\{(x_n^2\tilde{a}\tilde{X}_i^2u,R^*P_{2s-1}u)-(x_n^2\tilde{a}\tilde{X}_i^2P_{2s-1}^*Ru,u)\big\}\\
&\qquad+\left(\left[P_{2s-1}^*,x_n\chi\frac{\partial}{\partial x_n}+x_n\tilde{c}\right]Ru,u\right)-\big([x_n\tilde{c},R]u,P_{2s-1}u\big)\\
&\quad\le \mathrm{Re}\,\big(x_n\tilde{L}P_{2s-1}^*Ru,u\big)+\sum_{i=1}^r\big\{\mathrm{Re}\,(x_n^2\tilde{a}\tilde{X}_i^2u,R^*P_{2s-1}u)-\mathrm{Re}\,(x_n^2\tilde{a}\tilde{X}_i^2P_{2s-1}^*Ru,u)\big\}\\
&\qquad+\big\|R^*P_{2s-1}u\big\|_0^2+C_1\big(\|\tilde{L}u\|_0^2+\|Ru\|_{2s-1}^2+\|u\|_0^2+\|u\|_{2s-1}^2\big).
\end{aligned} \tag{5.13}$$

We estimate the first term in the right-hand side of (5.13). We see that with some $q_i\in C_0^\infty(R^n)$ $(1\le i\le r)$

$$\big(x_n^2\tilde{a}\tilde{X}_i^2\big)^*=x_n^2\tilde{a}\tilde{X}_i^2-2h_i\tilde{X}_i+q_i,\quad 1\le i\le r, \tag{5.14}$$

where $h_i$ $(1\le i\le r)$ are the functions defined by (5.5). Hence by using the relation: $-x_n\chi\partial/\partial x_n=-x_n\tilde{L}+\sum_{i=1}^r x_n^2\tilde{a}\tilde{X}_i^2+x_n\tilde{c}$, we have with some $q\in C_0^\infty(R^n)$

$$\big(x_n\tilde{L}\big)^*=-x_n\tilde{L}+2\sum_{i=1}^r x_n^2\tilde{a}\tilde{X}_i^2-2\sum_{i=1}^r h_i\tilde{X}_i+q. \tag{5.15}$$

Hence

$$\begin{aligned}
\big|(x_n\tilde{L}P_{2s-1}^*Ru,u)\big|&\le 2\sum_{i=1}^r\big|(P_{2s-1}^*Ru,x_n^2\tilde{a}\tilde{X}_i^2u)\big|+\sum_{i=1}^r\big\|h_i\tilde{X}_iu\big\|_0^2\\
&\quad+C_2\big(\|\tilde{L}u\|_0^2+\|u\|_0^2+\|Ru\|_{2s-1}^2\big).
\end{aligned}$$

We apply Lemma 5.3 with $A_t = P^*_{2s-1}R$ to the first term in the right-hand side of the above inequality, and (5.6) to the second one. Then we have

$$\left|(x_n\tilde{L}P^*_{2s-1}Ru, u)\right| \le C_3\big(\|\tilde{L}u\|_0^2 + \|u\|_0^2 + \|Ru\|^2_{2s-1} + \|Ru\|^2_{4s-1}\big). \tag{5.16}$$

We estimate the second term in the right-hand side of (5.13). By (5.12) there exists $N_{2s-1} \in \mathrm{Op}(S^{2s-1})$ such that $P^*_{2s-1}R = R^*P_{2s-1} + N_{2s-1}$. Then it follows from (5.14) that

$$\begin{aligned}\big(x_n^2\tilde{a}\tilde{X}_i^2P^*_{2s-1}Ru, u\big) &= \big(P^*_{2s-1}Ru, x_n^2\tilde{a}\tilde{X}_i^2u - 2h_i\tilde{X}_iu + q_iu\big)\\ &= \big(R^*P_{2s-1}u, x_n^2\tilde{a}\tilde{X}_i^2u\big) + \big(R^*P_{2s-1}u, -2h_i\tilde{X}_iu + q_iu\big)\\ &\quad + \big(N_{2s-1}u, x_n^2\tilde{a}\tilde{X}_i^2u - 2h_i\tilde{X}_iu + q_iu\big).\end{aligned} \tag{5.17}$$

Hence, noting that $\mathrm{Re}\,(v,w) = \mathrm{Re}\,(w,v)$, we have by (5.17)

$$\begin{aligned}&\sum_{i=1}^r \big\{\mathrm{Re}\,(x_n^2\tilde{a}\tilde{X}_i^2u, R^*P_{2s-1}u) - \mathrm{Re}\,(x_n^2\tilde{a}\tilde{X}_i^2P^*_{2s-1}Ru, u)\big\}\\ &\quad = \sum_{i=1}^r \mathrm{Re}\,\big(R^*P_{2s-1}u + N_{2s-1}u, 2h_i\tilde{X}_iu - q_iu\big) - \sum_{i=1}^r \mathrm{Re}\,\big(N_{2s-1}u, x_n^2\tilde{a}\tilde{X}_i^2u\big)\\ &\quad \le 2r\big\|R^*P_{2s-1}u\big\|_0^2 + 2\sum_{i=1}^r \big\|h_i\tilde{X}_iu\big\|_0^2 - \sum_{i=1}^r \mathrm{Re}\,\big(N_{2s-1}u, x_n^2\tilde{a}\tilde{X}_i^2u\big)\\ &\qquad + C_4\big(\|u\|_0^2 + \|u\|^2_{2s-1}\big).\end{aligned}$$

We apply (5.6) to the second term in the right-hand side of the above inequality, and Lemma 5.3 with $A_t = N_{2s-1}$ to the third one. Then we have

$$\begin{aligned}&\sum_{i=1}^r \big\{\mathrm{Re}\,(x_n^2\tilde{a}\tilde{X}_i^2u, R^*P_{2s-1}u) - \mathrm{Re}\,(x_n^2\tilde{a}\tilde{X}_i^2P^*_{2s-1}Ru, u)\big\}\\ &\qquad \le 2r\big\|R^*P_{2s-1}u\big\|_0^2 + C_5\big(\|\tilde{L}u\|_0^2 + \|u\|_0^2 + \|u\|^2_{2s-1} + \|u\|^2_{4s-2}\big).\end{aligned} \tag{5.18}$$

Now we have

$$\begin{aligned}&\big\|R^*P_{2s-1}u\big\|_0^2 + \|Ru\|^2_{2s-1} + \|u\|^2_{2s-1} + \|u\|^2_{4s-2}\\ &\quad \le \big\|P_{2s-1}R^*u + [R^*, P_{2s-1}]u\big\|_0^2 + \|Ru\|^2_{4s-1} + 2\|u\|_0^2\\ &\quad \le C_6\big(\|Ru\|^2_{4s-1} + \|u\|_0^2\big),\end{aligned} \tag{5.19}$$

because $0 \le s \le 1/2$ by hypothesis. In virtue of (5.13), (5.16), (5.18) and (5.19) we obtain (5.10). □

We define

$$\tilde{Z}_0 = x_n \frac{\partial}{\partial x_n}, \quad \tilde{Z}_1 = x_n \tilde{a} \tilde{X}_1, \ldots, \tilde{Z}_r = x_n \tilde{a} \tilde{X}_r. \tag{5.20}$$

Let $l$ be a positive integer. For every multi-index $J = (j_1, \ldots, j_l)$ $(0 \le j_k \le r,\ 1 \le k \le l)$ we define

$$\tilde{Z}_J = \big[\tilde{Z}_{j_1}, [\tilde{Z}_{j_2}, \ldots, [\tilde{Z}_{j_{l-1}}, \tilde{Z}_{j_l}] \ldots]\big]. \tag{5.21}$$

Let $\rho_0 > 0$ be the radius determined in Proposition 4.2 with $s = 0$. We recall that $B(p, \rho_0) \subset \omega_p$. Then we have the following

Lemma 5.5. *For any multi-index $J = (j_1, \ldots, j_l)$ $(0 \le j_k \le r,\ 1 \le k \le l)$, there exists a constant $C_J > 0$ such that*

$$\big\|\tilde{Z}_J u\big\|^2_{\tau(J)-1} \le C_J \big(\|\tilde{L}u\|_0^2 + \|u\|_0^2\big), \quad u \in C_0^\infty(B(p, \rho_0)), \tag{5.22}$$

*where $\tau(J)$ is the number defined by* (3.3).

Proof. We shall prove the lemma by induction on $l$. When $l = 1$ it follows from (3.3) that $\tau(J) = 1$ $(1 \le j_1 \le r)$ and $\tau(J) = 1/2$ $(j_1 = 0)$. Hence (5.22) holds by Corollary 4.2 and Corollary 4.3 respectively.

Let $m$ be a positive integer and assume that the lemma is valid for $l = m$. Let $J = (j_1, j_2, \ldots, j_{m+1})$ be any multi-index such that $0 \le j_k \le r$ $(1 \le k \le m+1)$. We put $J' = (j_2, \ldots, j_{m+1})$, and choose a function $\psi(x) \in C_0^\infty(R^n)$ such that $\psi(x) = 1$ in $B(p, \rho_0)$. Then

$$\big\|\tilde{Z}_J u\big\|_{\tau(J)-1} = \big\|[\tilde{Z}_{j_1}, \psi \tilde{Z}_{J'}]u\big\|_{\tau(J)-1}, \quad u \in C_0^\infty(B(p, \rho_0)). \tag{5.23}$$

First we consider the case of $1 \le j_1 \le r$. In this case $\tilde{Z}_{j_1} = x_n \tilde{a} \tilde{X}_{j_1}$, $0 < \tau(J) \le 1/2$ and $2\tau(J) = \tau(J')$. Hence we have (5.22) by (5.23), Lemma 5.1 and hypothesis of induction. Next we consider the case of $j_1 = 0$. In this case $\tilde{Z}_{j_1} = x_n \partial/\partial x_n$, $0 < \tau(J) \le 1/2$ and $4\tau(J) = \tau(J')$. Hence we have (5.22) by (5.23), Lemma 5.4 and hypothesis of induction. □

Lemma 5.6. *Let $l_0$ be the positive integer defined by* (3.14). *There exist constants $0 < \tau \le 1/2$ and $C > 0$ such that*

$$\left\| x_n^{l_0} \frac{\partial u}{\partial x_j} \right\|^2_{\tau - 1} \le C \big(\|\tilde{L}u\|_0^2 + \|u\|_0^2\big), \quad u \in C_0^\infty(B(p, \rho_0)),\ 1 \le j \le n.$$

Proof. It follows from (5.20), (3.5), (3.18) and (3.19) that $Z_j = \tilde{Z}_j$ in $\omega_p$, $1 \le j \le r$. We note that $B(p, \rho_0) \subset \omega_p$ by definition of $\rho_0$ in Proposition 4.2 with $s = 0$. Then we have by (3.16) and (3.17)

$$x_n^{l_0}\frac{\partial u}{\partial x_j}=\sum_{k=1}^{n}\sum_{I\subseteq J_k} g_{j,k,I}\tilde{Z}_I u,\quad u\in C_0^\infty(B(p,\rho_0)),\ 1\le j\le n. \tag{5.24}$$

In view of (3.3) we put

$$\tau=\min_{1\le k\le n}\tau(J_k). \tag{5.25}$$

Then $0<\tau\le 1/2$ by (3.3), (3.14) and (3.15), and the lemma follows from (5.24) and Lemma 5.5. □

Let $l_0$ be the positive integer defined by (3.14) and $\tau$ be the positive number determined in Lemma 5.6. We put

$$\sigma=\frac{\tau}{4^{l_0}}. \tag{5.26}$$

Here

$$0<\sigma\le\frac{1}{8}, \tag{5.27}$$

because $0<\tau\le 1/2$ by Lemma 5.6, and $l_0\ge 1$ by (3.15). Then we have the following

Proposition 5.1 (estimate of the subelliptic kind). *Let $\rho_0$ be the radius determined in Proposition* 4.2 *with $s=0$. There exists a constant $C>0$ such that*

$$\|u\|_\sigma^2\le C\big(\|\tilde{L}u\|_0^2+\|u\|_0^2\big),\quad u\in C_0^\infty\Big(B\Big(p,\frac{1}{2}\rho_0\Big)\Big).$$

Proof. To prove the proposition it is sufficient to show that

$$\left\|\frac{\partial u}{\partial x_j}\right\|_{\sigma-1}^2\le C\big(\|\tilde{L}u\|_0^2+\|u\|_0^2\big),\quad u\in C_0^\infty\Big(B\Big(p,\frac{1}{2}\rho_0\Big)\Big),\ 1\le j\le n. \tag{5.28}$$

For non-negative integers $q=l_0,l_0-1,\dots,0$ we put

$$\sigma_q=\frac{\tau}{4^{l_0-q}}. \tag{5.29}$$

We shall show by induction on $q$ that for non-negative integers $q=l_0,l_0-1,\dots,0$ there exist constants $C_q>0$ such that

$$\left\|x_n^q\frac{\partial u}{\partial x_j}\right\|_{\sigma_q-1}^2\le C_q\big(\|\tilde{L}u\|_0^2+\|u\|_0^2\big),\quad u\in C_0^\infty\Big(B\Big(p,\frac{1}{2}\rho_0\Big)\Big),\ 1\le j\le n. \tag{5.30}$$

Then (5.28) follows from (5.30) with $q=0$. We know from Lemma 5.6 that (5.30) is valid when $q = l_0$. Let $q$ be any integer such that $1 \le q \le l_0$, and assume that (5.30) holds. In the rest of the proof we shall show that

$$\left\| x_n^{q-1} \frac{\partial u}{\partial x_j} \right\|_{\sigma_{q-1}-1}^2 \le C_{q-1}\big(\|\tilde{L}u\|_0^2 + \|u\|_0^2\big), \quad u \in C_0^\infty\left(B\left(p, \frac{1}{2}\rho_0\right)\right),\ 1 \le j \le n. \tag{5.31}$$

Let $s = 0$ in Corollary 4.3. Then we see that (5.31) holds for $j = n$, because $\sigma_{q-1} = \tau/4^{l_0-q+1} < 1/8$ by definition of $\tau$ in Lemma 5.6. Hence it remains to prove (5.31) for $j = 1, \dots, n-1$. From now on we shall fix $j$ ($1 \le j \le n-1$), and denote by $C_1, C_2, \dots$ positive constants independent of $u \in C_0^\infty(B(p, \rho_0/2))$.

We put

$$s = \sigma_{q-1}, \quad R = \chi x_n^q \frac{\partial}{\partial x_j}, \tag{5.32}$$

where $\chi$ is the function defined by (3.18) in order to localize $X_i$ ($1 \le i \le r$) and $L$. We see that

$$0 < s = \frac{\tau}{4^{l_0-q+1}} \le \frac{1}{8}. \tag{5.33}$$

Since $\chi(x) = 1$ in $\omega_p$ and $B\big(p, \frac{1}{2}\rho_0\big) \subset \omega_p$, it follows that $x_n^q \frac{\partial u}{\partial x_j} = Ru$, $u \in C_0^\infty\big(B(p, \frac{1}{2}\rho_0)\big)$. Hence we have by (5.30)

$$\|Ru\|_{4s-1}^2 \le C_q\big(\|\tilde{L}u\|_0^2 + \|u\|_0^2\big). \tag{5.34}$$

Taking into account that $2s-1 < 4s-1 < 0$, we have also

$$\|Ru\|_{2s-1}^2 + \|u\|_{2s-1}^2 \le C_1\big(\|\tilde{L}u\|_0^2 + \|u\|_0^2\big). \tag{5.35}$$

Now we put

$$T_{2s-1} = E_{2s-2}\chi x_n^{q-1} \frac{\partial}{\partial x_j}. \tag{5.36}$$

Then $T_{2s-1} \in \mathrm{Op}(S^{2s-1})$, and by using the relations:

$$x_n^{q-1}\frac{\partial}{\partial x_j} = \chi x_n^{q-1}\frac{\partial}{\partial x_j} \quad \text{in } B\left(p, \frac{1}{2}\rho_0\right),$$

$$x_n^{q-1}\frac{\partial}{\partial x_j} = \frac{1}{q}\left[\frac{\partial}{\partial x_n}, x_n^q\frac{\partial}{\partial x_j}\right] = \frac{1}{q}\left[\chi\frac{\partial}{\partial x_n}, R\right] \quad \text{in } B\left(p, \frac{1}{2}\rho_0\right),$$

$$\chi\frac{\partial}{\partial x_n} = \tilde{L} - \sum_{i=1}^r x_n \tilde{a}\tilde{X}_i^2 - \tilde{c}$$

we have

$$\left\| x_n^{q-1} \frac{\partial u}{\partial x_j} \right\|^2_{\sigma_{q-1}-1} = \mathrm{Re}\left( x_n^{q-1} \frac{\partial u}{\partial x_j}, E_{2s-2}\chi x_n^{q-1} \frac{\partial u}{\partial x_j} \right)$$

$$\leq \frac{1}{q}\mathrm{Re}\left([\tilde{L}, R]u, T_{2s-1}u\right) - \frac{1}{q}\sum_{i=1}^{r} \mathrm{Re}\left([x_n \tilde{a}\tilde{X}_i^2, R]u, T_{2s-1}u\right)$$

$$+ C_2\left(\|u\|_0^2 + \|u\|_{2s-1}^2\right). \tag{5.37}$$

We estimate the first term in the right-hand side of (5.37). We have by (5.35)

$$\mathrm{Re}\left([\tilde{L}, R]u, T_{2s-1}u\right) = \mathrm{Re}\left(\tilde{L}Ru, T_{2s-1}u\right) - \mathrm{Re}\left(\tilde{L}u, R^* T_{2s-1}u\right)$$

$$\leq \mathrm{Re}\left(\tilde{L}Ru, T_{2s-1}u\right) + C_3\left(\|\tilde{L}u\|_0^2 + \|u\|_0^2\right). \tag{5.38}$$

It remains to estimate the first term in the right-hand side of (5.38). By using (4.13) we can write with some $\beta \in C_0^\infty(R^n)$

$$\tilde{L}^* = \sum_{i=1}^{r} x_n \tilde{a}\tilde{X}_i^2 + \sum_{i=1}^{r} \alpha_i \tilde{X}_i - \chi \frac{\partial}{\partial x_n} + \beta, \tag{5.39}$$

where

$$\alpha_i = 2\tilde{X}_i(x_n \tilde{a}) - 2f_i x_n \tilde{a}, \quad 1 \leq i \leq r. \tag{5.40}$$

Hence we have

$$\left(\tilde{L}Ru, T_{2s-1}u\right) = \sum_{i=1}^{r} \left(Ru, x_n \tilde{a}\tilde{X}_i^2 T_{2s-1}u\right) + \sum_{i=1}^{r} \left(Ru, \alpha_i \tilde{X}_i T_{2s-1}u\right)$$

$$- \left(Ru, T_{2s-1}\chi \frac{\partial}{\partial x_n} u\right) - \left(Ru, \left[\chi \frac{\partial}{\partial x_n}, T_{2s-1}\right] u\right) + (Ru, \beta T_{2s-1}u).$$

We apply the relation: $\chi\partial/\partial x_n = \tilde{L} - \sum_{i=1}^r x_n \tilde{a}\tilde{X}_i^2 - \tilde{c}$ to the third term in the right-hand side of the above equation and we rewrite $(\tilde{L}Ru, T_{2s-1}u)$ as follows.

$$\left(\tilde{L}Ru, T_{2s-1}u\right) = 2\sum_{i=1}^{r} \left(Ru, T_{2s-1} x_n \tilde{a}\tilde{X}_i^2 u\right) + \sum_{i=1}^{r} \left(Ru, [x_n \tilde{a}\tilde{X}_i^2, T_{2s-1}]u\right)$$

$$+ \sum_{i=1}^{r} \left(Ru, T_{2s-1}\alpha_i \tilde{X}_i u\right) - \left(Ru, T_{2s-1}\tilde{L}u\right) + (Ru, A_{2s-1}u), \tag{5.41}$$

where $A_{2s-1} \in \mathrm{Op}(S^{2s-1})$. We have

$$\sum_{i=1}^{r} \big|(Ru, T_{2s-1}\alpha_i \tilde{X}_i u)\big| + \big|(Ru, T_{2s-1}\tilde{L}u)\big| + \big|(Ru, A_{2s-1}u)\big| \\ \le C_4\bigg(\|Ru\|_{2s-1}^2 + \sum_{i=1}^{r} \|\alpha_i \tilde{X}_i u\|_0^2 + \|\tilde{L}u\|_0^2 + \|u\|_0^2\bigg). \tag{5.42}$$

It follows from (5.40) and (4.7) that there exists a constant $M' > 0$ such that $|\alpha_i(x)|^2 \le M' x_n^2 \tilde{a}(x)$, $x \in R^n$ $(1 \le i \le r)$. Hence we have by Corollary 4.1

$$\sum_{i=1}^{r} \big\|\alpha_i \tilde{X}_i u\big\|_0^2 \le C_5\big(\|\tilde{L}u\|_0^2 + \|u\|_0^2\big). \tag{5.43}$$

Combining (5.41)–(5.43) we have

$$\operatorname{Re}\big(\tilde{L}Ru, T_{2s-1}u\big) \le 2\sum_{i=1}^{r} \operatorname{Re}\big(Ru, T_{2s-1}x_n\tilde{a}\tilde{X}_i^2 u\big) + \sum_{i=1}^{r} \operatorname{Re}\big(Ru, [x_n\tilde{a}\tilde{X}_i^2, T_{2s-1}]u\big) \\ + C_6\big(\|\tilde{L}u\|_0^2 + \|u\|_0^2 + \|Ru\|_{2s-1}^2\big). \tag{5.44}$$

We estimate the first term in the right-hand side of (5.44). Since $j \ne n$, it follows from (5.36) and (5.32) that $T_{2s-1}x_n\tilde{a}\tilde{X}_i^2 = E_{2s-2}R\tilde{a}\tilde{X}_i^2$ $(1 \le i \le r)$. Then we have by (4.12)

$$\begin{aligned} &\big(Ru, T_{2s-1}x_n\tilde{a}\tilde{X}_i^2 u\big) \\ &\quad = \big(E_{s-1}Ru, \tilde{a}\tilde{X}_i^2 E_{s-1}Ru\big) + \big(E_{s-1}Ru, [E_{s-1}R, \tilde{a}\tilde{X}_i^2]u\big) \\ &\quad = -\big(\tilde{X}_i E_{s-1}Ru, \tilde{a}\tilde{X}_i E_{s-1}Ru\big) + \big(E_{s-1}Ru, f_i\tilde{a}\tilde{X}_i E_{s-1}Ru\big) \\ &\qquad -\big(E_{s-1}Ru, \tilde{X}_i(\tilde{a})\tilde{X}_i E_{s-1}Ru\big) + \big(E_{s-1}Ru, [E_{s-1}R, \tilde{a}\tilde{X}_i^2]u\big). \end{aligned} \tag{5.45}$$

We note that

$$-\big(\tilde{X}_i E_{s-1}Ru, \tilde{a}\tilde{X}_i E_{s-1}Ru\big) \le 0. \tag{5.46}$$

Since $R = \chi x_n^q \partial/\partial x_j$ by (5.32) and $q$ is a positive integer, there exist $F_s^{(i)}$, $G_s^{(i,k)}$, $H_s^{(i)} \in \operatorname{Op}(S^s)$ $(1 \le i \le r,\ 1 \le k \le n)$ such that

$$f_i\tilde{a}\tilde{X}_i E_{s-1}R - \tilde{X}_i(\tilde{a})\tilde{X}_i E_{s-1}R + \big[E_{s-1}R, \tilde{a}\tilde{X}_i^2\big] \\ \equiv F_s^{(i)} x_n\tilde{a}\tilde{X}_i + \sum_{k=1}^{n} G_s^{(i,k)} x_n\tilde{a}_{x_k}\tilde{X}_i + H_s^{(i)}\tilde{X}_i(x_n\tilde{a})\tilde{X}_i \pmod{\operatorname{Op}(S^s)}. \tag{5.47}$$

It follows from (5.45)–(5.47) and Corollary 4.2 that

$$\operatorname{Re}\big(Ru, T_{2s-1}x_n\tilde{a}\tilde{X}_i^2 u\big) \le C_7\big(\|\tilde{L}u\|_0^2 + \|u\|_0^2 + \|Ru\|_{2s-1}^2\big). \tag{5.48}$$

We estimate the second term in the right-hand side of (5.44). In the same way as the proof of Lemma 5.2 we can write

$$\big[x_n\tilde{a}\tilde{X}_i^2, T_{2s-1}\big] = J_{2s-1}^{(i)} x_n\tilde{a}\tilde{X}_i + \sum_{k=1}^{n} K_{2s-1}^{(i,k)} x_n\tilde{a}_{x_k}\tilde{X}_i + M_{2s-2}\tilde{a}\tilde{X}_i^2 + N_{2s-1}^{(i)},$$

where $J_{2s-1}^{(i)}$, $K_{2s-1}^{(i,k)}$, $N_{2s-1}^{(i)} \in \mathrm{Op}(S^{2s-1})$ $(1 \le i \le r,\ 1 \le k \le n)$, and $M_{2s-2} \in \mathrm{Op}(S^{2s-2})$. Hence it follows from Corollary 4.2 that

$$\begin{aligned}&\mathrm{Re}\,\big(Ru, [x_n\tilde{a}\tilde{X}_i^2, T_{2s-1}]u\big)\\ &\quad \le C_8\big(\|\tilde{L}u\|_0^2 + \|u\|_0^2 + \|Ru\|_{2s-1}^2\big) + \mathrm{Re}\,\big(Ru, M_{2s-2}\tilde{a}\tilde{X}_i^2 u\big).\end{aligned} \tag{5.49}$$

Let $\varepsilon_0$ be the positive number determined in Proposition 4.2 with $s = 0$, and let $\psi$ be a function belonging to $C_0^\infty(B(p,\rho_0))$ such that $\psi(x) = 1$ in $B\big(p, \frac{1}{2}\rho_0\big)$. We put $P_{2s-1} = M_{2s-2}E_1^{(\varepsilon_0)}$. Then $P_{2s-1} \in \mathrm{Op}(S^{2s-1})$ and we can write with some $Q_{2s-1}^{(i)} \in \mathrm{Op}(S^{2s-1})$

$$M_{2s-2}\tilde{a}\tilde{X}_i^2 u = M_{2s-2}\psi\tilde{a}\tilde{X}_i^2 u = -P_{2s-1}\psi\tilde{X}_i^* E_{-\frac{1}{2}}^{(\varepsilon_0)}\tilde{a}E_{-\frac{1}{2}}^{(\varepsilon_0)}\tilde{X}_i u + Q_{2s-1}^{(i)}u.$$

Hence we have

$$\begin{aligned}\mathrm{Re}\,\big(Ru, M_{2s-2}\tilde{a}\tilde{X}_i^2 u\big) &\le \Big(\tilde{a}E_{-\frac{1}{2}}^{(\varepsilon_0)}\tilde{X}_i\psi P_{2s-1}^* Ru, E_{-\frac{1}{2}}^{(\varepsilon_0)}\tilde{X}_i\psi P_{2s-1}^* Ru\Big)\\ &\quad + \Big(\tilde{a}E_{-\frac{1}{2}}^{(\varepsilon_0)}\tilde{X}_i u, E_{-\frac{1}{2}}^{(\varepsilon_0)}\tilde{X}_i u\Big) + C_9\big(\|u\|_0^2 + \|Ru\|_{2s-1}^2\big).\end{aligned} \tag{5.50}$$

We apply Proposition 4.2 to the first two terms in the right-hand side of (5.50). Then we have

$$\begin{aligned}&\mathrm{Re}\,\big(Ru, M_{2s-2}\tilde{a}\tilde{X}_i^2 u\big)\\ &\quad \le -2\mathrm{Re}\,\big(x_n\tilde{L}\psi P_{2s-1}^* Ru, \psi P_{2s-1}^* Ru\big) + 2\mathrm{Re}\,\Big(\tilde{L}\psi P_{2s-1}^* Ru, \frac{\partial}{\partial x_n}\psi P_{2s-1}^* Ru\Big)_{-\frac{1}{2},\varepsilon_0}\\ &\qquad + C_{10}\big(\|\tilde{L}u\|_0^2 + \|u\|_0^2 + \|Ru\|_{2s-1}^2\big)\\ &\quad = \mathrm{Re}\,\big(\tilde{L}U_{2s-1}Ru, V_{2s-1}Ru\big) + C_{10}\big(\|\tilde{L}u\|_0^2 + \|u\|_0^2 + \|Ru\|_{2s-1}^2\big),\end{aligned} \tag{5.51}$$

where $U_{2s-1} = \psi P_{2s-1}^*$, and $V_{2s-1} = -2x_n\psi P_{2s-1}^* + 2E_{-1}^{(\varepsilon_0)}\frac{\partial}{\partial x_n}\psi P_{2s-1}^*$. Then $U_{2s-1}$, $V_{2s-1} \in \mathrm{Op}(S^{2s-1})$ and it follows that

$$\big(\tilde{L}U_{2s-1}Ru, V_{2s-1}Ru\big) = \big(\tilde{L}u, (U_{2s-1}R)^* V_{2s-1}Ru\big) + \big([\tilde{L}, U_{2s-1}R]u, V_{2s-1}Ru\big).$$

Let $U_{2s-1}(x,\xi)$ be the symbol of $U_{2s-1}$. Since the principal symbol of $U_{2s-1}R$ is equal to $\sqrt{-1}\chi(x)x_n^q\xi_j U_{2s-1}(x,\xi)$ and $q$ is a positive integer, it follows that

$$\big[\tilde{L}, U_{2s-1}R\big] = \sum_{i=1}^{r} W_{2s}^{(i)} x_n \tilde{a} \tilde{X}_i + \sum_{i=1}^{r}\sum_{k=1}^{n} W_{2s}^{(i,k)} x_n \tilde{a}_{x_k} \tilde{X}_i + X_{2s},$$

where $W_{2s}^{(i)}$, $W_{2s}^{(i,k)}$, $X_{2s} \in \mathrm{Op}(S^{2s})$ $(1 \le i \le r,\ 1 \le k \le n)$. Hence we have by Corollary 4.2

$$\big|(\tilde{L}U_{2s-1}Ru, V_{2s-1}Ru)\big| \le C_{11}\big(\|\tilde{L}u\|_0^2 + \|u\|_0^2 + \|Ru\|_{4s-1}^2\big). \tag{5.52}$$

It follows from (5.49), (5.51) and (5.52) that

$$\sum_{i=1}^{r} \mathrm{Re}\,\big(Ru, [x_n\tilde{a}\tilde{X}_i^2, T_{2s-1}]u\big) \le C_{12}\big(\|\tilde{L}u\|_0^2 + \|u\|_0^2 + \|Ru\|_{2s-1}^2 + \|Ru\|_{4s-1}^2\big). \tag{5.53}$$

In virtue of (5.38), (5.44), (5.48) and (5.53) we obtain

$$\frac{1}{q}\,\mathrm{Re}\,\big([\tilde{L}, R]u, T_{2s-1}u\big) \le C_{13}\big(\|\tilde{L}u\|_0^2 + \|u\|_0^2 + \|Ru\|_{2s-1}^2 + \|Ru\|_{4s-1}^2\big). \tag{5.54}$$

We estimate the second term in the right-hand side of (5.37). By definition (5.36) of $T_{2s-1}$ we can write

$$T_{2s-1}^* = -T_{2s-1} + Y_{2s-2}, \quad Y_{2s-2} \in \mathrm{Op}(S^{2s-2}). \tag{5.55}$$

We write $R^* = -R + f$, $f \in C_0^\infty(R^n)$. Then we have by (4.13)

$$\big[x_n\tilde{a}\tilde{X}_i^2, R\big]^* = \big[x_n\tilde{a}\tilde{X}_i^2, R\big] + \big[\alpha_i\tilde{X}_i, R\big] + \big[f, x_n\tilde{a}\tilde{X}_i^2\big] + h_i, \quad 1 \le i \le r, \tag{5.56}$$

where $\alpha_i$ is the function defined by (5.40), and $h_i \in C_0^\infty(R^n)$. It follows from (5.55) and (5.56) that

$$2\mathrm{Re}\,\big([x_n\tilde{a}\tilde{X}_i^2, R]u, T_{2s-1}u\big) = J_1^{(i)} + J_2^{(i)}, \quad u \in C_0^\infty(R^n),\ 1 \le i \le r, \tag{5.57}$$

where

$$J_1^{(i)} = \big(Y_{2s-2}u, [x_n\tilde{a}\tilde{X}_i^2, R]u\big) + \big(u, [[x_n\tilde{a}\tilde{X}_i^2, R], T_{2s-1}]u\big), \tag{5.58}$$

$$J_2^{(i)} = \big(u, [\alpha_i\tilde{X}_i, R]T_{2s-1}u\big) + \big(u, [f, x_n\tilde{a}\tilde{X}_i^2]T_{2s-1}u\big) + (u, h_iT_{2s-1}u). \tag{5.59}$$

We fix $i$ $(1 \le i \le r)$ and estimate $J_1^{(i)}$. We recall that $R = \chi x_n^q \partial/\partial x_j$ by (5.32), $q$ is a positive integer, and $\tilde{X}_i(x_n)\tilde{a}(x)|_{x_n=0} = 0$ by (4.7). Then we have

$$\big[x_n\tilde{a}\tilde{X}_i^2, R\big] = L_1^{(i)} x_n^{q+1}\tilde{X}_i + M_1^{(i)}, \tag{5.60}$$

where $L_1^{(i)}$ and $M_1^{(i)}$ are first order linear partial differential operators with coefficients belonging to $C_0^\infty(R^n)$. Hence there exist $B_{2s-1}^{(i)}$, $C_{2s-1}^{(i)}$, $D_{2s-1}^{(i)} \in \mathrm{Op}(S^{2s-1})$ such that

$$\big[[x_n\tilde{a}\tilde{X}_i^2, R], T_{2s-1}\big] = B_{2s-1}^{(i)} x_n^{q+1}\tilde{X}_i + C_{2s-1}^{(i)} x_n^q L_1^{(i)} + D_{2s-1}^{(i)}. \tag{5.61}$$

It follows from (5.58), (5.60) and (5.61) that

$$\big|J_1^{(i)}\big| \le C_{14}\Big(\big\|x_n^q\tilde{X}_i u\big\|_{2s-1}^2 + \big\|x_n^q L_1^{(i)} u\big\|_{2s-1}^2 + \|u\|_0^2 + \|u\|_{2s-1}^2\Big).$$

Since $2s-1 < 4s-1 = \sigma_q - 1$ by (5.32) and (5.29), it follows from hypothesis (5.30) of induction that

$$\big|J_1^{(i)}\big| \le C_{15}\big(\|\tilde{L}u\|_0^2 + \|u\|_0^2 + \|u\|_{2s-1}^2\big). \tag{5.62}$$

We fix $i$ $(1 \le i \le r)$ and estimate $J_2^{(i)}$. We can write with some $p_i, q_i \in C_0^\infty(R^n)$

$$\big[f, x_n\tilde{a}\tilde{X}_i^2\big] = p_i x_n\tilde{a}\tilde{X}_i + q_i.$$

Hence it follows from (5.59) that

$$\begin{aligned} J_2^{(i)} &= \big((\alpha_i\tilde{X}_i)^*u, T_{2s-1}Ru\big) + \big((\alpha_i\tilde{X}_i)^*u, [R, T_{2s-1}]u\big) - \big(R^*u, T_{2s-1}\alpha_i\tilde{X}_i u\big) \\ &\quad - \big(R^*u, [\alpha_i\tilde{X}_i, T_{2s-1}]u\big) + \big((p_i x_n\tilde{a}\tilde{X}_i)^*u, T_{2s-1}u\big) + \big(u, (q_i+h_i)T_{2s-1}u\big). \end{aligned}$$

Then we have by (5.43) and Corollary 4.2

$$\begin{aligned} \big|J_2^{(i)}\big| &\le C_{16}\big(\|\alpha_i\tilde{X}_i u\|_0^2 + \|Ru\|_{2s-1}^2 + \|x_n\tilde{a}\tilde{X}_i u\|_0^2 + \|u\|_{2s-1}^2 + \|u\|_0^2\big) \\ &\le C_{17}\big(\|\tilde{L}u\|_0^2 + \|u\|_0^2 + \|u\|_{2s-1}^2 + \|Ru\|_{2s-1}^2\big). \end{aligned} \tag{5.63}$$

Combining (5.57), (5.62) and (5.63) we have

$$-\frac{1}{q}\sum_{i=1}^r \mathrm{Re}\,\big([x_n\tilde{a}\tilde{X}_i^2, R]u, T_{2s-1}u\big) \le C_{18}\big(\|\tilde{L}u\|_0^2 + \|u\|_0^2 + \|u\|_{2s-1}^2 + \|Ru\|_{2s-1}^2\big). \tag{5.64}$$

In virtue of (5.37), (5.54), (5.64), (5.34) and (5.35) we obtain (5.31) for $j = 1, \ldots, n-1$. □

## 6. Estimate of the subelliptic kind in $H_s^{loc}$ space.

In this section we shall fix $p \in \{(x', x_n) \mid |x'| < T,\ x_n = 0\}$ and extend Proposition 5.1 to $H_s^{loc}$ space. To this end we introduce smoothing operators as follows. For every positive number $0 < \kappa < 1$ we define $T^{(\kappa)}$ as the pseudo-differential operator with the symbol:

$$T^{(\kappa)}(\xi) = e^{-|\kappa\xi|^2}, \tag{6.1}$$

and for every positive numbers $0 < \kappa$, $\gamma < 1$ and $0 < \varepsilon \leq 1$ we define $T^{(\kappa,\gamma,\varepsilon)}$ as the pseudo-differential operator with the symbol:

$$T^{(\kappa,\gamma,\varepsilon)}(\xi) = e^{-|\kappa\xi'|^2 - |\kappa\varepsilon\xi_n|^2}\left(1 + |\gamma\xi'|^2 + |\gamma\varepsilon\xi_n|^2\right)^{-\frac{1}{2}}, \tag{6.2}$$

where $\xi' = (\xi_1, \dots, \xi_{n-1})$. Then $T^{(\kappa)} \in \mathrm{Op}(S^{-\infty})$ for any fixed $0 < \kappa < 1$, and $T^{(\kappa,\gamma,\varepsilon)} \in \mathrm{Op}(S^{-\infty})$ for any fixed $0 < \kappa$, $\gamma < 1$ and $0 < \varepsilon \leq 1$. Now we define $T^{(\kappa,\varepsilon)}$ and $E^{(\gamma,\varepsilon)}$ as the pseudo-differential operators with the symbols:

$$T^{(\kappa,\varepsilon)}(\xi) = e^{-|\kappa\xi'|^2 - |\kappa\varepsilon\xi_n|^2}, \tag{6.3}$$

$$E^{(\gamma,\varepsilon)}(\xi) = \left(1 + |\gamma\xi'|^2 + |\gamma\varepsilon\xi_n|^2\right)^{-\frac{1}{2}} \tag{6.4}$$

respectively. Then $T^{(\kappa,\varepsilon)} \in \mathrm{Op}(S^{-\infty})$ and $E^{(\gamma,\varepsilon)} \in \mathrm{Op}(S^{-1})$ for any fixed $0 < \kappa$, $\gamma < 1$ and $0 < \varepsilon \leq 1$, and $T^{(\kappa,\gamma,\varepsilon)}$ is decomposed to

$$T^{(\kappa,\gamma,\varepsilon)} = T^{(\kappa,\varepsilon)}E^{(\gamma,\varepsilon)}. \tag{6.5}$$

Here we summarize well-known facts about uniform boundedness of families of pseudo-differential operators. Let $m$ be a real number and let $\{A_m^{(\lambda)}\}_{\lambda\in\Lambda}$ be a family of pseudo-differential operators belonging to $\mathrm{Op}(S^m)$. We say that the family of the symbols $\{A_m^{(\lambda)}(x,\xi)\}_{\lambda\in\Lambda}$ of $\{A_m^{(\lambda)}\}_{\lambda\in\Lambda}$ is bounded in $S^m$ if for any multi-indices $\alpha = (\alpha_1, \dots, \alpha_n)$ and $\beta = (\beta_1, \dots, \beta_n)$ there exists a constant $C_{\alpha,\beta} > 0$ independent of $\lambda \in \Lambda$ such that

$$\left|\frac{\partial^\alpha}{\partial x^\alpha}\frac{\partial^\beta}{\partial \xi^\beta}A_m^{(\lambda)}(x,\xi)\right| \leq C_{\alpha,\beta}(1 + |\xi|^2)^{\frac{1}{2}(m-|\beta|)}, \quad (x,\xi) \in R^n \times R^n.$$

For real numbers $s$ and $t$, let $B(H_s, H_t)$ be the space of all bounded linear operators from $H_s$ to $H_t$. It is known that if $\{A_m^{(\lambda)}(x,\xi)\}_{\lambda\in\Lambda}$ is bounded in $S^m$, then $\{A_m^{(\lambda)}\}_{\lambda\in\Lambda}$ is uniformly bounded in $B(H_{s+m}, H_s)$ for any fixed $s \in R$. For the proof see Theorem 2.7 in Chapter 3 of [**7**]. In addition, Corollary 2 of Lemma 2.3, Theorem 3.1 and Lemma 2.4 in Chapter 2 of [**7**] imply the following

Lemma 6.1. *For real numbers $m_1$ and $m_2$, let $\{A_{m_1}^{(\lambda)}(x,\xi)\}_{\lambda\in\Lambda}$ and $\{B_{m_2}^{(\mu)}(x,\xi)\}_{\mu\in M}$ be bounded families in $S^{m_1}$ and $S^{m_2}$ respectively. Then the symbol of the operator $A_{m_1}^{(\lambda)}B_{m_2}^{(\mu)}$ is of the form*:

$$A_{m_1}^{(\lambda)}(x,\xi)B_{m_2}^{(\mu)}(x,\xi) + \frac{1}{\sqrt{-1}}\sum_{k=1}^{n}\frac{\partial A_{m_1}^{(\lambda)}(x,\xi)}{\partial \xi_k}\frac{\partial B_{m_2}^{(\mu)}(x,\xi)}{\partial x_k} + C_{m_1+m_2-2}^{(\lambda,\mu)}(x,\xi),$$

*where* $\{C^{(\lambda,\mu)}_{m_1+m_2-2}(x,\xi)\}_{(\lambda,\mu)\in\Lambda\times M}$ *is bounded in* $S^{m_1+m_2-2}$.

The above lemma yields the following corollary.

COROLLARY 6.1. *For real numbers* $m_1$ *and* $m_2$, *let* $\{A^{(\lambda)}_{m_1}(x,\xi)\}_{\lambda\in\Lambda}$ *and* $\{B^{(\mu)}_{m_2}(x,\xi)\}_{\mu\in M}$ *be bounded families in* $S^{m_1}$ *and* $S^{m_2}$ *respectively. Then,* $\{[A^{(\lambda)}_{m_1}, B^{(\mu)}_{m_2}]\}_{(\lambda,\mu)\in\Lambda\times M}$ *is uniformly bounded in* $B(H_{s+m_1+m_2-1}, H_s)$ *for any fixed* $s$.

Concerning the boundedness of the families $\{T^{(\kappa)}(\xi)\}_{0<\kappa<1}$, $\{T^{(\kappa,\varepsilon)}(\xi)\}_{0<\kappa<1}$ and $\{E^{(\gamma,\varepsilon)}(\xi)\}_{0<\gamma<1}$ with fixed $\varepsilon$ $(0<\varepsilon\le 1)$, the following two lemmas hold.

LEMMA 6.2. $\{T^{(\kappa)}(\xi)\}_{0<\kappa<1}$ *is bounded in* $S^0$.

LEMMA 6.3. *Fix any* $\varepsilon$, $0<\varepsilon\le 1$. *Then*

$$\big\{T^{(\kappa,\varepsilon)}(\xi)\big\}_{0<\kappa<1} \text{ is bounded in } S^0, \tag{6.6}$$

$$\big\{\kappa^2 T^{(\kappa,\varepsilon)}(\xi)\big\}_{0<\kappa<1} \text{ is bounded in } S^{-2}, \tag{6.7}$$

$$\big\{E^{(\gamma,\varepsilon)}(\xi)\big\}_{0<\gamma<1} \text{ is bounded in } S^0, \tag{6.8}$$

$$\big\{\gamma E^{(\gamma,\varepsilon)}(\xi)\big\}_{0<\gamma<1} \text{ is bounded in } S^{-1}, \tag{6.9}$$

$$\big\{T^{(\kappa,\gamma,\varepsilon)}(\xi)\big\}_{0<\kappa,\gamma<1} \text{ is bounded in } S^0. \tag{6.10}$$

PROOF. We put $E(\xi)=e^{-|\xi|^2}$. Then $T^{(\kappa,\varepsilon)}(\xi)=E(\kappa\xi',\kappa\varepsilon\xi_n)$, and we have for any multi-index $\alpha=(\alpha_1,\dots,\alpha_{n-1},\alpha_n)$

$$\begin{aligned}
(1+|\xi|^2)^{\frac{|\alpha|}{2}}\left|\frac{\partial^\alpha}{\partial\xi^\alpha}T^{(\kappa,\varepsilon)}(\xi)\right| &= (1+|\xi|^2)^{\frac{|\alpha|}{2}}\kappa^{|\alpha|}\varepsilon^{\alpha_n}\left|\frac{\partial^\alpha E}{\partial\xi^\alpha}(\kappa\xi',\kappa\varepsilon\xi_n)\right| \\
&\le (1+|\kappa\xi|^2)^{\frac{|\alpha|}{2}}\left|\frac{\partial^\alpha E}{\partial\xi^\alpha}(\kappa\xi',\kappa\varepsilon\xi_n)\right| \\
&\le \varepsilon^{-|\alpha|}\big(1+|\kappa\xi'|^2+|\kappa\varepsilon\xi_n|^2\big)^{\frac{|\alpha|}{2}}\left|\frac{\partial^\alpha E}{\partial\xi^\alpha}(\kappa\xi',\kappa\varepsilon\xi_n)\right| \\
&\le \varepsilon^{-|\alpha|}C_\alpha, \quad \xi\in R^n,
\end{aligned}$$

where $C_\alpha$ is a positive constant depending only on $\alpha$. Thus (6.6) has been proved. In a similar way we can prove (6.7), (6.8) and (6.9). Since $T^{(\kappa,\gamma,\varepsilon)}(\xi)=T^{(\kappa,\varepsilon)}(\xi)E^{(\gamma,\varepsilon)}(\xi)$ by (6.2), (6.3) and (6.4), (6.10) follows from (6.6) and (6.8). □

NOTATION 6.1. For a real number $s$ and a positive number $\delta$ we define

$$H^{loc}_s\big(B(p,\delta);\tilde L\big)=\big\{u\mid u\in H^{loc}_s(B(p,\delta)) \text{ and } \tilde Lu\in H^{loc}_s(B(p,\delta))\big\}.$$

NOTATION 6.2. Let $\phi(x),\ \psi(x)\in C^\infty_0(R^n)$. We write $\phi\subset\subset\psi$ if $0\le\psi(x)\le 1$ and

$\psi(x) = 1$ in a neighborhood of supp $\phi$.

Lemma 6.4. *Let $A(x) \in C_0^\infty(R^n)$ and assume that*

$$|A(x)|^2 \leq K\tilde{a}(x), \quad x \in R^n, \tag{6.11}$$

*where $K > 0$ is a constant. For any real number $s$, any positive number $\delta$, and any $\phi(x)$, $\psi(x) \in C_0^\infty(B(p,\delta))$, $\phi \subset\subset \psi$, there exists a constant $C = C(s,\delta,\phi,\psi) > 0$ such that*

$$\sum_{i=1}^{r} \left\| x_n A\phi\tilde{X}_i u \right\|_s^2 \leq C\big(\|\psi\tilde{L}u\|_s^2 + \|\psi u\|_s^2\big), \quad u \in H_s^{loc}\big(B(p,\delta);\tilde{L}\big).$$

Proof. We shall denote by $C_1, C_2, \ldots$ positive constants depending on $s$, $\delta$, $\phi$ and $\psi$, and independent of $\kappa$ $(0 < \kappa < 1)$, and $u \in H_s^{loc}(B(p,\delta);\tilde{L})$. Let $T^{(\kappa)}$, $0 < \kappa < 1$, be the pseudo-differential operator with the symbol $T^{(\kappa)}(\xi)$ defined by (6.1). Since $\phi\tilde{X}_i u = \phi\tilde{X}_i\psi u$, we have

$$\sum_{i=1}^{r} \left\| x_n A\phi\tilde{X}_i u \right\|_s^2 = \lim_{\kappa \to +0} \sum_{i=1}^{r} \left\| T^{(\kappa)} E_s x_n A\phi\tilde{X}_i\psi u \right\|_0^2. \tag{6.12}$$

It follows from Lemmm 6.2, Corollary 6.1 and (6.11) that

$$\begin{aligned}\sum_{i=1}^{r} \left\| T^{(\kappa)} E_s x_n A\phi\tilde{X}_i\psi u \right\|_0^2 &\leq 2\left\| x_n A E_s\tilde{X}_i\phi T^{(\kappa)}\psi u \right\|_0^2 + C_1\left\|\psi u\right\|_s^2 \\ &\leq 2K\sum_{i=1}^{r}\big(x_n^2\tilde{a}E_s\tilde{X}_i\phi T^{(\kappa)}\psi u, E_s\tilde{X}_i\phi T^{(\kappa)}\psi u\big) + C_1\|\psi u\|_s^2.\end{aligned} \tag{6.13}$$

Since $\phi T^{(\kappa)}\psi u \in C_0^\infty(R^n)$, we can apply Proposition 4.1 to the first term in the right-hand side of (6.13) and we have by Lemma 6.2

$$\begin{aligned}&\sum_{i=1}^{r}\big(x_n^2\tilde{a}E_s\tilde{X}_i\phi T^{(\kappa)}\psi u, E_s\tilde{X}_i\phi T^{(\kappa)}\psi u\big) \\ &\quad\leq -\mathrm{Re}\,\big(x_n\tilde{L}\phi T^{(\kappa)}\psi u, E_{2s}\phi T^{(\kappa)}\psi u\big) + C_s\left\|\phi T^{(\kappa)}\psi u\right\|_s^2 \\ &\quad\leq -\mathrm{Re}\,\big(T^{(\kappa)}\phi x_n\tilde{L}\psi u, E_{2s}\phi T^{(\kappa)}\psi u\big) - \mathrm{Re}\,\big([\phi, T^{(\kappa)}]x_n\tilde{L}\psi u, E_{2s}\phi T^{(\kappa)}\psi u\big) \\ &\qquad -\mathrm{Re}\,\big([x_n\tilde{L}, \phi T^{(\kappa)}]\psi u, E_{2s}\phi T^{(\kappa)}\psi u\big) + C_2\|\psi u\|_s^2.\end{aligned} \tag{6.14}$$

We note that $\phi x_n\tilde{L}\psi u = \phi x_n\psi\tilde{L}u$, because $\psi(x) = 1$ in a neighborhood of supp $\phi$ by hypothesis. Then we have

$$\big|(T^{(\kappa)}\phi x_n\tilde{L}\psi u, E_{2s}\phi T^{(\kappa)}\psi u)\big| \leq C_3\big(\|\psi\tilde{L}u\|_s^2 + \|\psi u\|_s^2\big). \tag{6.15}$$

It follows from Lemma 6.1 and Lemma 6.2 that $[\phi, T^{(\kappa)}] = \sum_{k=1}^n A_{-1}^{(\kappa,k)}\phi_{x_k} + B_{-2}^{(\kappa)}$, where $\{A_{-1}^{(\kappa,k)}(\xi)\}_{0<\kappa<1}$ is bounded in $S^{-1}$ for any $k$ $(1 \le k \le n)$, and $\{B_{-2}^{(\kappa)}(x,\xi)\}_{0<\kappa<1}$ is bounded in $S^{-2}$. Hence, taking into account that $\phi_{x_k}x_n\tilde{L}\psi u = \phi_{x_k}x_n\psi\tilde{L}u$ we have

$$\begin{aligned} &\big|([\phi, T^{(\kappa)}]x_n\tilde{L}\psi u, E_{2s}\phi T^{(\kappa)}\psi u)\big| \\ &\quad \le \sum_{k=1}^n \Big|\big(A_{-1}^{(\kappa,k)}\phi_{x_k}x_n\tilde{L}\psi u, E_{2s}\phi T^{(\kappa)}\psi u\big)\Big| + \Big|\big(B_{-2}^{(\kappa)}x_n\tilde{L}\psi u, E_{2s}\phi T^{(\kappa)}\psi u\big)\Big| \\ &\quad \le C_4\big(\|\psi\tilde{L}u\|_{s-1}^2 + \|\psi u\|_s^2\big). \end{aligned} \tag{6.16}$$

In the same way as the proof of Lemma 5.2 we have by Lemma 6.1 and Lemma 6.2

$$\big[x_n\tilde{L}, \phi T^{(\kappa)}\big] = \sum_{i=1}^r x_n\tilde{a}\tilde{X}_i D_0^{(\kappa,i)} + \sum_{i=1}^r\sum_{k=1}^n x_n\tilde{a}_{x_k}\tilde{X}_i E_0^{(\kappa,i,k)} + F_0^{(\kappa)},$$

where $\{D_0^{(\kappa,i)}(x,\xi)\}_{0<\kappa<1, 1\le i\le r}$, $\{E_0^{(\kappa,i,k)}(x,\xi)\}_{0<\kappa<1,1\le i\le r,1\le k\le n}$ and $\{F_0^{(\kappa)}(x,\xi)\}_{0<\kappa<1}$ are bounded in $S^0$. Hence we have for any $\varepsilon > 0$

$$\begin{aligned} &\big|\big([x_n\tilde{L}, \phi T^{(\kappa)}]\psi u, E_{2s}\phi T^{(\kappa)}\psi u\big)\big| \\ &\le \sum_{i=1}^r \varepsilon\big\|x_n\tilde{a}E_s\tilde{X}_i\phi T^{(\kappa)}\psi u\big\|_0^2 + \sum_{i=1}^r\sum_{k=1}^n \varepsilon\big\|x_n\tilde{a}_{x_k}E_s\tilde{X}_i\phi T^{(\kappa)}\psi u\big\|_0^2 + \left(\frac{1}{\varepsilon}C_5 + C_6\right)\|\psi u\|_s^2. \end{aligned} \tag{6.17}$$

Let $M > 0$ be the constant determined in Lemma 4.1. Now we put $\varepsilon = 1/2M(1+n)$. Then it follows from (6.17) that

$$\begin{aligned} &\big|\big([x_n\tilde{L}, \phi T^{(\kappa)}]\psi u, E_{2s}\phi T^{(\kappa)}\psi u\big)\big| \\ &\quad \le \varepsilon M(1+n)\sum_{i=1}^r \big(x_n^2\tilde{a}E_s\tilde{X}_i\phi T^{(\kappa)}\psi u, E_s\tilde{X}_i\phi T^{(\kappa)}\psi u\big) + C_7\|\psi u\|_s^2 \\ &\quad = \frac{1}{2}\sum_{i=1}^r \big(x_n^2\tilde{a}E_s\tilde{X}_i\phi T^{(\kappa)}\psi u, E_s\tilde{X}_i\phi T^{(\kappa)}\psi u\big) + C_7\|\psi u\|_s^2. \end{aligned} \tag{6.18}$$

In virtue of (6.14), (6.15), (6.16) and (6.18) we obtain

$$\sum_{i=1}^r \big(x_n^2\tilde{a}E_s\tilde{X}_i\phi T^{(\kappa)}\psi u, E_s\tilde{X}_i\phi T^{(\kappa)}\psi u\big) \le C_8\big(\|\psi\tilde{L}u\|_s^2 + \|\psi u\|_s^2\big).$$

Hence we have by (6.13)

$$\sum_{i=1}^r \big\|T^{(\kappa)}E_s x_n A\phi\tilde{X}_i\psi u\big\|_0^2 \le C_9\big(\|\psi\tilde{L}u\|_s^2 + \|\psi u\|_s^2\big). \tag{6.19}$$

Thus the lemma follows from (6.19) and (6.12). □

Corollary 6.2. *Let* $A(x) \in C_0^\infty(R^n)$ *and assume that*

$$|A(x)|^2 \le K\tilde{a}(x), \quad x \in R^n,$$

*where* $K > 0$ *is a constant. For any real number* $s$*, any positive number* $\delta$*, and any* $\phi(x), \psi(x) \in C_0^\infty(B(p,\delta))$*,* $\phi \subset\subset \psi$*, there exists a constant* $C = C(s,\delta,\phi,\psi) > 0$ *such that*

$$\sum_{i=1}^r \big\| x_n A \tilde{X}_i(\phi u)\big\|_s^2 \le C\big(\|\psi \tilde{L} u\|_s^2 + \|\psi u\|_s^2\big), \quad u \in H_s^{loc}\big(B(p,\delta);\tilde{L}\big).$$

Proof. Since $\phi \subset\subset \psi$, it follows that $x_n A \tilde{X}_i(\phi u) = x_n A \phi \tilde{X}_i u + x_n A \tilde{X}_i(\phi)\psi u$. Then the corollary follows from Lemma 6.4. □

Corollary 6.3. *For any real number* $s$*, any positive number* $\delta$*, and any* $\phi(x)$*,* $\psi(x) \in C_0^\infty(B(p,\delta))$*,* $\phi \subset\subset \psi$*, there exists a constant* $C = C(s,\delta,\phi,\psi) > 0$ *such that*

$$\big\|\tilde{L}(\phi u)\big\|_s^2 \le C\big(\|\psi \tilde{L} u\|_s^2 + \|\psi u\|_s^2\big), \quad u \in H_s^{loc}(B(p,\delta);\tilde{L}).$$

Proof. Since $\phi \subset\subset \psi$, it follows that

$$\tilde{L}(\phi u) = \phi\psi\tilde{L}u + 2\sum_{i=1}^r x_n \tilde{a}\tilde{X}_i(\phi)\tilde{X}_i u + \left[\sum_{i=1}^r x_n\tilde{a}\tilde{X}_i^2(\phi) + \chi\frac{\partial\phi}{\partial x_n}\right]\psi u.$$

Then the corollary follows from Lemma 6.4, because $\tilde{X}_i(\phi) \subset\subset \psi$. □

For a real number $s$, let $\rho_s$ be the radius determined in Proposition 4.2. On the other hand, let $M > 0$ and $C > 0$ be the constants determined in Lemma 4.1 and (4.24) respectively. We define another radius $\delta_s$ by

$$\delta_s = \min\big(\rho_s, \rho_{s-\frac{1}{2}}, 2^{-1}\rho_0, 2^{-6}(nr)^{-1}(M+1)^{-\frac{1}{2}}C^{-\frac{1}{2}}\big). \tag{6.20}$$

Then the following Lemma 6.5, Lemma 6.6 and Proposition 6.1 hold.

Lemma 6.5. *For any real number* $s$*, any positive number* $\delta$ $(0 < \delta \le \delta_s)$*, and any* $\phi(x), \psi(x) \in C_0^\infty(B(p,\delta))$*,* $\phi \subset\subset \psi$*, there exists a constant* $C = C(s,\delta,\phi,\psi) > 0$ *such that*

$$\sum_{i=1}^r \big\|\tilde{a}\tilde{X}_i(\phi u)\big\|_{s-\frac{1}{2},\varepsilon_s}^2 + \left\|\frac{\partial}{\partial x_n}(\phi u)\right\|_{s-\frac{1}{2},\varepsilon_s}^2 \le C\big(\|\psi\tilde{L}u\|_s^2 + \|\psi u\|_s^2\big),$$

$$u \in H_s^{loc}\big(B(p,\delta);\tilde{L}\big),$$

*where $\varepsilon_s$ is the positive number determined in Proposition* 4.2.

PROOF. We shall denote by $C_1^{(\varepsilon)}, C_2^{(\varepsilon)}, \ldots$ positive constants depending on $\varepsilon$ ($0 < \varepsilon \le 1$), $s$, $\delta$, $\phi$ and $\psi$, and independent of $\kappa, \gamma$ ($0 < \kappa, \gamma < 1$), and $u \in H_s^{loc}(B(p,\delta); \tilde{L})$. Since $\{T^{(\kappa,\gamma,\varepsilon)}(\xi)\}_{0<\kappa,\gamma<1}$ is bounded in $S^0$ for any $\varepsilon$ ($0 < \varepsilon \le 1$) by (6.10), it follows from Corollary 6.1 and (4.6) that

$$\begin{aligned}
&\sum_{i=1}^{r} \left\| T^{(\kappa,\gamma,\varepsilon)} \tilde{a} \tilde{X}_i(\phi u) \right\|_{s-\frac{1}{2},\varepsilon_s}^2 + \left\| T^{(\kappa,\gamma,\varepsilon)} \frac{\partial}{\partial x_n}(\phi u) \right\|_{s-\frac{1}{2},\varepsilon_s}^2 \\
&\quad = \sum_{i=1}^{r} \left\| E_{s-\frac{1}{2}}^{(\varepsilon_s)} T^{(\kappa,\gamma,\varepsilon)} \tilde{a} \psi \tilde{X}_i(\phi u) \right\|_0^2 + \left\| T^{(\kappa,\gamma,\varepsilon)} \psi \frac{\partial}{\partial x_n}(\phi u) \right\|_{s-\frac{1}{2},\varepsilon_s}^2 \\
&\quad \le 2 \sum_{i=1}^{r} \left\| \tilde{a} E_{s-\frac{1}{2}}^{(\varepsilon_s)} \tilde{X}_i \psi T^{(\kappa,\gamma,\varepsilon)}(\phi u) \right\|_0^2 + 2 \left\| \frac{\partial}{\partial x_n} \psi T^{(\kappa,\gamma,\varepsilon)}(\phi u) \right\|_{s-\frac{1}{2},\varepsilon_s}^2 + C_1^{(\varepsilon)} \|\phi u\|_{s-\frac{1}{2}}^2 \\
&\quad \le 2M \sum_{i=1}^{r} \left( \tilde{a} E_{s-\frac{1}{2}}^{(\varepsilon_s)} \tilde{X}_i \psi T^{(\kappa,\gamma,\varepsilon)}(\phi u), E_{s-\frac{1}{2}}^{(\varepsilon_s)} \tilde{X}_i \psi T^{(\kappa,\gamma,\varepsilon)}(\phi u) \right) \\
&\qquad + 2 \left\| \frac{\partial}{\partial x_n} \psi T^{(\kappa,\gamma,\varepsilon)}(\phi u) \right\|_{s-\frac{1}{2},\varepsilon_s}^2 + C_1^{(\varepsilon)} \|\phi u\|_{s-\frac{1}{2}}^2. \qquad (6.21)
\end{aligned}$$

Now we put

$$w = \psi T^{(\kappa,\gamma,\varepsilon)}(\phi u). \qquad (6.22)$$

Since $w \in C_0^\infty(B(p,\delta)) \subset C_0^\infty(B(p,\delta_s)) \subset C_0^\infty(B(p,\rho_s))$ by (6.20), it follows from Proposition 4.2, (6.22) and (6.10) that

$$\begin{aligned}
&\sum_{i=1}^{r} \left( \tilde{a} E_{s-\frac{1}{2}}^{(\varepsilon_s)} \tilde{X}_i w, E_{s-\frac{1}{2}}^{(\varepsilon_s)} \tilde{X}_i w \right) + \left\| \frac{\partial}{\partial x_n} w \right\|_{s-\frac{1}{2},\varepsilon_s}^2 \\
&\quad \le -2\mathrm{Re} \left( x_n \tilde{L} w, w \right)_s + 2\mathrm{Re} \left( \tilde{L} w, \frac{\partial}{\partial x_n} w \right)_{s-\frac{1}{2},\varepsilon_s} + C_2^{(\varepsilon)} \|\phi u\|_s^2 \\
&\quad \equiv I_1 + I_2 + C_2^{(\varepsilon)} \|\phi u\|_s^2. \qquad (6.23)
\end{aligned}$$

First we estimate $I_1$. In virtue of Lemma 5.2 and Lemma 6.1, we can write

$$\left[ x_n \tilde{L}, \psi T^{(\kappa,\gamma,\varepsilon)} \right] = \sum_{i=1}^{r} C_0^{(\kappa,\gamma,\varepsilon,i)} x_n \tilde{a} \tilde{X}_i + \sum_{i=1}^{r} \sum_{k=1}^{n} D_0^{(\kappa,\gamma,\varepsilon,i,k)} x_n \tilde{a}_{x_k} \tilde{X}_i + E_0^{(\kappa,\gamma,\varepsilon)},$$

where $\{C_0^{(\kappa,\gamma,\varepsilon,i)}(x,\xi)\}_{0<\kappa,\gamma<1}$, $\{D_0^{(\kappa,\gamma,\varepsilon,i,k)}(x,\xi)\}_{0<\kappa,\gamma<1}$ and $\{E_0^{(\kappa,\gamma,\varepsilon)}(x,\xi)\}_{0<\kappa,\gamma<1}$ are bounded in $S^0$ for any fixed $\varepsilon$ ($0 < \varepsilon \le 1$), $i$ ($1 \le i \le r$) and $k$ ($1 \le k \le n$). Hence we have by (6.22), (6.10), Corollary 6.2 and Corollary 6.3

$$
\begin{aligned}
|I_1| &\le 2\big|(x_n\tilde{L}w, w)_s\big| \\
&\le 2\big|(\psi T^{(\kappa,\gamma,\varepsilon)}x_n\tilde{L}(\phi u), w)_s\big| + 2\big|([x_n\tilde{L}, \psi T^{(\kappa,\gamma,\varepsilon)}](\phi u), w)_s\big| \\
&\le C_3^{(\varepsilon)}\big(\|\psi\tilde{L}u\|_s^2 + \|\psi u\|_s^2 + \|\phi u\|_s^2\big). \qquad (6.24)
\end{aligned}
$$

Next we estimate $I_2$. It follows from Lemma 6.1 and (6.10) that

$$
\begin{aligned}
\big[\tilde{L}, \psi T^{(\kappa,\gamma,\varepsilon)}\big] = &\sum_{i=1}^{r} F_0^{(\kappa,\gamma,\varepsilon,i)} x_n \tilde{a}\tilde{X}_i + \sum_{i=1}^{r}\sum_{k=1}^{n} G_0^{(\kappa,\gamma,\varepsilon,i,k)} x_n \tilde{a}_{x_k}\tilde{X}_i \\
&+ \sqrt{-1}\psi P_2 T_{\xi_n}^{(\kappa,\gamma,\varepsilon)} + H_0^{(\kappa,\gamma,\varepsilon)}, \qquad (6.25)
\end{aligned}
$$

where $P_2$ is the principal part of $\tilde{a}\sum_{i=1}^{r}\tilde{X}_i^2$, $T_{\xi_n}^{(\kappa,\gamma,\varepsilon)}$ is the pseudo-differential operator with the symbol $(T^{(\kappa,\gamma,\varepsilon)}(\xi))_{\xi_n}$, and $\{F_0^{(\kappa,\gamma,\varepsilon,i)}(x,\xi)\}_{0<\kappa,\gamma<1}$, $\{G_0^{(\kappa,\gamma,\varepsilon,i,k)}(x,\xi)\}_{0<\kappa,\gamma<1}$, and $\{H_0^{(\kappa,\gamma,\varepsilon)}(x,\xi)\}_{0<\kappa,\gamma<1}$ are bounded in $S^0$ for any fixed $\varepsilon$ $(0<\varepsilon\le 1)$, $i$ $(1\le i\le r)$ and $k$ $(1\le k\le n)$. Hence, taking (6.22) into account we have by Crollary 6.2 and Corollary 6.3

$$
\begin{aligned}
|I_2| = &2\bigg|\mathrm{Re}\left(E_s^{(\varepsilon_s)}\psi T^{(\kappa,\gamma,\varepsilon)}\tilde{L}(\phi u), E_{s-1}^{(\varepsilon_s)}\frac{\partial}{\partial x_n}w\right) \\
&+ \mathrm{Re}\left(E_s^{(\varepsilon_s)}[\tilde{L}, \psi T^{(\kappa,\gamma,\varepsilon)}](\phi u), E_{s-1}^{(\varepsilon_s)}\frac{\partial}{\partial x_n}w\right)\bigg| \\
\le &C_4^{(\varepsilon)}\big(\|\psi\tilde{L}u\|_s^2 + \|\psi u\|_s^2 + \|\phi u\|_s^2 + \|w\|_s^2\big) + 2\left|\left(\psi P_2 T_{\xi_n}^{(\kappa,\gamma,\varepsilon)}(\phi u), \frac{\partial}{\partial x_n}w\right)_{s-\frac{1}{2},\varepsilon_s}\right| \\
\le &C_5^{(\varepsilon)}\big(\|\psi\tilde{L}u\|_s^2 + \|\psi u\|_s^2 + \|\phi u\|_s^2\big) + 8\left\|\psi P_2 T_{\xi_n}^{(\kappa,\gamma,\varepsilon)} E_{s-\frac{1}{2}}^{(\varepsilon_s)}(\phi u)\right\|_0^2 + \frac{1}{2}\left\|\frac{\partial}{\partial x_n}w\right\|_{s-\frac{1}{2},\varepsilon_s}^2. \qquad (6.26)
\end{aligned}
$$

Combining (6.23), (6.24) and (6.26) we have

$$
\begin{aligned}
&\sum_{i=1}^{r}\left(\tilde{a}E_{s-\frac{1}{2}}^{(\varepsilon_s)}\tilde{X}_i w, E_{s-\frac{1}{2}}^{(\varepsilon_s)}\tilde{X}_i w\right) + \frac{1}{2}\left\|\frac{\partial}{\partial x_n}w\right\|_{s-\frac{1}{2},\varepsilon_s}^2 \\
&\quad\le C_6^{(\varepsilon)}\big(\|\psi\tilde{L}u\|_s^2 + \|\psi u\|_s^2 + \|\phi u\|_s^2\big) + 8\left\|\psi P_2 T_{\xi_n}^{(\kappa,\gamma,\varepsilon)} E_{s-\frac{1}{2}}^{(\varepsilon_s)}(\phi u)\right\|_0^2. \qquad (6.27)
\end{aligned}
$$

We estimate the last term in the right-hand side of (6.27). Since $P_2$ is the principal part of $\tilde{a}\sum_{i=1}^{r}\tilde{X}_i^2$, we see from (4.17) that $P_2 = \tilde{a}\sum_{i=1}^{r}\sum_{k,l=1}^{n}\tilde{c}_{ik}\tilde{c}_{il}\partial^2/\partial x_k\partial x_l$. Hence

$$\left\|\psi P_2 T^{(\kappa,\gamma,\varepsilon)}_{\xi_n} E^{(\varepsilon_s)}_{s-\frac{1}{2}}(\phi u)\right\|_0^2 \le \frac{n^2 r}{(2\pi)^n}\sum_{i=1}^r \sum_{k,l=1}^n \left\{\sup_{x\in R^n}|\psi(x)\tilde{a}(x)\tilde{c}_{ik}(x)\tilde{c}_{il}(x)|^2\right\}$$
$$\times \int \left|\xi_k\xi_l (T^{(\kappa,\gamma,\varepsilon)}(\xi))_{\xi_n} E^{(\varepsilon_s)}_{s-\frac{1}{2}}(\xi)\widehat{\phi u}(\xi)\right|^2 d\xi.$$

Here we have by (6.2) and (6.4)

$$\begin{aligned}\left|(T^{(\kappa,\gamma,\varepsilon)}(\xi))_{\xi_n}\right| &= \left(2\kappa^2 + \frac{1}{\gamma^{-2}+|\xi'|^2+|\varepsilon\xi_n|^2}\right)e^{-|\kappa\xi'|^2-|\kappa\varepsilon\xi_n|^2}\varepsilon^2|\xi_n|E^{(\gamma,\varepsilon)}(\xi)\\ &\le \left\{2(1+|\kappa\xi'|^2+|\kappa\varepsilon\xi_n|^2)+1\right\}e^{-|\kappa\xi'|^2-|\kappa\varepsilon\xi_n|^2}E^{(\varepsilon)}_{-2}(\xi)\varepsilon^2|\xi_n|E^{(\gamma,\varepsilon)}(\xi)\\ &\le 3E^{(\varepsilon)}_{-2}(\xi)\varepsilon^2|\xi_n|E^{(\gamma,\varepsilon)}(\xi).\end{aligned}$$

We note that $p \in \{(x', x_n) \mid x_n = 0\}$, $\psi(x) \in C_0^\infty(B(p,\delta))$ and $0 \le \psi(x) \le 1$ in $R^n$. Then, using the same method as the proof of (4.24) we have

$$\left\|\psi P_2 T^{(\kappa,\gamma,\varepsilon)}_{\xi_n} E^{(\varepsilon_s)}_{s-\frac{1}{2}}(\phi u)\right\|_0^2 \le n^2r^2C(\delta^2+\varepsilon^4)\left\|E^{(\gamma,\varepsilon)}\frac{\partial}{\partial x_n}(\phi u)\right\|^2_{s-\frac{1}{2},\varepsilon_s}, \tag{6.28}$$

where $C$ is the positive constant determined in (4.24). Here we note that

$$\left\|E^{(\gamma,\varepsilon)}\frac{\partial}{\partial x_n}(\phi u)\right\|_{s-\frac{1}{2},\varepsilon_s} < \infty, \tag{6.29}$$

because $\phi u \in H_s$ and $E^{(\gamma,\varepsilon)}\frac{\partial}{\partial x_n} \in \mathrm{Op}(S^0)$ by (6.4).

In virtue of (6.21), (6.22), (6.27) and (6.28) we obtain

$$\begin{aligned}&\sum_{i=1}^r \left\|T^{(\kappa,\gamma,\varepsilon)}\tilde{a}\tilde{X}_i(\phi u)\right\|^2_{s-\frac{1}{2},\varepsilon_s} + \left\|T^{(\kappa,\gamma,\varepsilon)}\frac{\partial}{\partial x_n}(\phi u)\right\|^2_{s-\frac{1}{2},\varepsilon_s}\\ &\quad\le C_7^{(\varepsilon)}\left(\|\psi\tilde{L}u\|_s^2 + \|\psi u\|_s^2 + \|\phi u\|_s^2\right)\\ &\qquad + 32(M+1)n^2r^2C(\delta^2+\varepsilon^4)\left\|E^{(\gamma,\varepsilon)}\frac{\partial}{\partial x_n}(\phi u)\right\|^2_{s-\frac{1}{2},\varepsilon_s}.\end{aligned} \tag{6.30}$$

Let $\kappa \to +0$ in (6.30). Then we have by (6.2) and (6.4)

$$\begin{aligned}&\sum_{i=1}^r \left\|E^{(\gamma,\varepsilon)}\tilde{a}\tilde{X}_i(\phi u)\right\|^2_{s-\frac{1}{2},\varepsilon_s} + \left\|E^{(\gamma,\varepsilon)}\frac{\partial}{\partial x_n}(\phi u)\right\|^2_{s-\frac{1}{2},\varepsilon_s}\\ &\quad\le C_7^{(\varepsilon)}\left(\|\psi\tilde{L}u\|_s^2 + \|\psi u\|_s^2 + \|\phi u\|_s^2\right)\\ &\qquad + 32(M+1)n^2r^2C(\delta^2+\varepsilon^4)\left\|E^{(\gamma,\varepsilon)}\frac{\partial}{\partial x_n}(\phi u)\right\|^2_{s-\frac{1}{2},\varepsilon_s}.\end{aligned} \tag{6.31}$$

We see from (6.20) that $32(M+1)n^2r^2C\delta^2 \le 32(M+1)n^2r^2C\delta_s^2 \le 1/4$. Now we choose $\varepsilon$ $(0<\varepsilon\le 1)$ so small that $32(M+1)n^2r^2C\varepsilon^4 \le 1/4$. Then we have by (6.31) and (6.29)

$$\sum_{i=1}^r \big\|E^{(\gamma,\varepsilon)}\tilde{a}\tilde{X}_i(\phi u)\big\|_{s-\frac12,\varepsilon_s}^2 + \frac12\left\|E^{(\gamma,\varepsilon)}\frac{\partial}{\partial x_n}(\phi u)\right\|_{s-\frac12,\varepsilon_s}^2 \le C_7^{(\varepsilon)}\big(\|\psi\tilde{L}u\|_s^2+\|\psi u\|_s^2+\|\phi u\|_s^2\big).$$

Let $\gamma\to +0$ in the above inequality. Then we have by (6.4)

$$\sum_{i=1}^r \big\|\tilde{a}\tilde{X}_i(\phi u)\big\|_{s-\frac12,\varepsilon_s}^2 + \frac12\left\|\frac{\partial}{\partial x_n}(\phi u)\right\|_{s-\frac12,\varepsilon_s}^2 \le C_7^{(\varepsilon)}\big(\|\psi u\|_s^2+\|\psi\tilde{L}u\|_s^2+\|\phi u\|_s^2\big).$$

Since $\phi u=\phi\psi u$, the lemma follows from the above inequality. □

Lemma 6.6. *For any real number $s$, any positive number $\delta$ $(0<\delta\le\delta_s)$, and any $\phi(x),\psi(x)\in C_0^\infty(B(p,\delta))$, $\phi\subset\subset\psi$, there exists a constant $C=C(s,\delta,\phi,\psi)>0$ such that*

$$\sum_{i=1}^r\left\|\frac{\partial}{\partial x_n}\tilde{a}\tilde{X}_i(\phi u)\right\|_{s-1,\varepsilon_{s-\frac12}}^2 \le C\big(\|\psi\tilde{L}u\|_s^2+\|\psi u\|_s^2\big),\quad u\in H_s^{loc}\big(B(p,\delta);\tilde{L}\big),$$

*where $\varepsilon_{s-\frac12}$ is the positive number determined in Proposition 4.2 with $s$ replaced by $s-\frac12$.*

Proof. It is sufficient to show that for any $i$ $(1\le i\le r)$

$$\left\|\frac{\partial}{\partial x_n}\tilde{a}\tilde{X}_i(\phi u)\right\|_{s-1,\varepsilon_{s-\frac12}}^2 \le C\big(\|\psi\tilde{L}u\|_s^2+\|\psi u\|_s^2\big),\quad u\in H_s^{loc}\big(B(p,\delta);\tilde{L}\big). \tag{6.32}$$

From now on we shall fix $i$ and denote by $C_1^{(\varepsilon)},C_2^{(\varepsilon)},\dots$, positive constants depending on $\varepsilon$ $(0<\varepsilon\le1)$, $s$, $\delta$, $\phi$ and $\psi$, and independent of $\kappa,\gamma$ $(0<\kappa,\gamma<1)$, and $u\in H_s^{loc}(B(p,\delta);\tilde{L})$. Since $\{T^{(\kappa,\gamma,\varepsilon)}(\xi)\}_{0<\kappa,\gamma<1}$ is bounded in $S^0$ for any $\varepsilon$ $(0<\varepsilon\le1)$ by (6.10), it follows from Corollary 6.1 that

$$\begin{aligned}\left\|T^{(\kappa,\gamma,\varepsilon)}\frac{\partial}{\partial x_n}\tilde{a}\tilde{X}_i(\phi u)\right\|_{s-1,\varepsilon_{s-\frac12}}^2 &= \left\|T^{(\kappa,\gamma,\varepsilon)}\frac{\partial}{\partial x_n}\tilde{a}\tilde{X}_i\psi(\phi u)\right\|_{s-1,\varepsilon_{s-\frac12}}^2\\ &\le 2\left\|\frac{\partial}{\partial x_n}\tilde{a}\tilde{X}_i\psi T^{(\kappa,\gamma,\varepsilon)}(\phi u)\right\|_{s-1,\varepsilon_{s-\frac12}}^2 + C_1^{(\varepsilon)}\|\phi u\|_s^2.\end{aligned} \tag{6.33}$$

Now we put

$$w=\psi T^{(\kappa,\gamma,\varepsilon)}(\phi u). \tag{6.34}$$

Since $\tilde{a}\tilde{X}_iw\in C_0^\infty(B(p,\delta))\subset C_0^\infty(B(p,\delta_s))\subset C_0^\infty(B(p,\rho_{s-\frac12}))$ by (6.20), it follows from

Proposition 4.2 with $u$ and $s$ repalced by $\tilde{a}\tilde{X}_i w$ and $s-1/2$ respectively that

$$\begin{aligned}&\left\|\frac{\partial}{\partial x_n}\tilde{a}\tilde{X}_i\psi T^{(\kappa,\gamma,\varepsilon)}(\phi u)\right\|^2_{s-1,\varepsilon_{s-\frac12}}\\&\quad\le -2\mathrm{Re}\left(x_n\tilde{L}\tilde{a}\tilde{X}_i w,\tilde{a}\tilde{X}_i w\right)_{s-\frac12}+2\mathrm{Re}\left(\tilde{L}\tilde{a}\tilde{X}_i w,\frac{\partial}{\partial x_n}\tilde{a}\tilde{X}_i w\right)_{s-1,\varepsilon_{s-\frac12}}\\&\qquad+C_{s-\frac12}\left\|\tilde{a}\tilde{X}_i w\right\|^2_{s-\frac12}. \qquad (6.35)\end{aligned}$$

On the other hand, since the norms $\|\cdot\|_{s-\frac12}$ and $\|\cdot\|_{s-\frac12,\varepsilon_s}$ are equivalent, it follows from (6.10) and Corollary 6.1 that

$$\begin{aligned}\left\|\tilde{a}\tilde{X}_i w\right\|^2_{s-\frac12}&=\left\|\psi T^{(\kappa,\gamma,\varepsilon)}\tilde{a}\tilde{X}_i(\phi u)+[\tilde{a}\tilde{X}_i,\psi T^{(\kappa,\gamma,\varepsilon)}](\phi u)\right\|^2_{s-\frac12}\\&\le C_2^{(\varepsilon)}\left(\|\tilde{a}\tilde{X}_i(\phi u)\|^2_{s-\frac12,\varepsilon_s}+\|\phi u\|^2_{s-\frac12}\right).\end{aligned}$$

Hence we have by Lemma 6.5

$$\left\|\tilde{a}\tilde{X}_i w\right\|^2_{s-\frac12}\le C_3^{(\varepsilon)}\left(\|\psi\tilde{L}u\|^2_s+\|\psi u\|^2_s+\|\phi u\|^2_{s-\frac12}\right). \qquad (6.36)$$

Combining (6.33), (6.35) and (6.36) we have

$$\begin{aligned}&\left\|T^{(\kappa,\gamma,\varepsilon)}\frac{\partial}{\partial x_n}\tilde{a}\tilde{X}_i(\phi u)\right\|^2_{s-1,\varepsilon_{s-\frac12}}\\&\quad\le -4\mathrm{Re}\left(x_n\tilde{L}\tilde{a}\tilde{X}_i w,\tilde{a}\tilde{X}_i w\right)_{s-\frac12}+4\mathrm{Re}\left(\tilde{L}\tilde{a}\tilde{X}_i w,\frac{\partial}{\partial x_n}\tilde{a}\tilde{X}_i w\right)_{s-1,\varepsilon_{s-\frac12}}\\&\qquad+C_4^{(\varepsilon)}\left(\|\psi\tilde{L}u\|^2_s+\|\psi u\|^2_s+\|\phi u\|^2_s\right)\\&\quad\equiv J_1+J_2+C_4^{(\varepsilon)}\left(\|\psi\tilde{L}u\|^2_s+\|\psi u\|^2_s+\|\phi u\|^2_s\right). \qquad (6.37)\end{aligned}$$

We estimate $J_1$. We have by (6.34), Corollary 6.2 and Corollary 6.3

$$\begin{aligned}|J_1|&=4\left|\mathrm{Re}\,(x_n\tilde{L}\tilde{a}\tilde{X}_i\psi T^{(\kappa,\gamma,\varepsilon)}(\phi u),E_{2s-1}\tilde{a}\tilde{X}_i w)\right|\\&\le 4\left|(\psi T^{(\kappa,\gamma,\varepsilon)}\tilde{L}(\phi u),(x_n\tilde{a}\tilde{X}_i)^*E_{2s-1}\tilde{a}\tilde{X}_i w)\right|\\&\quad+4\left|\left(E_{s-1}^{(\varepsilon_{s-\frac12})}[\tilde{L},\tilde{a}\tilde{X}_i\psi T^{(\kappa,\gamma,\varepsilon)}](\phi u),E_{-s+1}^{(\varepsilon_{s-\frac12})}x_nE_{2s-1}\tilde{a}\tilde{X}_i w\right)\right|\\&\le C_5^{(\varepsilon)}\left(\|\tilde{L}(\phi u)\|^2_s+\|x_n\tilde{a}\tilde{X}_i(\phi u)\|^2_s+\|\phi u\|^2_s\right)\\&\quad+4\left|\left(E_{s-1}^{(\varepsilon_{s-\frac12})}[\tilde{L},\tilde{a}\tilde{X}_i\psi T^{(\kappa,\gamma,\varepsilon)}](\phi u),E_{-s+1}^{(\varepsilon_{s-\frac12})}x_nE_{2s-1}\tilde{a}\tilde{X}_i w\right)\right|\end{aligned}$$

$$\leq C_6^{(\varepsilon)}\big(\|\psi\tilde{L}u\|_s^2+\|\psi u\|_s^2+\|\phi u\|_s^2\big)$$
$$+4\Big|\Big(E_{s-1}^{(\varepsilon_{s-\frac{1}{2}})}[\tilde{L},\tilde{a}\tilde{X}_i\psi T^{(\kappa,\gamma,\varepsilon)}](\phi u),E_{-s+1}^{(\varepsilon_{s-\frac{1}{2}})}x_nE_{2s-1}\tilde{a}\tilde{X}_iw\Big)\Big|. \tag{6.38}$$

Now it follows that

$$\begin{aligned}\big[\tilde{L},\tilde{a}\tilde{X}_i\psi T^{(\kappa,\gamma,\varepsilon)}\big]&=\big[\tilde{L},\tilde{a}\tilde{X}_i\big]\psi T^{(\kappa,\gamma,\varepsilon)}+\tilde{a}\tilde{X}_i\big[\tilde{L},\psi T^{(\kappa,\gamma,\varepsilon)}\big]\\&=\sum_{j=1}^r\big\{-\tilde{a}\tilde{X}_i(x_n\tilde{a})\tilde{X}_j^2+x_n\tilde{a}[\tilde{X}_j^2,\tilde{a}\tilde{X}_i]\big\}\psi T^{(\kappa,\gamma,\varepsilon)}\\&\quad+\Big[\chi\frac{\partial}{\partial x_n}+\tilde{c},\tilde{a}\tilde{X}_i\Big]\psi T^{(\kappa,\gamma,\varepsilon)}+\tilde{a}\tilde{X}_i\big[\tilde{L},\psi T^{(\kappa,\gamma,\varepsilon)}\big],\end{aligned}$$

where $\tilde{X}_i(x_n\tilde{a})|_{x_n=0}=0$ by (4.7). Hence we have by (6.25) and (6.10)

$$\big[\tilde{L},\tilde{a}\tilde{X}_i\psi T^{(\kappa,\gamma,\varepsilon)}\big]=\sum_{j=1}^r B_1^{(\kappa,\gamma,\varepsilon,j)}x_n\tilde{a}\tilde{X}_j+\sqrt{-1}\psi P_2T_{\xi_n}^{(\kappa,\gamma,\varepsilon)}\tilde{a}\tilde{X}_i+C_1^{(\kappa,\gamma,\varepsilon)}, \tag{6.39}$$

where $\{B_1^{(\kappa,\gamma,\varepsilon,j)}(x,\xi)\}_{0<\kappa,\gamma<1}$ and $\{C_1^{(\kappa,\gamma,\varepsilon)}(x,\xi)\}_{0<\kappa,\gamma<1}$ are bounded in $S^1$ for any fixed $\varepsilon$ $(0<\varepsilon\leq 1)$ and $j$ $(1\leq j\leq r)$. It follows from (6.38) and (6.39) that

$$\begin{aligned}|J_1|&\leq C_6^{(\varepsilon)}\big(\|\psi\tilde{L}u\|_s^2+\|\psi u\|_s^2+\|\phi u\|_s^2\big)+\Big\|\psi P_2T_{\xi_n}^{(\kappa,\gamma,\varepsilon)}E_{s-1}^{(\varepsilon_{s-\frac{1}{2}})}\tilde{a}\tilde{X}_i(\phi u)\Big\|_0^2\\&\quad+C_7^{(\varepsilon)}\bigg(\sum_{j=1}^r\|x_n\tilde{a}\tilde{X}_j(\phi u)\|_s^2+\Big\|E_{-s+1}^{(\varepsilon_{s-\frac{1}{2}})}x_nE_{2s-1}\tilde{a}\tilde{X}_iw\Big\|_0^2+\|\phi u\|_s^2\bigg).\end{aligned}$$

Hence we have by (6.34), (6.10) and Corollary 6.2

$$|J_1|\leq C_8^{(\varepsilon)}\big(\|\psi\tilde{L}u\|_s^2+\|\psi u\|_s^2+\|\phi u\|_s^2\big)+\Big\|\psi P_2T_{\xi_n}^{(\kappa,\gamma,\varepsilon)}E_{s-1}^{(\varepsilon_{s-\frac{1}{2}})}\tilde{a}\tilde{X}_i(\phi u)\Big\|_0^2.$$

In the same way as the proof of (6.28) we have

$$\Big\|\psi P_2T_{\xi_n}^{(\kappa,\gamma,\varepsilon)}E_{s-1}^{(\varepsilon_{s-\frac{1}{2}})}\tilde{a}\tilde{X}_i(\phi u)\Big\|_0^2\leq n^2r^2C(\delta^2+\varepsilon^4)\Big\|E^{(\gamma,\varepsilon)}\frac{\partial}{\partial x_n}\tilde{a}\tilde{X}_i(\phi u)\Big\|_{s-1,\varepsilon_{s-\frac{1}{2}}}^2. \tag{6.40}$$

Here we note that

$$\Big\|E^{(\gamma,\varepsilon)}\frac{\partial}{\partial x_n}\tilde{a}\tilde{X}_i(\phi u)\Big\|_{s-1,\varepsilon_{s-\frac{1}{2}}}<\infty, \tag{6.41}$$

because $\phi u\in H_s$ and $E^{(\gamma,\varepsilon)}\frac{\partial}{\partial x_n}\tilde{a}\tilde{X}_i\in \mathrm{Op}(S^1)$ by (6.4). Hence

$$
\begin{aligned}
|J_1| \le\, & C_8^{(\varepsilon)}\big(\|\psi\tilde{L}u\|_s^2+\|\psi u\|_s^2+\|\phi u\|_s^2\big)\\
&+n^2r^2C(\delta^2+\varepsilon^4)\Big\|E^{(\gamma,\varepsilon)}\frac{\partial}{\partial x_n}\tilde{a}\tilde{X}_i(\phi u)\Big\|^2_{s-1,\varepsilon_{s-\frac12}}.
\end{aligned}
\tag{6.42}
$$

We estimate $J_2$. We have by (6.34), Corollary 6.3, (6.39), Corollary 6.2, and (6.40)

$$
\begin{aligned}
|J_2| =\, & 4\Big|\mathrm{Re}\Big(E_{s-1}^{(\varepsilon_{s-\frac12})}\tilde{L}\tilde{a}\tilde{X}_i\psi T^{(\kappa,\gamma,\varepsilon)}(\phi u), E_{s-1}^{(\varepsilon_{s-\frac12})}\frac{\partial}{\partial x_n}\tilde{a}\tilde{X}_i w\Big)\Big|\\
\le\, & 4\Big|\Big(E_{s-1}^{(\varepsilon_{s-\frac12})}\tilde{a}\tilde{X}_i\psi T^{(\kappa,\gamma,\varepsilon)}\tilde{L}(\phi u), E_{s-1}^{(\varepsilon_{s-\frac12})}\frac{\partial}{\partial x_n}\tilde{a}\tilde{X}_i w\Big)\Big|\\
&+4\Big|\Big(E_{s-1}^{(\varepsilon_{s-\frac12})}[\tilde{L},\tilde{a}\tilde{X}_i\psi T^{(\kappa,\gamma,\varepsilon)}](\phi u), E_{s-1}^{(\varepsilon_{s-\frac12})}\frac{\partial}{\partial x_n}\tilde{a}\tilde{X}_i w\Big)\Big|\\
\le\, & C_9^{(\varepsilon)}\big(\|\psi\tilde{L}u\|_s^2+\|\psi u\|_s^2+\|\phi u\|_s^2\big)+\frac18\Big\|\frac{\partial}{\partial x_n}\tilde{a}\tilde{X}_i w\Big\|_{s-1,\varepsilon_{s-\frac12}}\\
&+4\Big|\Big(E_{s-1}^{(\varepsilon_{s-\frac12})}\psi P_2 T_{\xi_n}^{(\kappa,\gamma,\varepsilon)}\tilde{a}\tilde{X}_i(\phi u), E_{s-1}^{(\varepsilon_{s-\frac12})}\frac{\partial}{\partial x_n}\tilde{a}\tilde{X}_i w\Big)\Big|\\
\le\, & C_{10}^{(\varepsilon)}\big(\|\psi\tilde{L}u\|_s^2+\|\psi u\|_s^2+\|\phi u\|_s^2\big)+\frac14\Big\|\frac{\partial}{\partial x_n}\tilde{a}\tilde{X}_i w\Big\|^2_{s-1,\varepsilon_{s-\frac12}}\\
&+256\Big\|\psi P_2 T_{\xi_n}^{(\kappa,\gamma,\varepsilon)}E_{s-1}^{(\varepsilon_{s-\frac12})}\tilde{a}\tilde{X}_i(\phi u)\Big\|_0^2\\
\le\, & C_{10}^{(\varepsilon)}\big(\|\psi\tilde{L}u\|_s^2+\|\psi u\|_s^2+\|\phi u\|_s^2\big)+\frac14\Big\|\frac{\partial}{\partial x_n}\tilde{a}\tilde{X}_i w\Big\|^2_{s-1,\varepsilon_{s-\frac12}}\\
&+256n^2r^2C(\delta^2+\varepsilon^4)\Big\|E^{(\gamma,\varepsilon)}\frac{\partial}{\partial x_n}\tilde{a}\tilde{X}_i(\phi u)\Big\|^2_{s-1,\varepsilon_{s-\frac12}}.
\end{aligned}
\tag{6.43}
$$

Here it follows from (6.34) and the equality: $\psi\phi u=\phi u$ that

$$
\Big\|\frac{\partial}{\partial x_n}\tilde{a}\tilde{X}_i w\Big\|^2_{s-1,\varepsilon_{s-\frac12}}\le 2\Big\|T^{(\kappa,\gamma,\varepsilon)}\frac{\partial}{\partial x_n}\tilde{a}\tilde{X}_i(\phi u)\Big\|^2_{s-1,\varepsilon_{s-\frac12}}+C_{11}^{(\varepsilon)}\|\phi u\|_s^2. \tag{6.44}
$$

In virtue of (6.43) and (6.44) we obtain

$$
\begin{aligned}
|J_2| \le\, & C_{12}^{(\varepsilon)}\big(\|\psi\tilde{L}u\|_s^2+\|\psi u\|_s^2+\|\phi u\|_s^2\big)+\frac12\Big\|T^{(\kappa,\gamma,\varepsilon)}\frac{\partial}{\partial x_n}\tilde{a}\tilde{X}_i(\phi u)\Big\|^2_{s-1,\varepsilon_{s-\frac12}}\\
&+256n^2r^2C(\delta^2+\varepsilon^4)\Big\|E^{(\gamma,\varepsilon)}\frac{\partial}{\partial x_n}\tilde{a}\tilde{X}_i(\phi u)\Big\|^2_{s-1,\varepsilon_{s-\frac12}}.
\end{aligned}
\tag{6.45}
$$

Combining (6.37), (6.42) and (6.45) we obtain

$$\frac{1}{2}\left\|T^{(\kappa,\gamma,\varepsilon)}\frac{\partial}{\partial x_n}\tilde{a}\tilde{X}_i(\phi u)\right\|^2_{s-1,\varepsilon_{s-\frac{1}{2}}} \le C_{13}^{(\varepsilon)}\left(\|\psi\tilde{L}u\|_s^2+\|\psi u\|_s^2+\|\phi u\|_s^2\right) + 257n^2r^2C(\delta^2+\varepsilon^4)\left\|E^{(\gamma,\varepsilon)}\frac{\partial}{\partial x_n}\tilde{a}\tilde{X}_i(\phi u)\right\|^2_{s-1,\varepsilon_{s-\frac{1}{2}}}.$$

Let $\kappa \to +0$ in the above inequality. Then we have by (6.2) and (6.4)

$$\frac{1}{2}\left\|E^{(\gamma,\varepsilon)}\frac{\partial}{\partial x_n}\tilde{a}\tilde{X}_i(\phi u)\right\|^2_{s-1,\varepsilon_{s-\frac{1}{2}}} \le C_{13}^{(\varepsilon)}\left(\|\psi\tilde{L}u\|_s^2+\|\psi u\|_s^2+\|\phi u\|_s^2\right) + 257n^2r^2C(\delta^2+\varepsilon^4)\left\|E^{(\gamma,\varepsilon)}\frac{\partial}{\partial x_n}\tilde{a}\tilde{X}_i(\phi u)\right\|^2_{s-1,\varepsilon_{s-\frac{1}{2}}}. \tag{6.46}$$

We see from (6.20) that $257n^2r^2C\delta^2 \le 257n^2r^2C\delta_s^2 \le 1/8$. Now we choose $\varepsilon$ $(0<\varepsilon\le 1)$ so small that $257n^2r^2C\varepsilon^4 \le 1/8$. Then we have by (6.46)

$$\frac{1}{2}\left\|E^{(\gamma,\varepsilon)}\frac{\partial}{\partial x_n}\tilde{a}\tilde{X}_i(\phi u)\right\|^2_{s-1,\varepsilon_{s-\frac{1}{2}}} \le C_{13}^{(\varepsilon)}\left(\|\psi\tilde{L}u\|_s^2+\|\psi u\|_s^2+\|\phi u\|_s^2\right) + \frac{1}{4}\left\|E^{(\gamma,\varepsilon)}\frac{\partial}{\partial x_n}\tilde{a}\tilde{X}_i(\phi u)\right\|^2_{s-1,\varepsilon_{s-\frac{1}{2}}}.$$

Hence we have by (6.41)

$$\left\|E^{(\gamma,\varepsilon)}\frac{\partial}{\partial x_n}\tilde{a}\tilde{X}_i(\phi u)\right\|^2_{s-1,\varepsilon_{s-\frac{1}{2}}} \le 4C_{13}^{(\varepsilon)}\left(\|\psi\tilde{L}u\|_s^2+\|\psi u\|_s^2+\|\phi u\|_s^2\right).$$

Let $\gamma \to +0$ in the above inequality. Then we have by (6.4)

$$\left\|\frac{\partial}{\partial x_n}\tilde{a}\tilde{X}_i(\phi u)\right\|^2_{s-1,\varepsilon_{s-\frac{1}{2}}} \le 4C_{13}^{(\varepsilon)}\left(\|\psi\tilde{L}u\|_s^2+\|\psi u\|_s^2+\|\phi u\|_s^2\right).$$

Since $\phi u = \phi\psi u$, we obtain (6.32) by the above inequality. □

Proposition 6.1 (estimate of the subelliptic kind in $H_s^{loc}$ space). *For any real number $s$, any positive number $\delta$ $(0<\delta\le\delta_s)$, and any $\phi(x),\psi(x)\in C_0^\infty(B(p,\delta))$, $\phi\subset\subset\psi$, there exists a constant $C=C(s,\delta,\phi,\psi)>0$ such that*

$$\|\phi u\|^2_{s+\sigma} \le C\left(\|\psi\tilde{L}u\|_s^2+\|\psi u\|_s^2\right), \quad u\in H_s^{loc}\left(B(p,\delta);\tilde{L}\right).$$

PROOF. In the proof we shall denote by $C_1, C_2, \ldots$, positive constants depending on $s$, $\delta$, $\phi$ and $\psi$, and independent of $\kappa$ $(0<\kappa<1)$ and $u \in H_s^{loc}(B(p,\delta);\tilde{L})$. It follows from (6.1) that

$$\|\phi u\|_{s+\sigma}^2 = \lim_{\kappa\to+0} \big\|T^{(\kappa)} E_{s+\sigma}(\phi u)\big\|_0^2. \tag{6.47}$$

Since $s+\sigma-1<s$ by (5.27) and $\phi u = \psi\phi u$, we have by Lemma 6.2

$$\begin{aligned}\big\|T^{(\kappa)} E_{s+\sigma}(\phi u)\big\|_0^2 &= \big\|E_\sigma \psi T^{(\kappa)} E_s(\phi u) + [T^{(\kappa)}E_s, E_\sigma\psi](\phi u)\big\|_0^2 \\ &\le 2\big\|\psi T^{(\kappa)} E_s(\phi u)\big\|_\sigma^2 + C_1\|\phi u\|_s^2. \end{aligned}\tag{6.48}$$

Since $T^{(\kappa)} \in \mathrm{Op}(S^{-\infty})$ for any fixed $\kappa$ $(0<\kappa<1)$ and $\delta \le \delta_s \le \rho_0/2$ by (6.20), we see that $\psi T^{(\kappa)} E_s(\phi u) \in C_0^\infty(B(p,\delta)) \subset C_0^\infty(B(p,\rho_0/2))$. Hence we can apply Proposition 5.1 to $\psi T^{(\kappa)} E_s(\phi u)$ and we have by Lemma 6.2

$$\begin{aligned}\big\|\psi T^{(\kappa)} E_s(\phi u)\big\|_\sigma^2 &\le C_2\big(\|\tilde{L}\psi T^{(\kappa)} E_s(\phi u)\|_0^2 + \|\psi T^{(\kappa)} E_s(\phi u)\|_0^2\big) \\ &\le C_3\big(\|\tilde{L}\psi T^{(\kappa)} E_s(\phi u)\|_0^2 + \|\phi u\|_s^2\big). \end{aligned}\tag{6.49}$$

It follows from (6.48), (6.49), Lemma 6.2 and Corollary 6.3 that

$$\begin{aligned}\big\|T^{(\kappa)} E_{s+\sigma}(\phi u)\big\|_0^2 &\le C_4\big(\|\psi T^{(\kappa)} E_s \tilde{L}(\phi u)\|_0^2 + \|[\tilde{L}, \psi T^{(\kappa)} E_s](\phi u)\|_0^2 + \|\phi u\|_s^2\big) \\ &\le C_5\big(\|\psi\tilde{L}u\|_s^2 + \|\psi u\|_s^2 + \|\phi u\|_s^2 + \|[\tilde{L}, \psi T^{(\kappa)} E_s](\phi u)\|_0^2\big). \end{aligned}\tag{6.50}$$

In the same way as the proof of (6.25) we have by Lemma 6.1 and Lemma 6.2

$$\begin{aligned}\big[\tilde{L}, \psi T^{(\kappa)} E_s\big] &= \sum_{i=1}^r F_s^{(\kappa,i)} x_n \tilde{a}\tilde{X}_i + \sum_{i=1}^r\sum_{k=1}^n G_s^{(\kappa,i,k)} x_n \tilde{a}_{x_k}\tilde{X}_i \\ &\quad + \sqrt{-1}\psi P_2\big(T^{(\kappa)}E_s\big)_{\xi_n} + H_s^{(\kappa)}, \end{aligned}$$

where $\{F_s^{(\kappa,i)}(x,\xi)\}_{0<\kappa<1}$, $\{G_s^{(\kappa,i,k)}(x,\xi)\}_{0<\kappa<1}$ and $\{H_s^{(\kappa)}(x,\xi)\}_{0<\kappa<1}$ are bounded in $S^s$ for any $i$ $(1\le i\le r)$ and $k$ $(1\le k\le n)$, and $(T^\kappa E_s)_{\xi_n}$ is the pseudo-differential operator with the symbol: $(T^{(\kappa)}(\xi)E_s(\xi))_{\xi_n} = (s(1+|\xi|^2)^{-1} - 2\kappa^2)e^{-|\kappa\xi|^2}E_s(\xi)\xi_n$. We see from Lemma 6.2 and (6.7) with $\varepsilon=1$ that $\{(s(1+|\xi|^2)^{-1}-2\kappa^2)e^{-|\kappa\xi|^2}\}_{0<\kappa<1}$ is bounded in $S^{-2}$. Hence we have

$$\big[\tilde{L}, \psi T^{(\kappa)} E_s\big] = \sum_{i=1}^r F_s^{(\kappa,i)} x_n \tilde{a}\tilde{X}_i + \sum_{i=1}^r\sum_{k=1}^n G_s^{(\kappa,i,k)} x_n \tilde{a}_{x_k}\tilde{X}_i + \sum_{i=1}^r I_{s-1}^{(\kappa,i)} \frac{\partial}{\partial x_n}\tilde{a}\tilde{X}_i + J_s^{(\kappa)},$$

where $\{I_{s-1}^{(\kappa,i)}(x,\xi)\}_{0<\kappa<1}$ is bounded in $S^{s-1}$ for any fixed $i$ $(1\le i\le r)$, and

$\{J_s^{(\kappa)}(x,\xi)\}_{0<\kappa<1}$ is bounded in $S^s$. Hence we have by Corollary 6.2 and Lemma 6.6

$$\begin{aligned}\big\|[\tilde{L},\psi T^{(\kappa)}E_s](\phi u)\big\|_0^2 &\le C_6\bigg(\sum_{i=1}^r\big\|x_n\tilde{a}\tilde{X}_i(\phi u)\big\|_s^2+\sum_{i=1}^r\sum_{k=1}^n\big\|x_n\tilde{a}_{x_k}\tilde{X}_i(\phi u)\big\|_s^2\\&\qquad+\sum_{i=1}^r\bigg\|\frac{\partial}{\partial x_n}\tilde{a}\tilde{X}_i(\phi u)\bigg\|_{s-1}^2+\|\phi u\|_s^2\bigg)\\&\le C_7\big(\|\psi\tilde{L}u\|_s^2+\|\psi u\|_s^2+\|\phi u\|_s^2\big).\end{aligned}\tag{6.51}$$

It follows from (6.50) and (6.51) that

$$\big\|T^{(\kappa)}E_{s+\sigma}(\phi u)\big\|_0^2\le C_8\big(\|\psi\tilde{L}u\|_s^2+\|\psi u\|_s^2+\|\phi u\|_s^2\big).$$

Letting $\kappa\to+0$ in the above inequality we have by (6.47)

$$\|\phi u\|_{s+\sigma}^2\le C_8\big(\|\psi\tilde{L}u\|_s^2+\|\psi u\|_s^2+\|\phi u\|_s^2\big).$$

Since $\phi u=\phi\psi u$, the proposition follows from the above inequality. □

## 7. Proof of the hypoellipticity of $L$.

Let $\omega$ be an open subset of $\Omega$. Suppose that $u\in\mathscr{D}'(\omega)$ and $Lu\in C^\infty(\omega)$. We shall prove that $u\in C^\infty(\omega)$. We put

$$\omega_+=\{x\in\omega\mid x_n>0\},\quad \omega_-=\{x\in\omega\mid x_n<0\},\quad \omega_0=\{x\in\omega\mid x_n=0\}.$$

Then $x_na(x)\ge 0$ in $\omega_+$ and $x_na(x)\le 0$ in $\omega_-$. We write the operator $L$ in the divergent form:

$$L=\sum_{k=1}^n\frac{\partial}{\partial x_k}Y_k+Y_0+c(x),$$

where $Y_k$ $(1\le k\le n)$ and $Y_0$ are first order operators with coefficients belonging to $C^\infty(\omega)$. Then by the hypotheses (1.3), (1.4) and Theorem 1.1 of [**2**],

$$\operatorname{rank}\operatorname{Lie}(Y_0,Y_1,\dots,Y_n)(x)=n,\quad x\in\omega_+\cup\omega_-.$$

Hence $L$ is hypoelliptic in $\omega_+\cup\omega_-$ by Theorem 2.6.4 of [**9**], and so

$$u\in C^\infty(\omega_+\cup\omega_-).\tag{7.1}$$

Let $j$ be an arbitrary positive integer. We shall show that

$u$ is $j$-times continuously differentiable in an open neighborhood of $\omega_0$. (7.2)

Let $p$ be an arbitrary point of $\omega_0$. We choose a positive number $r_0$ so that $\overline{B(p,r_0)} \subset \omega_p \cap \omega$, where $\omega_p$ is the open neighborhood of $p$ defined by (3.17). Then there exists a real number $s_0$ such that $u \in H^{loc}_{s_0}(B(p,r_0))$. On the other hand, we see from (3.18)–(3.20) that $\tilde{L}u = Lu \in C^\infty(B(p,r_0))$. Let $\sigma$ be the positive number defined by (5.26), and let $\delta_{s_0}$ be the radius defined by (6.20) with $s$ replaced by $s_0$. We put $r_1 = \min(r_0, \delta_{s_0})$. Then $u \in H^{loc}_{s_0}(B(p,r_1);\tilde{L})$ and it follows from Proposition 6.1 that $u \in H^{loc}_{s_0+\sigma}(B(p,r_1))$. Repeating this argument we see that $u \in H^{loc}_{s_0+k\sigma}(B(p,r_k))$, $r_k = \min(r_{k-1}, \delta_{s_0+(k-1)\sigma})$, $k = 1,2,\ldots$. Now we take $k$ so large that $s_0 + k\sigma > n/2 + j$. Then it follows from Sobolev's imbedding theorem that $u \in C^j(B(p,r_k))$. Thus (7.2) has been proved.

It follows from (7.1) and (7.2) that $u \in C^j(\omega)$ for any positive integer $j$. Hence $u \in C^\infty(\omega)$.

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