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Special values of the spectral zeta functions for locally symmetric Riemannian manifolds

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Abstract. In this paper, we establish the formulas expressing the special values of the spectral zeta function $\zeta_\Delta(n)$ of the Laplacian Δ on some locally symmetric Riemannian manifold $\Gamma\backslash G/K$ in terms of the coefficients of the Laurent expansion of the corresponding Selberg zeta function. As an application, we give a numerical estimation of the first eigenvalue of Δ by computing the values $\zeta_\Delta(n)$ numerically, when $\Gamma\backslash G/K$ is a Riemann surface with Γ being the quaternion group.

1. Introduction.

Let G be a connected non-compact semisimple Lie group of real rank one with finite center, K a maximal compact subgroup of G, and Γ a discrete subgroup of G such that $\Gamma\backslash G/K < \infty$. We denote by λ_j the eigenvalue of the Laplacian on $\Gamma\backslash G/K$ such that $0 = \lambda_0 < \lambda_1 < \dots$ and n_j the multiplicity of λ_j. We define the spectral zeta function $\zeta_\Delta(s)$ by

$$\zeta_\Delta(s) = \sum_{j=1}^{\infty} n_j \lambda_j^{-s} \qquad \mathrm{Re}\, s > d/2, \tag{1.1}$$

where $d = \dim(G/K)$. When $\Gamma\backslash G/K$ is a compact Riemann surfaces of genus $g \geq 2$, the values $\{\zeta_\Delta(n)\}_{n\geq 2}$ satisfy a certain formula which assures that $\zeta_\Delta(n)$ is expressed by the (higher) Euler-Selberg constants and the special values of the Riemann zeta function $\zeta(s)$ (see, [**HIKW**] or [**St**]). Here the Euler-Selberg constants are defined as the coefficients of the Laurent expansion of the Selberg zeta function $Z_\Gamma(s)$ (for the definition, see (2.5)). These are analogues to the Euler constant γ which is a constant term of the Laurent expansion of the Riemann zeta function. Similar to the expression $\gamma = \lim_{x\to\infty}(\sum_{n<x} 1/n - \log x)$, the Euler-Selberg constants are expressed as the sum over the hyperbolic conjugacy classes of Γ(see (2.8); see also [**H**]).

The aim of this paper is to establish an explicit description of $\zeta_\Delta(n)$ by the Euler-Selberg constants for locally symmetric Riemannian manifolds. In principal, this relation can be obtained by the determinant expression of the Selberg zeta function (see Remark 2.2). In fact, using the trace formula here, we first show that the value $\zeta_\Delta(n)$ is explicitly written in terms of the Euler-Selberg constants and the special values of the Riemann zeta function $\zeta(s)$ for a compact locally symmetric Riemannian manifold (see Theorem 2.1). Furthermore, we also deal with some non-compact cases. Actually, we establish the formulas of $\zeta_\Delta(n)$ when Γ is either $SL_2(\boldsymbol{Z})$ or the congruence subgroup of $SL_2(\boldsymbol{Z})$ (see Theorem 4.2). Since no explicit descriptions of the scattering matrices are known in general, it is hard to obtain a similar formula very explicitly for a non-compact case generally, though the idea for obtaining such a formula is the same as, e.g., the $SL_2(\boldsymbol{Z})$ case.

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In the last section we give a numerical computation of $\zeta_\Delta(n)$ for the quaternion groups. By virtue of Theorem 4.1, the calculation is reduced to that of the (higher) Euler-Selberg constants. In order to calculate the (higher) Euler-Selberg constants for the quaternion group cases, we use the arithmetic expression of the Selberg zeta function obtained in [**AKN**]. This method is regarded as a generalization of the discussion for calculation for $SL_2(\mathbf{Z})$ in [**H**]. Since the growth of the sequence $\{\zeta_\Delta(n)\}_{n\geq 2}$ depends mainly on the first eigenvalue λ_1 of the Laplacian Δ, we also discuss a numerical estimation of the eigenvalue λ_1.

2. Preliminaries and main result.

Let G be a connected non-compact semisimple Lie group with finite center, and K a maximal compact subgroup of G. We put $d=\dim(G/K)$. We denote by $\mathfrak{g}$, $\mathfrak{k}$ the Lie algebras of G, K respectively and $\mathfrak{g}=\mathfrak{k}+\mathfrak{p}$ a Cartan decomposition with respect to the Cartan involution θ. Let $\mathfrak{a}_\mathfrak{p}$ be a maximal abelian subspace of $\mathfrak{p}$. Throughout this paper we assume that $\mathrm{rank}(G/K)=1$, that is, $\dim\mathfrak{a}_\mathfrak{p}=1$. We extend $\mathfrak{a}_\mathfrak{p}$ to a θ-stable maximal abelian subalgebra $\mathfrak{a}$ of $\mathfrak{g}$, so that $\mathfrak{a}=\mathfrak{a}_\mathfrak{p}+\mathfrak{a}_\mathfrak{k}$, where $\mathfrak{a}_\mathfrak{p}=\mathfrak{a}\cap\mathfrak{p}$ and $\mathfrak{a}_\mathfrak{k}=\mathfrak{a}\cap\mathfrak{k}$. We put $A=\exp\mathfrak{a}$, $A_\mathfrak{p}=\exp\mathfrak{a}_\mathfrak{p}$ and $A_\mathfrak{k}=\exp\mathfrak{a}_\mathfrak{k}$.

We denote by $\mathfrak{g}^{\boldsymbol{C}}$, $\mathfrak{a}^{\boldsymbol{C}}$ the complexification of $\mathfrak{g}$, $\mathfrak{a}$ respectively. Let Φ be the set of roots of $(\mathfrak{g}^{\boldsymbol{C}},\mathfrak{a}^{\boldsymbol{C}})$, Φ^+ the set of positive roots in Φ, $P_+=\{\alpha\in\Phi^+|\alpha\not\equiv 0 \text{ on } \mathfrak{a}_\mathfrak{p}\}$, and $P_-=\Phi^+-P_+$. We put $\rho=1/2\sum_{\alpha\in P_+}\alpha$. For $h\in A$ and linear form λ on $\mathfrak{a}$, we denote by ξ_λ the character of $\mathfrak{a}$ given by $\xi_\lambda(h)=\exp\lambda(\log h)$. Let Σ be the set of restrictions to $\mathfrak{a}_\mathfrak{p}$ of the elements of P_+. Then the set Σ is either of the form $\{\beta\}$ or $\{\beta,2\beta\}$ with some element $\beta\in\Sigma$. We fix an element $H_0\in\mathfrak{a}_\mathfrak{p}$ such that $\beta(H_0)=1$, and put $\rho_0=\rho(H_0)$.

Let Γ be a co-compact torsion free discrete subgroup of G. We denote by $C(\Gamma)$ a complete set of representatives of Γ-conjugacy classes of semisimple elements in Γ, $\mathrm{Prim}(\Gamma)$ a set of primitive hyperbolic conjugacy classes of Γ, and $Z(\Gamma)$ a center of Γ. For $\gamma\in C(\Gamma)$, we denote by δ_γ an element of $\mathrm{Prim}(\Gamma)$ such that $\gamma=\delta_\gamma^j$ for some integer $j\geq 1$, $h(\gamma)$ an element of A which is conjugate to γ, and $h_\mathfrak{p}(\gamma)$, $h_\mathfrak{k}(\gamma)$ the elements of $A_\mathfrak{p}$, $A_\mathfrak{k}$ respectively such that $h(\gamma)=h_\mathfrak{p}(\gamma)h_\mathfrak{k}(\gamma)$. Let $N(\gamma)$ be a norm of γ given by $N(\gamma)=\exp(\beta(\log(h_\mathfrak{p}(\gamma))))$, and $D(\gamma)$ the function defined by $D(\gamma)=N(\gamma)^{2\rho_0}\prod_{\alpha\in P_+}|1-\xi_\alpha(h(\gamma))^{-1}|$.

We denote by $\mu(s)$ the Plancherel measure of G/K. For convenience, we write

$$\alpha=\begin{cases}1,\\ 2,\end{cases}\quad \hat{\rho}_0=\begin{cases}\rho_0,\\ \rho_0/2,\end{cases}\quad \bar{\mu}(r)=\begin{cases}\mu(r) & \text{if} \quad G=SO(n,1),\\ 2\mu(2r) & \text{if} \quad G\neq SO(n,1).\end{cases}\tag{2.1}$$

The function $\bar{\mu}(r)$ is expressed by the form

$$\bar{\mu}(s):=\pi C_G^{-1}P(r)\sigma(r),$$

where C_G, $P(r)$ and $\sigma(r)$ are as follows (see, e.g. [**Mi**] or [**Wi**]).

G	d	ρ_0	$\hat{\rho}_0$	C_G	$\sigma(r)$
$SO(2m-1,1)$	$2m-1$	$m-1$	$m-1$	$2^{4m-6}\Gamma(m-1/2)^2$	1
$SO(2m,1)$	$2m$	$m-1/2$	$m-1/2$	$2^{4m-4}\Gamma(m)^2$	$\tanh\pi r$
$SU(2m-1,1)$	$4m-2$	$2m-1$	$m-1/2$	$2^{4m-5}\Gamma(2m-1)^2$	$\tanh\pi r$
$SU(2m,1)$	$4m$	$2m$	m	$2^{4m-5}\Gamma(2m)^2$	$\coth\pi r$
$SP(m,1)$	$4m$	$2m+1$	$m+1/2$	$2^{4m-1}\Gamma(2m)^2$	$\tanh\pi r$
F_4	16	11	$11/2$	$2^{19}\Gamma(8)^2$	$\tanh\pi r$

$$\begin{array}{ll} G & P(r) \\ SO(2m-1,1) & r^2\prod_{j=1}^{m-2}(r^2+j^2) \\ SO(2m,1) & r\prod_{j=1}^{m-1}\{r^2+(j-1/2)^2\} \\ SU(2m-1,1) & r\prod_{j=1}^{m-1}\{r^2+(j-1/2)^2\}^2 \\ SU(2m,1) & r^3\prod_{j=1}^{m-1}(r^2+j^2)^2 \\ SP(m,1) & r\{r^2+(m-1/2)^2\}\prod_{j=1}^{m-1}\{r^2+(j-1/2)^2\}^2 \\ F_4 & r(r^2+1/4)^2(r^2+9/4)^2(r^2+25/4)(r^2+49/4)(r^2+81/4) \end{array} \tag{2.2}$$

Let λ_j be the eigenvalue of the Laplacian Δ on $\Gamma\backslash G/K$ such that $0=\lambda_0<\lambda_1<\lambda_2<\dots$, r_j the number given by $\lambda_j=\rho_0^2+r_j^2$, and n_j the multiplicity of λ_j. We define the spectral zeta function $\zeta_\Delta(s)$ of Δ by

$$\zeta_\Delta(s)=\sum_{j=1}^{\infty}n_j\lambda_j^{-s}\quad s>\frac{d}{2}. \tag{2.3}$$

We assume that f is a function whose Fourier transform $\hat f(r)=1/2\pi\int_{-\infty}^{\infty}f(x)e^{ixr}dx$ satisfies that $\hat f(r)=\hat f(-r)$, $\hat f$ is holomorphic in $\{|\mathrm{Im}\,r|\le\rho_0+\delta\}$, and $\hat f(r)=O(|r|^{-d-\delta})$ as $|r|\to\infty$ for some $\delta>0$. Then, the following formula (*the Selberg trace formula,* see, e.g. [**Ga**]) holds.

$$\begin{aligned}\sum_{j\ge 0}\hat f(r_j)=&\sum_{\gamma\in C(\Gamma)-Z(\Gamma)}\log N(\delta_\gamma)D(\gamma)^{-1}N(\gamma)^{\rho_0}f(\log N(\gamma))\\&+\frac{1}{4\pi}\mathrm{vol}(\Gamma\backslash G)[Z(\Gamma)]\int_{-\infty}^{\infty}\hat f(r)\mu(r)dr.\end{aligned} \tag{2.4}$$

Then the Selberg zeta function is defined as follows.

$$Z_\Gamma(s)=\prod_{\delta\in\mathrm{Prim}(\Gamma)}\prod_{\lambda\in L}\left(1-\xi_\lambda(h(\delta))^{-1}N(\delta)^{-s}\right)^{m_\lambda}\quad \mathrm{Re}\,s>2\rho_0, \tag{2.5}$$

where L is the semi-lattice of linear forms on $\mathfrak{a}$ given by $L=\left\{\sum_{i=1}^{l}m_i\alpha_i|\alpha_i\in P_+,m_i\in\mathbf{Z}_{\ge 0}\right\}$, m_λ denotes the number of distinct l-tuples $(m_1,\dots,m_l)$ such that $\lambda=\sum_{i=1}^{l}m_i\alpha_i\in L$. The logarithmic derivative of $Z_\Gamma(s)$ can be written by

$$\frac{Z'_\Gamma(s)}{Z_\Gamma(s)}=\sum_{\gamma\in C(\Gamma)-Z(\Gamma)}\log N(\delta_\gamma)D(\gamma)^{-1}N(\gamma)^{2\rho_0-s}\quad \mathrm{Re}\,s>2\rho_0. \tag{2.6}$$

It is known that $Z'_\Gamma(s)/Z_\Gamma(s)$ has a simple pole at $s=2\rho_0$, and the Laurent expansion at $s=2\rho_0$ is written as

$$\frac{Z'_\Gamma(s)}{Z_\Gamma(s)}=\frac{1}{s-2\rho_0}+\tilde\gamma_\Gamma^{(0)}+\sum_{k=1}^{\infty}\tilde\gamma_\Gamma^{(k)}(s-2\rho_0)^k. \tag{2.7}$$

Here, the coefficient $\tilde\gamma_\Gamma^{(0)}$ is called the Euler-Selberg constant, and $\tilde\gamma_\Gamma^{(k)}$ $(k\ge 1)$ the Euler-Selberg constant of order k or simply the higher Euler-Selberg constant. These values have the following expressions (see [**H**]).

$$\tilde{\gamma}_\Gamma^{(k)} = \frac{(-1)^k}{k!} \lim_{x\to\infty}\left\{ \sum_{\substack{\gamma\in C(\Gamma)-Z(\Gamma)\\ N(\gamma)<x}} \log N(\delta_\gamma) D(\gamma)^{-1} (\log N(\gamma))^k - \frac{(\log x)^{k+1}}{k+1}\right\}. \tag{2.8}$$

The main result of this paper is to provide expressions of the values of the spectral zeta function at $s = n > d/2$ in terms of the Euler-Selberg constants above and $\zeta(n)$'s.

THEOREM 2.1. *For $n > d/2$, we have*

$$(2\rho_0)^{2n}\zeta_\Delta(n) = \sum_{k=0}^{n-1}(-1)^k \binom{2n-k-2}{n-1}(2\rho_0)^{k+1}\tilde{\gamma}_\Gamma^{(k)} - \binom{2n-1}{n-1} + [Z(\Gamma)]\mathrm{vol}(\Gamma\backslash G) I_G^{(n)},$$

where

$$I_G^{(n)} := C_G^{-1}\times\begin{cases} \displaystyle\sum_{l=2}^{n} A_l^{(n)}\left(\zeta(l) - \frac{1}{2}\sum_{k=1}^{2\rho_0} k^{-l}\right) - \frac{1}{2}A_0^{(n)}, & \text{if } G\neq SO(2m-1,1),\\ \displaystyle\pi\sum_{m=1}^{(d-1)/2}(-1)^{m-1}p_{2m}\sum_{q=0}^{\min(2m,n-1)}\frac{(-2)^q}{(2m-q)!}\binom{2n-q-2}{n-1} & \\ & \text{if } G = SO(2m-1,1), \end{cases}$$

$$A_l^{(n)} := (2\hat{\rho}_0)^l \sum_{m=1}^{d/2}(-1)^{m-1}\hat{\rho}_0^{2m-1}p_{2m-1}\sum_{q=0}^{\min(n-l,2m-1)}\binom{2m-1}{q}\binom{2n-l-q-1}{n-1}(-2)^q,$$

and p_m's are determined by $P(r) = \sum_{m=1}^{d-1} p_m r^m$.

REMARK 2.1. By using the trace formula (2.4), the spectral zeta function $\zeta_\Delta(s)$ is analytically continued to the whole complex plane $\boldsymbol{C}$ as a function which is holomorphic except for possibly simple poles at $s = d/2-k$ $(0\le k\le [d/2])$ (see [**Ra**] and [**Wi**]). In this analytical continuation, the contribution of the hyperbolic elements vanishes at $s=-n$ $(n\ge 0)$, hence $\zeta_\Delta(-n)$ is expressed only by the contribution of the identity element, especially a sum of the Bernoulli numbers (see [**BW**]). On the other hand, at $s = n > d/2$, since the contribution of the hyperbolic elements does not disappear, $\zeta_\Delta(n)$ is expressed by $\tilde{\gamma}_\Gamma^{(k)}$.

REMARK 2.2. When we denote by $\zeta_\Delta(s,x) := \sum_{j\ge 0} n_j(\lambda_j+x)^{-s}$, it is known that there exists a meromorphic function $G_\Gamma(s)$ such that

$$\exp\left(-\frac{\partial}{\partial z}\zeta_\Delta(z, s(2\rho_0-s))\big|_{z=0}\right) = Z_\Gamma(s)G_\Gamma(s). \tag{2.9}$$

Here, $G_\Gamma(s)$ is called the Gamma factor and is explicitly calculated in [**Ku**]. Since the left hand side of (2.9) is interpreted as the zeta regularized determinant of $\Delta - s(2\rho_0 - s)$, the formula (2.9) is called the determinant expression of the Selberg zeta function (see, e.g. [**Vo**]). When we take the Laurent expression at $s = 2\rho_0$ of the logarithmic derivative of (2.9) and compare the coefficients of the both sides, we can obtain certain formulas which relate $\zeta_\Delta(n)$'s with $\tilde{\gamma}_\Gamma^{(k)}$'s. However, in this paper, we shall not use this idea but apply the trace formula (2.4) directly to prove Theorem 2.1, because the form $\zeta_\Delta(n) = \sum\tilde{\gamma}_\Gamma^{(k)}$ is immediately obtained by the trace formula.

3. Proof of Theorem 2.1.

Let $n > d/2$ and $a > \rho_0$. Putting

$$\hat{f}(r) = (r^2 + a^2)^{-n},$$
$$f(x) = e^{-a|x|} \sum_{k=0}^{n-1} \frac{1}{k!} \binom{2n-k-2}{n-1} (2a)^{-2n+k+1} |x|^k$$

into the trace formula (2.4), we obtain

$$(a^2 - \rho_0^2)^{-n} + \sum_{j \geq 1} (\lambda_j + a^2 - \rho_0^2)^{-n} = \sum_{k=0}^{n-1} \binom{2n-k-2}{n-1} (2a)^{-2n+k+1} \frac{(-1)^k}{k!} \left(\frac{Z'_\Gamma}{Z_\Gamma} \right)^{(k)} (a + \rho_0)$$
$$+ \frac{\mathrm{vol}(\Gamma \backslash G)[Z(\Gamma)]}{4\pi} \int_{-\infty}^{\infty} (r^2 + a^2)^{-n} \mu(r) dr. \tag{3.1}$$

Since

$$(a^2 - \rho_0^2)^{-n} = \sum_{k=0}^{n-1} \binom{2n-k-2}{n-1} (2a)^{-2n+k+1} \{ (a - \rho_0)^{-(k+1)} + (a + \rho_0)^{-(k+1)} \}, \tag{3.2}$$

it follows that

$$-(a^2 - \rho_0^2)^{-n} + \sum_{k=0}^{n-1} \binom{2n-k-2}{n-1} (2a)^{-2n+k+1} \frac{(-1)^k}{k!} \left(\frac{Z'_\Gamma}{Z_\Gamma} \right)^{(k)} (a + \rho_0)$$
$$= \sum_{k=0}^{n-1} \binom{2n-k-2}{n-1} (2a)^{-2n+k+1} \left\{ \frac{(-1)^k}{k!} \left(\frac{Z'_\Gamma}{Z_\Gamma} \right)^{(k)} (a + \rho_0) - (a - \rho_0)^{-k-1} - (a + \rho_0)^{-k-1} \right\}.$$

Hence if we take the limit $a \to \rho_0$ in (3.1), we obtain

$$\zeta_\Delta(n) = \sum_{k=0}^{n-1} \binom{2n-k-2}{n-1} (2\rho_0)^{-2n+k+1} (-1)^k \tilde{\gamma}_\Gamma^{(k)} - (2\rho_0)^{-2n} \binom{2n-1}{n-1}$$
$$+ \frac{\mathrm{vol}(\Gamma \backslash G)[Z(\Gamma)]}{4\pi} \int_{-\infty}^{\infty} (r^2 + \rho_0^2)^{-n} \mu(r) dr. \tag{3.3}$$

Now we calculate the following definite integral.

$$I_n = \int_{-\infty}^{\infty} (r^2 + \rho_0^2)^{-n} \mu(r) dr.$$

The case $G = SO(2m-1, 1)$: Since $\mu(r)$ is a polynomial (see (2.2)), I_n is a definite integral of a rational function. Hence, we can obtain the following formula easily.

$$I_n = 2\pi^2 C_G^{-1} (2\rho_0)^{-2n} \sum_{l=1}^{(d-1)/2} (-1)^{l-1} p_{2l} \sum_{q=0}^{\min(2l, n-1)} \frac{(-2)^{q+1}}{(2l-q)!} \binom{2n-q-2}{n-1}. \tag{3.4}$$

The case $G \neq SO(2m-1, 1)$: First, we rewrite I_n as

$$I_n = \alpha^{-2n} \int_{-\infty}^{\infty} (r^2 + \hat{\rho}_0^2)^{-n} \overline{\mu}(r) dr,$$

where α, $\hat{\rho}_0$ and $\overline{\mu}(r)$ are defined in (2.1). We calculate I_n by using the residue theorem. Since $\overline{\mu}(r)$ has simple poles at $s = \hat{\rho}_0 + k$ for $k \geq 0$ (see (2.2)), we have

$$\begin{aligned} I_n &= 2\pi i \alpha^{-2n} \sum_{k=0}^{\infty} \operatorname*{Res}_{s=i(\rho_0+k)} (s^2 + \hat{\rho}_0^2)^{-n} \overline{\mu}(s) \\ &= 2\pi C_G^{-1} \sum_{k=0}^{\infty} \pi i \alpha^{-2n} \operatorname*{Res}_{s=i(\rho_0+k)} (s^2 + \hat{\rho}_0^2)^{-n} P(s) \sigma(s) =: 2\pi C_G^{-1} \sum_{k=0}^{\infty} I_n^{(k)}. \end{aligned} \tag{3.5}$$

Since the order of the pole at $s = i\hat{\rho}_0$ of the integrand in I_n is $n+1$, $I_n^{(0)}$ is calculated as follows.

$$\begin{aligned} I_n^{(0)} &= \frac{\alpha^{-2n}}{n!} \lim_{s \to i\hat{\rho}_0} \frac{d^n}{ds^n} \left\{ (s + i\hat{\rho}_0)^{-n} iP(s) \pi (s - i\hat{\rho}_0) \sigma(s) \right\} \\ &= \frac{\alpha^{-2n}}{n!} \lim_{s \to 0} \left[\sum_{l=0}^{n} \binom{n}{l} \frac{d^l}{ds^l} (\pi s \coth \pi s) \sum_{m=0}^{n-l} \binom{n-l}{m} \frac{d^m}{ds^m} iP(s + i\hat{\rho}_0) \frac{d^{n-l-m}}{ds^{n-l-m}} (s + 2i\hat{\rho}_0)^{-n} \right]. \end{aligned}$$

It is easy to see that

$$(b_l :=) \frac{1}{l!} \lim_{s \to 0} \frac{d^l}{ds^l} (\pi s \coth \pi s) = \begin{cases} 1 & \text{if } l = 0, \\ 0 & \text{if } l \text{ is odd}, \\ -2(-1)^{l/2} \zeta(l) & \text{if } l \text{ is even}, \end{cases}$$

$$\lim_{s \to 0} \frac{d^m}{ds^m} iP(s + i\hat{\rho}_0) = \sum_{q=0}^{d-m-1} \frac{m!}{q!} p_{q+m} i (i\hat{\rho}_0)^q,$$

$$\lim_{s \to 0} \frac{d^{n-l-m}}{ds^{n-l-m}} (s + 2i\hat{\rho}_0)^{-n} = (-1)^{n-l-m} \frac{(2n-l-m-1)!}{(n-1)!} (2i\hat{\rho}_0)^{-2n+l+m}.$$

Hence we obtain

$$\begin{aligned} I_n^{(0)} &= \frac{\alpha^{-2n}}{n!} \sum_{l=0}^{n} \binom{n}{l} l! b_l \sum_{m=0}^{n-l} \binom{n-l}{m} \sum_{q=0}^{d-m-1} \frac{m!}{q!} p_{q+m} i (i\hat{\rho}_0)^q (-1)^{n-l-m} \\ &\quad \times \frac{(2n-l-m-1)!}{(n-1)!} (2i\hat{\rho}_0)^{-2n+l+m} \\ &= (2\rho_0)^{-2n} \sum_{l=0}^{n} i^l (-1)^l (2\hat{\rho}_0)^l b_l \sum_{m=0}^{d-1} i^{m+1} \hat{\rho}_0^m p_m \\ &\quad \times \sum_{q=0}^{\min(n-l,m)} \binom{m}{q} \binom{2n-l-q-1}{n-1} (-1)^q \hat{\rho}_0^{-q} (2\hat{\rho}_0)^q \\ &= (2\rho_0)^{-2n} \left\{ -A_0^{(n)} + \sum_{l=2}^{n} A_l^{(n)} (1 + (-1)^n) \zeta(l) \right\}. \end{aligned} \tag{3.6}$$

On the other hand, since $s = i(\hat{\rho}_0 + k)$ $(k \geq 1)$ is a simple pole, it follows that

$$I_n^{(k)} = \alpha^{-2n}\big(\hat{\rho}_0^2 - (\hat{\rho}_0+k)^2\big)^{-n} iP\big(i(\hat{\rho}_0+k)\big). \tag{3.7}$$

We now write $iP\big(i(\hat{\rho}_0+k)\big)$ by

$$iP\big(i(\hat{\rho}_0+k)\big) = \begin{cases} \displaystyle\sum_{m=0}^{d-1} p_m i^{m+1} \sum_{q=0}^{m} \binom{m}{q} \hat{\rho}_0^{m-q} k^q, \\ \displaystyle\sum_{m=0}^{d-1} p_m i^{m+1} \sum_{q=0}^{m} \binom{m}{q} (-\hat{\rho}_0)^{m-q} (k+2\hat{\rho}_0)^q. \end{cases} \tag{3.8}$$

Hence, using (3.2) and (3.8), we have

$$\begin{aligned} I_n^{(k)} &= \alpha^{-2n} \sum_{m=0}^{d-1} i^{m+1} p_m \sum_{q=0}^{m} \sum_{l=1}^{n} \binom{2n-l-1}{n-1} \binom{m}{q} (2\hat{\rho}_0)^{-2n+l} \hat{\rho}_0^{m-q} \\ &\quad \times \big\{(-1)^l k^{q-l} + (-1)^{m-q}(k+2\hat{\rho}_0)^{q-l}\big\} \\ &= (2\rho_0)^{-2n} \sum_{m=0}^{d-1} i(i\hat{\rho}_0)^m p_m \sum_{q=0}^{m} \sum_{l=1}^{n} \binom{2n-l-1}{n-1} \binom{m}{q} 2^l (-1)^q \hat{\rho}_0^{l-q} \\ &\quad \times \big\{(-k)^{q-l} - (k+2\hat{\rho}_0)^{q-l}\big\}. \end{aligned}$$

If we put

$$\begin{aligned} a_l &= \binom{2n-l-1}{n-1} 2^l, \\ b_q^{(m)} &= \binom{m}{q} (-1)^q, \\ c_t &= \hat{\rho}_0^t \big\{(-k)^{-t} - (k+2\hat{\rho}_0)^{-t}\big\}, \\ e_m &= i(i\hat{\rho}_0)^m p_m, \end{aligned}$$

we have

$$I_n^{(k)} = (2\rho_0)^{-2n} \sum_{m=0}^{d-1} e_m \sum_{l=1}^{n} \sum_{q=0}^{m} a_l b_q^{(m)} c_{l-q}. \tag{3.9}$$

We rewrite the sum (3.9) as

$$\begin{aligned} I_n^{(k)} = (2\rho_0)^{-2n} \Bigg[& \sum_{m\le n} e_m \Bigg\{ \sum_{l=1}^{m} c_{n-l} \sum_{q=0}^{l} a_{n-q} b_{l-q}^{(m)} + \sum_{l=m+1}^{n} c_{n-l} \sum_{q=0}^{m} a_{m+n-l-q} b_{m-q}^{(m)} \\ & + \sum_{l=n+1}^{n+m} c_{n-l} \sum_{q=0}^{m+n-l} a_{m+n-l-q} b_{m-q}^{(m)} \Bigg\} + \sum_{m>n} e_m \Bigg\{ \sum_{l=1}^{n} c_{l-m} \sum_{q=0}^{l} a_{l-q} b_{m-q}^{(m)} \\ & + \sum_{l=n+1}^{m} c_{l-m} \sum_{q=0}^{n} a_{n-q} b_{n+m-l-q}^{(m)} + \sum_{l=m+1}^{n+m} c_{l-m} \sum_{q=0}^{n+m-l} a_{n-q} b_{n+m-l-q}^{(m)} \Bigg\} \Bigg]. \end{aligned}$$

Since the sum $\sum_{k\ge1} I_n^{(k)}$ converges (see, (3.7)), the coefficients of c_t for $t \le 1$ must be disappeared. Hence we have

$$
\begin{aligned}
I_n^{(k)} &= (2\rho_0)^{-2n}\left[\sum_{m\le n} e_m\left\{\sum_{l=n-m}^{n} c_l \sum_{q=0}^{n-l} a_{q-l} b_q^{(m)} + \sum_{l=2}^{n-m-1} c_l \sum_{q=0}^{m} a_{q-l} b_q^{(m)}\right\}\right.\\
&\qquad\qquad \left.+\sum_{m>n} e_m \sum_{l=2}^{n} c_l \sum_{q=0}^{n-l} a_{q-l} b_q^{(m)}\right]\\
&= (2\rho_0)^{-2n}\sum_{l=2}^{n} c_l \sum_{m=1}^{d-1} e_m \sum_{q=0}^{\min(n-l,m)} a_{q-l} b_q^{(m)}\\
&= (2\rho_0)^{-2n}\sum_{l=2}^{n} A_l^{(n)}\{-(-k)^{-l}+(k+2\hat{\rho}_0)^{-l}\}. \qquad (3.10)
\end{aligned}
$$

Therefore, combining (3.6) and (3.10), we obtain

$$
\begin{aligned}
I_n^{(0)}+\sum_{k=1}^{\infty} I_n^{(k)} &= (2\rho_0)^{-2n}\left\{\sum_{l=2}^{n} A_l^{(n)}(1+(-1)^l)\zeta(l) - A_0^{(n)}\right.\\
&\qquad\qquad \left.+\sum_{l=2}^{n} A_l^{(n)}\left((1-(-1)^l)\zeta(l)-\sum_{k=1}^{2\hat{\rho}_0} k^{-l}\right)\right\}\\
&= (2\rho_0)^{-2n}\left\{\sum_{l=2}^{n} A_l^{(n)}\left(2\zeta(l)-\sum_{k=1}^{2\hat{\rho}_0} k^{-l}\right)-A_0^{(n)}\right\}. \qquad (3.11)
\end{aligned}
$$

This completes the proof of Theorem 2.1 □

4. The case $G = SL_2(\boldsymbol{R})$.

When $G = SL_2(\boldsymbol{R})$ and Γ is a co-compact torsion free discrete subgroup of $SL_2(\boldsymbol{R})$, Theorem 2.1 is written as

$$
\begin{aligned}
\zeta_\Delta(n) &= \sum_{k=0}^{n-1}(-1)^k\binom{2n-k-2}{n-1}\tilde{\gamma}_\Gamma^{(k)} - \binom{2n-1}{n-1}\\
&\quad + \frac{\mathrm{vol}(\Gamma\backslash H)}{2\pi}\left[\sum_{l=2}^{n-1}\left\{\binom{2n-l-1}{n-1}-2\binom{2n-l-2}{n-1}\right\}\zeta(l)+\zeta(n)\right], \qquad (4.1)
\end{aligned}
$$

where H is the upper half plane. In Section 4.1, we treat with the case when Γ is co-compact but have elliptic elements. Furthermore, in Section 4.2, we also consider the case when $\Gamma\backslash H$ is not compact, and give the formulas for $\Gamma = SL_2(\boldsymbol{Z})$ similar to (4.1).

4.1. The contributions of elliptic elements.

When Γ is co-compact and has elliptic elements, the trace formula is as follows.

$$
\begin{aligned}
\sum_{j=0}^{\infty}\hat{f}(r_j) &= \sum_{\gamma\in\mathrm{Hyp}(\Gamma)}\frac{\log N(\delta_\gamma)}{N(\gamma)^{1/2}-N(\gamma)^{-1/2}} f(\log N(\gamma)) + \frac{\mathrm{vol}(\Gamma\backslash H)}{4\pi}\int_{-\infty}^{\infty}\hat{f}(r) r\tanh\pi r\,dr\\
&\quad + \sum_{\sigma\in\mathrm{Ell}(\Gamma)}\sum_{k=1}^{\nu(\sigma)-1}\left(\nu(\sigma)\sin\frac{\pi k}{\nu(\sigma)}\right)^{-1}\int_{-\infty}^{\infty}\hat{f}(r)\frac{\cosh(1-2k/\nu(\sigma))\pi r}{\cosh\pi r}dr, \qquad (4.2)
\end{aligned}
$$

where $\mathrm{Ell}(\Gamma)$ is the set of the primitive elliptic conjugacy classes of Γ and $\nu(\sigma)$ is the order of $\sigma \in \mathrm{Ell}(\Gamma)$. Hence, similar to Theorem 2.1, we can obtain the relation between $\zeta_\Delta(n)$ and $\tilde{\gamma}_\Gamma^{(k)}$'s. In fact, when Γ has elliptic elements of order 2 and 3, the following formula holds.

THEOREM 4.1. *For $n \geq 2$, we have*

$$\begin{aligned}
\zeta_\Delta(n) &= \sum_{k=0}^{n-1}(-1)^k\binom{2n-k-2}{n-1}\tilde{\gamma}_\Gamma^{(k)} - \binom{2n-1}{n-1}\\
&\quad + \frac{\mathrm{vol}(\Gamma\backslash H)}{2\pi}\left[\sum_{l=2}^{n-1}\left\{\binom{2n-l-1}{n-1} - 2\binom{2n-l-2}{n-1}\right\}\zeta(l) + \zeta(n)\right]\\
&\quad + \frac{\nu_2}{2}\sum_{l=1}^{n}\binom{2n-l-1}{n-1}\xi(l) + \frac{\nu_3}{3\sqrt{3}}\left[\sum_{m=1}^{n}\frac{(-1)^{[m/2]}}{m!}\binom{2n-m-1}{n-1}\left(\frac{\pi}{3}\right)^m\alpha_m\right.\\
&\quad \left. + 2\sum_{l=1}^{[n/2]}\sum_{m=0}^{n-2l}\frac{(-1)^{[m/2]}}{m!}\alpha_m\binom{2n-2l-m-1}{n-1}\xi(2l) + 2\sum_{l=1}^{n}\binom{2n-l-1}{n-1}\alpha_{l+1}\eta(l)\right], \qquad (4.3)
\end{aligned}$$

where

$$\alpha_m = \begin{cases} \sqrt{3} & \text{if } m \text{ is even,}\\ 1 & \text{if } m \text{ is odd,}\end{cases}$$

$$\xi(l) = \sum_{k\geq 1}(-1)^{k-1}k^{-l} = \begin{cases} \log 2 & \text{if} \quad l=1,\\ (1-2^{1-l})\zeta(l) & \text{if} \quad l \geq 2,\end{cases}$$

$$\eta(l) = \begin{cases} \displaystyle\sum_{k\geq 1}(-1)^{k-1}\cos\frac{\pi k}{3}k^{-l} & \text{if } l \text{ is odd,}\\ \displaystyle\sum_{k\geq 1}(-1)^{k-1}\sin\frac{\pi k}{3}k^{-l} & \text{if } l \text{ is even,}\end{cases}$$

and ν_2, ν_3 are the number of the primitive elliptic conjugacy classes of order 2, 3 *respectively.*

4.2. Non-compact case.

When $\Gamma\backslash H$ is not compact but $\mathrm{vol}(\Gamma\backslash H) < \infty$, the trace formula reads as follows.

$$\begin{aligned}
\sum_{j=0}^{\infty}\hat{f}(r_j) &= \sum_{\gamma\in\mathrm{Hyp}(\Gamma)}\frac{\log N(\delta_\gamma)}{N(\gamma)^{1/2}-N(\gamma)^{-1/2}}f(\log N(\gamma)) + \frac{\mathrm{vol}(\Gamma\backslash H)}{4\pi}\int_{-\infty}^{\infty}\hat{f}(r)r\tanh\pi r\,dr\\
&\quad + \sum_{\sigma\in\mathrm{Ell}(\Gamma)}\sum_{k=1}^{\nu(\sigma)-1}\left(\nu(\sigma)\sin\frac{\pi k}{\nu(\sigma)}\right)^{-1}\int_{-\infty}^{\infty}\hat{f}(r)\frac{\cosh(1-2k/\nu(\sigma))\pi r}{\cosh\pi r}dr\\
&\quad -\nu_\infty f(0)\log 2 - \frac{\nu_\infty}{2\pi}\int_{-\infty}^{\infty}\hat{f}(r)\frac{\Gamma'(1+ir)}{\Gamma(1+ir)}dr\\
&\quad + \frac{\hat{f}(0)}{4}\mathrm{Tr}\big(I-\Phi(1/2)\big) + \frac{1}{4\pi}\int_{-\infty}^{\infty}\hat{f}(r)\frac{\varphi'(1/2+ir)}{\varphi(1/2+ir)}dr, \qquad (4.4)
\end{aligned}$$

where $\Phi(s)$ is the scattering matrix, $\varphi(s) = \det \Phi(s)$ and ν_∞ is the number of cusps. Since the explicit expressions of $\Phi(s)$ are not obtained in many cases, it is difficult to get the formulas similar to (4.1). However, when Γ is $SL_2(\boldsymbol{Z})$ and the congruence subgroup of $SL_2(\boldsymbol{Z})$, the scattering matrix is obtained explicitly (see [**He1**] and [**Hu**]), hence we can obtain the formulas. Now, we denote by

$$\Gamma_0(N) := \{(a_{ij})_{i,j=1,2} \in SL_2(\boldsymbol{Z}) | a_{21} \equiv 0 \bmod N\},$$

$$\Gamma_1(N) := \{(a_{ij})_{i,j=1,2} \in SL_2(\boldsymbol{Z}) | a_{11} \equiv a_{22} \equiv \pm 1, a_{21} \equiv 0 \bmod N\},$$

$$\Gamma(N) := \{(a_{ij})_{i,j=1,2} \in SL_2(\boldsymbol{Z}) | a_{11} \equiv a_{22} \equiv \pm 1, a_{12}, a_{21} \equiv 0 \bmod N\},$$

and, for simplicity, we assume that N is odd and square free.

THEOREM 4.2. *Let Γ be $SL_2(\boldsymbol{Z})$ or a congruence subgroup of $SL_2(\boldsymbol{Z})$. For $n \geq 2$, we have*

$$\begin{aligned}
\zeta_\Delta(n) = &\sum_{k=0}^{n-1} (-1)^k \binom{2n-k-2}{n-1} \tilde{\gamma}_\Gamma^{(k)} - \binom{2n-1}{n-1} \\
&+ \frac{\mathrm{vol}(\Gamma \backslash H)}{2\pi} \left[\sum_{l=2}^{n-1} \left\{ \binom{2n-l-1}{n-1} - 2\binom{2n-l-2}{n-1} \right\} \zeta(l) + \zeta(n) \right] \\
&+ \frac{\nu_2}{2} \sum_{l=1}^{n} \binom{2n-l-1}{n-1} \xi(l) + \frac{\nu_3}{3\sqrt{3}} \left[\sum_{m=1}^{n} \frac{(-1)^{[m/2]}}{m!} \binom{2n-m-1}{n-1} \left(\frac{\pi}{3}\right)^m \alpha_m \right. \\
&\left. + 2 \sum_{l=1}^{[n/2]} \sum_{m=0}^{n-2l} \frac{(-1)^{[m/2]}}{m!} \alpha_m \binom{2n-2l-m-1}{n-1} \xi(2l) + 2 \sum_{l=1}^{n} \binom{2n-l-1}{n-1} \alpha_{l+1} \eta(l) \right] \\
&+ \nu_\infty \left[\sum_{l=2}^{n} \binom{2n-l-1}{n-1} 2^l \zeta(l) + \binom{2n-2}{n-1} (\log 2\pi + \gamma) - 2^{2n-1} \right] + J_\Gamma^{(n)}, \qquad (4.5)
\end{aligned}$$

where ν_∞ is the number of cusps and $J_\Gamma^{(n)}$ is as follows.

$$J_{SL_2(\boldsymbol{Z})}^{(n)} = \sum_{l=0}^{n-1} \frac{2(-2)^l}{l!} \binom{2n-l-2}{n-1} \left(\frac{\zeta'}{\zeta}\right)^{(l)} (2),$$

$$J_{\Gamma_0(N)}^{(n)} = \nu_\infty \left[\sum_{l=0}^{n-1} \frac{2^l}{l!} \binom{2n-l-2}{n-1} \left\{ 2(-1)^l \left(\frac{\zeta'}{\zeta}\right)^{(l)} (2) - \sum_{p|N} \sum_{k=p^m} \frac{\Lambda(k)}{k^2} (\log k)^l \right\} - \binom{2n-2}{n-1} \log N \right],$$

$$\begin{aligned}
J_{\Gamma_1(N)}^{(n)} = &-2^{\omega(N)+2n-2} \\
&+ \nu_\infty \left[2^{2n-2} - \frac{3}{2} \binom{2n-2}{n-1} \log N + \sum_{l=0}^{n-1} \frac{2^{l+1}}{l!} \binom{2n-l-2}{n-1} \sum_{k \equiv \pm 1 \bmod N} \frac{\Lambda(k)}{k^2} (\log k)^l \right. \\
&\left. + \sum_{p|N} \frac{c(N/p)}{p-1} \left\{ \binom{2n-3}{n-1} \log p + \sum_{l=0}^{n-1} \frac{2^l}{l!} \binom{2n-l-2}{n-1} \sum_{\substack{k \equiv \pm 1 \bmod N/p \\ k=p^m}} \frac{\Lambda(k)}{k^2} (\log k)^l \right\} \right],
\end{aligned}$$

$$J^{(n)}_{\Gamma(N)} = -2^{2n-2}\prod_{p|N}(p+1)$$
$$+2\nu_\infty\Bigg[2^{2n-3} - \binom{2n-2}{n-1}\log N + \sum_{l=0}^{n-1}\frac{2^l}{l!}\binom{2n-l-2}{n-1}\sum_{k\equiv\pm1 \bmod N}\frac{\Lambda(k)}{k^2}(\log k)^l$$
$$+\sum_{p|N}\frac{c(N/p)}{p^2-1}\left\{\binom{2n-3}{n-1}\log p + \sum_{l=0}^{n-1}\frac{2^l}{l!}\binom{2n-l-2}{n-1}\sum_{\substack{k\equiv\pm1 \bmod N/p\\ k=p^m}}\frac{\Lambda(k)}{k^2}(\log k)^l\right\}\Bigg].$$

Here, $c(1)=2$ *and* $c(k)=1$ *when* $k\geq 2$.

5. Numerical estimates of $\boldsymbol{\lambda_1}$ for quaternion groups.

In this section, we give a numerical computation of $\zeta_\Delta(n)$ for the quaternion groups by using Theorem 4.1, and estimate the first eigenvalue λ_1. Now, we define the quaternion group.

Let a, b be positive integers which are relatively prime and square free, and B the quaternion algebra over $\boldsymbol{Q}$ defined by $B=\boldsymbol{Q}+\boldsymbol{Q}\alpha+\boldsymbol{Q}\beta+\boldsymbol{Q}\alpha\beta$, where $\alpha^2=a$, $\beta^2=b$, $\alpha\beta=-\beta\alpha$. For an element $q=q_0+q_1\alpha+q_2\beta+q_3\alpha\beta$ ($q_i\in\boldsymbol{Q}$), we define $\overline{q}=q_0-q_1\alpha-q_2\beta-q_3\alpha\beta$, $n(q)=q\overline{q}=q_0^2-q_1^2a-q_2^2b+q_3^2ab$ and $\mathrm{tr}\,q=q+\overline{q}=2q_0$. We choose and fix a maximal order $\mathscr{O}$ of B. Let B^1 (resp. $\mathscr{O}^1$) be the group consisting of all elements q of B (resp. $\mathscr{O}$) with $n(q)=1$. The group $\mathscr{O}^1$ can be identified with a discrete subgroup $\Gamma_{\mathscr{O}}$ of $SL_2(\boldsymbol{R})$ by the map.

$$q\mapsto\begin{pmatrix} q_0+q_1\sqrt{a} & q_2\sqrt{b}+q_3\sqrt{ab}\\ q_2\sqrt{b}-q_3\sqrt{ab} & q_0-q_1\sqrt{a}\end{pmatrix} \tag{5.1}$$

The discriminant d_B of B is defined by $d_B:=|\det(\mathrm{tr}(u_i,u_j))|^{1/2}$, where $\{u_i\}$ is the basis of $\mathscr{O}$ over $\boldsymbol{Z}$. The number d_B is independent of the choice of $\mathscr{O}$ and $\{u_i\}$, and equals the product of prime numbers which ramify at $B/\boldsymbol{Q}$. It is known that the number of prime factors of d_B is even. The group $\Gamma_{\mathscr{O}}$ is a co-compact subgroup of $SL_2(\boldsymbol{R})$ and may have elliptic elements of order 2 or 3. The values ν_2, ν_3 and $\mathrm{vol}(\Gamma_{\mathscr{O}}\backslash H)$ are respectively determined as follows (see, e.g. [**He2**], [**Sh**]).

$$\mathrm{vol}(\Gamma_{\mathscr{O}}\backslash H)=\frac{\pi}{3}\prod_{p|d_B}(p-1),\quad \nu_2=\prod_{p|d_B}\left(1-\left(\frac{-1}{p}\right)\right),\quad \nu_3=\prod_{p|d_B}\left(1-\left(\frac{-3}{p}\right)\right), \tag{5.2}$$

where

$$(-1/p)=\begin{cases} 0 & (p=2),\\ 1 & (p\equiv 1 \bmod 4),\\ -1 & (p\equiv 3 \bmod 4),\end{cases}\qquad (-3/p)=\begin{cases} 0 & (p=3),\\ 1 & (p\equiv 1 \bmod 3),\\ -1 & (p\equiv 2 \bmod 3).\end{cases}$$

Hence, Theorem 4.1 applies in this case. In the formula of Theorem 4.1, the terms other than $\tilde{\gamma}^{(k)}_\Gamma$ are exactly computable, the computation of $\zeta_\Delta(n)$ consequently is reduced to that of $\tilde{\gamma}^{(k)}_\Gamma$. The (higher) Euler-Selberg constants are also computed as follows.

When $G=SL_2(\boldsymbol{R})$, $\tilde{\gamma}^{(k)}_\Gamma$'s are expressed as

$$\tilde{\gamma}_{\Gamma}^{(k)} = \frac{(-1)^k}{k!} \lim_{x\to\infty} \left\{ \sum_{\substack{\gamma\in\mathrm{Hyp}(\Gamma)\\ N(\gamma)<x}} \frac{\log N(\delta_\gamma)}{N(\gamma)-1} (\log N(\gamma))^k - \frac{(\log x)^{k+1}}{k+1} \right\}, \tag{5.3}$$

where $\mathrm{Hyp}(\Gamma)$ is the set of the hyperbolic conjugacy classes of Γ (see [**HIKW**]). When Γ is the quaternion groups of discriminant d_B, according to [**AKN**], the expressions of $\tilde{\gamma}_{\Gamma}^{(k)}$ (5.3) are rewritten by

$$\begin{aligned}\tilde{\gamma}_{\Gamma}^{(k)} = \frac{(-1)^k}{k!} \lim_{T\to\infty} \Biggl\{ \sum_{t=3}^{T} \sum_{\substack{u;u^2|t^2-4\\ d(t,u)\equiv 0,1 \bmod 4}} & h\bigl(d(t,u)\bigr)\lambda\bigl(d(t,u)\bigr) \\ & \times \frac{2\log\varepsilon_0(t,u)}{\varepsilon(t)^2-1} (2\log\varepsilon(t))^k - \frac{(2\log T)^{k+1}}{k+1} \Biggr\},\end{aligned} \tag{5.4}$$

where

$$d(t,u) = \frac{t^2-4}{u^2},$$

$$\varepsilon(t) = \frac{1}{2}\bigl(t+\sqrt{t^2-4}\bigr) = \frac{1}{2}\bigl(t+u\sqrt{d(t,u)}\bigr),$$

$$\varepsilon_0(t,u) = \min\left\{ \frac{1}{2}\bigl(t_0+u_0\sqrt{d(t,u)}\bigr) \,\middle|\, \left(\frac{1}{2}\bigl(t_0+u_0\sqrt{d(t,u)}\bigr)\right)^k = \varepsilon(t), \quad \exists k\geq 1 \right\},$$

$$\lambda(d) := \begin{cases} 0 & \text{if } p^2|d \text{ and } d/p^2\equiv 0,1 \bmod 4 \text{ for some } p|d_B, \\ \displaystyle\prod_{p|d_B}\left(1-\left(\frac{\boldsymbol{Q}(\sqrt{d})}{p}\right)\right) & \text{otherwise } \bigl((*/p) \text{ is the Artin symbol}\bigr), \end{cases}$$

and $h(d)$ is the class number of the binary quadratic forms of discriminant $d>0$ in the narrow sense. Since the algorithm of computation of the class number is well-known (see, for example, [**Sc**] or [**Wa**]), we can compute the approximate value of $\tilde{\gamma}_{\Gamma}^{(k)}$. Hence, applicating Theorem 4.1, we can also compute $\zeta_\Delta(n)$. Now, we consider $\zeta_\Delta(n)^{-1/n}$ and $\zeta_\Delta(m)/\zeta_\Delta(m+1)$ for $n,m\geq 2$. Since

$$\begin{aligned}\zeta_\Delta(n)^{-1/n} &= \lambda_1\left(1+\left(\frac{\lambda_1}{\lambda_2}\right)^n+\left(\frac{\lambda_1}{\lambda_3}\right)^n+\dots\right)^{-1/n}, \\ \frac{\zeta_\Delta(m)}{\zeta_\Delta(m+1)} &= \lambda_1\frac{1+(\lambda_1/\lambda_2)^m+(\lambda_1/\lambda_3)^m+\dots}{1+(\lambda_1/\lambda_2)^{m+1}+(\lambda_1/\lambda_3)^{m+1}+\dots},\end{aligned}$$

we have

$$\zeta_\Delta(n)^{-1/n} < \lambda_1 < \frac{\zeta_\Delta(m)}{\zeta_\Delta(m+1)} \quad (\forall n,m\geq 2), \tag{5.5}$$

$$\zeta_\Delta(n)^{-1/n}, \frac{\zeta_\Delta(m)}{\zeta_\Delta(m+1)} \to \lambda_1 \quad \text{as} \quad n,m\to\infty. \tag{5.6}$$

Computing $\zeta_\Delta(n)$ and using the fact above, we can estimate λ_1 numerically.

We denote by

$$\tilde{\gamma}_{\Gamma}^{(k)}(T) := \frac{(-1)^k}{k!}\left\{\sum_{t=3}^{T} \sum_{\substack{u;u^2|t^2-4 \\ d(t,u)\equiv 0,1 \bmod 4}} h\big(d(t,u)\big)\lambda\big(d(t,u)\big) \right.$$
$$\left. \times \frac{2\log\varepsilon_0(t,u)}{\varepsilon(t)^2-1}(2\log\varepsilon(t))^k - \frac{(2\log T)^{k+1}}{k+1}\right\},$$
$$\zeta_\Delta(n,T) := \sum_{k=0}^{n-1}(-1)^k\binom{2n-k-2}{n-1}\tilde{\gamma}_{\Gamma}^{(k)}(T) - \binom{2n-1}{n-1} + \cdots \quad \text{(see Theorem 4.1)},$$

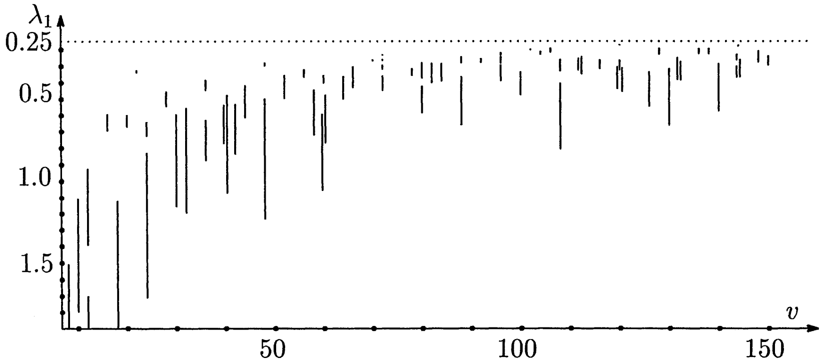
and compute these values for $T \leq 3.0\times 10^6$ (in this computation, we use 'C++'). When we write

$$d := d_B = p_1 p_2 \cdots p_{2l},$$
$$v := \mathrm{vol}(\Gamma\backslash H)/(\pi/3) = (p_1-1)(p_2-1)\cdots(p_{2l}-1),$$
$$N := \max\left\{ n \geq 2 \,\middle|\, 2.0\times 10^6 \leq \forall T \leq 3.0\times 10^6, \quad \left|\frac{\zeta_\Delta(n,3.0\times 10^6)-\zeta_\Delta(n,T)}{\zeta_\Delta(n,3.0\times 10^6)}\right| < 0.01 \right\},$$
$$L := \zeta_\Delta(N,3.0\times 10^6)^{-1/N},$$
$$R := \zeta_\Delta(N-1,3.0\times 10^6)/\zeta_\Delta(N,3.0\times 10^6),$$

the values above are computed as follows.

d	v	v_2	v_3	N	L	R	d	v	v_2	v_3	N	L	R
6	2	2	2	2	4.922	—	93	60	4	0	5	0.452	0.502
10	4	0	4	3	2.278	3.031	94	46	2	4	7	0.323	0.324
14	6	2	0	2	2.251	—	95	72	0	0	6	0.393	0.419
15	8	0	2	2	1.605	—	106	52	0	4	5	0.453	0.595
21	12	4	0	3	1.022	1.494	111	72	0	0	6	0.331	0.336
22	10	2	4	3	1.204	1.898	115	88	0	4	4	0.460	0.761
26	12	0	0	3	1.797	3.506	118	58	2	4	4	0.540	0.822
33	20	4	2	4	0.696	0.770	119	96	0	0	7	0.317	0.336
34	16	0	4	4	0.691	0.796	122	60	0	0	4	0.689	1.154
35	24	0	0	3	0.925	1.810	123	80	0	2	5	0.518	0.685
38	18	2	0	3	1.218	2.428	129	84	4	0	5	0.382	0.490
39	24	0	0	4	0.738	0.827	133	108	4	0	5	0.357	0.431
46	22	2	4	5	0.422	0.439	134	66	2	0	4	0.401	0.532
51	32	0	2	3	0.652	1.295	141	92	4	2	6	0.353	0.379
55	40	0	4	4	0.633	0.872	142	70	2	4	6	0.359	0.369
57	36	4	0	4	0.480	0.548	143	120	0	0	6	0.362	0.426
58	28	0	4	4	0.550	0.646	145	112	0	4	5	0.341	0.449
62	30	2	0	3	0.692	1.255	146	72	0	0	6	0.355	0.367
65	48	0	0	3	0.596	1.332	155	120	0	0	5	0.408	0.555
69	44	4	2	4	0.514	0.716	158	78	2	0	5	0.412	0.457
74	36	0	0	4	0.728	0.975	159	104	0	2	7	0.311	0.333
77	60	4	0	4	0.572	0.869	161	132	4	0	5	0.370	0.485
82	40	0	4	3	0.573	1.171	166	82	2	4	5	0.381	0.503
85	64	0	4	5	0.459	0.600	177	116	4	2	5	0.361	0.467
86	42	2	0	4	0.627	0.935	178	88	0	4	5	0.342	0.384
87	56	0	2	5	0.417	0.467	183	120	0	0	5	0.403	0.539
91	72	0	0	5	0.455	0.547	185	144	0	0	6	0.328	0.365

d	v	v_2	v_3	N	L	R	d	v	v_2	v_3	N	L	R
187	160	0	4	5	0.391	0.567	249	164	4	2	10	0.258	0.261
194	96	0	0	6	0.339	0.384	253	220	4	4	5	0.328	0.502
201	132	4	0	5	0.348	0.484	254	126	2	0	5	0.433	0.648
202	100	0	4	5	0.433	0.575	259	216	0	0	6	0.330	0.426
203	168	0	0	5	0.389	0.578	262	130	2	4	4	0.413	0.762
205	160	0	4	6	0.337	0.424	265	208	0	4	6	0.291	0.372
206	102	2	0	8	0.295	0.304	267	176	0	2	9	0.255	0.256
209	180	4	0	5	0.338	0.483	274	136	0	4	7	0.293	0.327
210	48	0	0	6	0.377	0.399	278	138	2	0	4	0.372	0.658
213	140	4	2	4	0.382	0.676	287	240	0	0	7	0.294	0.333
214	106	2	4	7	0.289	0.319	291	192	0	0	6	0.344	0.434
215	168	0	0	10	0.259	0.263	295	232	0	4	6	0.316	0.406
217	180	4	0	7	0.271	0.286	298	148	0	4	6	0.308	0.379
218	108	0	0	4	0.501	0.907	299	264	0	0	8	0.267	0.284
219	144	0	0	6	0.397	0.469	301	252	4	0	6	0.280	0.353
221	192	0	0	6	0.319	0.388	302	150	2	0	6	0.335	0.396
226	112	0	4	6	0.350	0.425	303	200	0	2	6	0.359	0.485
235	184	0	4	5	0.329	0.461	305	240	0	0	5	0.319	0.444
237	156	4	0	7	0.323	0.373	309	204	4	0	7	0.312	0.374



The dotted line $\cdots$ of the figure above expresses the bound of Selberg's conjecture $\lambda_1 \geq 1/4$ for the congruence subgroups of $SL_2(\boldsymbol{Z})$ (see [**Se**]). In the data above, all L's are bigger than $1/4$, hence it could be suggested that the Selberg conjecture holds for the quaternion groups of "small" discriminant (or "small" volume). On the other hand, in many case of the "large" discriminant (or "large" volume), we obtain the data that $L < 1/4$ and $R > 1/4$. For example, in the case $d = 30030 = 2 \times 3 \times 5 \times 7 \times 11 \times 13$ ($v = 5760$, $v_2 = v_3 = 0$), we have $L \fallingdotseq 0.214$ and $R \fallingdotseq 0.301$ for $N = 10$, and in the case $d = 255255 = 3 \times 5 \times 7 \times 11 \times 13 \times 17$ ($v = 92160$, $v_2 = v_3 = 0$), we have $L \fallingdotseq 0.189$ and $R \fallingdotseq 0.283$ for $N = 14$. In such cases, we cannot confirm whether the Selberg conjecture holds.

REMARK 5.1. In [**Se**], $\lambda_1 \geq 3/16$ was proved for the congruence subgroups. Then, several studies have been made on the estimate of λ_1 and the best estimate at the present time is $\lambda_1 \geq 975/4096 = 0.238\ldots$ by [**Ki**] (see, also [**LRS**] and [**KS**]).

REMARK 5.2. In [**H**], we compute $\tilde{\gamma}^{(0)}_{SL_2(\boldsymbol{Z})}$ numerically by using the correspondence between the primitive hyperbolic conjugacy classes of $SL_2(\boldsymbol{Z})$ and the equivalence classes of the primitive indefinite binary quadratic forms. Hence we can also compute $\zeta_\Delta(n,T)$ by using Theorem 4.2. However, the values $\zeta_\Delta(n)$ are very small, because λ_1, λ_2, ... are large:

$$\lambda_1 = 91.5229\cdots, \quad \lambda_2 = 148.4319\cdots, \quad \lambda_3 = 190.1315\cdots, \quad \ldots \quad \text{(see [}\mathbf{He1}\text{])}.$$

Thus it is hard to obtain the approximate values of $\zeta_\Delta(n)$'s by computing $\zeta_\Delta(n,T)$'s.

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