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Weak type (1, 1) bounds for a class of the Littlewood-Paley operators

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Abstract. In this paper the authors give the weak type (1,1) boundedness and the L^p boundedness of a class of the parametrized Littlewood-Paley operators. These conclusions improve and complete some known results.

1. Introduction.

It is well known that the Littlewood-Paley operators, such as the Littlewood-Paley g-function, the area integral, g_λ^* function and the Marcinkiewicz integral, play very important roles in harmonic analysis and PDE. Historically, the Littlewood-Paley operators of higher dimension were first introduced by Stein in 1958 ([**S1**]). In the same paper, Stein gave the weak type (1,1) boundedness of the Marcinkiewicz integral and area integral. In 1970, Fefferman [**Fe**] proved that the Littlewood-Paley g_λ^* function is weak type (p, p) if $1 < p < 2$ and $\lambda = 2/p$. Recently, Fan and Sato [**FS**] obtained the weak type (1,1) boundedness of the Marcinkiewicz integral with rough kernel.

The aim of this paper is to discuss the weak type (1,1) boundedness for the parametrized Littlewood-Paley g_λ^* function and the area integral under a weaker condition assumed on the kernel. As a corollary of the above conclusions, we will also give the type (p, p) boundedness of the parametrized g_λ^* function and the area integral for $1 < p < 2$ under a weaker condition. Let us first give some definitions and known results on the parametrized Littlewood-Paley operators. Suppose that $\Omega \in L^1(S^{n-1})$ is homogeneous of degree zero on $\boldsymbol{R}^n$ and satisfies

$$\int_{S^{n-1}} \Omega(x') d\sigma(x') = 0, \tag{1.1}$$

where S^{n-1} denotes the unit sphere of $\boldsymbol{R}^n (n \geq 2)$ equipped with the area measure $d\sigma = d\sigma(x')$. Let $\varphi^\rho(x) = \Omega(x)|x|^{-n+\rho}\chi_B(x)$, where $\rho > 0$ and B denotes the unit ball in $\boldsymbol{R}^n$. Then the parametrized Littlewood-Paley operator $\mu_\lambda^{*,\rho}$ is defined by

$$\mu_\lambda^{*,\rho}(f)(x) = \left(\iint_{\boldsymbol{R}_+^{n+1}} \left(\frac{t}{t + |x-y|} \right)^{\lambda n} \left| \varphi_t^\rho * f(y) \right|^2 \frac{dydt}{t^{n+1}} \right)^{1/2},$$

where $\lambda > 1$ and $\varphi_t^\rho(x) = (1/t^n)\varphi^\rho(x/t)$. Clearly, if take $\rho = 1$ and $\Omega \in \mathscr{C}^\infty(S^{n-1})$, then the operator $\mu_\lambda^{*,\rho}$ is the classical Littlewood-Paley g_λ^* function. In 1999, Sakamoto and Yabuta [**SY**] studied the $L^p(\boldsymbol{R}^n)$ boundedness of $\mu_\lambda^{*,\rho}$. Their results may be summarized as follows.

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THEOREM A ([**SY**]). *If Ω satisfies* (1.1) *and the* $\mathrm{Lip}_\alpha(S^{n-1})$ $(0<\alpha\le 1)$ *condition, then*
(i) *for* $\rho>0$ *and* $2\le p<\infty$, $\|\mu_\lambda^{*,\rho}(f)\|_p\le C_{n,p,\rho,\alpha}\|f\|_p$;
(ii) *for* $0<\rho\le n/2$, $2n/(n+2\rho)<p<2$ *and* $\lambda>2/p$, $\|\mu_\lambda^{*,\rho}(f)\|_p\le C_{n,p,\rho,\alpha}\|f\|_p$;
(iii) *for* $\rho>n/2$, $1<p<2$ *and* $\lambda>2/p$, $\|\mu_\lambda^{*,\rho}(f)\|_p\le C_{n,p,\rho,\alpha}\|f\|_p$;
(iv) *for* $0<\rho\le n/2$ *and* $1\le p\le 2n/(n+2\rho)$, *there exists a function* $\Omega\in\mathrm{Lip}_\alpha(S^{n-1})$ *satisfies* (1.1), *such that* $\mu_\lambda^{*,\rho}$ *is not bounded on* $L^p(\boldsymbol{R}^n)$.

In 2002, Ding, Lu and Yabuta [**DLY**] improved the conclusion (i) in Theorem A.

THEOREM B ([**DLY**]). *Suppose that* $\Omega\in L\log^+L(S^{n-1})$ *satisfies* (1.1). *Then for* $\rho>0$ *and* $2\le p<\infty$, $\|\mu_\lambda^{*,\rho}(f)\|_p\le C_{n,p,\rho}\|f\|_p$.

Now we give the definition of the integral modulus of continuity.

DEFINITION 1. Let $\Omega(x')\in L^q(S^{n-1})$, $q\ge 1$. Then the integral modulus $\omega_q(\delta)$ of continuity of order q of Ω is defined by

$$\omega_q(\delta)=\sup_{\|\gamma\|\le\delta}\left(\int_{S^{n-1}}|\Omega(\gamma x')-\Omega(x')|^q d\sigma(x')\right)^{1/q},$$

where γ denotes a rotation on S^{n-1} and $\|\gamma\|=\sup_{x'\in S^{n-1}}|\gamma x'-x'|$.

The main result in this paper is the following theorem.

THEOREM 1. *Let* $\Omega\in L^2(S^{n-1})$ *satisfies* (1.1) *and*

$$\int_0^1\frac{\omega_2(\delta)}{\delta}(1+|\log\delta|)^\sigma d\delta<\infty,\qquad \sigma>1. \tag{1.2}$$

Then for $\rho>n/2$ *and* $\lambda>2\rho/n+1/n+1$, *there exists a constant* $C=C_{n,\rho,\sigma}$ *such that for all* $\beta>0$ *and* $f\in L^1(\boldsymbol{R}^n)$,

$$|\{x\in\boldsymbol{R}^n:\ \mu_\lambda^{*,\rho}(f)(x)>\beta\}|\le\frac{C}{\beta}\|f\|_1.$$

REMARK 1. Notice that for any $0<\alpha\le 1$ and $1<q\le\infty$, the following including relationship

$$\mathrm{Lip}_\alpha(S^{n-1})\subsetneq L^q(S^{n-1})\subsetneq L\log^+L(S^{n-1})\subsetneq L^1(S^{n-1}) \tag{1.3}$$

holds. Hence, applying the Marcinkiewicz interpolation theorem (see [**S2**]) between Theorem 1 and Theorem B, we may obtain immediately the $L^p(\boldsymbol{R}^n)$ boundedness of the operator $\mu_\lambda^{*,\rho}$ for $1<p<2$.

COROLLARY 1. *Let* $\Omega\in L^2(S^{n-1})$ *satisfies* (1.1) *and* (1.2). *Then for* $\rho>n/2$ *and* $\lambda>2\rho/n+1/n+1$, $\mu_\lambda^{*,\rho}$ *is of type* (p,p) *for* $1<p<2$.

REMARK 2. The conclusion of Theorem 1 does not hold if relaxing the restriction $\rho>n/2$ to $0<\rho<n/2$ for $n\ge 2$. Otherwise, when Ω satisfies (1.1) and (1.2) $\mu_\lambda^{*,\rho}$ is of weak type (1,1) for $0<\rho<n/2$ and $n\ge 2$, then by the Marcinkiewicz interpolation theorem $\mu_\lambda^{*,\rho}$ is also of type (p,p) for $0<\rho<n/2$, $n\ge 2$ and $1<p<2$. However, this contradicts with the conclusions (iv) in Theorem A.

Another parametrized Littlewood-Paley operator μ_S^ρ, which is related to the area integral, is defined by

$$\mu_S^\rho(f)(x) = \left(\iint_{\Gamma(x)} \left|\varphi_t^\rho * f(y)\right|^2 \frac{dydt}{t^{n+1}} \right)^{1/2},$$

where $\Gamma(x) = \{(y,t) \in \boldsymbol{R}_+^{n+1} : |x-y| < t\}$. The following conclusions are about the $L^p\,(p>1)$ boundedness of the operator μ_S^ρ.

THEOREM C ([**SY**]). *If Ω satisfies* (1.1) *and the* $\mathrm{Lip}_\alpha(S^{n-1})\,(0<\alpha\le 1)$ *condition. Then*

(i) *for $\rho>0$ and $2\le p<\infty$, $\|\mu_S^\rho(f)\|_p \le C_{n,p,\rho,\alpha}\|f\|_p$;*
(ii) *for $0<\rho\le n/2$ and $2n/(n+2\rho)<p<2$, $\|\mu_S^\rho(f)\|_p \le C_{n,p,\rho,\alpha}\|f\|_p$;*
(iii) *for $\rho>n/2$ and $1<p<2$, $\|\mu_S^\rho(f)\|_p \le C_{n,p,\rho,\alpha}\|f\|_p$;*
(iv) *for $0<\rho\le n/2$ and $1\le p\le 2n/(n+2\rho)$, there exists a function $\Omega\in\mathrm{Lip}_\alpha(S^{n-1})$ satisfies* (1.1)*, such that μ_S^ρ is not bounded on $L^p(\boldsymbol{R}^n)$.*

Since for any $x\in\boldsymbol{R}^n$, $\mu_S^\rho(f)(x)\le 2^{\lambda n}\mu_\lambda^{*,\rho}(f)(x)$ (see [**SY**]), hence the following theorem is the direct result of Theorem 1 and Corollary 1.

THEOREM 2. *Let $\Omega\in L^2(S^{n-1})$ satisfies* (1.1) *and* (1.2)*. Then for $\rho>n/2$ the operator μ_S^ρ is both of weak type $(1,1)$ and of type (p,p) for $1<p<2$.*

REMARK 3. Notice that if $\Omega\in\mathrm{Lip}_\alpha(S^{n-1})$ for $0<\alpha\le 1$, then Ω satisfies also the condition (1.2) (see [**DLX**, Remark 2]). Therefore, Theorem 2 is improvement and extension of the conclusion (iii) in Theorem C.

REMARK 4. With the same reason shown in Remark 2, the operator μ_S^ρ is not of weak type $(1,1)$ if $\Omega\in L^2(S^{n-1})$ satisfies (1.1) and (1.2) and $0<\rho<n/2$ for $n\ge 2$.

2. Proof of Theorem 1.

We begin with recalling the following lemma.

LEMMA 2.1. *Suppose that $\rho>0$, Ω is homogeneous of degree zero and satisfies the L^2-Dini condition. If there exists a constant $0<\theta<1/2$ such that $|x|<\theta R$, then we have*

$$\left(\int_{R<|y|<2R} \left| \frac{\Omega(y-x)}{|y-x|^{n-\rho}} - \frac{\Omega(y)}{|y|^{n-\rho}} \right|^2 dy \right)^{1/2} \le CR^{n/2-(n-\rho)} \left\{ \frac{|x|}{R} + \int_{|x|/2R<\delta<|x|/R} \frac{\omega_2(\delta)}{\delta}\, d\delta \right\},$$

where the constant $C>0$ is independent of R and x.

See [**DL**] for the case $0<\rho<n$ and the proof is trivial for the case $\rho\ge n$.

Now let us turn to the proof of Theorem 1. For $f\in L^1(\boldsymbol{R}^n)$ and any $\beta>0$, by the Calderón-Zygmund decomposition (see [**S2**]) we have the following conclusions:

(i) $\boldsymbol{R}^n = F\bigcup E$ with $F\bigcap E=\varnothing$;
(ii) $E=\cup_k Q_k$, where $\{Q_k\}$ is a sequence of the cubes, whose interiors are disjoint;
(iii) $|f|\le\beta$ a.e. $x\in F$;

(iv) $\beta < (1/|Q_k|)\int_{Q_k}|f|dx \le 2^n\beta$ for every k;
(v) $|E| \le (C/\beta)\int_{\boldsymbol{R}^n}|f|dx$.

Denote

$$u(x) = \begin{cases} f(x), & \text{for} \quad x \in F \\ \dfrac{1}{|Q_k|}\displaystyle\int_{Q_k} f(y)dy, & \text{for} \quad x \in Q_k, \end{cases}$$

and set $b = f - u$, then $b(x) = 0$ for $x \in F$ and $\int_{Q_k} b(x)dx = 0$ for each k. Thus we have

$$|\{x : \mu_\lambda^{*,\rho}(f)(x) > \beta\}| \le |\{x : \mu_\lambda^{*,\rho}(u)(x) > \beta/2\}| + |\{x : \mu_\lambda^{*,\rho}(b)(x) > \beta/2\}|. \tag{2.1}$$

By the L^2 boundedness of $\mu_\lambda^{*,\rho}$(Theorem B) and (iii)–(v), it is easy to see that

$$\begin{aligned} |\{x : \mu_\lambda^{*,\rho}(u)(x) > \beta/2\}| &\le \frac{4}{\beta^2}\int_{\boldsymbol{R}^n} |\mu_\lambda^{*,\rho}(u)|^2 dx \le \frac{C}{\beta^2}\|u\|_2^2 \\ &\le \frac{C}{\beta^2}\left\{\int_F |f|^2 dx + \sum_k \int_{Q_k}\left(\frac{1}{|Q_k|}\int_{Q_k} f(y)dy\right)^2 dx\right\} \le \frac{C}{\beta}\|f\|_1. \end{aligned} \tag{2.2}$$

On the other hand, we denote by x_k and a_k the center and side length of Q_k respectively, and let B_k be a ball with center at x_k and radius $r_k = (\sqrt{n}/2)a_k$ for each k. Then

$$|\{x : \mu_\lambda^{*,\rho}(b)(x) > \beta/2\}| \le |\{x : \mu_\lambda^{*,\rho}(b)(x) > \beta/2\} \cap E^*| + |\{x : \mu_\lambda^{*,\rho}(b)(x) > \beta/2\} \cap (E^*)^c|,$$

where $E^* = \cup_k (16B_k)$. By (ii) and (v), we have

$$|\{x : \mu_\lambda^{*,\rho}(b)(x) > \beta/2\} \cap E^*| \le |E^*| \le C\sum_k |B_k| \le C_n|E| \le \frac{C_n}{\beta}\|f\|_1. \tag{2.3}$$

Note that

$$|\{x : \mu_\lambda^{*,\rho}(b)(x) > \beta/2\} \cap (E^*)^c| \le \frac{C}{\beta}\int_{(E^*)^c} \mu_\lambda^{*,\rho}(b)(x)dx$$

and $\int_{\boldsymbol{R}^n}|b(x)|dx \le C\|f\|_1$. Hence by (2.1)–(2.3), to complete the proof of Theorem 1, it remains to verify that

$$\int_{(E^*)^c} \mu_\lambda^{*,\rho}(b)(x)dx \le C\|b\|_1. \tag{2.4}$$

Denote

$$b_k(x) = \begin{cases} b(x), & \text{for} \quad x \in Q_k \\ 0, & \text{for} \quad x \notin Q_k, \end{cases}$$

then by the Minkowski inequality

$$\begin{aligned}
&\int_{(E^*)^c} \mu_\lambda^{*,\rho}(b)dx \\
&\quad = \int_{(E^*)^c} \left[\iint_{\mathbf{R}_+^{n+1}} \left(\frac{t}{t+|x-y|} \right)^{\lambda n} \left| \sum_k \int_{|y-z|<t} \frac{\Omega(y-z)}{|y-z|^{n-\rho}} b_k(z)dz \right|^2 \frac{dydt}{t^{n+2\rho+1}} \right]^{1/2} dx \\
&\quad \le \int_{(E^*)^c} \sum_k \left[\iint_{\mathbf{R}_+^{n+1}} \left(\frac{t}{t+|x-y|} \right)^{\lambda n} \left| \int_{|y-z|<t} \frac{\Omega(y-z)}{|y-z|^{n-\rho}} b_k(z)dz \right|^2 \frac{dydt}{t^{n+2\rho+1}} \right]^{1/2} dx.
\end{aligned}$$

Let

$$\begin{aligned}
J_1 &= \int_{(E^*)^c} \sum_k \left[\iint_{|y-x|<t} \left(\frac{t}{t+|x-y|} \right)^{\lambda n} \left| \int_{|y-z|<t} \frac{\Omega(y-z)}{|y-z|^{n-\rho}} b_k(z)dz \right|^2 \frac{dydt}{t^{n+2\rho+1}} \right]^{1/2} dx, \\
J_2 &= \int_{(E^*)^c} \sum_k \left[\iint_{\substack{|y-x| \ge t \\ y \in 4B_k}} \left(\frac{t}{t+|x-y|} \right)^{\lambda n} \left| \int_{|y-z|<t} \frac{\Omega(y-z)}{|y-z|^{n-\rho}} b_k(z)dz \right|^2 \frac{dydt}{t^{n+2\rho+1}} \right]^{1/2} dx
\end{aligned}$$

and

$$J_3 = \int_{(E^*)^c} \sum_k \left[\iint_{\substack{|y-x| \ge t \\ y \in (4B_k)^c}} \left(\frac{t}{t+|x-y|} \right)^{\lambda n} \left| \int_{|y-z|<t} \frac{\Omega(y-z)}{|y-z|^{n-\rho}} b_k(z)dz \right|^2 \frac{dydt}{t^{n+2\rho+1}} \right]^{1/2} dx.$$

Then

$$\int_{(E^*)^c} \mu_\lambda^{*,\rho}(b)(x)dx \le J_1 + J_2 + J_3. \tag{2.5}$$

Below we will respectively give the estimates of J_1, J_2 and J_3. First we have

$$\begin{aligned}
J_1 &\le \int_{(E^*)^c} \sum_k \left(\iint_{|y-x|<t} \left| \int_{|y-z|<t} \frac{\Omega(y-z)}{|y-z|^{n-\rho}} b_k(z)dz \right|^2 \frac{dydt}{t^{n+2\rho+1}} \right)^{1/2} dx \\
&\le \int_{(E^*)^c} \sum_k \left[\left(\iint_{\substack{|y-x|<t \\ y \in 4B_k}} + \iint_{\substack{|y-x|<t \\ y \in (4B_k)^c}} \right) \left| \int_{|y-z|<t} \frac{\Omega(y-z)}{|y-z|^{n-\rho}} b_k(z)dz \right|^2 \frac{dydt}{t^{n+2\rho+1}} \right]^{1/2} dx \\
&\le J_{11} + J_{12}.
\end{aligned}$$

By $x \in (E^*)^c$, $y \in 4B_k$ and $z \in Q_k$, we have $|x-x_k| - 4r_k \le |x-x_k| - |y-x_k| \le |x-y| < t$ and $|y-z| < 8r_k$. Applying Minkowski's inequality again, we get

$$\begin{aligned}
J_{11} &\le \int_{(E^*)^c} \sum_k \int_{Q_k} |b(z)| \left(\iint_{\substack{|y-x|<t \\ |y-z|<t \\ y \in 4B_k}} \frac{|\Omega(y-z)|^2}{|y-z|^{2n-2\rho}} \frac{dydt}{t^{n+2\rho+1}} \right)^{1/2} dzdx \\
&\le \int_{(E^*)^c} \sum_k \int_{Q_k} |b(z)| \left[\int_{|y-z|<8r_k} \frac{|\Omega(y-z)|^2}{|y-z|^{2n-2\rho}} \left(\int_{|x-x_k|-4r_k}^{\infty} \frac{dt}{t^{n+2\rho+1}} \right) dy \right]^{1/2} dzdx \\
&\le C \sum_k \int_{Q_k} |b(z)| \left(\int_{|y-z|<8r_k} \frac{|\Omega(y-z)|^2}{|y-z|^{2n-2\rho}} dy \right)^{1/2} dz \int_{(E^*)^c} \frac{1}{(|x-x_k|-4r_k)^{(n+2\rho)/2}} dx \\
&\le C\|b\|_1.
\end{aligned} \tag{2.6}$$

As for J_{12}, we have

$$J_{12}=\int_{(E^*)^c}\sum_k\Bigg[\Bigg(\iint_{\substack{|y-x|<t\\ t\le |y-x_k|+2r_k\\ y\in(4B_k)^c}}+\iint_{\substack{|y-x|<t\\ t> |y-x_k|+2r_k\\ y\in(4B_k)^c}}\Bigg)\Bigg|\int_{|y-z|<t}\frac{\Omega(y-z)}{|y-z|^{n-\rho}}b_k(z)dz\Bigg|^2\frac{dydt}{t^{n+2\rho+1}}\Bigg]^{1/2}dx$$
$$\le J_{12}^1+J_{12}^2.$$

By $z\in Q_k$, $x\in(E^*)^c$ and $y\in(4B_k)^c$ it is easy to see that $|y-z|\sim|y-x_k|$ and

$$|x-x_k|\le|x-y|+|y-x_k|\le t+|y-x_k|\le 2|y-x_k|+2r_k<3|y-x_k|.$$

Thus

$$\begin{aligned}
J_{12}^1&\le\int_{(E^*)^c}\sum_k\int_{Q_k}|b(z)|\Bigg(\iint_{\substack{|y-x|<t\\ t\le|y-x_k|+2r_k\\ |y-z|<t\\ y\in(4B_k)^c}}\frac{|\Omega(y-z)|^2}{|y-z|^{2n-2\rho}}\frac{dydt}{t^{n+2\rho+1}}\Bigg)^{1/2}dzdx\\
&\le\int_{(E^*)^c}\sum_k\int_{Q_k}|b(z)|\Bigg[\int_{y\in(4B_k)^c}\frac{|\Omega(y-z)|^2}{|y-z|^{2n-2\rho}}\Bigg(\int_{|y-z|}^{|y-x_k|+2r_k}\frac{dt}{t^{n+2\rho+1}}\Bigg)dy\Bigg]^{1/2}dzdx\\
&\le\int_{(E^*)^c}\sum_k\int_{Q_k}|b(z)|\Bigg(\int_{y\in(4B_k)^c}\frac{|\Omega(y-z)|^2}{|y-z|^{2n-2\rho}}\frac{Cr_k}{|y-z|^{n+2\rho+1}}dy\Bigg)^{1/2}dzdx\,(\text{since }|y-z|\sim|y-x_k|)\\
&\le C\int_{(E^*)^c}\sum_k\int_{Q_k}|b(z)|\Bigg(\int_{y\in(4B_k)^c}\frac{|\Omega(y-z)|^2}{|y-z|^{n+1/2}}\frac{r_k}{|y-x_k|^{2n+1/2}}dy\Bigg)^{1/2}dzdx\\
&\le C\sum_k\int_{Q_k}|b(z)|\Bigg(\int_{y\in(4B_k)^c}\frac{|\Omega(y-z)|^2r_k^{1/2}}{|y-z|^{n+1/2}}dy\Bigg)^{1/2}dz\int_{(E^*)^c}\frac{r_k^{1/4}}{|x-x_k|^{n+1/4}}dx\\
&\le C\|b\|_1. \qquad (2.7)
\end{aligned}$$

Now we give the estimate of J_{12}^2. Note that $Q_k\subset B_k\subset\{z:|y-z|<t\}$ since $y\in(4B_k)^c$ and $t>|y-x_k|+2r_k$. In addition, $|x-x_k|<|x-y|+|y-x_k|<3t$. Hence by the cancellation property of b on B_k, we have

$$\begin{aligned}
J_{12}^2&=\int_{(E^*)^c}\sum_k\Bigg[\iint_{\substack{|y-x|<t\\ t>|y-x_k|+2r_k\\ y\in(4B_k)^c}}\Bigg|\int_{|y-z|<t}\Bigg(\frac{\Omega(y-z)}{|y-z|^{n-\rho}}-\frac{\Omega(y-x_k)}{|y-x_k|^{n-\rho}}\Bigg)b_k(z)dz\Bigg|^2\frac{dydt}{t^{n+2\rho+1}}\Bigg]^{1/2}dx\\
&\le\int_{(E^*)^c}\sum_k\int_{Q_k}|b(z)|\Bigg[\int_{y\in(4B_k)^c}\Bigg|\frac{\Omega(y-z)}{|y-z|^{n-\rho}}-\frac{\Omega(y-x_k)}{|y-x_k|^{n-\rho}}\Bigg|^2\\
&\qquad\times\Bigg(\int_{\substack{|y-x|<t\\ t>|y-x_k|+2r_k\\ |y-z|<t}}\frac{dt}{t^{n+2\rho+1}}\Bigg)dy\Bigg]^{1/2}dzdx\\
&=\int_{(E^*)^c}\sum_k\int_{Q_k}|b(z)|\Bigg[\int_{y\in(4B_k)^c}\Bigg|\frac{\Omega(y-z)}{|y-z|^{n-\rho}}-\frac{\Omega(y-x_k)}{|y-x_k|^{n-\rho}}\Bigg|^2\\
&\qquad\times\Bigg(\int_{\substack{|y-x|<t\\ t>|y-x_k|+2r_k\\ |y-z|<t}}\frac{(\log(t/r_k))^{2+2\varepsilon}dt}{t^{2\rho-n+1}t^{2n}(\log(t/r_k))^{2+2\varepsilon}}\Bigg)dy\Bigg]^{1/2}dzdx
\end{aligned}$$

$$\leq \int_{(E^*)^c} \sum_k \int_{Q_k} |b(z)| \left[\int_{y\in(4B_k)^c} \left| \frac{\Omega(y-z)}{|y-z|^{n-\rho}} - \frac{\Omega(y-x_k)}{|y-x_k|^{n-\rho}} \right|^2 \right.$$
$$\left. \times \left(\int_{t>|y-x_k|+2r_k} \frac{(\log(t/r_k))^{2+2\varepsilon} dt}{t^{2\rho-n+1}(|x-x_k|/3)^{2n}(\log(|x-x_k|/(3r_k)))^{2+2\varepsilon}} \right) dy \right]^{1/2} dzdx, \quad (2.8)$$

where $0<\varepsilon<\min\{1/2,\rho-n/2,\sigma-1\}$. (We always take ε to satisfy this restriction in the whole paper). To complete the estimate of J_{12}^2, we need the following inequality.

LEMMA 2.2. For $|y-x_k|>4r_k$,

$$\int_{|y-x_k|+2r_k}^{\infty} \frac{(\log(t/r_k))^{2+2\varepsilon}}{t^{2\rho-n+1}} dt \leq C \frac{[\log((|y-x_k|/r_k)+2)]^{2+2\varepsilon}}{(|y-x_k|+2r_k)^{2\rho-n}}. \quad (2.9)$$

The proof of (2.9) is simple, we omit the details here.

Let us continue to estimate J_{12}^2. By (2.8), (2.9) and Lemma 2.1, we get

$$J_{12}^2 \leq C \sum_k \int_{(E^*)^c} \frac{1}{(|x-x_k|/3)^n (\log(|x-x_k|/(3r_k)))^{1+\varepsilon}} \int_{Q_k} |b(z)|$$
$$\times \left(\int_{y\in(4B_k)^c} \left| \frac{\Omega(y-z)}{|y-z|^{n-\rho}} - \frac{\Omega(y-x_k)}{|y-x_k|^{n-\rho}} \right|^2 \frac{[\log((|y-x_k|/r_k)+2)]^{2+2\varepsilon}}{(|y-x_k|+2r_k)^{2\rho-n}} dy \right)^{1/2} dzdx$$
$$\leq C \sum_k \int_{(E^*)^c} \frac{1}{(|x-x_k|/3)^n (\log(|x-x_k|/(3r_k)))^{1+\varepsilon}} \int_{Q_k} |b(z)|$$
$$\times \sum_{j=2}^{\infty} \left(\int_{2^j r_k \leq |y-x_k| < 2^{j+1} r_k} \left| \frac{\Omega(y-z)}{|y-z|^{n-\rho}} - \frac{\Omega(y-x_k)}{|y-x_k|^{n-\rho}} \right|^2 \right.$$
$$\left. \times \frac{[\log((2^{j+1} r_k/r_k)+2)]^{2+2\varepsilon}}{(2^j r_k+2r_k)^{2\rho-n}} dy \right)^{1/2} dzdx$$
$$\leq C \sum_k \int_{(E^*)^c} \frac{1}{(|x-x_k|/3)^n (\log(|x-x_k|/3r_k))^{1+\varepsilon}} \int_{Q_k} |b(z)|$$
$$\times \sum_{j=2}^{\infty} \frac{(j+1)^{1+\varepsilon}}{(2^j r_k+2r_k)^{\rho-n/2}} (2^j r_k)^{n/2-(n-\rho)} \left\{ \frac{|z-x_k|}{2^j r_k} + \int_{|z-x_k|/(2^{j+1}r_k)}^{|z-x_k|/(2^j r_k)} \frac{w_2(\delta)}{\delta} d\delta \right\} dzdx. \quad (2.10)$$

Note that

$$\int_{|z-x_k|/(2^{j+1}r_k)}^{|z-x_k|/(2^j r_k)} \frac{w_2(\delta)}{\delta} d\delta = \int_{|z-x_k|/(2^{j+1}r_k)}^{|z-x_k|/(2^j r_k)} \frac{\omega_2(\delta)(1+|\log\delta|)^{\sigma}}{\delta(1+|\log\delta|)^{\sigma}} d\delta$$
$$\leq \frac{C}{j^{\sigma}} \int_{|z-x_k|/(2^{j+1}r_k)}^{|z-x_k|/(2^j r_k)} \frac{\omega_2(\delta)(1+|\log\delta|)^{\sigma}}{\delta} d\delta. \quad (2.11)$$

By (2.10), (2.11) and the condition (1.2), since $0<\varepsilon<\sigma-1$, we get

$$J_{12}^2 \leq C \sum_k \int_{(E^*)^c} \frac{1}{(|x-x_k|/3)^n (\log(|x-x_k|/(3r_k)))^{1+\varepsilon}} \int_{Q_k} |b(z)|$$
$$\times \sum_{j=2}^{\infty} (j+1)^{1+\varepsilon} \left\{ \frac{1}{2^j} + \frac{C}{j^{\sigma}} \int_{|z-x_k|/(2^{j+1}r_k)}^{|z-x_k|/(2^j r_k)} \frac{\omega_2(\delta)(1+|\log\delta|)^{\sigma}}{\delta} d\delta \right\} dzdx$$

$$\le C\sum_k\int_{(E^*)^c}\frac{1}{(|x-x_k|/3)^n(\log(|x-x_k|/(3r_k)))^{1+\varepsilon}}\int_{Q_k}|b(z)|$$
$$\times\left(1+\sum_{j=2}^{\infty}\int_{|z-x_k|/(2^{j+1}r_k)}^{|z-x_k|/(2^j r_k)}\frac{\omega_2(\delta)(1+|\log\delta|)^\sigma}{\delta}d\delta\right)dzdx$$
$$\le C\sum_k\int_{(E^*)^c}\frac{1}{(|x-x_k|/3)^n(\log(|x-x_k|/(3r_k)))^{1+\varepsilon}}\int_{Q_k}|b(z)|$$
$$\times\left(1+\int_0^1\frac{\omega_2(\delta)(1+|\log\delta|)^\sigma}{\delta}d\delta\right)dzdx$$
$$\le C\sum_k\int_{Q_k}|b(z)|dz\int_{(E^*)^c}\frac{1}{(|x-x_k|/3)^n(\log(|x-x_k|/(3r_k)))^{1+\varepsilon}}dx$$
$$\le C\sum_k\int_{Q_k}|b(z)|dz\le C\|b\|_1. \tag{2.12}$$

Thus, by (2.6), (2.7) and (2.12) we obtain

$$J_1\le C\|b\|_1. \tag{2.13}$$

As for J_2, from $y\in 4B_k, x\in(E^*)^c$ and $z\in Q_k$, we see that $|y-x|>|x-x_k|-|y-x_k|>|x-x_k|/2$, $|y-z|<32r_k$ and $|x-y|\sim|x-x_k|$. By the Minkowski inequality we have

$$J_2\le\int_{(E^*)^c}\sum_k\int_{Q_k}|b(z)|\left[\iint_{\substack{|y-x|\ge t\\ |y-z|<32r_k\\ y\in 4B_k}}\left(\frac{t}{t+|x-y|}\right)^{\lambda n}\frac{|\Omega(y-z)|^2}{|y-z|^{2n-2\rho}}\frac{dydt}{t^{n+2\rho+1}}\right]^{1/2}dzdx$$
$$\le\int_{(E^*)^c}\sum_k\int_{Q_k}|b(z)|\left[\int_{|y-z|<32r_k}\frac{|\Omega(y-z)|^2}{(|x-x_k|/2)^{\lambda n}|y-z|^{2n-2\rho}}\right.$$
$$\left.\times\left(\int_0^{|y-x|}\frac{t^{\lambda n}}{t^{n+2\rho+1}}dt\right)dy\right]^{1/2}dzdx$$
$$\le\int_{(E^*)^c}\sum_k\int_{Q_k}|b(z)|\left(\int_{|y-z|<32r_k}\frac{|\Omega(y-z)|^2|y-x|^{\lambda n-n-2\rho}}{(|x-x_k|/2)^{\lambda n}|y-z|^{2n-2\rho}}dy\right)^{1/2}dzdx$$
$$\le C\int_{(E^*)^c}\sum_k\int_{Q_k}|b(z)|\left(\int_{|y-z|<32r_k}\frac{|\Omega(y-z)|^2}{(|x-x_k|)^{n+2\rho}|y-z|^{2n-2\rho}}dy\right)^{1/2}dzdx$$
$$\le C\sum_k\int_{Q_k}|b(z)|\left(\int_{|y-z|<32r_k}\frac{|\Omega(y-z)|^2}{|y-z|^{2n-2\rho}}dy\right)^{1/2}dz\int_{(E^*)^c}\frac{1}{(|x-x_k|)^{n/2+\rho}}dx$$
$$\le C\|b\|_1. \tag{2.14}$$

Now let us estimate J_3. Denote

$$J_{31}=\int_{(E^*)^c}\sum_k\left[\iint_{\substack{y\in(4B_k)^c\\ t\le|y-x_k|+8e^{3/(2\rho-n)}r_k\\ |y-x|\ge t}}\left(\frac{t}{t+|x-y|}\right)^{\lambda n}\left|\int_{|y-z|<t}\frac{\Omega(y-z)}{|y-z|^{n-\rho}}b_k(z)dz\right|^2\frac{dydt}{t^{n+2\rho+1}}\right]^{1/2}dx$$

and

$$J_{32}=\int_{(E^*)^c}\sum_k\left[\iint_{\substack{y\in(4B_k)^c\\ |y-x_k|+8e^{3/(2\rho-n)}r_k<t\\ |y-x|\ge t}}\left(\frac{t}{t+|x-y|}\right)^{\lambda n}\left|\int_{|y-z|<t}\frac{\Omega(y-z)}{|y-z|^{n-\rho}}b_k(z)dz\right|^2\frac{dydt}{t^{n+2\rho+1}}\right]^{1/2}dx,$$

then $J_3 \le J_{31} + J_{32}$. Notice that when $y \in (4B_k)^c$ and $z \in Q_k$, then $|y-z| \sim |y-x_k|$, $|y-x_k| \le |y-z| + |z-x_k| \le t + 2r_k$. Moreover, for $\alpha > 0$ we have

$$\int_{|y-x_k|-2r_k}^{|y-x_k|+8e^{3/(2\rho-n)}r_k} \frac{dt}{t^{\alpha+1}} \le \frac{Cr_k}{|y-x_k|^{\alpha+1}}. \tag{2.15}$$

Since

$$\begin{aligned} J_{31} \le \int_{(E^*)^c} \sum_k \int_{Q_k} |b(z)| \Bigg[\Bigg(\iint_{\substack{y \in (4B_k)^c \\ |y-x| \ge t \\ t \le |y-x_k| + 8e^{3/(2\rho-n)}r_k \\ |y-z| < t \\ |x-x_k| \le 2|y-x_k|}} + \iint_{\substack{y \in (4B_k)^c \\ |y-x| \ge t \\ t \le |y-x_k| + 8e^{3/(2\rho-n)}r_k \\ |y-z| < t \\ |x-x_k| > 2|y-x_k|}} \Bigg) \\ \left(\frac{t}{t+|x-y|}\right)^{\lambda n} \frac{|\Omega(y-z)|^2}{|y-z|^{2n-2\rho}} \frac{dtdy}{t^{n+2\rho+1}} \Bigg]^{1/2} dzdx \\ \le J_{31}^1 + J_{31}^2, \end{aligned}$$

by (2.15) we get

$$\begin{aligned} J_{31}^1 &\le \int_{(E^*)^c} \sum_k \int_{Q_k} |b(z)| \Bigg[\int_{\substack{y \in (4B_k)^c \\ |x-x_k| \le 2|y-x_k|}} \left(\int_{|y-x_k|-2r_k}^{|y-x_k|+8e^{3/(2\rho-n)}r_k} \frac{dt}{t^{n+2\rho+1}} \right) \\ &\qquad \times \frac{|\Omega(y-z)|^2}{|y-z|^{2n-2\rho}} dy \Bigg]^{1/2} dzdx \\ &\le C \int_{(E^*)^c} \sum_k \int_{Q_k} |b(z)| \left(\int_{\substack{y \in (4B_k)^c \\ |x-x_k| \le 2|y-x_k|}} \frac{|\Omega(y-z)|^2}{|y-z|^{2n-2\rho}} \frac{r_k}{|y-x_k|^{n+2\rho+1}} dy \right)^{1/2} dzdx. \end{aligned}$$

Using the same method of estimating J_{12}^1 in (2.7), we may get

$$J_{31}^1 \le C\|b\|_1. \tag{2.16}$$

Now we consider J_{31}^2. Note that $0 < \varepsilon < \min\{1/2, \rho - n/2, \sigma - 1\}$, $|y-x| \ge |x-x_k| - |y-x_k| \ge |x-x_k|/2$ and $|y-z| \sim |y-x_k|$, by (2.16) we have

$$\begin{aligned} J_{31}^2 &\le \int_{(E^*)^c} \sum_k \int_{Q_k} |b(z)| \Bigg[\int_{\substack{y \in (4B_k)^c \\ |x-x_k| > 2|y-x_k|}} \left(\frac{t}{t+|x-y|}\right)^{2n+2\varepsilon} \frac{|\Omega(y-z)|^2}{|y-z|^{2n-2\rho}} \\ &\qquad \times \left(\int_{|y-x_k|-2r}^{|y-x_k|+8e^{3/(2\rho-n)}r_k} \frac{dt}{t^{n+2\rho+1}} \right) dy \Bigg]^{1/2} dzdx \\ &\le C \int_{(E^*)^c} \sum_k \int_{Q_k} |b(z)| \Bigg[\int_{\substack{y \in (4B_k)^c \\ |x-x_k| > 2|y-x_k|}} \left(\int_{|y-x_k|-2r_k}^{|y-x_k|+8e^{3/(2\rho-n)}r_k} \frac{t^{2n+2\varepsilon-n-2\rho-1}}{(|x-y|)^{2n+2\varepsilon}} dt \right) \\ &\qquad \times \frac{|\Omega(y-z)|^2}{|y-z|^{2n-2\rho}} dy \Bigg]^{1/2} dzdx \\ &\le C \int_{(E^*)^c} \sum_k \int_{Q_k} |b(z)| \left(\int_{\substack{y \in (4B_k)^c \\ |x-x_k| \ge 2|y-x_k|}} \frac{|\Omega(y-z)|^2}{|y-z|^{2n-2\rho}} \frac{r_k}{|x-x_k|^{2n+2\varepsilon}|y-x_k|^{2\rho-n-2\varepsilon+1}} dy \right)^{1/2} dzdx \end{aligned}$$

$$
\begin{aligned}
&\le C\int_{(E^*)^c}\sum_k\int_{Q_k}|b(z)|\bigg(\int_{\substack{y\in(4B_k)^c\\|x-x_k|\ge 2|y-x_k|}}\frac{|\Omega(y-z)|^2}{|y-z|^{n-2\varepsilon+1}}\frac{r_k}{|x-x_k|^{2n+2\varepsilon}}dy\bigg)^{1/2}dzdx\\
&\le C\sum_k\int_{Q_k}|b(z)|\bigg(\int_{(4B_k)^c}\frac{|\Omega(y-z)|^2r_k^{1-2\varepsilon}}{|y-z|^{n-2\varepsilon+1}}dy\bigg)^{1/2}dz\int_{(E^*)^c}\frac{r_k^{\varepsilon}}{|x-x_k|^{n+\varepsilon}}dx\\
&\le C\|b\|_1. \qquad (2.17)
\end{aligned}
$$

Finally, let us estimate J_{32}. By $y\in(4B_k)^c$ and $t>|y-x_k|+8e^{3/(2\rho-n)}r_k$, we have $Q_k\subset B_k\subset\{z:|y-z|<t\}$. On the other hand, it is easy to see that

$$
\begin{aligned}
t+|x-y|&\ge t+|x-x_k|-|y-x_k|\\
&\ge|y-x_k|+8e^{3/(2\rho-n)}r_k+|x-x_k|-|y-x_k|\\
&\ge|x-x_k|+8e^{3/(2\rho-n)}r_k.
\end{aligned}
$$

Thus

$$
\begin{aligned}
J_{32}&\le\int_{(E^*)^c}\sum_k\int_{Q_k}|b(z)|\bigg[\iint_{\substack{y\in(4B_k)^c\\|y-x_k|+8e^{3/(2\rho-n)}r_k<t\\|y-z|<t\le|y-x|}}\bigg(\frac{t}{t+|x-y|}\bigg)^{\lambda n}\\
&\qquad\times\bigg|\frac{\Omega(y-z)}{|y-z|^{n-\rho}}-\frac{\Omega(y-x_k)}{|y-x_k|^{n-\rho}}\bigg|^2\frac{dtdy}{t^{n+2\rho+1}}\bigg]^{1/2}dzdx\\
&=\int_{(E^*)^c}\sum_k\int_{Q_k}|b(z)|\bigg(\iint_{\substack{y\in(4B_k)^c\\|y-x_k|+8e^{3/(2\rho-n)}r_k<t\\|y-z|<t\le|y-x|}}\frac{t^{\lambda n}}{(t+|x-y|)^{2n}[\log((t+|x-y|)/r_k)]^{2+2\varepsilon}}\\
&\qquad\times\frac{[\log((t+|x-y|)/r_k)]^{2+2\varepsilon}}{(t+|x-y|)^{\lambda n-2n}}\bigg|\frac{\Omega(y-z)}{|y-z|^{n-\rho}}-\frac{\Omega(y-x_k)}{|y-x_k|^{n-\rho}}\bigg|^2\frac{dtdy}{t^{n+2\rho+1}}\bigg)^{1/2}dzdx\\
&\le C\int_{(E^*)^c}\sum_k\int_{Q_k}\frac{|b(z)|}{(|x-x_k|+8e^{3/(2\rho-n)}r_k)^n[\log((|x-x_k|+8e^{3/(2\rho-n)}r_k)/r_k)]^{1+\varepsilon}}\\
&\qquad\times\bigg(\iint_{\substack{y\in(4B_k)^c\\|y-x|\ge t\\|y-x_k|+8e^{3/(2\rho-n)}r_k<t\\|y-z|<t}}\frac{t^{\lambda n}[\log((t+|x-y|)/r_k)]^{2+2\varepsilon}}{(t+|x-y|)^{\lambda n-2n}}\\
&\qquad\times\bigg|\frac{\Omega(y-z)}{|y-z|^{n-\rho}}-\frac{\Omega(y-x_k)}{|y-x_k|^{n-\rho}}\bigg|^2\frac{dtdy}{t^{n+2\rho+1}}\bigg)^{1/2}dzdx\\
&\le C\int_{(E^*)^c}\sum_k\int_{Q_k}\frac{|b(z)|}{(|x-x_k|+8e^{3/(2\rho-n)}r_k)^n[\log((|x-x_k|+8e^{3/(2\rho-n)}r_k)/r_k)]^{1+\varepsilon}}\\
&\qquad\times\bigg(\int_{\substack{y\in(4B_k)^c\\|y-x|\ge|y-x_k|+8e^{3/(2\rho-n)}r_k}}\bigg|\frac{\Omega(y-z)}{|y-z|^{n-\rho}}-\frac{\Omega(y-x_k)}{|y-x_k|^{n-\rho}}\bigg|^2\\
&\qquad\times\int_{|y-x_k|+8e^{3/(2\rho-n)}r_k}^{|y-x|}\frac{|y-x|^{\lambda n-n-2\rho-1}[\log((t+|x-y|)/r_k)]^{2+2\varepsilon}}{(t+|x-y|)^{\lambda n-2n}}dtdy\bigg)^{1/2}dzdx.
\end{aligned}
$$

Notice that

$$\begin{aligned}
&\int_{|y-x_k|+8e^{3/(2\rho-n)}r_k}^{|y-x|} \frac{[\log((t+|x-y|)/r_k)]^{2+2\varepsilon}}{(t+|x-y|)^{\lambda n-2n}}dt \\
&\quad = \int_{|y-x|+|y-x_k|+8e^{3/(2\rho-n)}r_k}^{2|y-x|} \frac{[\log(t/r_k)]^{2+2\varepsilon}}{t^{\lambda n-2n}}dt \\
&\quad \le \int_{|y-x|+|y-x_k|+8e^{3/(2\rho-n)}r_k}^{\infty} \frac{[\log(t/r_k)]^{2+2\varepsilon}}{t^{\lambda n-2n}}dt \\
&\quad \le C\frac{[\log((|y-x|+|y-x_k|+8e^{3/(2\rho-n)}r_k)/r_k)]^{2+2\varepsilon}}{(|y-x|+|y-x_k|+8e^{3/(2\rho-n)}r_k)^{\lambda n-2n-1}}. \qquad (2.18)
\end{aligned}$$

Applying the idea of proving (2.9), we may obtain the last inequality in (2.18). Here we omit the detail. Since the function $g(s)=(\log s)^{2+2\varepsilon}/s^{2\rho-n}$ is decreasing for $s>e^{3/(2\rho-n)}$ and $\lambda n > 2\rho+n+1$ and

$$\frac{|y-x|+|y-x_k|+8e^{3/(2\rho-n)}r_k}{r_k} > \frac{2|y-x_k|+8e^{3/(2\rho-n)}r_k}{r_k} > e^{3/(2\rho-n)},$$

therefore

$$\begin{aligned}
&\frac{|y-x|^{\lambda n-n-2\rho-1}[\log((|y-x|+|y-x_k|+8e^{3/(2\rho-n)}r_k)/r_k)]^{2+2\varepsilon}}{(|y-x|+|y-x_k|+8e^{3/(2\rho-n)}r_k)^{\lambda n-2n-1}} \\
&\quad \le \frac{[\log((|y-x|+|y-x_k|+8e^{3/(2\rho-n)}r_k)/r_k)]^{2+2\varepsilon}}{(|y-x|+|y-x_k|+8e^{3/(2\rho-n)}r_k)^{2\rho-n}} \\
&\quad \le \frac{[\log((2|y-x_k|+8e^{3/(2\rho-n)}r_k)/r_k)]^{2+2\varepsilon}}{(2|y-x_k|+8e^{3/(2\rho-n)}r_k)^{2\rho-n}}. \qquad (2.19)
\end{aligned}$$

By (2.18) and (2.19), we have

$$\begin{aligned}
J_{32} &\le C\int_{(E^*)^c}\sum_k\int_{Q_k}\frac{|b(z)|}{(|x-x_k|+8e^{3/(2\rho-n)}r)^n\left[\log((|x-x_k|+8e^{3/(2\rho-n)}r_k)/r_k)\right]^{1+\varepsilon}} \\
&\quad \times\Bigg(\int_{\substack{y\in(4B_k)^c\\ |y-x|\ge|y-x_k|+8e^{3/(2\rho-n)}r_k}}\left|\frac{\Omega(y-z)}{|y-z|^{n-\rho}}-\frac{\Omega(y-x_k)}{|y-x_k|^{n-\rho}}\right|^2 \\
&\quad\quad \times\frac{\left[\log(2|y-x_k|+8e^{3/(2\rho-n)}r_k)/r_k)\right]^{2+2\varepsilon}}{(2|y-x_k|+8e^{3/(2\rho-n)}r_k)^{2\rho-n}}dy\Bigg)^{1/2}dzdx \\
&\le C\int_{(E^*)^c}\sum_k\int_{Q_k}\frac{|b(z)|}{(|x-x_k|)^n[\log(|x-x_k|/r_k)]^{1+\varepsilon}}\Bigg(\sum_{j=3}^{\infty}\int_{2^jr_k\le|y-x_k|<2^{j+1}r_k} \\
&\quad \left|\frac{\Omega(y-z)}{|y-z|^{n-\rho}}-\frac{\Omega(y-x_k)}{|y-x_k|^{n-\rho}}\right|^2\frac{\left[\log((2|y-x_k|+8e^{3/(2\rho-n)}r_k)/r_k)\right]^{2+2\varepsilon}}{(2|y-x_k|)^{2\rho-n}}dy\Bigg)^{1/2}dzdx.
\end{aligned}$$

Applying the same method of estimating J_{12}^2 (see (2.10)–(2.12)), we may get $J_{32}\le C\|b\|_1$. From this and (2.16), (2.17), we see that

$$J_3 \le C\|b\|_1. \qquad (2.20)$$

Thus, (2.4) follows from (2.5), (2.13), (2.14) and (2.20) and we finish the proof of Theorem 1.

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