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On arithmetic subgroups of a $\boldsymbol{Q}$-rank 2 form of $SU(2,2)$ and their automorphic cohomology

By Takahiro HAYATA and Joachim SCHWERMER

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Abstract. The cohomology $H^*(\Gamma, E)$ of an arithmetic subgroup Γ of a connected reductive algebraic group G defined over $\boldsymbol{Q}$ can be interpreted in terms of the automorphic spectrum of Γ. In this frame there is a sum decomposition of the cohomology into the cuspidal cohomology (*i.e.*, classes represented by cuspidal automorphic forms for G) and the so called Eisenstein cohomology. The present paper deals with the case of a quasi split form G of $\boldsymbol{Q}$-rank two of a unitary group of degree four. We describe in detail the Eisenstein series which give rise to non-trivial cohomology classes and the cuspidal automorphic forms for the Levi components of parabolic $\boldsymbol{Q}$-subgroups to which these classes are attached. Mainly the generic case will be treated, *i.e.*, we essentially suppose that the coefficient system E is regular.

Introduction.

Let G be a connected semisimple algebraic group defined over an algebraic number field F. Suppose that the F-rank of G is greater than zero. Let $K_{\boldsymbol{R}}$ be a maximal compact subgroup of the real Lie group $G(\boldsymbol{R})$ of real points of G, and denote by $X_{G(\boldsymbol{R})} = G(\boldsymbol{R})/K_{\boldsymbol{R}}$ the associated symmetric space. Let (ν, E) be an irreducible algebraic representation of $G(\boldsymbol{R})$ on a finite dimensional complex vector space E.

An arithmetic torsion free subgroup Γ of $G(F)$ acts properly and freely on $X_{G(\boldsymbol{R})}$. The quotient $\Gamma\backslash X_{G(\boldsymbol{R})}$ is a non-compact complete Riemannian manifold of finite volume. Our object of concern is the cohomology $H^\bullet(\Gamma\backslash X_{G(\boldsymbol{R})}, E)$ of this arithmetically defined locally symmetric space, its relationship with the theory of automorphic forms and certain number theoretical questions embodied in this relation.

The present paper deals with this circle of questions in the case of a quasi split form G of $\boldsymbol{Q}$-rank two of a unitary group of degree four.

In the case of the group SL_2/F Harder initiated around 1970 a program to identify the cohomology of Γ "at infinity", *i.e.*, those phenomena in the cohomology which are due to the non compactness of the space $\Gamma\backslash X_{G(\boldsymbol{R})}$. Langlands' theory of Eisenstein series provided the methological tools. Harder showed in this case that there exists a sum decomposition of the cohomology of Γ

$$H^\bullet(\Gamma\backslash X_{G(\boldsymbol{R})}, E) = H^\bullet_{\mathrm{cusp}}(\Gamma\backslash X_{G(\boldsymbol{R})}, E) \oplus H^\bullet_{\mathrm{Eis}}(\Gamma\backslash X_{G(\boldsymbol{R})}, E)$$

into the space of classes represented by cuspidal automorphic forms for G and the so called Eisenstein cohomology.

By now, due to the work of Franke [**F**], this interpretation of the cohomology of Γ in terms

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of its automorphic spectrum is possible in the general case of an arithmetic group defined by congruence conditions in a connected reductive algebraic group G/F. In this frame there is a sum decomposition of the cohomology of Γ

$$H^\bullet(\Gamma\backslash X_{G(\boldsymbol{R})},E) = H^\bullet_{\mathrm{cusp}}(\Gamma\backslash X_{G(\boldsymbol{R})},E) \oplus \bigoplus_{\{P\}\in\mathscr{C}} H^\bullet_{\{P\}}(\Gamma\backslash X_{G(\boldsymbol{R})},E)$$

into the subspace of classes represented by cuspidal automorphic forms for G and the Eisenstein cohomology constructed as the space of appropriate residues or derivatives of Eisenstein series attached to cuspidal automorphic forms π on the Levi components of proper parabolic F-subgroups of G. Thus, there is a sum decomposition where the sum ranges over the set $\mathscr{C}$ of classes of associate proper parabolic F-subgroups. These Eisenstein cohomology classes can be arranged according to the notion of cuspidal support for the Eisenstein series involved. This leads to a refined decomposition of each of the subspaces $H^\bullet_{\{P\}}(\Gamma\backslash X_{G(\boldsymbol{R})},E)$ indexed by a class of associate proper parabolic subgroups of G where the sum ranges over a certain set of classes of associate irreducible cuspidal automorphic representations of the Levi components of elements of $\{P\}$ (cf. [**F-S**, Theorem 2.3]).

However these results do not give a description of the internal structure of the subspaces $H^\bullet_{\{P\}}(\Gamma\backslash X_{G(\boldsymbol{R})},E)$ or account for the arithmetic nature of the classes therein. Only for a few groups of small F-rank *e.g.*, GL_2/F, $GL_3/\boldsymbol{Q}$ or the symplectic group $Sp_2/\boldsymbol{Q}$ of $\boldsymbol{Q}$-rank two (cf. [**H87**], [**H93**], [**S83**], [**S86**], [**S95**], [**Li-S0**]), these questions are more closely examined. These results provide evidence that the occurrence of specific types of Eisenstein cohomology classes is related to the analytic properties of certain Euler products (or automorphic L-functions). Moreover, questions in and connections with arithmetical algebraic geometry play an important role in these investigations.

In the present paper we are concerned with the case of a unitary group of degree four. Let $F/\boldsymbol{Q}$ be an imaginary quadratic extension of $\boldsymbol{Q}$. Let $x\mapsto\bar{x}$ be the non-trivial automorphism of $F/\boldsymbol{Q}$ of order two. If V denotes the four dimensional vector space F^4 endowed with the nondegenerate Hermitian form $\langle x,y\rangle = xH^t\bar{y}$ where $H=\left(\begin{smallmatrix}0 & I_2\\ I_2 & 0\end{smallmatrix}\right)$, then the corresponding special unitary group

$$SU(V,H) = \{g\in SL(V)\mid gH^t\bar{g}=H\}$$

is a quasi-split semisimple algebraic group defined over $\boldsymbol{Q}$. The $\boldsymbol{Q}$-rank of $G=SU(V,H)$ is two. In this case, the set $\mathscr{C}$ of classes of associate parabolic $\boldsymbol{Q}$-subgroups of G coincides with the set of conjugacy classes. Thus, the class $\{G\}$, the two conjugacy classes $\{P_1\}$, $\{P_2\}$ of maximal parabolic $\boldsymbol{Q}$-subgroups and the class $\{P_0\}$ of the minimal parabolic $\boldsymbol{Q}$-subgroups account for the set $\mathscr{C}$. In the decomposition alluded to above the cuspidal cohomology corresponds to the class $\{G\}$.

As our main results we determine the individual constituents of the subspaces

$$H^\bullet_{\{P_j\}}(\Gamma\backslash X_{G(\boldsymbol{R})},E)\quad (j=0,1,2)$$

of the Eisenstein cohomology. We describe in detail the Eisenstein series which give rise to non-trivial cohomology classes and the cuspidal automorphic forms for the Levi components to which these classes are attached. In this paper, the generic case is mainly dealt with, *i.e.*, we suppose that the highest weight of the representation (ν,E) is regular. We will pursue the other case in a subsequent paper in which, in particular, as one part of the study the contribution of the residual spectrum (as described in [**Konno**]) is discussed. This makes an analysis of the analytic behavior of certain Euler products necessary which naturally come up in the constant terms of the Eisenstein series in question.

Finally we note as one example: our result in the case $j = 2$ (cf. Theorem 5.6) suggests that it might be possible to establish a relation by way of congruences between cuspidal automorphic forms for congruence subgroups of SL_2/F, $F/\boldsymbol{Q}$ an imaginary quadratic extension and forms for $SU(V,H)$. This would be very much in the spirit of the congruences between a Siegel modular form and an elliptic modular form recently described by Harder [**H03**].

We now give an overview of the structure of this paper. In Section 1, we review the description of the cohomology of congruence subgroups in a connected reductive $\boldsymbol{Q}$-group in terms of automorphic forms. We then describe the decomposition of the cohomology as alluded to above. In Section 2, we recall the actual construction of Eisenstein cohomology classes. We make essential use of results in [**S83**]. In this context, we speak of a cohomology class "at infinity" of type (π, w) where π is a cuspidal automorphic representation of the Levi component of P and w ranges over the set of minimal coset representatives for the left cosets of the Weyl group of P in the one of G. These two parameters play a decisive role in the following investigations.

In Section 3, the focus is on the unitary group $SU(V,H)$. We fix the notation for the root system, and we describe the Langlands decomposition of the parabolic subgroups. Finally, we explicitly determine the various restrictions of specific weights μ_w (depending on the parameter w as above) to the Levi Cartan algebras. This is the first step towards determining the points of evaluation for the Eisenstein series we have to consider.

Section 4 contains the results pertaining to the cohomology "at infinity". The possible types (π, w) and their corresponding Archimedean components π_∞ are given. Section 5 contains the complete structural description of the Eisenstein cohomology in the generic case.

In Section 6, for the sake of completeness, we discuss the cuspidal cohomology. It decomposes as a finite algebraic sum where the sum ranges over all cuspidal automorphic representations for which the infinitesimal character of its Archimedean component matches the one of the representation E^* contragradient to (ν, E). Thus we are led to make explicit the classification in [**V-Z**] of the irreducible unitary representation of the real Lie group $G(\boldsymbol{R})$ with non-vanishing cohomology. We list these representations (up to equivalence) and determine their cohomology.

NOTATIONS. We use almost the same notations with those in [**Li-S2**]. We denote by $\boldsymbol{A}$, $\boldsymbol{A}_{\mathrm{f}}$ and $\boldsymbol{A}_\infty$ the ring of adeles, finite adeles and infinite adeles, of $\boldsymbol{Q}$, respectively.

The algebraic groups considered are linear. If H is an algebraic group defined over a field k, and k' is a commutative k-algebra, we denote by $H(k')$ the group of k'-valued points of H except in §4. In §4, H denotes the group of real-valued points $H(\boldsymbol{R})$. The connected component of the identity of the group $H(\boldsymbol{R})$ is denoted by $H(\boldsymbol{R})^\circ$. The ring of k'-points of n by n matrices are denoted by $\mathrm{Mat}_n(k')$.

Let G be a connected reductive algebraic group defined over $\boldsymbol{Q}$. Suppose that a minimal parabolic $\boldsymbol{Q}$-subgroup P_0 of G and a Levi decomposition $P_0 = L_0N_0$ of P_0 over $\boldsymbol{Q}$ have been fixed. By definition, a *standard* parabolic $\boldsymbol{Q}$-subgroup of G is a parabolic $\boldsymbol{Q}$-subgroup P of G with $P_0 \subset P$. Then P has a unique Levi decomposition $P = L_PN_P$ over $\boldsymbol{Q}$ such that $L_P \supset L_0$. If $A_0 = A_{P_0}$ is the maximal $\boldsymbol{Q}$-split torus in the center of L_{P_0}, then there is a unique Langlands decomposition $P = M_PA_PN_P$ with $M_P \supset M_0$ and $A_P \subset A_0$.

The Lie algebras of the group of the real points (e.g. $G(\boldsymbol{R})$, $L(\boldsymbol{R}), \ldots$) are expressed by corresponding Euler Fraktur (e.g. $\mathfrak{g}$, $\mathfrak{l},\ldots$). The complexification of the real Lie algebra $\mathfrak{g}$ is denoted by $\mathfrak{g}_{\boldsymbol{C}}$. We put $\check{\mathfrak{a}}_P = X^*(P)\otimes\boldsymbol{R}$, where $X^*(P)$ denotes the group of $\boldsymbol{Q}$-rational characters of P. We let $\check{\mathfrak{a}}_0 = \check{\mathfrak{a}}_{P_0}$. Similarly $\mathfrak{a}_P = X_*(P)\otimes\boldsymbol{R}$, $\mathfrak{a}_0 = X_*(P_0)\otimes\boldsymbol{R}$ where $X_*(P)$ denotes the group of $\boldsymbol{Q}$-rational cocharacters. By the natural pairing of $\check{\mathfrak{a}}_0$ and $\mathfrak{a}_0$, one has direct sum decompositions $\mathfrak{a}_0 = \mathfrak{a}_P\oplus\mathfrak{a}_0^P$ and $\check{\mathfrak{a}}_0 = \check{\mathfrak{a}}_P\oplus\check{\mathfrak{a}}_0^P$, respectively. Let $\mathfrak{a}_P^Q$ be the intersection of $\mathfrak{a}_P$ and $\mathfrak{a}_0^Q$. Similar notation is used for $\check{\mathfrak{a}}_P^Q$. By M_P resp. $\mathfrak{m}_P$, we denote the intersection $\cap\ker(\chi)$ resp. $\cap\ker(d\chi)$ of the kernels of all rational characters χ of P. Then $\mathfrak{a}_P^G = \mathfrak{a}_P\cap\mathfrak{m}_G$. We denote by $\Phi_P^{\boldsymbol{Q}}\subset X^*(A_P)$ the set of $\boldsymbol{Q}$-roots of A_P and by $\Delta_P^{\boldsymbol{Q}}$ the set of simple roots in $\Phi_P^{\boldsymbol{Q}}$. Then the open Weyl chamber in $\check{\mathfrak{a}}_P^G$ is defined and is denoted by $(\check{\mathfrak{a}}_P^G)^+$. The set of absolute roots of $H(\boldsymbol{C})$, the set of its simple roots and the absolute Weyl group of H are denoted by $\Phi(H)$, $\Delta(H)$ and W_H respectively.

We choose a maximal compact subgroup K of $G(\boldsymbol{A})$ of the form $K = K_{\boldsymbol{R}}C$ with $C\subset G(\boldsymbol{A}_{\mathrm{f}})$, which we suppose in *good* position. Then G has the Iwasawa decomposition $G(\boldsymbol{A}) = L_P(\boldsymbol{A})N_P(\boldsymbol{A})K$ for a given standard parabolic $\boldsymbol{Q}$-subgroup P of G and the standard *height* function $H_P\colon G(\boldsymbol{A})\to\mathfrak{a}_P$ is defined in a usual way.

Let $\mathscr{U}(\mathfrak{g})$ be the universal enveloping algebra of $\mathfrak{g}$, and let $\mathscr{Z}(\mathfrak{g})$ be its center. Any element D in $\mathscr{U}(\mathfrak{g})$ defines a differential operator on the space $C^\infty(A_G(\boldsymbol{R})^\circ\backslash G(\boldsymbol{A}))$ of smooth complex valued functions on $A_G(\boldsymbol{R})^\circ\backslash G(\boldsymbol{A})$ by right differentiation with respect to the real component of $g\in G(\boldsymbol{A})$. This operator is denoted by $f\mapsto Df$. It commutes with the action of $G(\boldsymbol{R})$ given by left translation. If $D\in\mathscr{Z}(g)$, this operator also commutes with action of $G(\boldsymbol{R})$ by right translation.

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1. Cohomology and automorphic forms.

1.1.

Let G be a connected reductive algebraic group defined over $\boldsymbol{Q}$, let A_G denote the maximal $\boldsymbol{Q}$-split torus in the center Z_G of G, and let C be an open compact subgroup of $G(\boldsymbol{A}_{\mathrm{f}})$. Then the double coset space

$$X_C := G(\boldsymbol{Q})A_G(\boldsymbol{R})^\circ\backslash G(\boldsymbol{A})/K_{\boldsymbol{R}}C \tag{1}$$

has only finitely many connected components each of which has the form $\Gamma\backslash G(\boldsymbol{R})^\circ/K_{\boldsymbol{R}}$ for an appropriate arithmetic subgroup Γ of G. We fix a finite dimensional algebraic representation (ν,E) of $G(\boldsymbol{C})$ in a vector space E. Suppose (for the sake of simplicity) that A_G acts on E by a central character χ_E. We may consider the cohomology groups

$$H^\bullet(X_C,E) \tag{2}$$

of X_C with coefficients in the local system E defined by (ν,E). For example, these cohomology groups are defined as the cohomology of the de Rham complex of E-valued currents on X_C. By passing over the open compact subgroups $C\subset G(\boldsymbol{A}_{\mathrm{f}})$ one has a direct system of cohomology groups. The corresponding inductive limit

$$H^\bullet(G,E) = \varinjlim_{C} H^\bullet(X_C,E) \tag{3}$$

is defined and carries a natural structure as a $\pi_0(G(\boldsymbol{Q}))\times G(\boldsymbol{A}_{\mathrm{f}})$-module.

1.2.

The cohomology groups $H^\bullet(G,E)$ have an interpretation in relative Lie algebra cohomology. Let $C^\infty(G(\boldsymbol{Q})A_G(\boldsymbol{R})^\circ\backslash G(\boldsymbol{A}))$ be the space of smooth $K_{\boldsymbol{R}}$-finite functions on $G(\boldsymbol{Q})A_G(\boldsymbol{R})^\circ\backslash G(\boldsymbol{A})$; it carries a natural $(\mathfrak{g},K_{\boldsymbol{R}},G(\boldsymbol{A}_\mathrm{f}))$-module structure. There is an isomorphism between the $(\mathfrak{m}_G,K_{\boldsymbol{R}})$-cohomology of $C^\infty(G(\boldsymbol{Q})A_G(\boldsymbol{R})^\circ\backslash G(\boldsymbol{A}))\otimes E$ and the cohomology of the de Rham complex of E-valued currents on $G(\boldsymbol{Q})A_G(\boldsymbol{R})^\circ\backslash G(\boldsymbol{A})/K_{\boldsymbol{R}}$ which computes the inductive limit $H^\bullet(G,E)$. One has an isomorphism of $G(\boldsymbol{A}_\mathrm{f})$-modules

$$H^\bullet(G,E)\simeq H^\bullet(\mathfrak{m}_G,K_{\boldsymbol{R}},C^\infty(G(\boldsymbol{Q})A_G(\boldsymbol{R})^\circ\backslash G(\boldsymbol{A}))\otimes E)(\chi_E) \tag{4}$$

where the $G(\boldsymbol{A}_\mathrm{f})$-action on the right hand side is twisted by the character χ_E attached to (ν,E).

1.3.

Without altering the cohomology $H^\bullet(G,E)$, the space

$$C^\infty(G(\boldsymbol{Q})A_G(\boldsymbol{R})^\circ\backslash G(\boldsymbol{A}))$$

may be replaced by the subspace V_G of smooth complex valued functions of uniform moderate growth. By definition, a C^∞-function

$$f\colon G(\boldsymbol{Q})A_G(\boldsymbol{R})^\circ\backslash G(\boldsymbol{A})\to\boldsymbol{C}$$

is in V_G if

- f is $K_{\boldsymbol{R}}$-finite
- There exists a constant $c>0$, $c\in\boldsymbol{R}$, such that for all elements $D\in\mathscr{U}(\mathfrak{g})$ there is $r_D\in\boldsymbol{R}$ with

$$|Df(g)|\leq r_D||g||^c \quad \text{for all } g\in G(\boldsymbol{A}). \tag{5}$$

The space V_G carries in a natural way the structure of a $(\mathfrak{g},K_{\boldsymbol{R}},G(\boldsymbol{A}_\mathrm{f}))$-module.

Let $\mathscr{C}$ be the set of classes of associate parabolic $\boldsymbol{Q}$-subgroups of G. For $\{P\}\in\mathscr{C}$ denote by $V_G(\{P\})$ the space of elements in V_G which are negligible along Q for every parabolic $\boldsymbol{Q}$-subgroup $Q\in\mathscr{C}$, $Q\notin\{P\}$, *i.e.*, given $Q=LN$ for all $g\in G(\boldsymbol{A})$ the function $l\mapsto f_Q(lg)$ (where f_Q denotes the constant term of f with respect to Q) is orthogonal to the space of cuspidal functions on $A_G(\boldsymbol{R})^\circ L(\boldsymbol{Q})\backslash L(\boldsymbol{A})$. The spaces $V_G(\{P\})$, $\{P\}\in\mathscr{C}$, are submodules with respect to the $(\mathfrak{g},K_{\boldsymbol{R}},G(\boldsymbol{A}_\mathrm{f}))$-module structure. Finally, proved by Langlands [**L**] (see also [**B-L-S**, 2.4]), the space V_G has a decomposition as a direct sum of $(\mathfrak{g},K_{\boldsymbol{R}},G(\boldsymbol{A}_\mathrm{f}))$-modules

$$V_G=\bigoplus_{\{P\}\in\mathscr{C}}V_G(\{P\}). \tag{6}$$

The inclusion $V_G\to C^\infty(G(\boldsymbol{Q})A_G(\boldsymbol{R})^\circ\backslash G(\boldsymbol{A}))$ induces an isomorphism on the level of $(\mathfrak{g},K_{\boldsymbol{R}})$-cohomology, *i.e.*, one gets in view of the decomposition

$$H^\bullet(G,E)=\bigoplus_{\{P\}\in\mathscr{C}}H^\bullet(\mathfrak{m}_G,K_{\boldsymbol{R}},V_G(\{P\})\otimes E)(\chi_E). \tag{7}$$

Let $\mathscr{Z}(\mathfrak{g})$ be the center of the universal enveloping algebra of the Lie algebra $\mathfrak{g}$ of G and let $\mathscr{J}\subset\mathscr{Z}(\mathfrak{g})$ be the annihilator of the dual representation E^* in $\mathscr{Z}(\mathfrak{g})$. The space of $K_{\boldsymbol{R}}$-finite smooth functions of uniform moderate growth in V_G which are annihilated by a power of $\mathscr{J}$ is denoted by $\mathscr{A}_E$. For a given class $\{P\} \in \mathscr{C}$ we put

$$\mathscr{A}_{E,\{P\}} := \mathscr{A}_E \cap V_G(\{P\}). \tag{8}$$

The space $\mathscr{A}_E$ (resp. the spaces $\mathscr{A}_{E,\{P\}}$) are $(\mathfrak{g}, K_{\boldsymbol{R}}, G(\boldsymbol{A}_{\mathrm{f}}))$-modules. There is a decomposition of $\mathscr{A}_E$ as a disjoint sum of $(\mathfrak{g}, K_{\boldsymbol{R}}, G(\boldsymbol{A}_{\mathrm{f}}))$-modules

$$\mathscr{A}_E = \bigoplus_{\{P\}\in\mathscr{C}} \mathscr{A}_{E,\{P\}}. \tag{9}$$

The inclusion $\mathscr{A}_E \to V_G$ of the spaces of automorphic forms on G (with respect to (ν, E)) in the space V_G of functions of uniform moderate growth induces, by [**F**, Theorem 18], an isomorphism on the level of $(\mathfrak{m}_G, K_{\boldsymbol{R}})$-cohomology. One obtains the decomposition

$$H^\bullet(G,E) = \bigoplus_{\{P\}\in\mathscr{C}} H^\bullet(\mathfrak{m}_G, K_{\boldsymbol{R}}, \mathscr{A}_{E,\{P\}} \otimes E)(\chi_E). \tag{10}$$

As exhibited in [**F-S**] there is a refinement

$$\mathscr{A}_E = \bigoplus_{\{P\}\in\mathscr{C}} \bigoplus_{\phi\in\Phi_{E,\{P\}}} \mathscr{A}_{E,\{P\},\phi} \tag{11}$$

of the decomposition (9) where for a given class $\{P\}$ in the set $\mathscr{C}$ of associate parabolic $\boldsymbol{Q}$-subgroups of G the second sum ranges over the set $\Phi_{E,\{P\}}$ of classes $\phi = \{\phi_Q\}_{Q\in\{P\}}$ of associate irreducible automorphic representations of the Levi components of elements of $\{P\}$. One can give two alternative definitions of the spaces $\mathscr{A}_{E,\{P\},\phi}$ one by use of the concept of the constant term, the other one in terms of Eisenstein series or residues of such. For details we refer to Section 2.2, respectively [**F-S**, Section 1.5]. This decomposition of $\mathscr{A}_E$, along the cuspidal support implies a corresponding one in cohomology

$$H^\bullet(G,E) = \bigoplus_{\{P\}\in\mathscr{C}} \bigoplus_{\phi\in\Phi_{E,\{P\}}} H^\bullet(\mathfrak{m}_G, K_{\boldsymbol{R}}, \mathscr{A}_{E,\{P\},\phi} \otimes E)(\chi_E). \tag{12}$$

The cuspidal cohomology to be denoted by

$$H^\bullet_{\mathrm{cusp}}(G,E) := H^\bullet(\mathfrak{m}_G, K_{\boldsymbol{R}}, \mathscr{A}_{E,\{G\}} \otimes E)(\chi_E) \tag{13}$$

is, by definition, the summand indexed by the class $\{G\} \in \mathscr{C}$ in this decomposition. The natural complement to the cuspidal cohomology in $H^\bullet(G,E)$ is called the Eisenstein cohomology, *i.e.*, one has, by definition,

$$H^\bullet_{\mathrm{Eis}}(G,E) = \bigoplus_{\substack{\{P\}\in\mathscr{C},\\ \{P\}\neq\{G\}}} \bigoplus_{\phi\in\Phi_{E,\{P\}}} H^\bullet(\mathfrak{m}_G, K_{\boldsymbol{R}}, \mathscr{A}_{E,\{P\},\phi} \otimes E)(\chi_E). \tag{14}$$

2. Construction of Eisenstein cohomology classes.

2.1. Eisenstein series.

Let $\{P\} \in \mathscr{C}$ be a class of associate parabolic $\boldsymbol{Q}$-subgroups of G represented by a standard parabolic $\boldsymbol{Q}$-subgroup $P = L_P N_P$. Let $\phi = \{\phi_Q\}_{Q\in\{P\}}$ be a class of associate irreducible cuspidal automorphic representations of the Levi components of elements of $\{P\}$, *i.e.*, by definition, given $Q \in \{P\}$, ϕ_Q is a finite set of irreducible representations of $L_Q(\boldsymbol{A})$ which are unitary modulo the center, such that for each $\pi \in \phi_Q$ the central character $\chi_\pi \colon A_Q(\boldsymbol{A}) \to \boldsymbol{C}^\times$ is trivial on $A_Q(\boldsymbol{Q})$ and such that π occurs in the cuspidal spectrum $L^2_{\text{cusp}}(L_Q(\boldsymbol{Q})\backslash L_Q(\boldsymbol{A}))_{\chi_\pi}$ (with respect to χ_π). Moreover, certain compatibility conditions as specified in [**F-S**, 1.2], have to be satisfied, and we suppose that for each $\pi \in \phi_Q$ its infinitesimal character coincides with the one of E^*. Recall that the set of all collections $\phi = \{\phi_Q\}_{Q\in\{P\}}$ constitute $\Phi_{E,\{P\}}$ for a given $\{P\} \in \mathscr{C}$.

Consider an irreducible representation $\pi \in \phi_Q$ for a given $Q \in \{P\}$ and a given collection $\phi \in \Phi_{E,\{P\}}$. Let $\tilde{\pi}$ be the unitary representation

$$\tilde{\pi}(l) = e^{-\langle d\chi_\pi, H_Q(l)\rangle}\pi(l) \quad l \in L_Q(\boldsymbol{A}) \tag{15}$$

and let $W_{Q,\tilde{\pi}}$ be the space of smooth $K_{\boldsymbol{R}}$-finite functions

$$f \colon L_Q(\boldsymbol{Q})N_Q(\boldsymbol{A})A_Q(\boldsymbol{R})^\circ\backslash G(\boldsymbol{A}) \to \boldsymbol{C} \tag{16}$$

such that for any $g \in G(\boldsymbol{A})$ the function $f(lg)$ of the variable $l \in L_Q(\boldsymbol{A})$ belongs to the space $L^2_{\text{cusp},\tilde{\pi}}(L_Q(\boldsymbol{Q})A_Q(\boldsymbol{R})^\circ\backslash L_Q(\boldsymbol{A}))$ of cuspidal automorphic forms which transform according to $\tilde{\pi}$. For $f \in W_{Q,\tilde{\pi}}$, there is the Eisenstein series depending on the complex parameter $\lambda \in \check{\mathfrak{a}}^G_Q$ defined (at least formally) by

$$E(f,\lambda)(g) = E^G_Q(f,\lambda)(g) = \sum_{\gamma\in Q(\boldsymbol{Q})\backslash G(\boldsymbol{Q})} e^{\langle H_Q(\gamma g),\lambda+\rho_Q\rangle} f(\gamma g) \quad g \in G(\boldsymbol{A}). \tag{17}$$

If the real part of the complex parameter is sufficiently regular and lies inside the positive Weyl chamber $(\check{\mathfrak{a}}^G_Q)^+$ defined by Q the Eisenstein series $E(f,\lambda)(g)$ converges uniformly in g. The map $\lambda \mapsto E(f,\lambda)(g)$ is holomorphic in the region of absolute convergence and has a meromorphic continuation to all of $\check{\mathfrak{a}}^G_Q$. Its singularities lie along certain root hyperplanes.

2.2. The spaces $\mathscr{A}_{E,\{P\},\phi}$.

For a given collection $\phi = \{\phi_Q\} \in \Phi_{E,\{P\}}$ we consider $\pi \in \phi_Q$, $Q \in \{P\}$. Then there exists a polynomial function q on $\check{\mathfrak{a}}^G_Q$ such that for any $f \in W_{Q,\tilde{\pi}}$ the function $q(\lambda)E^G_Q(f,\lambda)$ is holomorphic in a neighborhood of χ_π in $\check{\mathfrak{a}}^G_Q$. If we choose Cartesian coordinates $z_1(\lambda),\ldots,z_r(\lambda)$ on $\check{\mathfrak{a}}^G_Q$ the function $q(\lambda)E^G_Q(f,\lambda)$ has a Taylor expansion near χ_π given by

$$q(\lambda)E^G_Q(f,\lambda) = \sum_{i_1,\ldots,i_r=0}^{\infty} a_{i_1,\ldots,i_r}(f) z_i(\lambda-\chi_\pi)^{i_1}\cdots z_r(\lambda-\chi_\pi)^{i_r}. \tag{18}$$

By definition, the space $\mathscr{A}_{E,\{P\},\phi}$ is the space of functions generated by the $a_{i_1,\ldots,i_r}(f)$ where the $i_1,\ldots,i_r$ run from zero to infinity and where f ranges through the space $W_{Q,\tilde{\pi}}$. Another choice of the polynomial function q does not alter the space $\mathscr{A}_{E,\{P\},\phi}$. By the functional equations of the Eisenstein series involved and the compatibility conditions imposed on the elements in ϕ_Q it is also independent of the choices of the representatives Q in $\{P\}$ resp. $\pi \in \phi_Q$. Note that $\mathscr{A}_{E,\{P\},\phi}$ is spanned by all possible residues and derivatives with respect to the parameter λ of Eisenstein series attached to cuspidal automorphic forms of type ϕ at values λ in the positive Weyl chamber defined by Q for which the infinitesimal character χ_{E^*} of E^* is matched.

2.3.

Let $Q = L_Q N_Q$ be a standard parabolic $\boldsymbol{Q}$-subgroup of G so that $Q \in \{P\}$ for a given choice of an associate class $\{P\} \in \mathscr{C}$. The symmetric algebra $S(\check{\mathfrak{a}}_Q^G)$ of $\check{\mathfrak{a}}_Q^G$ may be viewed as the space of polynomials on $\mathfrak{a}_Q^G$ and thus $\mathfrak{a}_Q^G$ acts on it by translations. The symmetric algebra may also be interpreted as the space of differential operators with constant coefficients on $\mathfrak{a}_Q^G$. For a given $\pi \in \phi_Q$ the space $W_{Q,\tilde{\pi}} \otimes_{\boldsymbol{C}} S(\check{\mathfrak{a}}_Q^G)$ carries the structure of a $(\mathfrak{g}, K_{\boldsymbol{R}}, G(\boldsymbol{A}_{\mathrm{f}}))$-module ([**F**, p. 218]). There is an isomorphism of $(\mathfrak{g}, K_{\boldsymbol{R}}, G(\boldsymbol{A}_{\mathrm{f}}))$-modules between the space $W_{Q,\tilde{\pi}} \otimes_{\boldsymbol{C}} S(\check{\mathfrak{a}}_Q^G)$ and the sum of m_π copies of the induced representation

$$\operatorname{Ind}_{Q(\boldsymbol{A}_{\mathrm{f}})}^{G(\boldsymbol{A}_{\mathrm{f}})} \operatorname{Ind}_{\mathfrak{g} \cap \mathfrak{l}_Q, K_{\boldsymbol{R}} \cap L_Q(\boldsymbol{R})}^{\mathfrak{g}, K_{\boldsymbol{R}}} \left[H_\pi \otimes_{\boldsymbol{C}} S(\check{\mathfrak{a}}_Q^G) \right] \tag{19}$$

where H_π denotes the representation space corresponding to $\pi \in \phi_Q$ and occurring with multiplicity m_π in $L^2_{\mathrm{cusp}}(L_Q(\boldsymbol{Q}) \backslash L_Q(\boldsymbol{A}))_{\chi_\pi}$. Then the map (as defined in 2.2)

$$\begin{aligned} W_{Q,\tilde{\pi}} \otimes_{\boldsymbol{C}} S(\check{\mathfrak{a}}_Q^G) &\to \mathscr{A}_{E,\{P\},\phi} \\ f \otimes \frac{\partial^\alpha}{\partial \lambda^\alpha} &\mapsto \left[\frac{\partial^\alpha}{\partial \lambda^\alpha} \big(q(\lambda) E_Q^G(f, \lambda) \big) \right] (\chi_\pi) \end{aligned} \tag{20}$$

becomes under this identification a homomorphism of $(\mathfrak{g}, K_{\boldsymbol{R}}, G(\boldsymbol{A}_{\mathrm{f}}))$-modules. For (20), α denotes a multi-index and $\partial^\alpha / \partial \lambda^\alpha$ denotes the derivative of order α with respect to a Cartesian system of coordinates on $\check{\mathfrak{a}}_Q^G$.

Thus as a first step, in order to understand the cohomological contribution of the spaces $\mathscr{A}_{E,\{P\},\phi}$ in the decomposition

$$H^\bullet(G, E) \simeq \bigoplus_{\{P\} \in \mathscr{C}} \bigoplus_{\phi \in \Phi_{E,\{P\}}} H^\bullet(\mathfrak{m}_G, K_{\boldsymbol{R}}, \mathscr{A}_{E,\{P\},\phi} \otimes E)(\chi_E) \tag{21}$$

it is necessary to determine the cohomological spaces

$$H^\bullet\big(\mathfrak{m}_G, K_{\boldsymbol{R}}, W_{Q,\tilde{\pi}} \otimes S(\check{\mathfrak{a}}_Q^P) \otimes E\big) \quad Q \in \{P\},\, \pi \in \phi_Q. \tag{22}$$

The second step regards a careful analysis of the map induced by (20) on the cohomological level.

2.4. Classes of type $(\boldsymbol{\pi}, \boldsymbol{w})$.

Using the interpretation of the $(\mathfrak{g}, K_{\boldsymbol{R}}, G(\boldsymbol{A}_{\mathrm{f}}))$-module $W_{Q,\tilde{\pi}} \otimes S(\check{\mathfrak{a}}_Q^G)$ as the sum of m_π copies of the induced representation

$$\operatorname{Ind}_{Q(\boldsymbol{A}_{\mathrm{f}})}^{G(\boldsymbol{A}_{\mathrm{f}})} \operatorname{Ind}_{\mathfrak{g} \cap \mathfrak{l}_Q, K_{\boldsymbol{R}} \cap L_Q(\boldsymbol{R})}^{\mathfrak{g}, K_{\boldsymbol{R}}} \left[H_\pi \otimes_{\boldsymbol{C}} S(\check{\mathfrak{a}}_Q^G) \right] \tag{23}$$

as given in 2.3, the analysis of the cohomology of the $W_{Q,\tilde{\pi}} \otimes S(\check{\mathfrak{a}}_Q^G)$ leads to a study of the cohomology of

$$H^\bullet\Big(\mathfrak{m}_G, K_{\boldsymbol{R}}, \operatorname{Ind}_{Q(\boldsymbol{A}_{\mathrm{f}})}^{G(\boldsymbol{A}_{\mathrm{f}})} \operatorname{Ind}_{\mathfrak{g} \cap \mathfrak{l}_Q, K_{\boldsymbol{R}} \cap L_Q(\boldsymbol{R})}^{\mathfrak{g}, K_{\boldsymbol{R}}} \left[H_\pi \otimes_{\boldsymbol{C}} S(\check{\mathfrak{a}}_Q^G) \otimes E \right]\Big). \tag{24}$$

By use of Frobenius reciprocity this space is equal to (cf. [**F**, p. 256])

$$\operatorname{Ind}_{Q(\boldsymbol{A}_{\mathrm{f}})}^{G(\boldsymbol{A}_{\mathrm{f}})} H^\bullet\big(\mathfrak{l}_Q, K_{\boldsymbol{R}} \cap L_Q(\boldsymbol{R}), H_\pi \otimes H^\bullet(\mathfrak{n}_Q, E) \otimes S(\check{\mathfrak{a}}_Q^G)\big). \tag{25}$$

The $\mathfrak{n}_Q$-cohomology is well understood by the following result of Kostant ([**K**, 5.1]). Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$ contained in $\mathfrak{l}_{P_0}$. The highest weight of the given irreducible finite dimensional representation (ν,E) of $G(\boldsymbol{C})$ is denoted by Λ. Then the Lie algebra cohomology of $\mathfrak{n}_Q$ with coefficients in E is given as a $(\mathfrak{l}_Q, K_{\boldsymbol{R}} \cap L_Q(\boldsymbol{R}))$-module by the sum

$$H^q(\mathfrak{n}_Q, E) = \bigoplus_{\substack{w \in W^Q \\ \ell(w)=q}} F_{\mu_w} \tag{26}$$

where the sum ranges over the elements w in the set

$$W^Q = \{w \in W_G \mid w^{-1}(\Delta_{L_Q}) \subset \Phi^+\} \tag{27}$$

of minimal coset representatives for the left coset $W_Q \backslash W_G$ with length $\ell(w) = q$, and where F_{μ_w} denotes the irreducible finite dimensional $(\mathfrak{l}_Q, K_{\boldsymbol{R}} \cap L_Q(\boldsymbol{R}))$-module of highest weight

$$\mu_w = w(\Lambda + \rho) - \rho. \tag{28}$$

The weights μ_w are all dominant and distinct.

This result implies a further decomposition of (25) as a sum over W^Q. One obtains

$$\bigoplus_{w \in W^Q} \mathrm{Ind}_{Q(\boldsymbol{A}_{\mathrm{f}})}^{G(\boldsymbol{A}_{\mathrm{f}})} H^\bullet\big(\mathfrak{l}_Q, K_{\boldsymbol{R}} \cap L_Q(\boldsymbol{R}), H_\pi \otimes F_{\mu_w} \otimes S(\check{\mathfrak{a}}_Q^G)\big). \tag{29}$$

This allows us to introduce the notion of a cohomology class of type (π, w), $w \in W^Q$. By definition, it is a class contained in the summand in (29) indexed by $w \in W^Q$. Note that the infinitesimal character of the representation contragradient to F_{μ_w} is given by $\chi = -w(\Lambda+\rho)|_{\mathfrak{b}}$, $\mathfrak{b}$ a Cartan subalgebra in $\mathfrak{m}_Q$ so that $\mathfrak{b} + \mathfrak{a}_Q = \mathfrak{h}$. Thus, for given non-trivial cohomology class of type (π, w) the infinitesimal character χ_π of the Archimedean component of π is

$$\chi_\pi = \chi_{-w(\Lambda+\rho)|_{\mathfrak{b}}}. \tag{30}$$

3. The unitary group.

Let F be a quadratic imaginary extension of $\boldsymbol{Q}$. Let $x \mapsto \overline{x}$ be a nontrivial involution of $F/\boldsymbol{Q}$. Let V_F be an n-dimensional vector space endowed with non-degenerate hermitian form

$$\langle x, y \rangle = xH^t\overline{y}.$$

The group $U(H)$ (resp. $SU(H)$) is a unitary group (resp. special unitary group) defined over $\boldsymbol{Q}$, that is,

$$U(V,H) = \{g \in GL(V_F) \mid gH^t\overline{g} = H\},$$

$$SU(V,H) = \{g \in SL(V_F) \mid gH^t\overline{g} = H\}.$$

Let us assume $n = 2m$ is even. If we take $H_m = \begin{pmatrix} & I_m \\ I_m & \end{pmatrix}$, then $SU(H_m)$ becomes a quasi-split algebraic group of rank m which we denote $SU(m,m)$. In the following we let $m = 2$, *i.e.*, we consider the algebraic $\boldsymbol{Q}$-group $G = SU(2,2)$ of rank two.

3.1. Roots and parabolic subgroups.

Let

$$T = \{t_{a,b} = \operatorname{diag}(a,b,a^{-1},b^{-1}) \in G \mid a,b \in \boldsymbol{Q}^{\times}\}$$

be a $\boldsymbol{Q}$-split torus and let

$$e_1(t_{a,b}) = a, \quad e_2(t_{a,b}) = b.$$

Put $\alpha_1 = e_1/e_2$, $\alpha_2 = e_2^2$. Then the root system Φ_0 of G becomes $\{\alpha_1^{\pm 1}, \alpha_2^{\pm 1}, (\alpha_1\alpha_2)^{\pm 1}, (\alpha_1^2\alpha_2)^{\pm 1}\}$ which is of type C_2. We fix once and for all the simple roots $\Delta_0^{\boldsymbol{Q}} = \{\alpha_1, \alpha_2\}$. The conjugacy classes of $\boldsymbol{Q}$-parabolic subgroups of G are parametrized by the subsets of $\Delta_0^{\boldsymbol{Q}}$. Namely let the subsets be $J_0 = \varnothing$, $J_1 = \{\alpha_2\}$, $J_2 = \{\alpha_1\}$ and $J_3 = \Delta_0^{\boldsymbol{Q}}$ and let the standard parabolic subgroups to be $P_j = P_{J_j}$ where P_J is described as follows: We let $S_J = (\bigcap_{\alpha \in J} \ker \alpha)^{\circ}$, and let $Z(S_J)$ be the centralizer of S_J. Then P_J is the semidirect of its unipotent radical U_J by $Z(S_J)$. Note that the characters of T in U_J are exactly the positive roots which contain at least one simple root not in J. In particular P_0 is a minimal parabolic subgroup and $P_3 = G$ itself. The Langlands decomposition of P_j is denoted by $P_j = L_j N_j$.

Now let $J = \varnothing$. Then P_0 is described as follows.

$$\begin{aligned}
A_0 &= T, \\
L_0 &= \{\operatorname{diag}(a,b,\bar{a}^{-1},\bar{b}^{-1}) \mid a,b \in F,\, ab \in \boldsymbol{Q}^{\times}\}, \\
M_0 &= \{\operatorname{diag}(a,a^{-1},a,a^{-1}) \in L_0 \mid \mathcal{N}_{F/\boldsymbol{Q}}(a) = 1\}, \\
N_0 &= \left\{ \begin{pmatrix} 0 & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & * & 0 \end{pmatrix} \right\}.
\end{aligned} \tag{31}$$

The case of $J_1 = \{\alpha_2\}$ is as follows.

$$\begin{aligned}
A_1 &= \{\operatorname{diag}(a,1,a^{-1},1)\}, \\
L_1 &= \left\{ \begin{pmatrix} x & 0 & 0 & 0 \\ 0 & u_{11} & 0 & u_{12} \\ 0 & 0 & \bar{x}^{-1} & 0 \\ 0 & u_{21} & 0 & u_{22} \end{pmatrix} \;\middle|\; \begin{array}{l} u = (u_{ij}) \in U(1,1), \\ x\bar{x}^{-1} \cdot \det u = 1, x \in F \end{array} \right\}, \\
M_1 &= \left\{ \begin{pmatrix} x & 0 & 0 & 0 \\ 0 & u_{11} & 0 & u_{12} \\ 0 & 0 & x & 0 \\ 0 & u_{21} & 0 & u_{22} \end{pmatrix} \in L_1 \mid \mathcal{N}_{F/\boldsymbol{Q}}(x) = 1 \right\}, \\
N_1 &= \left\{ \begin{pmatrix} 0 & * & * & * \\ 0 & 0 & * & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & * & 0 \end{pmatrix} \right\}, \quad N_1(\boldsymbol{R}) \text{ is 5-dimensional, non-abelian.}
\end{aligned} \tag{32}$$

The case of $J_2 = \{\alpha_1\}$ is as follows.

$$\begin{aligned}
A_2 &= \{\mathrm{diag}(a,a,a^{-1},a^{-1})\},\\
L_2 &= \left\{ \begin{pmatrix} A & 0 \\ 0 & {}^t\overline{A}^{-1} \end{pmatrix} \mid A \in \mathrm{Mat}_2(F),\, \det(A) \in \boldsymbol{Q}^\times \right\},\\
M_2 &= \left\{ \begin{pmatrix} A & 0 \\ 0 & {}^t\overline{A}^{-1} \end{pmatrix} \mid A \in \mathrm{Mat}_2(F),\, \det(A) = \pm 1 \right\}, \qquad (33)\\
N_2 &= \left\{ \begin{pmatrix} 0 & 0 & * & * \\ 0 & 0 & * & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \right\}, \quad N_2(\boldsymbol{R}) \text{ is 4-dimensional abelian.}
\end{aligned}$$

In particular, we have

$$\begin{aligned}
L_0(\boldsymbol{R}) &\simeq \boldsymbol{Z}/2\boldsymbol{Z} \times (\boldsymbol{R}_{>0})^{\oplus 2} \times \boldsymbol{C}^{(1)},\\
L_1(\boldsymbol{R}) &\simeq \boldsymbol{R}^\times \times \boldsymbol{C}^{(1)} \times SU(1,1), \qquad (34)\\
L_2(\boldsymbol{R}) &\simeq \boldsymbol{R}_{>0} \times SL_2^{\pm}(\boldsymbol{C}).
\end{aligned}$$

3.2. The absolute root system and compatible order.

Let $\mathfrak{h}$ be the Lie algebra of $L_0(\boldsymbol{R})$. Then it is a Cartan algebra containing $\mathfrak{t}$. Let $\varepsilon_j(\mathrm{diag}(a_1,a_2,a_3,a_4)) = a_j$. Put

$$\beta_1 = \varepsilon_1 - \varepsilon_2, \quad \beta_2 = \varepsilon_2 - \varepsilon_4, \quad \beta_3 = \varepsilon_4 - \varepsilon_3.$$

Then the absolute root system $\Phi(\mathfrak{g}_{\boldsymbol{C}}, \mathfrak{h}_{\boldsymbol{C}})$ of $\mathfrak{g}_{\boldsymbol{C}}$ with respect to $\mathfrak{h}_{\boldsymbol{C}}$ is given by

$$\Phi(\mathfrak{g}_{\boldsymbol{C}}, \mathfrak{h}_{\boldsymbol{C}}) = \pm\{\beta_1, \beta_2, \beta_3, \beta_1+\beta_2, \beta_2+\beta_3, \beta_1+\beta_2+\beta_3\}.$$

This is of A_3-type. By definition, a compatible order with respect to $\Delta_0^{\boldsymbol{Q}}$ forces the simple roots of $\mathfrak{g}_{\boldsymbol{C}}, \mathfrak{h}_{\boldsymbol{C}}$ to be

$$\Delta = \{\beta_1, \beta_2, \beta_3\}.$$

The compatible simple root Δ also fixes the positive system $\Phi^+(\mathfrak{g}_{\boldsymbol{C}}, \mathfrak{h}_{\boldsymbol{C}})$. The root systems $\Phi(P_j)$ of the parabolic subgroups $P_j(\boldsymbol{C})$ are then

$$\begin{aligned}
&\Phi(P_0) = \Phi^+(\mathfrak{g}_{\boldsymbol{C}}, \mathfrak{h}_{\boldsymbol{C}}), \quad \Phi(P_1) = \Phi(P_0) \setminus \{\beta_2\},\\
&\Phi(P_2) = \{\beta_2, \beta_1+\beta_2, \beta_2+\beta_3, \beta_1+\beta_2+\beta_3\}.
\end{aligned}$$

The Weyl group W_G of $G(\boldsymbol{C})$ is generated by the simple reflections s_1, s_2, s_3 defined by the simple roots $\beta_1, \beta_2, \beta_3$, respectively. If we identify W_G with the symmetric group of degree 4 by

$$\text{“a reflection with respect to } \varepsilon_i - \varepsilon_j\text{”} \mapsto (i\,j) \quad \text{(a mutual permutation)},$$

then s_1, s_2, s_3 are given by $(12), (24), (34)$ respectively. The Weyl group W_j of $L_j(\boldsymbol{C})$ $(j = 0, 1, 2)$ are identified with subgroups of W_G,

$$W_0 = 1, W_1 = \langle s_2 \rangle, W_2 = \langle s_1, s_3 \rangle.$$

Let W^P be the set of minimal coset representatives of $W_P \backslash W_G$ defined in 2.4. A computation shows

$$\begin{aligned} W^{P_0} &= W_G, \quad W^{P_1} \cap W^{P_2} = \{1\}, \\ W^{P_1} &= \{1\} \cup \{s_1, s_3\} \cup \{s_1 s_3, s_1 s_2, s_3 s_2\} \cup \{s_1 s_3 s_2, s_1 s_2 s_3, s_3 s_2 s_1\} \\ &\quad \cup \{s_1 s_2 s_3 s_2, s_1 s_3 s_2 s_1\} \cup \{s_1 s_2 s_3 s_2 s_1\}, \\ W^{P_2} &= \{1\} \cup \{s_2\} \cup \{s_2 s_1, s_2 s_3\} \cup \{s_2 s_3 s_1\} \cup \{s_2 s_1 s_3 s_2\}. \end{aligned} \tag{35}$$

3.3. Fundamental weights.

Let $\omega_1, \omega_2, \omega_3$ be the fundamental weights defined by the simple roots Δ, namely

$$\begin{aligned} \omega_1 &= 1/4(3\varepsilon_1 - \varepsilon_2 - \varepsilon_3 - \varepsilon_4) = 1/4(3\beta_1 + 2\beta_2 + \beta_3), \\ \omega_2 &= 1/2(\varepsilon_1 + \varepsilon_2 - \varepsilon_3 - \varepsilon_4) = 1/2(\beta_1 + 2\beta_2 + \beta_3), \\ \omega_3 &= 1/4(\varepsilon_1 + \varepsilon_2 - 3\varepsilon_3 + \varepsilon_4) = 1/4(\beta_1 + 2\beta_2 + 3\beta_3). \end{aligned} \tag{36}$$

In the following, we express the weights by using coordinates

$$(c_1, c_2, c_3) := c_1\omega_1 + c_2\omega_2 + c_3\omega_3.$$

It is regular if and only if $c_1 > 0$, $c_2 > 0$ and $c_3 > 0$. The simple roots are in Δ are given in these coordinates by

$$\beta_1 = (2, -1, 0), \quad \beta_2 = (-1, 2, -1), \quad \beta_3 = (0, -1, 2).$$

Let ρ_P denote the half sum of the roots $\Phi(P)$ of the parabolic subgroup $P(\boldsymbol{C})$. One obtains in terms of these coordinates

$$\rho = \rho_{P_0} = (1, 1, 1), \quad \rho_{P_1} = (3/2, 0, 3/2), \quad \rho_{P_2} = (0, 2, 0).$$

3.4. The restriction of weights to the Levi Cartan subalgebra.

By the theorem of Kostant, we have an interest on the restriction of weights $\mu_w = w(\Lambda + \rho) - \rho$ for $w \in W^P$ to the Cartan subalgebra of the each Levi subalgebra.

Take a Levi Cartan subalgebra $\mathfrak{b}_j$ by $\mathfrak{b}_j = \mathfrak{m}_j \cap \mathfrak{h}$ such that $\mathfrak{h} = \mathfrak{b}_j + \mathfrak{t}_j$. Identify $\mathfrak{b}_j^*$ with its image in $\mathfrak{h}^*$ via the restriction map. Through this identification the root system $\Phi(M_j)$ of $M_j(\boldsymbol{C})$ is defined by the following simple roots.

$$\Delta(M_0) = \varnothing, \quad \Delta(M_1) = \{\beta_2\}, \quad \Delta(M_2) = \{\beta_1, \beta_3\}.$$

We take as a fundamental weight of Levi subalgebras in a usual manner.

$$\begin{aligned}
\omega_c &= (1/2, 0, -1/2) \quad (\in \mathfrak{b}_0^*, \mathfrak{b}_1^*) \\
\omega_{01} &= (1/2, 0, 1/2) \quad (\in \mathfrak{t}_0^*) \\
\omega_{02} &= (-1/2, 1, -1/2) \quad (\in \mathfrak{t}_0^*, \mathfrak{b}_1^*) \\
\omega_{21} &= (1, -1/2, 0) \quad (\in \mathfrak{b}_2^*) \\
\omega_{22} &= \mathrm{diag}(0, 0, -1/2, 1/2) = (0, -1/2, 1) \quad (\in \mathfrak{b}_2^*)
\end{aligned}$$

Then it follows

$$(c_1, c_2, c_3)|_{\mathfrak{b}_j^*} = \begin{cases} (c_1 - c_3)\omega_c & (j = 0) \\ c_2\omega_{02} + (c_1 - c_3)\omega_c & (j = 1) \\ c_1\omega_{21} + c_3\omega_{22} & (j = 2) \end{cases} \tag{37}$$

and for the split part

$$(c_1, c_2, c_3)|_{\mathfrak{t}_j^*} = \begin{cases} (c_1 + c_2 + c_3)\omega_{01} + c_2\omega_{02} & (j = 0) \\ (c_1 + c_2 + c_3) \cdot \rho_{P_1}/3 & (j = 1) \\ (c_1 + 2c_2 + c_3) \cdot \rho_{P_2}/4 & (j = 2). \end{cases} \tag{38}$$

The coordinates of $\mu_w(\Lambda) = w(\Lambda + \rho) - \rho$ as well as its restriction to each Cartan subalgebra are shown in Table 1, 2 and 3.

4. Cohomology at infinity.

As explained in Section 2, we should determine which cohomology class of type (π, w) remain non zero in the summand of (29). In this section we compute the cohomology $H^\bullet(\mathfrak{m}_j, K_{L_j}, H_\pi \otimes F_\mu)$ where $K_{L_j} = K_{\boldsymbol{R}} \cap L_j(\boldsymbol{R})$ and $\mu = \mu_w$ for all standard parabolic subgroups P_j and for all irreducible cohomological unitary representations (π, H_π) of M_j. For notation, G, L, M are the groups of real points and $G_{\boldsymbol{C}} = G(\boldsymbol{C})$, $L_{\boldsymbol{C}} = L(\boldsymbol{C})$, $M_{\boldsymbol{C}} = M(\boldsymbol{C})$.

4.1. The minimal parabolic P_0 $(j = 0)$.

Recall that L_0 consists of diagonal elements and M_0 has the identification

$$M_0 \ni \mathrm{diag}(z, \varepsilon \cdot z^{-1}, z, \varepsilon \cdot z^{-1}) \mapsto (\varepsilon, z) \in \{\pm 1\} \times \boldsymbol{C}^{(1)}$$

which is compact. Let $\chi_{e,n}$ be the unitary character of M_0 defined by

$$\chi_{e,n}(\varepsilon, z) = \varepsilon^e z^n \quad (e = 0, 1,\ n \in \boldsymbol{Z}).$$

Given $\mu \in \mathfrak{h}_{\boldsymbol{C}}^*$ with $\mu = (\mu_1, \mu_2, \mu_3)$ as in 3.3, we have an isomorphism

$$F_{(\mu_1, \mu_2, \mu_3)} \simeq \chi_{\mu_2, \mu_1 - \mu_3}.$$

LEMMA 4.2. *Let $\mu = (\mu_1, \mu_2, \mu_3) \in \mathfrak{h}_{\boldsymbol{C}}^*$ be the highest weight of the irreducible $L_{0,\boldsymbol{C}}$-module F_μ. Then*

$$H^q(\mathfrak{m}_0, M_0, \chi_{e,n} \otimes F_\mu) = \begin{cases} \boldsymbol{C} & q = 0,\ n = \mu_3 - \mu_1,\ e \equiv \mu_2 \pmod 2, \\ 0 & \textit{otherwise.} \end{cases}$$

Table 1. μ_w for $w \in W^{P_0}$.

$w \in W^{P_0}$	μ_w $? \cdot \omega_c$	 $? \cdot \omega_{01}$	 $? \cdot \omega_{02}$
1	(c_1, c_2, c_3)		
	$c_1 - c_3,$	$c_1 + c_2 + c_3,$	c_2
s_1	$(-c_1 - 2, c_1 + c_2 + 1, c_3)$		
	$-c_1 - c_3 - 2,$	$c_2 + c_3 - 1,$	$c_1 + c_2 + 1$
s_2	$(c_1 + c_2 + 1, -c_2 - 2, c_2 + c_3 + 1)$		
	$c_1 - c_3,$	$c_1 + c_2 + c_3,$	$-c_2 - 2$
s_3	$(c_1, c_2 + c_3 + 1, -c_3 - 2)$		
	$c_1 + c_3 + 2,$	$c_1 + c_2 - 1,$	$c_2 + c_3 + 1$
$s_1 s_3$	$(-c_1 - 2, c_1 + c_2 + c_3 + 2, -c_3 - 2)$		
	$-c_1 + c_3,$	$c_2 - 2,$	$c_1 + c_2 + c_3 + 2$
$s_1 s_2$	$(-c_1 - c_2 - 3, c_1, c_2 + c_3 + 1)$		
	$-c_1 - 2c_2 - c_3 - 4,$	$c_3 - 2,$	c_1
$s_2 s_1$	$(c_2, -c_1 - c_2 - 3, c_1 + c_2 + c_3 + 2)$		
	$-c_1 - c_3 - 2,$	$c_2 + c_3 - 1,$	$-c_1 - c_2 - 3$
$s_3 s_2$	$(c_1 + c_2 + 1, c_3, -c_2 - c_3 - 3)$		
	$c_1 + 2c_2 + c_3 + 4,$	$c_1 - 2,$	c_3
$s_2 s_3$	$(c_1 + c_2 + c_3 + 2, -c_2 - c_3 - 3, c_2)$		
	$c_1 + c_3 + 2,$	$c_1 + c_2 - 1,$	$-c_2 - c_3 - 3$
$s_1 s_2 s_1$	$(-c_2 - 2, -c_1 - 2, c_1 + c_2 + c_3 + 2)$		
	$-c_1 - 2c_2 - c_3 - 4,$	$c_3 - 2,$	$-c_1 - 2$
$s_2 s_3 s_2$	$(c_1 + c_2 + c_3 + 2, -c_3 - 2, -c_2 - 2)$		
	$c_1 + 2c_2 + c_3 + 4,$	$c_1 - 2,$	$-c_3 - 2$
$s_1 s_3 s_2$	$(-c_1 - c_2 - 3, c_1 + c_2 + c_3 + 2, -c_2 - c_3 - 3)$		
	$-c_1 + c_3,$	$-c_2 - 4,$	$c_1 + c_2 + c_3 + 2$
$s_1 s_2 s_3$	$(-c_1 - c_2 - c_3 - 4, c_1, c_2)$		
	$-c_1 - 2c_2 - c_3 - 4,$	$-c_3 - 4,$	c_1
$s_3 s_2 s_1$	$(c_2, c_3, -c_1 - c_2 - c_3 - 4)$		
	$c_1 + 2c_2 + c_3 + 4,$	$-c_1 - 4,$	c_3
$s_2 s_3 s_1$	$(c_2 + c_3 + 1, -c_1 - c_2 - c_3 - 4, c_1 + c_2 + 1)$		
	$-c_1 + c_3,$	$c_2 - 2,$	$-c_1 - c_2 - c_3 - 4$
$s_1 s_2 s_3 s_2$	$(-c_1 - c_2 - c_3 - 4, c_1 + c_2 + 1, -c_2 - 2)$		
	$-c_1 - c_3 - 2,$	$-c_2 - c_3 - 5,$	$c_1 + c_2 + 1$
$s_2 s_3 s_2 s_1$	$(c_2 + c_3 + 1, -c_3 - 2, -c_1 - c_2 - 3)$		
	$c_1 + 2c_2 + c_3 + 4,$	$-c_1 - 4,$	$-c_3 - 2$
$s_1 s_3 s_2 s_1$	$(-c_2 - 2, c_2 + c_3 + 1, -c_1 - c_2 - c_3 - 4)$		
	$c_1 + c_3 + 2,$	$-c_1 - c_2 - 5,$	$c_2 + c_3 + 1$
$s_2 s_1 s_3 s_2$	$(c_3, -c_1 - c_2 - c_3 - 4, c_1)$		
	$-c_1 + c_3,$	$-c_2 - 4,$	$-c_1 - c_2 - c_3 - 4$
$s_1 s_2 s_3 s_1$	$(-c_2 - c_3 - 3, -c_1 - 2, c_1 + c_2 + 1)$		
	$-c_1 - 2c_2 - c_3 - 4,$	$-c_3 - 4,$	$-c_1 - 2$
$s_1 s_2 s_3 s_2 s_1$	$(-c_2 - c_3 - 3, c_2, -c_1 - c_2 - 3)$		
	$c_1 - c_3,$	$-c_1 - c_2 - c_3 - 6,$	c_2
$s_2 s_1 s_3 s_2 s_1$	$(c_3, -c_2 - c_3 - 3, -c_1 - 2)$		
	$c_1 + c_3 + 2,$	$-c_1 - c_2 - 5,$	$-c_2 - c_3 - 3$
$s_1 s_2 s_1 s_3 s_2$	$(-c_3 - 2, -c_1 - c_2 - 3, c_1)$		
	$-c_1 - c_3 - 2,$	$-c_2 - c_3 - 5,$	$-c_1 - c_2 - 3$
$s_1 s_2 s_3 s_2 s_1 s_2$	$(-c_3 - 2, -c_2 - 2, -c_1 - 2)$		
	$c_1 - c_3,$	$-c_1 - c_2 - c_3 - 6,$	$-c_2 - 2$

Table 2. μ_w for $w \in W^{P_1}$.

$w \in W^{P_1}$	μ_w		
	$? \cdot \omega_{02}$	$? \cdot \omega_c$	$? \cdot \rho_1$
1	(c_1, c_2, c_3)		
	$c_2,$	$c_1 - c_3,$	$(c_1 + c_2 + c_3)/3$
s_1	$(-c_1 - 2, c_1 + c_2 + 1, c_3)$		
	$c_1 + c_2 + 1,$	$-c_1 - c_3 - 2,$	$(c_2 + c_3 - 1)/3$
s_3	$(c_1, c_2 + c_3 + 1, -c_3 - 2)$		
	$c_2 + c_3 + 1,$	$c_1 + c_3 + 2,$	$(c_1 + c_2 - 1)/3$
$s_1 s_3$	$(-c_1 - 2, c_1 + c_2 + c_3 + 2, -c_3 - 2)$		
	$c_1 + c_2 + c_3 + 2,$	$-c_1 + c_3,$	$(c_2 - 2)/3$
$s_1 s_2$	$(-c_1 - c_2 - 3, c_1, c_2 + c_3 + 1)$		
	$c_1,$	$-c_1 - 2c_2 - c_3 - 4,$	$(c_3 - 2)/3$
$s_3 s_2$	$(c_1 + c_2 + 1, c_3, -c_2 - c_3 - 3)$		
	$c_3,$	$c_1 + 2c_2 + c_3 + 4,$	$(c_1 - 2)/3$
$s_1 s_3 s_2$	$(-c_1 - c_2 - 3, c_1 + c_2 + c_3 + 2, -c_2 - c_3 - 3)$		
	$c_1 + c_2 + c_3 + 2,$	$-c_1 + c_3,$	$(-c_2 - 4)/3$
$s_1 s_2 s_3$	$(-c_1 - c_2 - c_3 - 4, c_1, c_2)$		
	$c_1,$	$-c_1 - 2c_2 - c_3 - 4,$	$(-c_3 - 4)/3$
$s_3 s_2 s_1$	$(c_2, c_3, -c_1 - c_2 - c_3 - 4)$		
	$c_3,$	$c_1 + 2c_2 + c_3 + 4,$	$(-c_1 - 4)/3$
$s_1 s_2 s_3 s_2$	$(-c_1 - c_2 - c_3 - 4, c_1 + c_2 + 1, -c_2 - 2)$		
	$c_1 + c_2 + 1,$	$-c_1 - c_3 - 2,$	$(-c_2 - c_3 - 5)/3$
$s_1 s_3 s_2 s_1$	$(-c_2 - 2, c_2 + c_3 + 1, -c_1 - c_2 - c_3 - 4)$		
	$c_2 + c_3 + 1,$	$c_1 + c_3 + 2,$	$(-c_1 - c_2 - 5)/3$
$s_1 s_2 s_3 s_2 s_1$	$(-c_2 - c_3 - 3, c_2, -c_1 - c_2 - 3)$		
	$c_2,$	$c_1 - c_3,$	$(-c_1 - c_2 - c_3 - 6)/3$

Table 3. μ_w for $w \in W^{P_2}$.

$w \in W^{P_2}$	μ_w		
	$? \cdot \omega_{21}$	$? \cdot \omega_{22}$	$? \cdot \rho_2$
1	(c_1, c_2, c_3)		
	$c_1,$	$c_3,$	$(c_1 + 2c_2 + c_3)/4$
s_2	$(c_1 + c_2 + 1, -c_2 - 2, c_2 + c_3 + 1)$		
	$c_1 + c_2 + 1,$	$c_2 + c_3 + 1,$	$(c_1 + c_3 - 2)/4$
$s_2 s_1$	$(c_2, -c_1 - c_2 - 3, c_1 + c_2 + c_3 + 2)$		
	$c_2,$	$c_1 + c_2 + c_3 + 2,$	$(-c_1 + c_3 - 4)/4$
$s_2 s_3$	$(c_1 + c_2 + c_3 + 2, -c_2 - c_3 - 3, c_2)$		
	$c_1 + c_2 + c_3 + 2,$	$c_2,$	$(c_1 - c_3 - 4)/4$
$s_2 s_3 s_1$	$(c_2 + c_3 + 1, -c_1 - c_2 - c_3 - 4, c_1 + c_2 + 1)$		
	$c_2 + c_3 + 1,$	$c_1 + c_2 + 1,$	$(-c_1 - c_3 - 6)/4$
$s_2 s_1 s_3 s_2$	$(c_3, -c_1 - c_2 - c_3 - 4, c_1)$		
	$c_3,$	$c_1,$	$(-c_1 - 2c_2 - c_3 - 8)/4$

4.3. The maximal parabolic P_1 $(j=1)$.

The group M_1 is identified with $M_0 \times SU(1,1)$ via the map

$$M_1 \ni \begin{pmatrix} z & & & \\ & \varepsilon\cdot z^{-1}\cdot u_{11} & & \varepsilon\cdot z^{-1}\cdot u_{12} \\ & & z & \\ & \varepsilon\cdot z^{-1}\cdot u_{21} & & \varepsilon\cdot z^{-1}\cdot u_{22} \end{pmatrix} \mapsto (\varepsilon, z, (u_{ij})) \in M_0 \times SU(1,1).$$

The finite dimensional representation F_μ restricted to M_1 is given by

$$F_{(\mu_1,\mu_2,\mu_3)} \simeq \chi_{\mu_2,\mu_1-\mu_3} \otimes \mathrm{sym}^{\mu_2}.$$

In order to detect the representations of M_1 with non-zero Lie algebra cohomology we proceed as follows: we take $\alpha^* = \left(\begin{smallmatrix} & i \\ i & \end{smallmatrix}\right) \in \mathrm{Lie}(SU(1,1))$. Then $\boldsymbol{R}\cdot\alpha^*$ is the compact Cartan subalgebra. Take the dual element α of α^*. Then the absolute root system becomes $\{\pm 2\alpha\}$. We fix the positive system so that α becomes the fundamental weight. Then the discrete series representation D_k of $SU(1,1)$ is described by its Blattner parameter $k\alpha$ $(|k| \geq 2, k \in \boldsymbol{Z})$, whose infinitesimal character is given by $(|k|-1)\alpha$. By conjugation of $SU(1,1)_{\boldsymbol{C}}$, it is identified with $(|k|-1)\omega_{02}$. Because the non trivial cohomological unitary representations of $SU(1,1)$ are nothing but the discrete series representations, those of M_1 with non-zero cohomology are then

$$\{\chi_{e,n}\otimes D_k, \chi_{e,n}\otimes \mathbf{1} \mid |k|\geq 2,\, k\in\boldsymbol{Z},\, e=0,1,\, n\in\boldsymbol{Z}\}.$$

So we have the lemma.

LEMMA 4.4. *Let $\mu=(\mu_1,\mu_2,\mu_3)\in\mathfrak{h}^*_{\boldsymbol{C}}$ be the highest weight of the irreducible $L_{1,\boldsymbol{C}}$-module F_μ. Then we have*

$$H^q(\mathfrak{m}_1, K_{L_1}, \chi_{e,n}\otimes D_k\otimes F_\mu) = \begin{cases} \boldsymbol{C} & q=1,\, n=\mu_3-\mu_1,\, |k|=\mu_2+2, \\ & \text{and } e\equiv\mu_2 \pmod 2 \\ 0 & \text{otherwise.}\end{cases}$$

$$H^q(\mathfrak{m}_1, K_{L_1}, \chi_{e,n}\otimes \mathbf{1}\otimes F_\mu) = \begin{cases} \boldsymbol{C} & q=0,2,\, e=\mu_2=0,\, n=\mu_3-\mu_1, \\ 0 & \text{otherwise.}\end{cases}$$

4.5. The maximal parabolic P_2 $(j=2)$.

For the Levi subgroup L_2 we consider the identification

$$L_2 = \mathrm{diag}(A, {}^t\overline{A}^{-1}) \mapsto (A, {}^t\overline{A}^{-1}) \in S(GL_2(\boldsymbol{C})\times GL_2(\boldsymbol{C}))(\simeq L_{2,\boldsymbol{C}}).$$

Then M_2 can be seen as the subgroup of elements $(A, {}^t\overline{A}^{-1})$ with $\det A = \pm 1$. Take the two fold covering

$$p\colon SL_2(\boldsymbol{C})\times SL_2(\boldsymbol{C})\times \boldsymbol{C}^\times \to S(GL_2(\boldsymbol{C})\times GL_2(\boldsymbol{C}))$$

by $p(g_1,g_2,z)=(z\cdot g_1, z\cdot g_2)$. Since a representation of the left-hand side factors through p only when the kernel acts trivially, the irreducible finite dimensional representations of $S(GL_2(\boldsymbol{C})\times GL_2(\boldsymbol{C}))$ are given by symmetric tensor representations

$$\tau_{[u_1,u_2;u_3]} := \mathrm{sym}^{u_1}(g_1)\otimes \mathrm{sym}^{u_2}(g_2)\otimes z^{u_3}$$

with condition $u_1,u_2\in\boldsymbol{Z}_{\geq 0}, u_3\in\boldsymbol{Z}, u_1+u_2+u_3\in 2\boldsymbol{Z}$. The representations of M_2 are given by their restriction. For given the highest weight $\mu=(\mu_1,\mu_2,\mu_3)$, we have

$$F_{(\mu_1,\mu_2,\mu_3)} = F_{\mu_1\omega_{21}+\mu_2\omega_{22}+(\mu_1+2\mu_2+\mu_3)\rho_{P_2}/4} \simeq \tau_{[\mu_1,\mu_3;\mu_1+2\mu_2+\mu_3]}$$

as $L_{2,\boldsymbol{C}}$-module.

To describe cohomological representations, we prepare the notation for principal series representations of $SL_2(\boldsymbol{C})^2$ (cf. [**G-S**]). Let Q be the minimal parabolic subgroup of L_2 with its split torus A_Q with Lie algebra $\mathfrak{a}_Q=\boldsymbol{R}\,\mathrm{diag}(1,-1,-1,1)$. The root system of Q is given by $\{\pm(\omega_{01}+\omega_{02})\}$. Along the Levi decomposition $Q=L_QN_Q$, the principal series $I(e,n,\nu)$ is defined by the underlying $(\mathfrak{g}_{\boldsymbol{C}},K)$-module of the set of L^2-functions f on M_2 satisfying

$$f(mang) = (\chi_{e,n}(m)\otimes a^{\nu+\rho_Q}\otimes 1)f(g), \quad man\in M_QA_Q^{\circ}N_Q$$

with $e=0,1$, $n\in\boldsymbol{Z}$, $\nu\in\boldsymbol{C}$. Note that $M_Q=M_0$. The only equivalence is the case $I(e_1,n_1,\nu_1)\simeq I(e_1,-n_1,-\nu_1)$.

The computation of the cohomology of the principal series is outlined as follows ([**B-W**, III.3.3]). Because L_2 is of $A_1\times A_1$-type, the Weyl group of L_2 is isomorphic to $\boldsymbol{Z}/2\boldsymbol{Z}^{\oplus 2}$ and contains the elements of length $0,1,1$ and 2. Let K_Q be the maximal compact subgroup of Q. Let $V_{[u_1,u_2;u_3]}$ be the $L_{Q,\boldsymbol{C}}$-module of highest weight $[u_1,u_2;u_3]$. In fact, as one-dimensional M_Q-module we have

$$V_{[u_1,u_2;u_3]} \simeq \chi_{(u_1+u_2-u_3)/2,u_1-u_2}.$$

If the $(\mathfrak{m}_2,K_{L_2})$-cohomology for principal series is non-zero, then it holds

$$H^q(\mathfrak{m}_2,K_{L_2},I(e,n,\nu)\otimes\tau_{[u_1,u_2;u_3]}) = (\chi_{e,n}\otimes V_{u'})^{K_Q}$$

and one of the following set of conditions is satisfied:

(1) $q=0,1$, $\nu=-u_1-u_2-2$, $n=-u_1+u_2$, $u'=[u_1,u_2;u_3]$,
(2) $q=1,2$, $\nu=u_1-u_2$, $n=u_1+u_2+2$, $u'=[-u_1-2,u_2;u_3]$,
(3) $q=1,2$, $\nu=u_2-u_1$, $n=-u_1-u_2-2$, $u'=[u_1,-u_2-2;u_3]$,
(4) $q=2,3$, $\nu=u_1+u_2+2$, $n=u_1-u_2$, $u'=[-u_1-2,-u_2-2;u_3]$.

As a conclusion we have the lemma.

LEMMA 4.6. *Given the highest weight $\mu=(\mu_1,\mu_2,\mu_3)\in\mathfrak{h}_{\boldsymbol{C}}^*$ of the irreducible $L_{2,\boldsymbol{C}}$ module F_μ we have*

(1) *If π is one-dimensional,*

$$H^q(\mathfrak{m}_2,K_{L_2},H_\pi\otimes F_\mu) = \begin{cases} \boldsymbol{C} & q=0,3,\ \pi=\mathbf{1},\ \mu_1=\mu_3=0,\ \mu_2\in 2\boldsymbol{Z}, \\ \boldsymbol{C} & q=0,3,\ \pi=\det,\ \mu_1=\mu_3=0,\ \mu_2\in 2\boldsymbol{Z}+1, \\ 0 & \textit{otherwise}. \end{cases}$$

(2) *If $\pi=I(e,n,\nu)$, $H^q(\mathfrak{m}_2,K_{L_2},H_\pi\otimes F_\mu)$ is isomorphic to*

$$\begin{cases} \boldsymbol{C} & q=0,1,\ \nu=-\mu_1-\mu_3-2, n=-\mu_1+\mu_3, e\equiv\mu_2 \pmod 2 \\ & q=1,2,\ \nu=\mu_1-\mu_3, n=\mu_1+\mu_3+2, e\equiv\mu_1+\mu_2+1 \pmod 2 \\ & q=1,2,\ \nu=\mu_3-\mu_1, n=-(\mu_1+\mu_3+2), e\equiv\mu_2+\mu_3+1 \pmod 2 \\ & q=2,3,\ \nu=\mu_1+\mu_3+2, n=\mu_1-\mu_3, e\equiv\mu_1+\mu_2+\mu_3 \pmod 2 \\ 0 & \textit{otherwise.} \end{cases}$$

Furthermore if we assume that π is unitary, it follows

LEMMA 4.7. (1) *If π is one-dimensional, the conclusion is the same as in Lemma* 4.6(1).

(2) *If $\pi = I(e,n,\nu)$, the equality $\mu_1=\mu_3$ is necessary for the non-vanishing of the cohomology. Precisely,*

$$H^{0|3}(\mathfrak{m}_2, K_{L_2}, H_\pi\otimes F_\mu)=0$$

$$H^{1|2}(\mathfrak{m}_2, K_{L_2}, H_\pi\otimes F_\mu)=\begin{cases} \boldsymbol{C} & \nu=0, n=\pm(2\mu_1+2), \mu_1=\mu_3, \\ & \textit{and } e\equiv\mu_1+\mu_2+1 \pmod 2 \\ 0 & \textit{otherwise.}\end{cases}$$

This assertion also follows from the use of the Cartan involution ([**B-W**, II.6.12]). The Cartan involution $\theta_2=\theta_{L_2}$ of L_2 acts on the fundamental Cartan subalgebra, and so on its dual. In this case $\theta_2(\omega_{01})=-\omega_{02}$, $\theta_2(\omega_{02})=-\omega_{01}$. Modulo by the Weyl group, it determines an automorphism on the weight space.

4.8. (D_k, w)-type for $w\in W^{P_1}$.

Let $\Lambda=(c_1,c_2,c_3)$ be the highest weight of E. Abbreviate $H^q(\mathfrak{m}_1, K_{L_1}, V)$ to $H^q(V)$ for simplicity. By Lemma 4.4, one can deduce which type of classes (π,w) contributes to the Eisenstein cohomology. In particular, if we take π as a discrete series D_k of M_1, the non-zero classes are listed as follows.

- $\ell(w)=5$ $w=s_1s_2s_3s_2s_1$, $\ell(w)=0$ $w=1$.

$$H^1(\chi_{c_2,-c_1+c_3}\otimes D_{\pm(c_2+2)}\otimes F_{\mu_w})=\boldsymbol{C}.$$

- $\ell(w)=4$ $w=s_1s_2s_3s_2$, $\ell(w)=1$ $w=s_1$.

$$H^1(\chi_{c_1+c_2+1,c_1+c_3+2}\otimes D_{\pm(c_1+c_2+3)}\otimes F_{\mu_w})=\boldsymbol{C}.$$

- $\ell(w)=4$ $w=s_1s_3s_2s_1$, $\ell(w)=1$ $w=s_3$.

$$H^1(\chi_{c_2+c_3+1,-c_1-c_3-2}\otimes D_{\pm(c_2+c_3+3)}\otimes F_{\mu_w})=\boldsymbol{C}.$$

- $\ell(w)=3$ $w=s_1s_3s_2$, $\ell(w)=2$ $w=s_1s_3$.

$$H^1(\chi_{c_1+c_2+c_3+2,c_1-c_3}\otimes D_{\pm(c_1+c_2+c_3+4)}\otimes F_{\mu_w})=\boldsymbol{C}.$$

- $\ell(w)=3$ $w=s_1s_2s_3$, $\ell(w)=2$ $w=s_1s_2$.

$$H^1(\chi_{c_1,c_1+2c_2+c_3+4}\otimes D_{\pm(c_1+2)}\otimes F_{\mu_w})=\boldsymbol{C}.$$

- $\ell(w)=3$ $w=s_3s_2s_1$, $\ell(w)=2$ $w=s_3s_2$.

$$H^1(\chi_{c_3,-c_1-2c_2-c_3-4}\otimes D_{\pm(c_3+2)}\otimes F_{\mu_w})=\boldsymbol{C}.$$

4.9. $(I(e,n,\nu),w)$-type for $w \in W^{P_2}$.

From Lemma 4.7, we find the cohomology classes which will not vanish. We take π as a principal series of M_2. Then the nonzero classes are listed below. For simplicity we write $H^q(V) := H^q(\mathfrak{m}_2, K_{L_2}, V)$.

- $\ell(w) = 4$ $w = s_2s_1s_3s_2$, $\ell(w) = 0$ $w = 1$. $c = c_1 = c_3$ is the only case
$$H^{1|2}(I(c+c_2+1, \pm 2(c+1), 0) \otimes F_{\mu_w}) = \boldsymbol{C}.$$
- $\ell(w) = 3$ $w = s_2s_3s_1$, $\ell(w) = 1$ $w = s_2$. $c = c_1 = c_3$ is the only case
$$H^{1|2}(I(c, \pm 2(c+c_2+1), 0) \otimes F_{\mu_w}) = \boldsymbol{C}.$$
- $\ell(w) = 2$ $w = s_2s_1$ $w = s_2s_3$. There is no cohomology of $(I(e,n,\nu),w)$-type which is nonzero.

We remark the case $w \in W^{P_2}$ with length 2. By Table 3, we have $\mu_w|_{\mathfrak{b}_2^*} = c_2\omega_{21} + (c_1+c_2+c_3+2)\omega_{22}$ or $(c_1+c_2+c_3+2)\omega_{21} + c_2\omega_{22}$. In both cases, $\mu = \mu_w$ does not satisfy the necessary condition in Lemma 4.7 (2). Thus holds the vanishing of the cohomology classes of type $(I(e,n,\nu),w)$.

5. Eisenstein cohomology — The generic case.

5.1.

The algebraic $\boldsymbol{Q}$-group $G = SU(2,2)$ has $\boldsymbol{Q}$-rank two. There are four conjugacy classes of parabolic $\boldsymbol{Q}$-subgroups of G. These can be represented by the standard parabolic $\boldsymbol{Q}$-subgroups as defined in Section 3. Since a maximal parabolic $\boldsymbol{Q}$-subgroup of G is conjugate to its opposite the associate class of P coincides with its conjugacy class. Thus, the associate classes $\{G\}$, $\{P_1\}$, $\{P_2\}$, $\{P_0\}$ account for the set $\mathscr{C}$. By 1.3, the cohomology space $H^\bullet(G,E)$ decomposes into the cuspidal cohomology $H^\bullet_{\mathrm{cusp}}(G,E)$ and the Eisenstein cohomology
$$H^\bullet_{\mathrm{Eis}}(G,E) = \bigoplus_{\substack{\{P\}\in\mathscr{C},\\ \{P\}\neq\{G\}}} \bigoplus_{\phi\in\Phi_{E,\{P\}}} H^\bullet(\mathfrak{m}_G, K_{\boldsymbol{R}}, \mathscr{A}_{E,\{P\},\phi} \otimes E). \tag{14 bis.}$$

If $C \subset G(\boldsymbol{A}_{\mathrm{f}})$ is an open compact subgroup, one gets the cohomology of the space
$$X_C := G(\boldsymbol{Q})A_G(\boldsymbol{R})^\circ \backslash G(\boldsymbol{A})/K_{\boldsymbol{R}}C \tag{39}$$
by taking the C-invariants $H^\bullet(G,E)^C = H^\bullet(X_C,E)$ in $H^\bullet(G,E)$ under the $G(\boldsymbol{A}_{\mathrm{f}})$-module structure. The space X_C may be viewed as the interior of a compact space $\overline{X}_C$ with corners called the Borel-Serre compactification ([**Rohlfs**] §1 in the adelic frame work). The inclusion $X_C \to \overline{X}_C$ is a homotopy equivalence so that the corresponding cohomology spaces coincide. The boundary of $\overline{X}_C$ is denoted by $\partial(\overline{X}_C)$. The pair $(\overline{X}_C, \partial(\overline{X}_C))$ gives rise to a long exact sequence in cohomology
$$\cdots \to H^\bullet_c(X_C,E) \xrightarrow{j^\bullet} H^\bullet(\overline{X}_C,E) \xrightarrow{r^\bullet} H^\bullet(\partial(\overline{X}_C),E) \to \cdots. \tag{40}$$

The interior cohomology is, by definition, the image of the cohomology with compact supports under the natural map $j^\bullet$. It is denoted by $H^\bullet_!(X_C,E)$. The interior cohomology contains the cuspidal cohomology, *i.e.*, $H^\bullet_{\mathrm{cusp}}(X_C,E) \subset H^*_!(X_C,E)$.

The absolute rank of the real Lie groups $G(\boldsymbol{R})$ and $K_{\boldsymbol{R}}$ coincide, and the dimension of the corresponding symmetric space $X_{G(\boldsymbol{R})} = G(\boldsymbol{R})/K_{\boldsymbol{R}}$ is equal

$$\dim X_{G(\boldsymbol{R})} = \dim G(\boldsymbol{R}) - \dim K_{\boldsymbol{R}} = 8.$$

Thus the virtual cohomological dimension of an arithmetic subgroup Γ of G is equal

$$\mathrm{vcd}(\Gamma) = \dim X_{G(\boldsymbol{R})} - \mathrm{rank}_{\boldsymbol{Q}}(G) = 6.$$

The constant $q_0(G(\boldsymbol{R}))$, in general defined by

$$q_0(G(\boldsymbol{R})) = 1/2(\dim X_{G(\boldsymbol{R})} - (\mathrm{rank}\, G(\boldsymbol{R}) - \mathrm{rank}\, K_{\boldsymbol{R}}))$$

turns out to be equal four. Note that $l_0(G(\boldsymbol{R})) = \mathrm{rank}\, G(\boldsymbol{R}) - \mathrm{rank}\, K_{\boldsymbol{R}} = 0$. As a consequence of the general result in [**Li-S2**, 5.6], we obtain the following.

THEOREM 5.2. *Let $G = SU(2,2)$ be the quasi-split special unitary group of $\boldsymbol{Q}$-rank two defined in Section 3, and suppose that the highest weight of the given finite dimensional representation (ν, E) of $G(\boldsymbol{C})$ is regular. Let $C \subset G(\boldsymbol{A}_{\mathrm{f}})$ be an open compact subgroup. Then one has*

$$H^j(X_C, E) = 0 \quad (j < q_0(G(\boldsymbol{R})) = 4)$$

and the restriction map

$$r^j \colon H^j(\overline{X}_C, E) \to H^j(\partial(\overline{X}_C), E)$$

is an isomorphism for $j > q_0(G(\boldsymbol{R})) + l_0(G(\boldsymbol{R})) = 4$.

In these degrees (*i.e.*, $j = 5, 6$, note that otherwise the cohomology vanishes above the virtual cohomological dimension) the cohomology $H^j(X_C, E)$ is spanned by regular Eisenstein cohomology classes.

5.3.

Given an associate class $\{P\} \in \mathscr{C}$ of a proper parabolic $\boldsymbol{Q}$-subgroup of G, we now determine the internal structure of the corresponding subspace

$$\bigoplus_{\phi \in \Phi_{E,\{P\}}} H^\bullet(\mathfrak{m}_G, K_{\boldsymbol{R}}, \mathscr{A}_{E,\{P\},\phi} \otimes E) \tag{41}$$

in the decomposition (14bis.) of the Eisenstein cohomology $H^\bullet_{\mathrm{Eis}}(G, E)$. Let $\phi = \{\phi_Q\}_{Q \in \{P\}}$ be a class of associate irreducible cuspidal automorphic representations of the Levi components of elements in $\{P\}$. The actual construction of the elements in $\mathscr{A}_{E,\{P\},\phi}$ is given by the map

$$W_{Q,\tilde{\pi}} \otimes_{\boldsymbol{C}} S(\check{\mathfrak{a}}^G_Q) \to \mathscr{A}_{E,\{P\},\phi} \tag{20 bis.}$$

where $\pi \in \phi_Q$ denotes an irreducible cuspidal automorphic representation. By assumption the highest weight Λ of the representation (ν, E) (determining the coefficient system in cohomology) is regular. This implies, by [**S94**, 4.9] that the highest weight

$$\mu_w = w(\Lambda+\rho)-\rho, \quad w \in W^Q$$

of each of the modules F_{μ_w}, $w \in W^Q$, in the decomposition of

$$H^q(\mathfrak{n}_Q, E) = \bigoplus_{\substack{w\in W^Q\\ \ell(w)=q}} F_{\mu_w} \tag{26 bis.}$$

as a $(\mathfrak{l}_Q, K_{\boldsymbol{R}} \cap L_Q(\boldsymbol{R}))$-module is regular. Thus, as proved in [**S94**, Section 2], a non-trivial cohomology class in the summand

$$\bigoplus_{w\in W^Q} \mathrm{Ind}_{Q(\boldsymbol{A}_\mathrm{f})}^{G(\boldsymbol{A}_\mathrm{f})} H^\bullet\big(\mathfrak{l}_Q, K_{\boldsymbol{R}} \cap L_Q(\boldsymbol{R}), H_\pi \otimes F_{\mu_w} \otimes S(\check{\mathfrak{a}}_Q^G)\big) \tag{29 bis.}$$

of type (π, w) indexed by $w \in W^Q$ corresponds to a cuspidal representation $\pi \in \phi_Q$ whose Archimedean component is tempered.

We consider an Eisenstein series $E(f,\lambda)$ attached to a non-trivial cohomology class of type (π, w), $\pi \in \phi_Q$, $w \in W^Q$, with $f \in W_{Q,\tilde{\pi}}$. As shown in [**S83**, 3.4, 4.3] the analytic behavior of the Eisenstein series $E(f,\lambda)$ at the point

$$\lambda_w = -w(\Lambda+\rho)|_{\mathfrak{a}_Q}$$

is decisive in order to get a non-trivial cohomology class contained in $H^\bullet(\mathfrak{m}_G, K_{\boldsymbol{R}}, \mathscr{A}_{E,\{P\},\phi} \otimes E)$. The element λ_w is real and uniquely determined by (π, w). One has

THEOREM 5.4 ([**S83**, 4.11]). *If the Eisenstein series $E(f,\lambda)$ attached to a class of type (π, w), $\pi \in \phi_Q$, $w \in W^Q$ with $f \in W_{Q,\tilde{\pi}}$ is regular at the point λ_w then the Eisenstein series evaluated at λ_w gives rise to a non-trivial cohomology class $[E(f,\lambda_w)] \in H^\bullet(\mathfrak{m}_G, K_{\boldsymbol{R}}, \mathscr{A}_{E,\{P\},\phi} \otimes E)$.*

Such a class is called a regular Eisenstein cohomology class.

As for the case of $SU(2,2)$ we compute λ_w for standard parabolic subgroups using (38). See Table 4, 5 and 6.

THEOREM 5.5. *Let $\{P_1\} \in \mathscr{C}$ be the associate class represented by the standard maximal parabolic $\boldsymbol{Q}$-subgroup P_1 of G. Suppose that the highest weight Λ of the representation (ν, E) is regular. Then the summand*

$$\bigoplus_{\phi\in\Phi_{E,\{P_1\}}} H^q(\mathfrak{m}_G, K_{\boldsymbol{R}}, \mathscr{A}_{E,\{P_1\},\phi} \otimes E) \tag{42}$$

in the Eisenstein cohomology $H^\bullet_{\mathrm{Eis}}(G,E)$

(1) *is built up in the degrees $q = 6,5,4$ by regular Eisenstein cohomology classes attached to classes (π, w), $w \in W^Q$, $Q \in \{P_1\}$, $l(w) = q-1$ and π a cuspidal automorphic representation of L_Q. The Archimedean component of π is the form $\pi_\infty = \chi_{\varepsilon,n} \otimes D_k$ where the parameters (ε, n) are uniquely determined by the highest weight μ_w depending on w. The discrete series representation D_k has parameter $k \in \boldsymbol{Z}$ uniquely determined up to sign by μ_w (cf. §4.8).*

(2) *vanishes otherwise.*

Table 4. $\lambda_w = a_{01}\omega_{01} + a_{02}\omega_{02}$ for $\Lambda = (c_1, c_2, c_3)$, $w \in W^{P_0}$.

w	a_{01}	a_{02}
1	$-c_1-c_2-c_3-3$	$-c_2-1$
s_1	$-c_2-c_3-2$	$-c_1-c_2-2$
s_2	$-c_1-c_2-c_3-3$	c_2+1
s_3	$-c_1-c_2-2$	$-c_2-c_3-2$
s_1s_3	$-c_2-1$	$-c_1-c_2-c_3-3$
s_1s_2	$-c_3-1$	$-c_1-1$
s_2s_1	$-c_2-c_3-2$	c_1+c_2+2
s_3s_2	$-c_1-1$	$-c_3-1$
s_2s_3	$-c_1-c_2-2$	c_2+c_3+2
$s_1s_2s_1$	$-c_3-1$	c_1+1
$s_2s_3s_2$	$-c_1-1$	c_3+1
$s_1s_3s_2$	c_2+1	$-c_1-c_2-c_3-3$
$s_1s_2s_3$	c_3+1	$-c_1-1$
$s_3s_2s_1$	c_1+1	$-c_3-1$
$s_2s_3s_1$	$-c_2-1$	$c_1+c_2+c_3+3$
$s_1s_2s_3s_2$	c_2+c_3+2	$-c_1-c_2-2$
$s_2s_3s_2s_1$	c_1+1	c_3+1
$s_1s_3s_2s_1$	c_1+c_2+2	$-c_2-c_3-2$
$s_2s_1s_3s_2$	c_2+1	$c_1+c_2+c_3+3$
$s_1s_2s_3s_1$	c_3+1	c_1+1
$s_1s_2s_3s_2s_1$	$c_1+c_2+c_3+3$	$-c_2-1$
$s_2s_1s_3s_2s_1$	c_1+c_2+2	c_2+c_3+2
$s_1s_2s_1s_3s_2$	c_2+c_3+2	c_1+c_2+2
$s_1s_2s_3s_2s_1s_2$	$c_1+c_2+c_3+3$	c_2+1

Table 5. $\lambda_w = a_1 \cdot \rho_1$ for $\Lambda = (c_1, c_2, c_3)$, $w \in W^{P_1}$.

w	a_1
1	$(-c_1-c_2-c_3-3)/3$
s_1	$(-c_2-c_3-2)/3$
s_3	$(-c_1-c_2-2)/3$
s_1s_3	$(-c_2-1)/3$
s_1s_2	$(-c_3-1)/3$
s_3s_2	$(-c_1-1)/3$
$s_1s_3s_2$	$(c_2+1)/3$
$s_1s_2s_3$	$(c_3+1)/3$
$s_3s_2s_1$	$(c_1+1)/3$
$s_1s_2s_3s_2$	$(c_2+c_3+2)/3$
$s_1s_3s_2s_1$	$(c_1+c_2+2)/3$
$s_1s_2s_3s_2s_1$	$(c_1+c_2+c_3+3)/3$

Table 6. $\lambda_w = a_2 \cdot \rho_2$ for $\Lambda = (c_1, c_2, c_3)$, $w \in W^{P_2}$.

w	a_2
1	$(-c_1-2c_2-c_3-4)/4$
s_2	$(-c_1-c_3-2)/4$
s_2s_1	$(c_1-c_3)/4$
s_2s_3	$(-c_1+c_3)/4$
$s_2s_3s_1$	$(c_1+c_3+2)/4$
$s_2s_1s_3s_2$	$(c_1+2c_2+c_3+4)/4$

PROOF. Let Λ be the highest weight of (ν, E); it is given in coordinates with respect to the fundamental weights ω_i $(i = 1,2,3)$ by (c_1, c_2, c_3). The assumption that Λ is regular is expressed by $c_i \geq 1$ for $i = 1,2,3$.

The set W^{P_1} of minimal coset representatives for the left cosets $W_{P_1} \backslash W_G$ contains twelve elements altogether. Of interest for us are only the elements

(1) $s_1 s_2 s_3 s_2 s_1$ of length 5
(2) $s_1 s_2 s_3 s_2$, resp. $s_1 s_3 s_2 s_1$ of length 4
(3) $s_1 s_3 s_2$, resp. $s_1 s_2 s_3$, resp. $s_3 s_2 s_1$ of length 2.

In the first resp. second case the point of evaluation λ_w for the Eisenstein series $E(f,\lambda)$ attached to the non-trivial cohomology classes of type (π, w), π an irreducible cuspidal representation in ϕ_Q $(Q \in \{P_1\})$ whose Archimedean component is tempered is given by

$$\begin{aligned}
\lambda_w &= [(c_1 + c_2 + c_3 + 3)/3] \cdot \rho_Q, && \ell(w) = 5\\
\lambda_w &= [(c_2 + c_3 + 2)/3] \cdot \rho_Q, && \ell(w) = 4\\
\lambda_w &= [(c_1 + c_2 + 2)/3] \cdot \rho_Q, && \ell(w) = 4.
\end{aligned}$$

Since Λ is regular, these points lie inside the region of absolute convergence of the Eisenstein series so that $E(f,\lambda)$ is holomorphic at λ_w. Then the class $[E(f,\lambda_w)]$ is a non-trivial cohomology class in $H^{\ell(w)+1}(\mathfrak{m}_G, K_{\boldsymbol{R}}, \mathscr{A}_{E,\{P\},\phi} \otimes E)$ of degree $\ell(w)+1$. For a given $w \in W^{P_1}$ there is only one possible choice for the Archimedean component $\pi_\infty = \chi_{\varepsilon,n} \otimes D_k$ of the representation π occurring in the type (π, w) of the class we started with. The parameters (ε, n) resp. the minimal $K_{\boldsymbol{Q}}$-type of the discrete series representation D_k can be read off from the list given in 4.8.

In the third case of an element of length 3 the points λ_w are given by

$$\lambda_w = [(c_2+1)/3]\rho_Q, \text{ resp. } [(c_3+1)/3]\rho_Q, \text{ resp. } [(c_1+1)/3]\rho_Q.$$

If λ_w lies inside the region of absolute convergence of the defining series (*i.e.* if $c_i > 2$ $i = 1,2,3$) one gets as above a non-trivial class $[E(f,\lambda_w)]$ in $H^4(\mathfrak{m}_G, K_{\boldsymbol{R}}, \mathscr{A}_{E,\{P\},\phi} \otimes E)$ of degree 4.

If the Eisenstein series $E(f,\lambda)$ has a pole at λ_w one gets by taking the residue a closed form $\operatorname{Res} E(f,\lambda)$ representing a cohomology class in $H^\bullet(G,E)$ which is square integrable. Under the regularity condition such a class is an element in the cuspidal cohomology (cf. [**S94**, Section 2]). But a residue of an Eisenstein series cannot represent a cuspidal class. Thus the class represented by $\operatorname{Res} E(f,\lambda)$ is trivial. □

THEOREM 5.6. *Let $\{P_2\} \in \mathscr{C}$ be the associate class represented by the standard maximal parabolic $\boldsymbol{Q}$-subgroup P_2 of G. Suppose that the highest weight $\Lambda = (c_1, c_2, c_3)$ of the representation (ν, E) is regular. Then the summand*

$$\bigoplus_{\phi \in \Phi_{E,\{P_2\}}} H^q(\mathfrak{m}_G, K_{\boldsymbol{R}}, \mathscr{A}_{E,\{P_2\},\phi} \otimes E)$$

in the Eisenstein cohomology $H^\bullet_{\mathrm{Eis}}(G,E)$

(1) *is built up in the degrees $q = 6,5,4$ and in the case $c_1 = c_3$ by regular Eisenstein cohomology classes attached to classes (π, w), $w \in W^Q$, $Q \in \{P_2\}$, $\ell(w) = 4$ or $\ell(w) = 3$, and π a cuspidal automorphic representation π of L_Q. The Archimedean component of π is of the form $\pi_\infty = I(e,n,\nu)$ where the parameters of the principal series representation is uniquely determined* (*up to the sign of* n) *by the highest weight μ_w* (*cf.* §4.9).

(2) *vanishes otherwise.*

PROOF. The set W^{P_2} contains six elements altogether. Of interest for us are only the elements

(1) $s_2s_1s_3s_2$ of length 4
(2) $s_2s_3s_1$ of length 3
(3) s_2s_1, resp. s_2s_3 of length 2.

Note that by 4.9, cohomology classes of type (π,w) can only occur in the cases (1) and (2) if the highest weight $\Lambda = (c_1,c_2,c_3)$ satisfies the condition $c_1 = c_3$. For Λ regular, as we suppose, there are no cohomology classes of type (π,w) with $\ell(w) = 2$. Thus, we are reduced to the first two cases. Then the point of evaluation λ_w for the Eisenstein series $E(f,\lambda)$ attached to a non-trivial class of type (π,w), π an irreducible cuspidal representation in ϕ_Q $(Q\in\{P_2\})$ whose Archimedean component is tempered is given by

$$\lambda_w = [(c_1+2c_2+c_3+4)/4]\cdot\rho_Q, \qquad \ell(w) = 4$$

$$\lambda_w = [(c_1+c_3+2)/4]\cdot\rho_Q, \qquad \ell(w) = 3.$$

Except for $c_1 = c_3 = 1$ this point lies inside the region of absolute convergence of the Eisenstein series so that $E(f,\lambda)$ is holomorphic at λ_w. Then $[E(f,\lambda)]$ give rise to non-trivial cohomology classes in $H^\bullet(\mathfrak{m}_G,K_{\boldsymbol{R}},\mathscr{A}_{E,\{P_2\},\phi}\otimes E)$. For a given $w\in W^{P_2}$ there are only two possible choices for the Archimedean component $\pi_\infty = I(\varepsilon,n,w)$ of π occurring in the type (π,w) of the class we stated with. The corresponding parameters are given in 4.9. □

5.7.
In the case of the associate class $\{P_0\}$ represented by the standard minimal parabolic $\boldsymbol{Q}$-subgroup P_0 of G we have to assume some familiarity with the results obtained in [**S94**].

Given a pair $Q_0\subset Q$ of parabolic $\boldsymbol{Q}$-subgroups in G with $Q_0\in\{P_0\}$ and a maximal parabolic $\boldsymbol{Q}$-subgroup Q there is the following relation between the corresponding sets W^{Q_0} resp. W^Q of minimal coset representatives for the left cosets $W_{Q_0}\backslash W_G$ resp. $W_Q\backslash W_G$ ([**S94**, 4.7]). Let W^{Q/Q_0} be the set of representatives of minimal length for the left cosets of W_{Q_0} in W_Q. For $w\in W^{Q_0}$ there exist uniquely determined elements w^{Q/Q_0} in W^{Q/Q_0} and w^Q in W^Q such that

$$w = w^{Q/Q_0}\cdot w^Q \quad\text{and}\quad \ell(w) = \ell(w^{Q/Q_0}) + \ell(w^Q). \tag{43}$$

Under the assumption that the highest weight Λ of (ν,E) is regular it is shown ([**S94**, 6.3,7]) that a necessary condition for a class of type (π,w) to give rise to a non-trivial Eisenstein cohomology class in $H^\bullet(G,E)$ is that the inequality

$$\ell(w^{Q/Q_0}) \geq (1/2)(\dim N_{Q_0}(\boldsymbol{R})/\dim N_Q(\boldsymbol{R})) \tag{44}$$

is satisfied. As usual $N_S(\boldsymbol{R})$ denotes the group of real points of the unipotent radical N_S of a parabolic $\boldsymbol{Q}$-subgroup S. In our case, the right-hand side of (44) takes the value $3/5$ for Q of P_1 resp. $3/4$ for Q of type P_2. Thus, the condition reads as $\ell(w^{Q/Q_0})\geq 1$. As a consequence, if a given $w\in W^{Q_0}$ is also an element in W^Q, *i.e.*, one has $w^{Q/Q_0} = 1$ in the decomposition (43), the condition is not satisfied.

This argument leads us to compare the set W^{P_0} with the sets W^{P_1} resp. W^{P_2} as given in 3.2 when we analyse the possible Eisenstein cohomology classes attached to a non-trivial cohomology class of type (π, w), $w \in W^{P_0}$. Of interest for us are only the elements of length 6, 5 and 4 in W^{Q_0}.

The unique element w_G in W^{P_0} of length 6 is not contained in W^{P_i} $i = 1, 2$. The corresponding point of evaluation is

$$\lambda_{w_G} = (c_1 + c_2 + c_3 + 3)\omega_{01} + (c_2 + 1)\omega_{02}.$$

One has $\rho_{P_0} = 3\omega_{01} + \omega_{02}$ so that λ_{w_G} lies in the region of absolute convergence of the corresponding Eisenstein series. By 5.4, $[E(f, \lambda_{w_G})]$ is a non-trivial regular Eisenstein cohomology class in $H^6(\mathfrak{m}_G, K_{\boldsymbol{R}}, \mathscr{A}_{E,\{P_0\},\phi} \otimes E)$.

There are three elements in W^{P_0} of length 5. The element $s_1s_2s_3s_2s_1 =: s$ also lies in W^{P_1} so that a class of type (π, w) with $w = s$ cannot be lifted to a class in $H^\bullet(G, E)$. The two other ones

$$w' = s_1s_2s_1s_3s_2 \quad w'' = s_2s_1s_3s_2s_1$$

are not contained in any of the sets W^{P_i}, $i = 1, 2$. The corresponding points λ_w are given by

$$\lambda_{w'} = (c_2 + c_3 + 2)\omega_{01} + (c_1 + c_2 + 2)\omega_{02}$$

$$\lambda_{w''} = (c_1 + c_2 + 2)\omega_{01} + (c_2 + c_3 + 2)\omega_{02}.$$

As above the classes of type (π, w) with $w = w'$ or $w = w''$ give rise to Eisenstein cohomology classes in $H^5(\mathfrak{m}_G, K_{\boldsymbol{R}}, \mathscr{A}_{E,\{P\},\phi} \otimes E)$ in the generic case.

There are five elements in W^{P_0} of length 4. The element $s_2s_1s_3s_2$ also lies in W^{P_2}, the elements $s_1s_3s_2s_1$ resp. $s_1s_2s_3s_2$ also lie in W^{P_1}. The remaining elements in W^{P_0} of length 4 are

$$s' = s_1s_2s_3s_1 \quad \text{and} \quad s'' = s_2s_3s_2s_1,$$

they are not contained in any of the sets W^{P_i}, $(i = 1, 2)$. The corresponding points of evaluation are given by

$$\lambda_{s'} = (c_3 + 1)\omega_{01} + (c_1 + 1)\omega_{02}$$

$$\lambda_{s''} = (c_1 + 1)\omega_{01} + (c_3 + 1)\omega_{02}.$$

For $c_1 > 2$ and $c_3 > 2$ classes of type (π, w) with $w = s'$ or $w = s''$ give rise to Eisenstein cohomology classes in $H^4(\mathfrak{m}_G, K_{\boldsymbol{R}}, \mathscr{A}_{E,\{P\},\phi} \otimes E)$. For $c_1 = 1$ or $c_3 = 1$ possible residues of Eisenstein series cannot contribute to the total cohomology $H^\bullet(G, E)$ by an argument similar to the one in 5.5 or 5.6: They cannot be lifted to the intermediate strata corresponding to the classes of maximal parabolic subgroups. We summarize this discussion in the following theorem.

THEOREM 5.8. *Let $\{P_0\} \in \mathscr{C}$ be the associate class represented by the standard minimal parabolic $\boldsymbol{Q}$-subgroup P_0 of G. Suppose that the highest weight $\Lambda = (c_1, c_2, c_3)$ of the representation (ν, E) is regular. Then the summand*

$$\bigoplus_{\phi \in \Phi_{E,\{P_0\}}} H^q(\mathfrak{m}_G, K_{\boldsymbol{R}}, A_{E,\{P_0\},\phi} \otimes E)$$

in the Eisenstein cohomology $H^*_{\mathrm{Eis}}(G, E)$

(1) *is built up in degree 6 by regular Eisenstein cohomology classes attached to classes* (π, w_G)*,* w_G *the unique longest element in* W^Q*,* $Q \in \{P_0\}$*,* $\ell(w_G) = 6$*, and* π *a cuspidal automorphic representation of* L_Q*. The Archimedean component of* π *is determined via* 4.1 *by the highest weight* μ_{w_G} *as given in Table* 1*.*

(2) *is built up in degree* $q = 5, 4$ *by regular Eisenstein cohomology classes attached to classes* (π, w) *with* w *equals*

$$w' = s_1 s_2 s_1 s_3 s_2 \text{ or } w'' = s_2 s_1 s_3 s_2 s_1 \text{ for } q = 5,$$

resp.

$$s' = s_1 s_2 s_3 s_1 \text{ or } s'' = s_2 s_3 s_2 s_1 \text{ for } q = 4.$$

In each case, the Archimedean component of π *is determined via* 4.1 *by the corresponding weight* μ_w *as given in Table* 1*.*

(3) *vanishes otherwise.*

6. Cuspidal cohomology.

For the sake of completeness we also consider the case $\{P\} = \{G\}$ with $G = SU(2,2)$. By the very definition the corresponding summand is cuspidal cohomology

$$H^\bullet_{\text{cusp}}(G, E) = \bigoplus_{\phi \in \Phi_{E,\{G\}}} H^\bullet(\mathfrak{m}_G, K_{\boldsymbol{R}}, \mathscr{A}_{E,\{G\},\phi} \otimes E).$$

It may be rewritten by using the irreducible unitary representations $\pi \in \phi$ as a finite algebraic sum

$$\bigoplus_{\substack{\pi \in \Phi_{E,\{G\}} \\ \chi_{\pi_\infty} = \chi_{E^*}}} [H^\bullet(\mathfrak{m}_G, K_{\boldsymbol{R}}, H_{\pi_\infty} \otimes E) \otimes H_{\pi_{\mathrm{f}}}]^{m(\pi)}$$

where the sum ranges over all cuspidal automorphic representations $\pi \in \Phi_{E,\{G\}}$ for which the infinitesimal character χ_{π_∞} of its Archimedean component matches the one of the representation E^* contragradient to E. Thus we are led to determine all irreducible unitary representations of the real Lie group $G(\boldsymbol{R})$ with non-vanishing $(\mathfrak{m}_G, K_{\boldsymbol{R}})$-cohomology. These are classified (up to equivalence) in [**V-Z**]. The representations in question are associated to various θ-stable parabolic subalgebras $\mathfrak{q}$ of $\mathfrak{g}_{\boldsymbol{C}}$. Consider one of these and let $\mathfrak{q} = \mathfrak{l}_{\boldsymbol{C}} \oplus \mathfrak{u}$ be a θ-stable Levi decomposition. Then $\mathfrak{l}_{\boldsymbol{C}}$ is the complexification of a real subalgebra $\mathfrak{l}$ of $\mathfrak{g}$. The normalizer of $\mathfrak{q}$ in $G(\boldsymbol{R})$ is connected, it coincides with the connected Lie subgroup of $G(\boldsymbol{R})$ with Lie algebra $\mathfrak{l}$. Starting off from one dimensional unitary representation λ of this group one obtains (if a positivity condition for the differential is satisfied) via cohomological induction an irreducible unitary representation to be denoted $A_{\mathfrak{q}}(\lambda)$. The representations so obtained exhaust (up to infinitesimal equivalence) all irreducible unitary representations of $G(\boldsymbol{R})$ with non-zero cohomology.

Following [**V-Z**], we list the θ-stable parabolic subalgebras and the cohomological representations of $SU(2,2)$ in §6.1.

6.1. $\boldsymbol{\theta}$-stable parabolic subalgebras.

In the setting of Section 3, we specify the Cartan involution θ by $\theta(g) = {}^t\overline{g}^{-1}$ and the maximal compact subgroup $K_{\boldsymbol{R}}$ by the set of θ-fixed elements. Put $Y = 1/\sqrt{2}\left(\begin{smallmatrix} 1_2 & -1_2 \\ 1_2 & 1_2 \end{smallmatrix}\right)$. Let

Table 7. $K_{\boldsymbol{C}}$-conjugacy classes of θ-stable parabolic subalgebras of $\mathfrak{g}$.

Ξ_J	$x^* = \sum x_j\varepsilon_j$	$\rho_{\mathfrak{u}_{x^*}\cap\mathfrak{p}_{\boldsymbol{C}}}$	Ξ_J	$x^* = \sum x_j\varepsilon_j$	$\rho_{\mathfrak{u}_{x^*}\cap\mathfrak{p}_{\boldsymbol{C}}}$
0	$x^* = 0$	0	III(ac)	$x_1 = x_4 > x_2 = x_3$	$[1,1;0]$
I	$x_1 > x_2 > x_4 > x_3$	$[0,0;4]$	VI	$x_4 > x_3 > x_1 > x_2$	$[0,0;-4]$
I(a)	$x_1 = x_2 > x_4 > x_3$	$[0,0;4]$	VI(a)	$x_4 = x_3 > x_1 > x_2$	$[0,0;-4]$
I(b)	$x_1 > x_2 = x_4 > x_3$	$[1/2,1/2;3]$	VI(b)	$x_4 > x_3 = x_1 > x_2$	$[1/2,1/2;-3]$
I(c)	$x_1 > x_2 > x_4 = x_3$	$[0,0;4]$	VI(c)	$x_4 > x_3 > x_1 = x_2$	$[0,0;-4]$
I(ab)	$x_1 = x_2 = x_4 > x_3$	$[0,1;2]$	VI(ab)	$x_4 = x_3 = x_1 > x_2$	$[1,0;-2]$
I(ac)	$x_1 = x_2 > x_4 = x_3$	$[0,0;4]$	VI(ac)	$x_4 = x_3 > x_1 = x_2$	$[0,0;-4]$
I(bc)	$x_1 > x_2 = x_4 = x_3$	$[1,0;2]$	VI(bc)	$x_4 > x_3 = x_1 = x_2$	$[0,1;-2]$
II	$x_1 > x_4 > x_2 > x_3$	$[1,1;2]$	V	$x_4 > x_1 > x_3 > x_2$	$[1,1;2]$
II(a)	$x_1 = x_4 > x_2 > x_3$	$[1/2,3/2;1]$	V(a)	$x_4 = x_1 > x_3 > x_2$	$[3/2,1/2;-1]$
II(c)	$x_1 > x_4 > x_2 = x_3$	$[3/2,1/2;1]$	V(c)	$x_4 > x_1 > x_3 = x_2$	$[1/2,3/2;-1]$
III	$x_1 > x_4 > x_3 > x_2$	$[2,0;0]$	IV	$x_4 > x_1 > x_2 > x_3$	$[0,2;0]$
III(b)	$x_1 > x_4 = x_3 > x_2$	$[2,0;0]$	IV(b)	$x_4 > x_1 = x_2 > x_3$	$[0,2;0]$

$\mathrm{Ad}(Y)$ be a Cayley transform defined by Y. Then $\mathfrak{h}' := \mathrm{Ad}(Y)\mathfrak{h}$ becomes a compact Cartan subalgebra in the Lie algebra $\mathfrak{k}$ of $K_{\boldsymbol{R}}$. We identify $(\mathfrak{h}')^*$ with $\mathfrak{h}^*$ by the Cayley transform and so with the root system $\Phi = \Phi(\mathfrak{g}_{\boldsymbol{C}}, \mathfrak{h}_{\boldsymbol{C}})$. Then a computation shows that the compact roots (resp. noncompact roots) are $\pm\{\beta_1, \beta_3\}$ (resp. $\pm\{\beta_1+\beta_2+\beta_3, \beta_2, \beta_2+\beta_3, \beta_1+\beta_2\}$). We define a θ-stable parabolic subalgebra in the following way. Note that the fundamental Cartan subalgebra in our case is itself compact. Take $x^* \in \sqrt{-1}(\mathfrak{h}')^*$. Put $(\mathfrak{l}_{x^*})_{\boldsymbol{C}}$ (resp. $\mathfrak{u}_{x^*}$) be the sum of root spaces $\mathfrak{g}_\beta$ of roots β such that $\langle\beta, x^*\rangle = 0$ (resp. $\langle\beta, x^*\rangle > 0$). Then $\mathfrak{q}_{x^*} = (\mathfrak{l}_{x^*})_{\boldsymbol{C}} + \mathfrak{u}_{x^*}$ becomes a θ-stable parabolic subalgebra determined by x^*.

The $K_{\boldsymbol{C}}$-conjugacy classes of the θ-stable parabolic subgroups are found in Table 7 ([**H-M**]). It says the space $\sqrt{-1}(\mathfrak{h}')^*$ is divided into 26 $K_{\boldsymbol{C}}$-conjugacy classes modulo the compact Weyl group; there are 6 open $K_{\boldsymbol{C}}$-orbits Ξ_j $(j = \mathrm{I}, \ldots, \mathrm{VI})$, while the others appear in their closures $\Xi_{j(*)}$ $(* = (\mathrm{a}), (\mathrm{b}), (\mathrm{c}), (\mathrm{ab}), (\mathrm{ac}), (\mathrm{bc}), (\mathrm{abc}))$. These orbits correspond to the $K_{\boldsymbol{C}}$-conjugacy classes of θ-stable parabolic subalgebras one by one, independent of the choice of $x^* \in \Xi_J$.

6.2. Hodge type of the cohomological representations.

Let $\mathfrak{q}$ be a θ-stable parabolic subalgebra determined by Ξ_J. Assume that the unitary character $\lambda \in \Xi_J$ of $\mathfrak{l}_{\boldsymbol{C}}$ is good and integral with respect to $\mathfrak{q}$ and $\mathfrak{g}$ ([**K-V**, 0.49]). Then there is an irreducible unitary representation determined by $(\mathfrak{q}, \lambda)$, whose underlying $(\mathfrak{g}, K)$-module is denoted by $A_{\mathfrak{q}}(\lambda)$. It belongs to the set of discrete series representations if and only if $\mathfrak{l}$ is compact. Its infinitesimal character and the minimal K-type are given by $\lambda + \rho_G$ and $\lambda + 2\rho_{\mathfrak{u}\cap\mathfrak{p}_{\boldsymbol{C}}}$ respectively.

Let Λ be the highest weight of the finite dimensional representation (ν, E) of G, taken as a dominant integral weight with respect to the given positive system by $\mathfrak{q}$. Assume that λ coincides with the highest weight of the contragradient representation of E, namely, $\lambda = -w_G\Lambda$ with the longest element w_G. In our setting of coordinates (§3.3), the dual of (c_1, c_2, c_3) is (c_3, c_2, c_1). By [**V-Z**, Theorem 5.5, Proposition 6.13], the $(\mathfrak{g}, K)$-cohomology of $A_{\mathfrak{q}}(\lambda)$ and its Hodge type are detected. We list the result in Table 8. (Note that not all representations are going to appear for a given E, and that they are not necessarily distinct, for example, I and I(a) of $\lambda = 0$ give an equivalent representation.) By Poincaré duality, $A_{\mathfrak{q}}(\lambda)$ which contributes to

Table 8. The degree $d(\leq 4)$ and the Hodge type (p,q) of $(\mathfrak{g},K)$-cohomology $H^d(\mathfrak{g},K,A_{\mathfrak{q}}(\lambda)\otimes E)$.

H^d	(p,q)	types of $A_{\mathfrak{q}}(\lambda)$
H^4	$(4,0)$	I, I(a), I(c), I(ac)
	$(3,1)$	II, I(ab), I(bc)
	$(2,2)$	III, III(b), IV, IV(b), III(ac), triv.
	$(1,3)$	V, VI(ab), VI(bc)
	$(0,4)$	VI, VI(a), VI(c), VI(ac)
H^3	$(3,0)$	I(b)
	$(2,1)$	II(a), II(c)
	$(1,2)$	V(a), V(c)
	$(0,3)$	VI(b)
H^2	$(2,0)$	I(ab), I(bc)
	$(1,1)$	III(ac), triv.
	$(0,2)$	VI(ab), VI(bc)
H^0	$(0,0)$	triv.

$H^{(p,q)}$ also contributes to $H^{(4-q,4-p)}$. Note that, if the highest weight Λ is regular, the cuspidal cohomology coincides with L^2-cohomology and the discrete series ($J=\mathrm{I},\mathrm{II},\ldots,\mathrm{VI}$) only happen as non-trivial cohomology classes.

6.3. The minimal K-type of $A_{\mathfrak{q}}(\lambda)$.

Because the minimal K-types of the representations $A_{\mathfrak{q}}(\lambda)$ are usually of interest, we summarize the coordinate expression of its highest weights. The maximal compact group K is isomorphic to $S(U(2)\times U(2))$, so the set of the isomorphism classes of irreducible representations $\hat{K}$ of K is by use of Weyl's trick the same as the set of the classes of irreducible finite dimensional representations of $S(GL_2(\boldsymbol{C})\times GL_2(\boldsymbol{C}))$ which has already appeared in §4.5. Take the basis of the Cartan subalgebra $\mathfrak{h}'_{\boldsymbol{C}}$ as

$$U_1=\frac{1}{2}\begin{pmatrix} I' & I' \\ I' & I' \end{pmatrix}, \quad U_2=\frac{1}{2}\begin{pmatrix} -I' & I' \\ I' & -I' \end{pmatrix}, \quad U_3=\begin{pmatrix} 0 & 1_2 \\ 1_2 & 0 \end{pmatrix},$$

with $I'=\mathrm{diag}(1,-1)$. Then we can relate the elements β of $\sqrt{-1}(\mathfrak{h}')^*$ with the highest weights $[u_1,u_2;u_3]$ of $\tau_{[u_1,u_2;u_3]}$ by

$$u_j=\beta(U_j) \quad (j=1,2,3).$$

For example, the compact roots β_1, β_3 and the noncompact roots are represented by $[2,0;0]$, $[0,2;0]$ and $[\pm 1,\pm 1;2]$, respectively. The highest weight of the minimal K-type of $A_{\mathfrak{q}}(\lambda)$ is given by $\lambda+2\rho_{\mathfrak{u}\cap\mathfrak{p}_{\boldsymbol{C}}}$ and the $\rho_{\mathfrak{u}\cap\mathfrak{p}_{\boldsymbol{C}}}$ can be read off from Table 7.

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Takahiro HAYATA

Department of Informatics
Yamagata University
4-3-16 Jyonan, Yonezawa, 992-8501
Japan
E-mail: hayata@yz.yamagata-u.ac.jp

Joachim SCHWERMER

Institut für Mathematik
Universität Wien
Nordbergstrasse 15, 1090 Wien
Austria
E-mail: joachim.schwermer@univie.ac.at