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A model of 3D shape memory alloy materials

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Abstract. It is a crucial step how to describe the relationship between the strain, the stress and the temperature field, when we consider the mathematical modelling for shape memory alloy materials. From the experimental results we know that the relationship can be described by the hysteresis operators. In this paper we propose a new system consisting of differential equations as a mathematical model for shape memory alloy materials occupying the three dimensional domain. The key of the modelling is the characterization for the generalized stop operators by using the ordinary differential equations including the subdifferential of the indicator function for the closed interval. Also, we give a proof of the well-posedness of the system.

1. Introduction.

In our previous works [**3**], [**4**], [**2**] we consider the one-dimensional shape memory alloy problems. The main idea of our modelling is the characterization for the generalized stop operators, which was already introduced by Visintin [**32**]. First, we approximate the relationship between the stress σ, the strain ε and the temperature field θ by the generalized stop operator defined by Figure 1, where f_l and f_u are given smooth curves with $f_l \leq f_u$ on $\boldsymbol{R}$. From engineering point of view f_l and f_u can be defined from data obtained by some experimental results.

In this case σ is determined by the operator with the input function ε if and only if σ is a solution of the following ordinary differential equation:

$$\sigma_t + \partial I(\theta, \varepsilon; \sigma) \ni c\varepsilon_t, \tag{1.1}$$

where $I(\theta, \varepsilon; \cdot)$ is the indicator function of the closed interval $[f_l(\theta, \varepsilon), f_u(\theta, \varepsilon)]$, ∂I is the subdifferential of I and c is a positive constant corresponding to the slope of the line in the hysteresis loop. In case f_l and f_u are independent of the input function, the operator, which is called a stop operator, was dealt by Krejci in [**20**]. We note that the equation (1.1) with $c = 0$ represents the generalized play operator, which appears in real-time control problems. Kenmochi, Koyama and Meyer studied the system including an approximation of the generalized play operator in [**19**]. Also, by the generalized play operator the solid-liquid phase transition phenomena can be described. The mathematical model for such a phenomena was investigated by Colli, Kenmochi and Kubo [**12**] and Minchev, Okazaki and Kenmochi [**23**].

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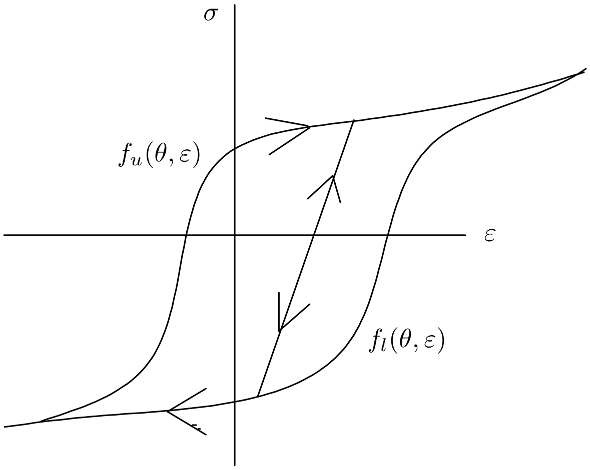

Figure 1.

From now on, by using the above characterization we propose a mathematical model of the dynamics for three dimensional shape memory alloy materials occupying a bounded domain $\Omega \subset \boldsymbol{R}^3$ with the smooth boundary $\partial\Omega$. We refer for the physical background Brokate-Sprekels [**5**] and Pawlow-Zochowski [**26**], [**27**]. Before the derivation of the model we introduce the following notations. Let $\boldsymbol{A} = (a_{ij})$ and $\boldsymbol{B} = (b_{ij})$ be tensors in $\boldsymbol{R}^3$. We write $\boldsymbol{A}_i = (a_{1i}, a_{2i}, a_{3i})$ for each i and $\boldsymbol{A} : \boldsymbol{B} = \sum_{i,j} a_{ij} b_{ij}$.

First, we use the following ordinary equation as the description for the relationship between the stress tensor $\boldsymbol{\sigma} = (\sigma_{ij})$, the strain tensor $\boldsymbol{\varepsilon} = (\varepsilon_{ij})$ and the temperature field θ:

$$\sigma_{ijt} + \partial I(\theta, \boldsymbol{\varepsilon}; \sigma_{ij}) \ni c\varepsilon_{ijt} \quad \text{on } [0, T] \text{ and for each } i, j = 1, 2, 3, \tag{1.2}$$

where $c \geq 0$ and $I(\theta, \boldsymbol{\varepsilon}; \cdot)$ is the indicator function of the closed interval $[f_*(\theta, \boldsymbol{\varepsilon}), f^*(\theta, \boldsymbol{\varepsilon})]$, and f^* and f_* are given continuous functions on $\boldsymbol{R} \times \boldsymbol{R}^9$ with $f_* \leq f^*$ on $\boldsymbol{R} \times \boldsymbol{R}^9$. Even if upper and lower curves are different with respect to each i and j, we can obtain the same results. In this paper, in order to avoid surplus notations we assume the common lower and upper curves. By some mathematical reasons we assume the viscosity for the stress, that is,

$$\hat{\boldsymbol{\sigma}} = \boldsymbol{\sigma} + \mu \nabla \boldsymbol{u}_t, \tag{1.3}$$

where $\hat{\boldsymbol{\sigma}}$ is the total stress, μ is a positive constant and $\boldsymbol{u} = (u_1, u_2, u_3)$ is the deformation vector. Moreover, we assume the linearized strain,

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \quad \text{for each } i \text{ and } j.$$

The momentum balance law leads to a basic equation of the dynamics of elastic materials:

$$u_{itt} = \mathrm{div}\hat{\boldsymbol{\sigma}}_i \quad \text{in } Q(T) := (0,T) \times \Omega \text{ for } i = 1,2,3.$$

By substituting (1.3) into the momentum balance law and adding the fourth-order term of $\boldsymbol{u}$ in order to get the regularity of solutions we obtain the following equation:

$$u_{itt} + \gamma\Delta(\Delta u_i) - \mu\Delta u_{it} = \mathrm{div}\boldsymbol{\sigma}_i \quad \text{in } Q(T) \text{ and for each } i. \tag{1.4}$$

Here, we note that the above fourth-order term is ascribed to the presence of a couple stress in the material so that systems including this term have been studied in previous works for shape memory alloys. The heat equation for elastic materials is

$$\theta_t - \kappa\Delta\theta = \hat{\boldsymbol{\sigma}} : \boldsymbol{\varepsilon}_t \quad \text{in } Q(T),$$

where κ is a positive constant. Hence, the viscosity for the stress implies

$$\theta_t - \kappa\Delta\theta = \boldsymbol{\sigma} : \boldsymbol{\varepsilon}_t + \mu\nabla\boldsymbol{u}_t : \boldsymbol{\varepsilon}_t \quad \text{in } Q(T). \tag{1.5}$$

Moreover, in order to obtain the regularity of $\boldsymbol{\sigma}$ we approximate (1.2) as follows:

$$\sigma_{ijt} - \nu\Delta\sigma_{ij} + \partial I(\theta, \boldsymbol{\varepsilon}; \sigma_{ij}) \ni c\varepsilon_{ijt} \quad \text{in } Q(T) \text{ and for each } i,j, \tag{1.6}$$

where $\nu > 0$. Also, we consider the homogeneous boundary conditions and the initial conditions:

$$u_i = 0,\ \Delta u_i = 0,\ \frac{\partial\theta}{\partial n} = 0 \text{ and } \frac{\partial\sigma_{ij}}{\partial n} = 0 \quad \text{on } \Sigma(T) := (0,T) \times \partial\Omega, \tag{1.7}$$

$$\boldsymbol{u}(0,\cdot) = \boldsymbol{u}_0, \boldsymbol{u}_t(0,\cdot) = \boldsymbol{v}_0, \theta(0,\cdot) = \theta_0 \text{ and } \boldsymbol{\sigma}(0,\cdot) = \boldsymbol{\sigma}_0 \quad \text{on } \Omega, \tag{1.8}$$

where n is the outward normal vector on $\partial\Omega$, and $\boldsymbol{u}_0$, $\boldsymbol{v}_0$, θ_0 and $\boldsymbol{\sigma}_0$ are given initial functions.

Here, we note the previous works related to shape memory alloys. Beginning from the one-dimensional theory due to Falk [**14**], [**15**], the model based on the Ginzburg-Landau free energy theory has been the subject of extensive studies, Niezgódka and Sprekels [**24**], [**25**], Sprekels and Zheng [**30**], Hoffmann and Zochowski [**17**], Hoffmann, Niezgódka and Songmu [**18**], Brokate and Sprekels [**5**], Bubner and Sprekels [**6**], [**7**], Sprekels, Zheng and Zhu [**31**], and Aiki [**1**]. Frémond also has proposed the other one-dimensional model, which was studied by Colli and Sprekels [**9**] and Shemetov [**29**].

In three dimensions there exist different approaches to thermomechanical of shape memory alloys. The well-known due to Frémond has been studied by Colli, Frémond and Visintin [**11**], Hoffmann, Niezgódka and Zheng [**18**], Colli and Sprekels [**10**] and Colli [**8**]. Three dimensional Falk's model was dealt by Falk and Konopka [**13**], Pawlow [**26**] and Pawlow and Zochowski [**27**], [**28**].

Our main purpose of this paper is to give a theorem, which guarantees the existence and uniqueness of the system (1.4)–(1.8) under the condition $\mu^2 > 4\gamma$. By the experimental results it is known that shape memory alloys do not exhibit viscosity which means that $\mu = 0$ so that this condition $\mu^2 > 4\gamma$ is likely to be not satisfied by a true shape memory alloy. However, by some mathematical difficulty we need to assume this condition.

At the end of this section we show notations, which are used throughout the present paper.

i) Let V be a Banach space with a norm $|\cdot|_V$ and $\boldsymbol{w} \in V^3$ or $\boldsymbol{w} \in V^9$. For simplicity we write

$$|\boldsymbol{w}|_V = \Big(\sum_{i=1}^{3} |w_i|_V^2\Big)^{1/2} \Big(\text{resp. } |\boldsymbol{w}|_V = \Big(\sum_{i,j=1}^{3} |w_{ij}|_V^2\Big)^{1/2}\Big),$$

as the norm of V^3 (resp. V^9), in case $\boldsymbol{w} = (w_1, w_2, w_3)$ (resp. $\boldsymbol{w} = (w_{ij})$).

ii) We put $X = H_0^1(\Omega)$, X^* is the dual space of X and $\langle \cdot, \cdot \rangle$ is a duality pairing between X and X^*.

iii) For $T > 0$ we put

$$V(T) = L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H^1(\Omega)),$$

and

$$|w|_{V(T)} := |w|_{L^\infty(0,T;L^2(\Omega))} + \Big(\int_{Q(T)} |\nabla w|^2 dxdt\Big)^{1/2} \quad \text{for } w \in V(T).$$

Immediately, $V(T)$ becomes a Banach space with norm $|\cdot|_{V(T)}$. The following inequality will play a very important role in our proof:

$$|w|_{L^{10/3}(Q(t))} \le C_0 |w|_{V(t)} \quad \text{for } w \in V(t) \text{ and } 0 \le t \le T, \tag{1.9}$$

where C_0 is a positive constant depending only on Ω and T (cf. [**21**, Chapter 2, Section 3]).

iv) Let $T > 0$, $\kappa > 0$, $f \in L^2(Q(T))$ and $\theta_0 \in L^2(\Omega)$. Now, we denote by $\mathrm{P}_1(\kappa; f, \theta_0)$ (resp. $\mathrm{P}_2(\kappa; f, \theta_0)$) the following initial boundary value problem:

$$\begin{aligned} \theta_t - \kappa \Delta\theta = f \quad & \text{in } Q(T), \\ \frac{\partial\theta}{\partial n} = 0 \text{ (resp. } \theta = 0) \quad & \text{on } \Sigma(T), \\ \theta(0) = \theta_0 \quad & \text{on } \Omega. \end{aligned} \tag{1.10}$$

On account of the classical theory [**21**, Chapter 3] we know: If $f \in L^2(Q(T))$ and $\theta_0 \in H^1(\Omega)$ (resp. $\theta_0 \in X$), then there exists a unique solution $\theta \in W^{1,2}(0,T;L^2(\Omega)) \cap L^\infty(0,T;H^1(\Omega))$.

v) Let $T > 0$, $\alpha > 0$, $\boldsymbol{f} = (f_1, f_2, f_3) \in L^2(Q(T))^3$ and $u_0 \in L^2(\Omega)$, and denote by

$\mathrm{P}_3(\alpha; \boldsymbol{f}, u_0)$ the following initial boundary value problem:

$$u_t - \alpha \Delta u = \mathrm{div}\boldsymbol{f} \quad \text{in } Q(T), \tag{1.11}$$
$$u = 0 \quad \text{on } \Sigma(T),$$
$$u(0) = u_0 \quad \text{on } \Omega.$$

Clearly, $\mathrm{P}_3(\alpha; \boldsymbol{f}, u_0)$ has a unique weak solution $u \in V(T)$ in the variational sense.
vi) Let E be a measurable subset in Ω. We denote by $|E|$ the Lebesgue measure of E.

2. A main result.

We denote by (SMAP) the initial boundary value problem, (1.4)–(1.8). Now, we begin with the precise assumptions for data.

(A1) κ, μ, γ, ν and c are positive constants.

(A2) $f_*, f^* \in C^1(\boldsymbol{R} \times \boldsymbol{R}^9) \cap W^{1,\infty}(\boldsymbol{R} \times \boldsymbol{R}^9)$ with $f_* \le f^*$ on $\boldsymbol{R} \times \boldsymbol{R}$. We denote by L the common Lipschitz constant of f_* and f^* and put

$$L_0 = \max\left\{|f_*|_{L^\infty(\boldsymbol{R}\times\boldsymbol{R}^9)}, |f^*|_{L^\infty(\boldsymbol{R}\times\boldsymbol{R}^9)}\right\}.$$

(A3) For given $\theta \in L^2(\Omega)$ and $\boldsymbol{\varepsilon} \in L^2(\Omega)^9$ we denote by $I(\theta, \boldsymbol{\varepsilon}; \cdot)$ the function on $L^2(\Omega)$ defined by

$$I(\theta, \boldsymbol{\varepsilon}; w) = \begin{cases} 0 & \text{if } w \in K(\theta, \boldsymbol{\varepsilon}), \\ +\infty & \text{otherwise,} \end{cases}$$

where $K(\theta, \boldsymbol{\varepsilon}) = \{w \in L^2(\Omega) : f_*(\theta, \boldsymbol{\varepsilon}) \le w \le f^*(\theta, \boldsymbol{\varepsilon}) \text{ a.e. on } \Omega\}$.

Clearly, $I(\theta, \boldsymbol{\varepsilon}; \cdot)$ is proper, l.s.c. and convex on $L^2(\Omega)$, the effective domain $D(I(\theta, \boldsymbol{\varepsilon}; \cdot)) = K(\theta, \boldsymbol{\varepsilon})$, and its subdifferential $\partial I(\theta, \boldsymbol{\varepsilon}; \cdot)$ is a multivalued operator in $L^2(\Omega)$ which has the following property: $\xi \in \partial I(\theta, \boldsymbol{\varepsilon}; w)$ if and only if $w \in L^2(\Omega)$ with $f_*(\theta, \boldsymbol{\varepsilon}) \le w \le f^*(\theta, \boldsymbol{\varepsilon})$ a.e. on Ω and $\xi \in L^2(\Omega)$ satisfying

$$\int_\Omega \xi(z - w)dx \le 0 \quad \text{for any } z \in K(\theta, \boldsymbol{\varepsilon}). \tag{2.1}$$

(A4) $\boldsymbol{u}_0 = (u_{01}, u_{02}, u_{03}) \in H^4(\Omega)^3 \cap X^3$, $\Delta \boldsymbol{u}_0 \in X^3$, $\boldsymbol{v}_0 \in X^3 \cap H^2(\Omega)^3$, $\theta_0 \in H^1(\Omega)$, $\boldsymbol{\sigma}_0 = (\sigma_{0ij}) \in H^1(\Omega)^9$, and

$$f_*(\theta_0, \boldsymbol{\varepsilon}_0) \le \sigma_{0ij} \le f^*(\theta_0, \boldsymbol{\varepsilon}_0) \text{ on } \Omega \text{ and } \sigma_{0ij} = \sigma_{0ji} \text{ for each } i, j.$$

Now, we give a definition of a solution to (SMAP).

Definition 2.1. We say that a triplet $\{\boldsymbol{u}, \theta, \boldsymbol{\sigma}\}$ of functions $\boldsymbol{u} : Q(T) \to \boldsymbol{R}^3$, $\theta : Q(T) \to \boldsymbol{R}$ and $\boldsymbol{\sigma} : Q(T) \to \boldsymbol{R}^9$ is a solution of (SMAP) on $[0, T]$, $T > 0$, if the following conditions hold:

(S1) $\boldsymbol{u} \in L^\infty(0,T;H^4(\Omega)^3) \cap W^{1,\infty}(0,T;H^2(\Omega)^3) \cap W^{1,2}(0,T;H^3(\Omega)^3)$.
(S2) $\theta \in W^{1,2}(0,T;L^2(\Omega)) \cap L^\infty(0,T;H^1(\Omega))$.
(S3) $\boldsymbol{\sigma} \in W^{1,2}(0,T;L^2(\Omega)^9) \cap L^\infty(0,T;H^1(\Omega)^9)$.
(S4) $\boldsymbol{u} \in L^2(0,T;X^3)$ and $\Delta\boldsymbol{u} \in L^2(0,T;X^3)$.
(S5) $u_{itt} + \gamma\Delta(\Delta u_i) - \mu\Delta u_{it} = \mathrm{div}\boldsymbol{\sigma}_i$ a.e. on $Q(T)$ for each i.
(S6) $$\int_{Q(T)} \theta_t \eta dxdt + \kappa \int_{Q(T)} \nabla\theta\cdot\nabla\eta dxdt = \int_{Q(T)} (\boldsymbol{\sigma} : \boldsymbol{\varepsilon}_t + \mu\nabla\boldsymbol{u}_t : \boldsymbol{\varepsilon}_t)\eta dxdt \quad \text{for } \eta \in L^2(0,T;H^1(\Omega)).$$
(S7) For each i and j there exists $\xi_{ij} \in L^2(0,T;L^2(\Omega))$ such that

$\xi_{ij}(t) \in \partial I(\theta(t),\boldsymbol{\varepsilon}(t);\sigma_{ij}(t))$ for a.e. $t \in [0,T]$ and

$$\int_{Q(T)} \sigma_{ijt}\eta dxdt + \nu\int_{Q(T)} \nabla\sigma_{ij}\cdot\nabla\eta dxdt + \int_{Q(T)} \xi_{ij}\eta dxdt = c\int_{Q(T)} \varepsilon_{ijt}\eta dxdt \quad \text{for } \eta \in L^2(0,T;H^1(\Omega)).$$

(S8) $\boldsymbol{u}(0) = \boldsymbol{u}_0, \boldsymbol{u}_t(0) = \boldsymbol{v}_0, \theta(0) = \theta_0$ and $\boldsymbol{\sigma}(0) = \boldsymbol{\sigma}_0$ a.e. on Ω.

The following theorem is concerned with the existence and the uniqueness of a solution to (SMAP).

Theorem 2.1. *Assume $T > 0$, (A1)–(A4) and $\mu^2 > 4\gamma$. Then we have*:
(i) (*Uniqueness*) *If $\Delta\boldsymbol{u}_0 \in W^{2-2/p,p}(\Omega)^3$ and $\boldsymbol{v}_0 \in W^{2-2/p,p}(\Omega)^3$ where $p = 30$, then (SMAP) has at most one solution on $[0,T]$.*
(ii) (*Existence*) *(SMAP) has at least one solution on $[0,T]$.*

We shall prove Theorem 2.1 in the following way. In Section 3 we investigate the properties concerned with estimates for initial boundary value problems P_1, P_2 and P_3. By decomposing 4th order equation to two parabolic equations we can apply the properties to (1.4) and obtain some a priori estimates. The estimates will play a very important role in the proof. The aim of Section 4 is to give a proof of the uniqueness in a similar way to those of [**3**], [**4**], [**2**]. In order to prove the existence we consider the following approximate problem (SMAP)(M, λ) for $M > 0$ and $\lambda > 0$:

$$\boldsymbol{u}_{tt} + \gamma\Delta(\Delta\boldsymbol{u}) - \mu\Delta\boldsymbol{u}_t = \mathrm{div}\boldsymbol{\sigma} \quad \text{in } Q(T), \tag{2.2}$$

$$\theta_t - \kappa\Delta\theta = \boldsymbol{\sigma} : \boldsymbol{\varepsilon}_t + \mu\nabla\boldsymbol{u}_t : \boldsymbol{\varepsilon}_t \quad \text{in } Q(T), \tag{2.3}$$

$$\sigma_{ijt} - \nu\Delta\sigma_{ij} + M\partial I_\lambda(\theta,\boldsymbol{\varepsilon};\sigma_{ij}) = c\varepsilon_{ijt} \quad \text{in } Q(T) \text{ and for each } i,j, \tag{2.4}$$

$$u_i = 0, \Delta u_i = 0, \frac{\partial\theta}{\partial n} = 0 \text{ and } \frac{\partial\sigma_{ij}}{\partial n} = 0 \quad \text{on } \Sigma(T), \tag{2.5}$$

$$\boldsymbol{u}(0,\cdot) = \boldsymbol{u}_0, \boldsymbol{u}_t(0,\cdot) = \boldsymbol{v}_0, \theta(0,\cdot) = \theta_0 \text{ and } \boldsymbol{\sigma}(0,\cdot) = \boldsymbol{\sigma}_0 \quad \text{on } \Omega, \tag{2.6}$$

where I_λ is the Yosida-approximation of I, which will be discussed in Section 4, precisely. Let $\{\boldsymbol{u}_\lambda, \theta_\lambda, \boldsymbol{\sigma}_\lambda\}$ be a solution of (SMAP)(M, λ) for $M > 0$ and $\lambda > 0$. In Section 4 we give uniform estimates for approximate solutions with respect to $\lambda \in (0,1]$ for sufficiently large M. The uniform estimates imply the existence of a solution (SMAP)(M), which is the system (2.2)–(2.6) with (2.7) instead of (2.4):

$$\sigma_{ijt} - \nu \Delta \sigma_{ij} + M\partial I(\theta, \boldsymbol{\varepsilon}; \sigma_{ij}) \ni c\varepsilon_{ijt} \quad \text{in } Q(T) \text{ and for each } i, j. \tag{2.7}$$

Since $M\partial I = \partial I$, we can obtain a solution of (SMAP).

3. Auxiliary lemmas.

The aim of this section is to give useful inequalities on the estimate for solutions of P_1, P_2 and (1.4). The following three lemmas are the classical results concerned with parabolic equation.

LEMMA 3.1 ([**21**, Chapter 4, Corollary of Theorem 9.1]). *Let $\kappa > 0$, $f \in L^q(Q(T))$, $q \geq 2$ and θ be a solution of $\mathrm{P}_1(\kappa; f, 0)$ on $[0,T]$. If $q > \frac{5}{2}$, then there exists a positive constant C_{1q} such that*

$$|\theta|_{L^\infty(Q(t))} \leq C_{1q} |f|_{L^q(Q(t))} \quad \text{for } 0 \leq t \leq T.$$

LEMMA 3.2 ([**16**, Lemma 2.1]). *Let $T > 0$, $\alpha > 0$, $q \geq 2$, $\boldsymbol{f} \in L^q(Q(T))^3$ and $u_0 \in W^{2-2/q,q}(\Omega)$, and denote by u a weak solution of $\mathrm{P}_3(\alpha; \boldsymbol{f}, u_0)$ on $[0,T]$. Then there exists a positive constant C_{2q} such that*

$$|\nabla u|_{L^q(Q(t))} \leq C_{2q}(|\boldsymbol{f}|_{L^q(Q(t))} + |u_0|_{W^{2-2/q,q}(\Omega)}) \quad \text{for } 0 \leq t \leq T.$$

Moreover, let $r > 1$ and $p > 1$ with

$$\frac{1}{r} = \frac{1}{p} + \frac{1}{q} - 1. \tag{3.1}$$

If $p < \frac{5}{4}$ and $u_0 = 0$, then there exists a positive constant $C_{r,q}$ such that

$$|u|_{L^r(Q(T))} \leq C_{r,q} |\boldsymbol{f}|_{L^q(Q(T))} \quad \text{for } 0 \leq t \leq T.$$

PROOF. First, let u_1 and u_2 be solutions of $\mathrm{P}_3(\alpha; \boldsymbol{f}, 0)$ and $\mathrm{P}_3(\alpha; \boldsymbol{0}, u_0)$ on $[0,T]$. The uniqueness of $\mathrm{P}_3(\alpha; \boldsymbol{f}, u_0)$ leads to $u = u_1 + u_2$. Here, by applying the classical theory [**16**, Lemma 2.1] and [**21**, Chapter 4, Theorem 9.1] it holds that

$$|\nabla u_1|_{L^q(Q(T))} \leq C_2 |\boldsymbol{f}|_{L^q(Q(T))} \text{ and } |\nabla u_0|_{L^q(Q(T))} \leq C_2 |u_0|_{W^{2-2/q,q}(\Omega)},$$

where C_2 is a positive constant.

In order to prove the second assertion we consider first the case $\Omega = \boldsymbol{R}^3$ and $\boldsymbol{f} \in C_0^\infty((0,T) \times \boldsymbol{R}^3)$. Let u be a solution of

$$u_t - \alpha \Delta u = \mathrm{div}\boldsymbol{f} \quad \text{in } (0,T) \times \boldsymbol{R}^3,$$

$$u = 0 \text{ at } |x| = \infty \text{ and } u(0,x) = 0 \text{ for } x \in \boldsymbol{R}^3.$$

Since $\mathrm{div}\boldsymbol{f} \in C_0^\infty(Q(T))$, u can be represented by the Green function of the heat equation as follows:

$$u(t,x) = \int_0^t \int_{\boldsymbol{R}^3} \Gamma(x-y, t-\tau)\mathrm{div}\boldsymbol{f}(\tau,y)dyd\tau \quad \text{for } (t,x) \in (0,T) \times \boldsymbol{R}^3,$$

where $\Gamma(x,t)$ is the Green function. See [**21**, Chapter 4] for the precise definition and basic properties of $\Gamma(x,t)$. Let $1/q + 1/q' = 1$ and $sq' = p$. By using Hölder's inequality we obtain

$$\begin{aligned} |u(t,x)| &\le \int_0^t \int_{\boldsymbol{R}^3} \sum_{i=1}^3 \left| \frac{\partial \Gamma(x-y,t-\tau)}{\partial y_i} f_i(\tau,y) \right| dyd\tau \\ &\le \sum_{i=1}^3 \left(\int_0^t \int_{\boldsymbol{R}^3} \left| \frac{\partial \Gamma(x-y,t-\tau)}{\partial x_i} \right|^{(1-s)q} |f_i(\tau,y)|^q dyd\tau \right)^{1/q} \left| \frac{\partial \Gamma}{\partial y_i} \right|^s_{L^p((0,T)\times \boldsymbol{R}^3)} \end{aligned}$$

for $(t,x) \in (0,T) \times \boldsymbol{R}^3$. Here, we put $\rho = r/q$ so that $\rho > 1$. Accordingly,

$$\begin{aligned} &\left(\int_0^T \int_{\boldsymbol{R}^3} |u(t,x)|^{\rho q} dxdt \right)^{1/\rho} \\ &\le 3^q \sum_{i=1}^3 \left| \frac{\partial \Gamma}{\partial y_i} \right|^{sq}_{L^p((0,T)\times \boldsymbol{R}^3)} \left| \int_0^T \int_{\boldsymbol{R}^3} \left| \frac{\partial \Gamma(x-y,t-\tau)}{\partial x_i} \right|^{(1-s)q} |f_i(\tau,y)|^q dyd\tau \right|_{L^\rho((0,T)\times \boldsymbol{R}^3)} \\ &\le 3^q \sum_{i=1}^3 \left| \frac{\partial \Gamma}{\partial y_i} \right|^{sq}_{L^p((0,T)\times \boldsymbol{R}^3)} \int_0^T \int_{\boldsymbol{R}^3} \left\| \frac{\partial \Gamma(x-y,t-\tau)}{\partial x_i} \right|^{(1-s)q} |f_i(\tau,y)|^q \right|_{L^\rho((0,T)\times \boldsymbol{R}^3)} dyd\tau \\ &\le 3^q \sum_{i=1}^3 \left| \frac{\partial \Gamma}{\partial y_i} \right|^{sq}_{L^p((0,T)\times \boldsymbol{R}^3)} \left(\int_0^T \int_{\boldsymbol{R}^3} \left| \frac{\partial \Gamma(x,t)}{\partial x_i} \right|^{(1-s)r} dxdt \right)^{q/r} |f_i|^q_{L^q((0,T)\times \boldsymbol{R}^3)}. \end{aligned}$$

Clearly, $r(1-s) = p$. Therefore,

$$|u|_{L^r((0,T)\times \boldsymbol{R}^3)} \le 3 \sum_{i=1}^3 |f_i|_{L^q((0,T)\times \boldsymbol{R}^3)} \left| \frac{\partial \Gamma}{\partial y_i} \right|_{L^p((0,T)\times \boldsymbol{R}^3)}.$$

The assumption $1 < p < \frac{5}{4}$ implies $\left|\frac{\partial \Gamma}{\partial y_i}\right|_{L^p((0,t)\times \boldsymbol{R}^3)} \le C$ for $0 \le t \le T$ where C is a positive constant.

In order to apply the above results to general domains, we use partition of unity and proceed as in the derivation of L^q estimates for parabolic equations (see [**21**, Chapter 4]). □

Lemma 3.3 ([**21**, Chapter 4, Corollary of Theorem 9.1]). *Let $T > 0$, $\alpha > 0$, $f \in L^q(Q(T))$, $q \ge 2$ and u be a solution of $\mathrm{P}_2(\alpha; f, 0)$ on $[0,T]$. If $q > 5$, then there exists a positive constant C_{3q} such that*

$$|\nabla u|_{L^\infty(Q(t))} \le C_{3q}|f|_{L^q(Q(t))} \quad \text{for } 0 \le t \le T.$$

The next lemma will be applied when we prove Lemma 3.5.

LEMMA 3.4. *Let $T > 0$ and $\alpha > 0$, $\boldsymbol{f} \in L^4((Q(T))^3$, $\mathrm{div}\boldsymbol{f} \in L^2(Q(T))$, $\Delta u_0 \in X$, and u be a solution of $\mathrm{P}_3(\alpha; \boldsymbol{f}, u_0)$ on $[0,T]$. Then there exists a positive constant C_4 such that*

$$|\nabla u|_{L^4(Q(t))} \le C_4\big(|\boldsymbol{f}|_{L^4(Q(t))} + |u_0|_{H^2(\Omega)}\big) \quad \text{for } 0 \le t \le T.$$

PROOF. Let u_1 and u_2 be solutions of $\mathrm{P}_3(\alpha; \boldsymbol{f}, 0)$ and $\mathrm{P}_2(\alpha; 0, u_0)$, respectively. According to the uniqueness of $\mathrm{P}_3(\alpha; \boldsymbol{f}, u_0)$, we have $u = u_1 + u_2$. By Lemma 3.2 it holds that

$$|\nabla u_1|_{L^4(Q(T))} \le C_{2p}|\boldsymbol{f}|_{L^4(Q(T))}.$$

Hence, it is sufficient to show

$$|\nabla u_2|_{L^4(Q(T))} \le C'|u_0|_{H^2(\Omega)}, \tag{3.2}$$

where C' is a positive constant.

First, we note that $u_2 \in W^{1,2}(0,T; L^2(\Omega)) \cap L^\infty(0,T;X) \cap L^2(0,T;H^2(\Omega))$ and

$$|u_{2t}|_{L^2(Q(T))} + |u_2|_{L^\infty(0,T;X)} + |u_2|_{L^2(0,T;H^2(\Omega))} \le C''|u_0|_X,$$

where C'' is a positive constant. Also, on account of the maximum principle it is clear that

$$|u_2|_{L^\infty(Q(T))} \le |u_0|_{L^\infty(\Omega)} \le C_\Omega |u_0|_{H^2(\Omega)},$$

where C_Ω is a positive constant determined by Sobolev's embedding Theorem.

Secondly, putting $v = u_{2t}$, we know that v is a weak solution of $\mathrm{P}_3(\alpha; \mathbf{0}, \alpha\Delta u_0)$ on $[0,T]$. Then we obtain

$$\nabla u_{2t} \in L^2(0,T;X). \tag{3.3}$$

Next, we multiply $u_{2t} - \alpha\Delta u_2 = 0$ by $|\nabla u_2|^2 u_2$ and integrate it over Ω. Thus we see that

$$\int_\Omega u_{2t}(t)|\nabla u_2(t)|^2 u_2(t)dx - \alpha\int_\Omega \Delta u_2(t)|\nabla u_2(t)|^2 u_2(t)dx = 0 \quad \text{for a.e. } t \in [0,T]. \tag{3.4}$$

Here, the left hand side in (3.4) is well-defined because $u_2 \in L^\infty(Q(T))$ and $\nabla u_2(t) \in L^6(\Omega)^3$ for a.e. $t \in [0,T]$. Also, it is easy to see that

$$
\begin{aligned}
&\frac{d}{dt}\int_\Omega |\nabla u_2(t)|^2 u_2(t)^2 dx \\
&\quad = 2\int_\Omega (\nabla u_{2t}(t)\cdot\nabla u_2(t))u_2(t)^2 dx + 2\int_\Omega |\nabla u_2(t)|^2 u_2(t)u_{2t}(t)dx \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{for a.e. } t\in[0,T].
\end{aligned}
$$

By (3.3) the first term of the right hand side in the above equation has a meaning. We continue to calculate only the first term in the following way:

$$
\begin{aligned}
&2\int_\Omega (\nabla u_{2t}(t)\cdot\nabla u_2(t))u_2(t)^2 dx \\
&\quad = -4\int_\Omega |\nabla u_2(t)|^2 u_2(t)u_{2t}(t)dx - 2\int_\Omega \Delta u_2(t)u_{2t}(t)|u_2(t)|^2 dx \quad \text{for a.e. } t\in[0,T].
\end{aligned}
$$

From the above equations we have

$$
\begin{aligned}
&\frac{d}{dt}\int_\Omega |\nabla u_2(t)|^2 u_2(t)^2 dx \\
&\quad = -2\int_\Omega |\nabla u_2(t)|^2 u_2(t)u_{2t}(t)dx - 2\int_\Omega \Delta u_2(t)u_2(t)|u_2(t)|^2 dx \quad \text{for a.e. } t\in[0,T].
\end{aligned}
$$

On the other hand, using integrating by parts, we observe that

$$
\begin{aligned}
&\int_\Omega \Delta u_2(t)|\nabla u_2(t)|^2 u_2(t)dx \\
&\quad = -\int_\Omega |\nabla u_2(t)|^4 dx - \int_\Omega \nabla u_2(t)\cdot\nabla(|\nabla u_2(t)|^2)u_2(t)dx \quad \text{for a.e. } t\in[0,T].
\end{aligned}
$$

From the above argument we have

$$
\begin{aligned}
&\alpha\int_\Omega |\nabla u_2(t)|^4 dx \\
&\quad = \frac{d}{dt}\int_\Omega |\nabla u_2(t)|^2|u_2(t)|^2 dx + 2\int_\Omega \Delta u_2(t)u_{2t}(t)|u_2(t)|^2 dx \\
&\qquad - \alpha\int_\Omega \nabla u_2(t)\cdot\nabla(|\nabla u_2(t)|^2)u_2(t)dx \\
&\quad \le \frac{d}{dt}\int_\Omega |\nabla u_2(t)|^2|u_2(t)|^2 dx + 2\int_\Omega |\Delta u_2(t)||u_{2t}(t)||u_2(t)|^2 dx \\
&\qquad + \frac{\alpha}{2}|\nabla u_2(t)|^4_{L^4(\Omega)} + \frac{36}{2\alpha}|u_2|^2_{L^\infty(Q(T))}|u_2(t)|^2_{H^2(\Omega)} \quad \text{for a.e. } t\in[0,T].
\end{aligned}
$$

Integrating it over $[0,T]$, we obtain

$$
\begin{aligned}
&\frac{\alpha}{2}\int_0^T\int_\Omega |\nabla u_2(t)|^4 dxdt \\
&\quad \le \int_\Omega |\nabla u_2(T)|^2|u_2(T)|^2dx + 2|u_2|^2_{L^\infty(Q(T))}\int_0^T |u_2(t)|_{L^2(\Omega)}|\Delta u_2(t)|_{L^2(\Omega)}dt \\
&\qquad + \frac{36}{2\alpha}|u_2|^2_{L^\infty(Q(T))}\int_0^T |u_2(t)|^2_{H^2(\Omega)}dt.
\end{aligned}
$$

Hence, we can show that this lemma is true. □

At the end of this section we consider the fourth order equation (1.4).

LEMMA 3.5. *Let $T>0$, $\gamma>0$, $\mu>0$, $q\ge 2$, $\boldsymbol{f}=(f_1,f_2,f_3)\in L^q(Q(T))^3$, $u_0\in H^4(\Omega)$ and $v_0\in H^2(\Omega)$, and assume that $u: Q(T)\to \boldsymbol{R}$ satisfies $u\in L^\infty(0,T;H^4(\Omega))\cap W^{1,\infty}(0,T;H^2(\Omega))\cap W^{1,2}(0,T;H^3(\Omega))$ and*

$$
\begin{cases}
u_{tt}+\gamma\Delta(\Delta u)-\mu\Delta u_t = \operatorname{div}\boldsymbol{f} \quad \text{in } Q(T),\\
u=0 \text{ and } \Delta u = 0 \quad \text{on } \Sigma(T),\\
u(0,x)=u_0 \text{ and } u_t(t,0)=v_0 \quad \text{for } x\in\Omega.
\end{cases}
\tag{3.5}
$$

(i) *If $\mu^2>4\gamma$, then there exists a positive constant C_{5q} such that*

$$
|\nabla u_t|_{L^q(Q(s))} \le C_{5q}\big(|\boldsymbol{f}|_{L^q(Q(s))}+|\Delta u_0|_{W^{2-2/q,q}(\Omega)}+|v_0|_{W^{2-2/q,q}(\Omega)}\big) \quad \text{for } 0\le s\le T.
$$

(ii) *If $\mu^2>4\gamma$, and positive constants p, q and r satisfy* (3.1) *with $r>5$ and $p<\frac{5}{4}$, $u_0=0$ and $v_0=0$, then there exists a positive constant C_6 such that*

$$
|\nabla u|_{L^\infty(Q(s))} \le C_6|\boldsymbol{f}|_{L^q(Q(s))} \quad \text{for } 0\le s\le T.
$$

(iii) *If $\mu^2>4\gamma$, then there exists a positive constant C_7 such that*

$$
|\nabla u_t|_{L^4(Q(s))} \le C_7\big(|\boldsymbol{f}|_{L^4(Q(s))}+|v_0|_{H^2(\Omega)}+|\Delta u_0|_{H^2(\Omega)}\big) \quad \text{for } 0\le s\le T.
$$

PROOF. By the assumption $\mu^2>4\gamma$ there exist positive numbers α and β satisfying $\alpha+\beta=\mu$ and $\alpha\beta=\gamma$ with $\alpha>\beta$. Here, we put $w=u_t-\alpha\Delta u$. Then we observe that

$$
w_t-\beta\Delta w=\operatorname{div}\boldsymbol{f} \text{ in } Q(T),\ w=0 \text{ on } \Sigma(T),\ w(0,\cdot)=v_0-\alpha\Delta u_0 \text{ on } \Omega
$$

so that w is a solution of $\mathrm{P}_3(\beta;\boldsymbol{f},v_0-\alpha\Delta u_0)$. Hence, Lemma 3.2 guarantees

$$
|\nabla w|_{L^q(Q(T))} \le C_{2q}\big(|\boldsymbol{f}|_{L^q(Q(T))}+|v_0|_{W^{2-2/q,q}(\Omega)}+\alpha|\Delta u_0|_{W^{2-2/q,q}(\Omega)}\big). \tag{3.6}
$$

Obviously,

$$u_{tt} - \alpha\Delta u_t = \mathrm{div}(\beta\nabla w + \boldsymbol{f}) \text{ in } Q(T),\ u_t = 0 \text{ on } \Sigma(T) \text{ and } u_t(0) = v_0 \text{ on } \Omega. \tag{3.7}$$

It follows from Lemma 3.2 that

$$|\nabla u_t|_{L^q(Q(T))} \le C_{2q}\big(\beta|\nabla w|_{L^q(Q(T))} + |\boldsymbol{f}|_{L^q(Q(T))} + |v_0|_{W^{2-2/q,q}(\Omega)}\big).$$

By substituting (3.6) into the above inequality we have proved the first assertion of this lemma.

Now, we shall show (ii). As mentioned before w is a solution of $\mathrm{P}_3(\beta; \boldsymbol{f}, 0)$. Then Lemma 3.2 implies

$$|w|_{L^r(Q(T))} \le C_{r,q}|\boldsymbol{f}|_{L^q(Q(T))}.$$

Also, u is a solution of $\mathrm{P}_2(\alpha; w, 0)$. It follows from Lemma 3.3 that

$$|\nabla u|_{L^\infty(Q(T))} \le C_{3r}|w|_{L^r(Q(T))}.$$

By combing the above two inequalities, we know that (ii) is valid.

Finally, we prove (iii). In this proof we also use the decomposition of 4th order differential equation. Lemma 3.4 implies

$$|\nabla w|_{L^4(Q(T))} \le C_4\big(|\boldsymbol{f}|_{L^4(Q(T))} + |v_0 - \alpha\Delta u_0|_{H^2(\Omega)}\big).$$

Since u_t satisfies (3.7), by using Lemma 3.4, again, we can infer that (iii) is true. □

4. Proof of uniqueness.

The aim of this section is to prove the uniqueness of (SMAP). The main idea of the proof is due to [**19**]. The proof is rather long so that we divide it into several steps. Throughout this section we use the following notations. Let $\{\boldsymbol{u}_1, \theta_1, \boldsymbol{\sigma}_1\}$ and $\{\boldsymbol{u}_2, \theta_2, \boldsymbol{\sigma}_2\}$ be solutions of (SMAP) on $[0, T]$ and put $\boldsymbol{u} = \boldsymbol{u}_1 - \boldsymbol{u}_2$, $\boldsymbol{u} = (u_1, u_2, u_3)$, $\boldsymbol{u}_\ell = (u_{\ell 1}, u_{\ell 2}, u_{\ell 3})$, $\theta = \theta_1 - \theta_2$, $\boldsymbol{\sigma} = \boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2$, $\boldsymbol{\sigma} = (\sigma_{ij})$, $\boldsymbol{\sigma}_\ell = (\sigma_{\ell ij})$, $\boldsymbol{\varepsilon}_\ell = \frac{1}{2}(\nabla\boldsymbol{u}_\ell + {}^t\nabla\boldsymbol{u}_\ell)$, $\boldsymbol{\varepsilon}_\ell = (\varepsilon_{\ell ij})$, $\ell = 1, 2$, $\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2$, and

$$M(s) = \max\big\{|f_*(\theta_1, \varepsilon_1) - f_*(\theta_2, \boldsymbol{\varepsilon}_2)|_{L^\infty(Q(s))}, |f^*(\theta_1, \varepsilon_1) - f^*(\theta_2, \boldsymbol{\varepsilon}_2)|_{L^\infty(Q(s))}\big\} \quad \text{for } 0 \le s \le T.$$

Moreover, let $\xi_{\ell ij} \in L^2(Q(T))$ satisfying $\sigma_{\ell it} - \nu\Delta\sigma_{ij} + \xi_{\ell ij} = c\varepsilon_{\ell ijt}$, $\ell = 1, 2$ and i, j.

1st step. *It holds that*

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |[\sigma_{ij}(t) - M(s)]^+|^2 dx + \nu\int_\Omega |\nabla[\sigma_{ij}(t) - M(s)]^+|^2 dx$$
$$\le c\int_\Omega \varepsilon_{ijt}[\sigma_{ij}(t) - M(s)]^+ dx;$$

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |[-\sigma_{ij}(t)-M(s)]^+|^2dx+\nu\int_\Omega|\nabla[-\sigma_{ij}(t)-M(s)]^+|^2dx$$
$$\leq -c\int_\Omega \varepsilon_{ijt}(t)[-\sigma_{ij}(t)-M(s)]^+dx \quad \textit{for a.e. } t\in[0,s], 0\leq s\leq T \textit{ and } i,j.$$

Proof. We fix $s\in(0,T]$, i and j, and put

$$z_{1ij}=\sigma_{1ij}-[\sigma_{ij}-M(s)]^+, z_{2ij}=\sigma_{2ij}+[\sigma_{ij}-M(s)]^+.$$

Clearly, $z_{1ij}(t)\in K(\theta_1(t),\boldsymbol{\varepsilon}_1(t))$ and $z_{2ij}(t)\in K(\theta_2(t),\boldsymbol{\varepsilon}_2(t))$ for $0\leq t\leq s$. Then we can multiply $\sigma_{1it}-\nu\Delta\sigma_{ij}+\xi_{1ij}=c\varepsilon_{1ijt}$ by $\sigma_{1ij}-z_{1ij}$ and integrate it over Ω. Thus by (2.1) we obtain

$$\int_\Omega \sigma_{1ijt}(t)(\sigma_{1ij}(t)-z_{1ij}(t))dx+\nu\int_\Omega\nabla\sigma_{1ij}\cdot\nabla(\sigma_{1ij}(t)-z_{1ij}(t))dx$$
$$\leq c\int_\Omega\varepsilon_{1ijt}(t)(\sigma_{1ij}(t)-z_{1ij}(t))dx \quad \text{for a.e. } t\in(0,s].$$

Similarly, we have

$$\int_\Omega \sigma_{2ijt}(t)(\sigma_{2ij}(t)-z_{2ij}(t))dx+\nu\int_\Omega\nabla\sigma_{2ij}\cdot\nabla(\sigma_{2ij}(t)-z_{2ij}(t))dx$$
$$\leq c\int_\Omega\varepsilon_{2ijt}(t)(\sigma_{1ij}(t)-z_{2ij}(t))dx \quad \text{for a.e. } t\in(0,s].$$

By adding two inequalities it follows that

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |[\sigma_{ij}(t)-M(s)]^+|dx+\nu\int_\Omega|\nabla[\sigma_{ij}(t)-M(s)]^+|^2dx$$
$$\leq c\int_\Omega\varepsilon_{ijt}(t)[\sigma_{ij}(t)-M(s)]^+dx \quad \text{for a.e. } t\in(0,s].$$

We can obtain the second inequality in the assertion of this step in a similar way. □

2nd step. *It holds that*

$$\frac{1}{2}\frac{d}{dt}\int_\Omega|\boldsymbol{u}_t(t)|^2dx+\frac{\gamma}{2}\frac{d}{dt}\int_\Omega|\Delta\boldsymbol{u}(t)|^2dx+\mu\int_\Omega|\nabla\boldsymbol{u}_t(t)|^2dx$$
$$=-\sum_{i,j=1}^3\int_\Omega\sigma_{ij}(t)\frac{\partial u_{jt}}{\partial x_i}(t)dx \quad \textit{for a.e. } t\in[0,T]. \tag{4.1}$$

Proof. It is clear that

$$\boldsymbol{u}_{tt} + \gamma \Delta(\Delta \boldsymbol{u}) - \mu \Delta \boldsymbol{u}_t = \mathrm{div}\boldsymbol{\sigma} \quad \text{in } Q(T). \tag{4.2}$$

We multiply (4.2) by $\boldsymbol{u}_t$ and integrate it over Ω. Thus we can obtain (4.1). □

3rd step. *$\boldsymbol{\sigma}_1$ and $\boldsymbol{\sigma}_2$ are symmetric tensors, that is, $\sigma_{\ell ij} = \sigma_{\ell ji}$ for $\ell = 1, 2$ and $i, j = 1, 2, 3$.*

Proof. Immediately, for each ℓ, i and j we have

$$\left.\begin{aligned} &\sigma_{\ell jit} - \nu \Delta \sigma_{\ell ji} + \partial I(\theta_\ell, \boldsymbol{\varepsilon}_\ell; \sigma_{\ell ji}) \ni c\varepsilon_{\ell ijt} \quad \text{in } Q(T), \\ &\frac{\partial \sigma_{ji}}{\partial n} = 0 \quad \text{on } \Sigma(T) \text{ and } \sigma_{\ell ji}(0) = \sigma_{0ij}, \end{aligned}\right\} \tag{4.3}$$

because $\boldsymbol{\sigma}_0$ and $\boldsymbol{\varepsilon}_\ell$ are symmetric tensors. According to the uniqueness of the initial boundary value problem (4.3), we show that the assertion of this step holds. □

4th step. *There exists a positive constant K_1 depending only on μ, c and $|\Omega|$ such that*

$$\begin{aligned} &\sum_{i,j=1}^{3} \left(\frac{1}{2}\frac{d}{dt}\int_\Omega |[\sigma_{ij}(t) - M(s)]^+|^2 dx + \nu \int_\Omega |\nabla [\sigma_{ij}(t) - M(s)]^+|^2 dx \right. \\ &\qquad \left. + \frac{1}{2}\frac{d}{dt}\int_\Omega |[-\sigma_{ij}(t) - M(s)]^+|^2 dx + \nu \int_\Omega |\nabla [-\sigma_{ij}(t) - M(s)]^+|^2 dx \right) \\ &\quad + \frac{c}{2}\frac{d}{dt}\int_\Omega |\boldsymbol{u}_t(t)|^2 dx + \frac{c\gamma}{2}\frac{d}{dt}\int_\Omega |\Delta \boldsymbol{u}(t)|^2 dx + \frac{c\mu}{2}\int_\Omega |\nabla \boldsymbol{u}_t(t)|^2 dx \\ &\leq K_1 M(s)^2 \quad \textit{for a.e. } t \in [0, s] \textit{ and } 0 < s \leq T. \end{aligned}$$

Proof. Let $0 < s \leq T$. It follows from 1st and 2nd steps that

$$\begin{aligned} &\sum_{i,j=1}^{3} \left(\frac{1}{2}\frac{d}{dt}\int_\Omega |[\sigma_{ij}(t) - M(s)]^+|^2 dx + \nu \int_\Omega |\nabla [\sigma_{ij}(t) - M(s)]^+|^2 dx \right. \\ &\qquad \left. + \frac{1}{2}\frac{d}{dt}\int_\Omega |[-\sigma_{ij}(t) - M(s)]^+|^2 dx + \nu \int_\Omega |\nabla [-\sigma_{ij}(t) - M(s)]^+|^2 dx \right) \\ &\quad + \frac{c}{2}\frac{d}{dt}\int_\Omega |\boldsymbol{u}_t(t)|^2 dx + \frac{c\gamma}{2}\frac{d}{dt}\int_\Omega |\Delta \boldsymbol{u}(t)|^2 dx + c\mu \int_\Omega |\nabla \boldsymbol{u}_t(t)|^2 dx \\ &\leq c \sum_{i,j=1}^{3} \left(\int_\Omega \varepsilon_{ijt}(t) [\sigma_{ij}(t) - M(s)]^+ dx - \int_\Omega \varepsilon_{ijt}(t)[-\sigma_{ij}(t) - M(s)]^+ dx \right) \\ &\quad - c \sum_{i,j=1}^{3} \int_\Omega \sigma_{ij}(t) \frac{\partial u_{jt}}{\partial x_i}(t) dx \quad \text{for a.e. } t \in [0, s]. \end{aligned} \tag{4.4}$$

Here, 3rd step implies that

$$\sum_{i,j=1}^{3}\Big(\int_{\Omega}\varepsilon_{ijt}(t)[\sigma_{ij}(t)-M(s)]^{+}dx-\int_{\Omega}\varepsilon_{ijt}(t)[-\sigma_{ij}(t)-M(s)]^{+}dx\Big)$$
$$=\sum_{i,j=1}^{3}\int_{\Omega}\Big(\frac{\partial u_{jt}}{\partial x_i}(t)[\sigma_{ij}(t)-M(s)]^{+}-\frac{\partial u_{jt}}{\partial x_i}(t)[-\sigma_{ij}(t)-M(s)]^{+}\Big)dx$$
$$\text{for a.e. } t\in[0,s].$$

Then we can calculate the right hand side of (4.4) in the following way.

$$c\sum_{i,j=1}^{3}\Big(\int_{\Omega}\varepsilon_{ijt}(t)[\sigma_{ij}(t)-M(s)]^{+}dx-\int_{\Omega}\varepsilon_{ijt}(t)[-\sigma_{ij}(t)-M(s)]^{+}dx\Big)$$
$$-c\sum_{i,j=1}^{3}\int_{\Omega}\sigma_{ij}(t)\frac{\partial u_{jt}}{\partial x_i}(t)dx$$
$$\le c\sum_{i,j=1}^{3}\int_{\Omega}\Big|\frac{\partial u_{it}}{\partial x_j}(t)\Big|\big|[\sigma_{ij}(t)-M(s)]^{+}-[-\sigma_{ij}(t)-M(s)]^{+}-\sigma_{ij}(t)\big|dx$$
$$\text{for a.e. } t\in[0,s].$$

It is easy to obtain

$$|[\sigma_{ij}-M(s)]^{+}-[-\sigma_{ij}-M(s)]^{+}-\sigma_{ij}|\le M(s)\quad\text{a.e. on } Q(s) \text{ for } i,j. \tag{4.5}$$

By applying Hölder's inequality to (4.5) we can get the assertion of this step. □

5TH STEP. *If $q>\frac{5}{2}$, then the following inequality holds:*

$$|\theta|_{L^\infty(Q(s))}\le C_{1q}\big(\mu|\nabla\boldsymbol{u}_t:\boldsymbol{\varepsilon}_{1t}|_{L^q(Q(s))}+\mu|\nabla\boldsymbol{u}_{2t}:\boldsymbol{\varepsilon}_t|_{L^q(Q(s))}$$
$$+|\boldsymbol{\sigma}:\boldsymbol{\varepsilon}_{1t}|_{L^q(Q(s))}+|\boldsymbol{\sigma}_2:\boldsymbol{\varepsilon}_t|_{L^q(Q(s))}\big)\quad \textit{for } 0<s\le T. \tag{4.6}$$

PROOF. Since the left hand side of (1.10) is linear, it holds that

$$\theta_t-\kappa\Delta\theta=\boldsymbol{\sigma}_{1t}:\boldsymbol{\varepsilon}_{1t}-\boldsymbol{\sigma}_{2t}:\boldsymbol{\varepsilon}_{2t}+\mu(\nabla\boldsymbol{u}_{1t}:\boldsymbol{\varepsilon}_{1t}-\nabla\boldsymbol{u}_{2t}:\boldsymbol{\varepsilon}_{2t})\quad\text{in } Q(T).$$

Therefore, this step is a direct consequence of Lemma 3.1. □

The next step is obtained from Lemma 3.5(i) and (4.2).

6TH STEP. *For $p\ge 2$ and each i it holds that*

$$|\nabla u_{it}|_{L^p(Q(s))}\le C_{5p}|\boldsymbol{\sigma}_i|_{L^p(Q(s))}\quad \textit{for } 0\le s\le T.$$

From now on, we fix positive numbers p_0, q_0 and r_0 as follows:

$$q_0 = \frac{10}{3}, \quad r_0 = \frac{11}{2}, \quad p_0 = \frac{110}{97}.$$

Clearly, p_0, q_0 and r_0 satisfy (3.1), $p_0 < \frac{5}{4}$, $q_0 > \frac{5}{2}$ and $r_0 > 5$. Obviously, by Lemma 3.5(ii) we have:

7th step. *There exists a positive constant K_2 such that*

$$|\nabla \boldsymbol{u}|_{L^\infty(Q(s))} \le K_2 |\boldsymbol{\sigma}|_{L^{q_0}(Q(s))} \quad \textit{for } 0 \le s \le T. \tag{4.7}$$

For simplicity, we put

$$\begin{aligned} E_0(t) = &\sum_{i,j=1}^{3} \left(\frac{1}{2} \int_\Omega |[\sigma_{ij}(t) - M(s)]^+|^2 dx + \frac{1}{2} \int_\Omega |[-\sigma_{ij}(t) - M(s)]^+|^2 dx \right) \\ &+ \frac{c}{2} \int_\Omega |\boldsymbol{u}_t(t)|^2 dx + \frac{c\gamma}{2} \int_\Omega |\Delta \boldsymbol{u}(t)|^2 dx, \\ E_1(t) = &\sum_{i,j=1}^{3} \left(\nu \int_\Omega |\nabla[\sigma_{ij}(t) - M(s)]^+|^2 dx + \nu \int_\Omega |\nabla[-\sigma_{ij}(t) - M(s)]^+|^2 dx \right) \\ &+ \frac{c\mu}{2} \int_\Omega |\nabla \boldsymbol{u}_t(t)|^2 dx \quad \text{for } 0 \le t \le s \le T. \end{aligned}$$

8th step. *For $q > \frac{5}{2}$ there exists a positive constant K_3 such that*

$$\begin{aligned} \frac{d}{dt} E_0(t) + E_1(t) \le K_3 \Big\{ &|\nabla \boldsymbol{u}_t : \boldsymbol{\varepsilon}_{1t}|^2_{L^q(Q(s))} + |\nabla \boldsymbol{u}_{2t} : \boldsymbol{\varepsilon}_t|^2_{L^q(Q(s))} + |\boldsymbol{\sigma} : \boldsymbol{\varepsilon}_{1t}|^2_{L^q(Q(s))} \\ &+ |\boldsymbol{\sigma}_2 : \boldsymbol{\varepsilon}_t|^2_{L^q(Q(s))} + \sum_{i,j=1}^{3} |\sigma_{ij}|^2_{L^{q_0}(Q(s))} \Big\} \\ &\textit{for a.e. } t \in [0, s] \textit{ and } 0 < s \le T. \end{aligned} \tag{4.8}$$

Proof. From 4th step it follows

$$\frac{d}{dt} E_0(t) + E_1(t) \le K_1 M(s)^2 \quad \text{for a.e. } t \in [0, s] \text{ and } 0 < s \le T. \tag{4.9}$$

Due to (A2) we observe that

$$M(s) \le L\big(|\theta|_{L^\infty(Q(s))} + |\boldsymbol{\varepsilon}|_{L^\infty(Q(s))}\big) \le L\big(|\theta|_{L^\infty(Q(s))} + |\nabla \boldsymbol{u}|_{L^\infty(Q(s))}\big) \text{ for } 0 < s \le T.$$

By substituting (4.6) and (4.7) into (4.9) we get (4.8). □

9TH STEP. *For $\ell = 1, 2$ and $1 < p \le 30$ $\boldsymbol{\varepsilon}_{\ell t} \in L^p(Q(T))^9$.*

PROOF. For $\ell = 1, 2$ the definition of a solution shows that

$$f_*(\theta_\ell, \boldsymbol{\varepsilon}_\ell) \le \sigma_{\ell ij} \le f^*(\theta_\ell, \boldsymbol{\varepsilon}_\ell) \quad \text{on } Q(T) \text{ for } i, j.$$

Thus $\boldsymbol{\sigma}_\ell \in L^\infty(Q(T))^9$ because of the assumption (A2). Hence, in case $p = 30$ Lemma 3.5(i) and the assumption imply

$$|\nabla u_{\ell t}|_{L^p(Q(T))} \le C_{5p}\big(|\boldsymbol{\sigma}_\ell|_{L^p(Q(T))} + |\Delta \boldsymbol{u}_0|_{W^{2-2/p,p}(\Omega)} + |\boldsymbol{v}_0|_{W^{2-2/p,p}(\Omega)}\big)$$

so that this step holds. $\square$

Here, in order to apply the Hölder inequality we set

$$q = 3, \rho = \frac{10}{9} \text{ and } \frac{1}{\rho'} + \frac{1}{\rho} = 1.$$

Clearly, $\rho q = 10/3 = q_0$ and $\rho' q = 30$.

10TH STEP. *There exists a positive constant K_4 depending on $|\boldsymbol{\varepsilon}_{1t}|_{L^{\rho' q}(Q(s))}$, $|\boldsymbol{\sigma}_2|_{L^\infty(Q(T))}$ and $|\nabla \boldsymbol{u}_{2t}|_{L^{\rho' q}(Q(T))}$ such that*

$$|\boldsymbol{\sigma} : \boldsymbol{\varepsilon}_{1t}|^2_{L^q(Q(s))}$$
$$\le K_4 \sum_{i,j=1}^3 \big(|[\sigma_{ij} - M(s)]^+|^2_{L^{q_0}(Q(s))} + |[-\sigma_{ij} - M(s)]^+|^2_{L^{q_0}(Q(s))}\big) + K_4 M(s)^2 s^{2/(\rho' q)};$$

$$|\boldsymbol{\sigma}_2 : \boldsymbol{\varepsilon}_t|^2_{L^q(Q(s))}$$
$$\le K_4 \sum_{i,j=1}^3 \big(|[\sigma_{ij} - M(s)]^+|^2_{L^{q_0}(Q(s))} + |[-\sigma_{ij} - M(s)]^+|^2_{L^{q_0}(Q(s))}\big) + K_4 M(s)^2 s^{2/q};$$

$$|\nabla \boldsymbol{u}_t : \boldsymbol{\varepsilon}_{1t}|^2_{L^q(Q(s))} \le K_4 |\nabla \boldsymbol{u}_t|^2_{L^{q_0}(Q(s))};$$

$$|\nabla \boldsymbol{u}_{2t} : \boldsymbol{\varepsilon}_t|^2_{L^q(Q(s))} \le K_4 \sum_{i,j=1}^3 |\sigma_{ij}|^2_{L^{q_0}(Q(s))} \quad \textit{for } 0 \le s \le T.$$

PROOF. First let $s \in (0, T]$. (4.5) implies

$$|\boldsymbol{\sigma} : \boldsymbol{\varepsilon}_{1t}|^2_{L^q(Q(s))} \le 9^2 \sum_{i,j} \Big(|[\sigma_{ij} - M(s)]^+ \varepsilon_{1ijt}|^2_{L^q(Q(s))}$$
$$+ |[-\sigma_{ij} - M(s)]^+ \varepsilon_{1ijt}|^2_{L^q(Q(s))} + M(s)^2 |\varepsilon_{1ijt}|^2_{L^q(Q(s))}\Big).$$

By applying Hölder's inequality for i and j we have

$$|[\sigma_{ij}-M(s)]^+\varepsilon_{1ijt}|^2_{L^q(Q(s))}\le|[\sigma_{ij}-M(s)]^+|^2_{L^{\rho q}(Q(s))}|\varepsilon_{1ijt}|^2_{L^{\rho' q}(Q(s))},$$

and

$$|\varepsilon_{1ijt}|^2_{L^q(Q(s))}\le|\Omega|^{2/(\rho' q)}s^{2/(\rho' q)}|\varepsilon_{1ijt}|^2_{L^{\rho' q}(Q(s))}.$$

Thus on account of 9th step and $\rho q=\frac{10}{3}$, $\rho' q=30$ we obtain the first inequality of this step.

Easily from 6th step, we show

$$\begin{aligned}|\boldsymbol{\sigma}_2:\boldsymbol{\varepsilon}_t|^2_{L^q(Q(s))}&\le|\boldsymbol{\sigma}_2|^2_{L^\infty(Q(T))}\bigg(\sum_{i,j=1}^3|\varepsilon_{ijt}|_{L^q(Q(s))}\bigg)^2\\&\le|\boldsymbol{\sigma}_2|^2_{L^\infty(Q(T))}\bigg(\sum_{i,j=1}^3\left|\frac{\partial u_{it}}{\partial x_j}\right|_{L^q(Q(s))}\bigg)^2\\&\le|\boldsymbol{\sigma}_2|^2_{L^\infty(Q(T))}\bigg(\sum_{i=1}^3|\boldsymbol{\sigma}_i|_{L^q(Q(s))}\bigg)^2\\&\le 9^2|\boldsymbol{\sigma}_2|^2_{L^\infty(Q(T))}\bigg(\sum_{i,j=1}^3\big(|[\sigma_{ij}-M(s)]^+|^2_{L^q(Q(s))}\\&\qquad+|[-\sigma_{ij}-M(s)]^+|^2_{L^q(Q(s))}\big)+M(s)^2|\Omega|^{2/q}s^{2/q}\bigg).\end{aligned}$$

This is the second inequality of this step since $q_0>q$. The rest assertions of this step can be proved, similarly. □

11th step. *Put $\lambda=\min\left\{\frac{1}{\rho' q},\frac{2}{q_0}\right\}$. Then there exists a positive constant K_5 such that*

$$\begin{aligned}\frac{d}{dt}E_0(t)+E_1(t)\le K_5\sum_{i,j=1}^3\big(|[\sigma_{ij}-M(s)]^+|^2_{L^{q_0}(Q(s))}+|[-\sigma_{ij}-M(s)]^+|^2_{L^{q_0}(Q(s))}\big)\\+s^\lambda K_5M(s)^2\quad\text{for a.e. } t\in[0,s]\text{ and }0<s\le T.\end{aligned}\tag{4.10}$$

Proof. By elementary calculations we can prove this step together with help of 6th, 8th and 10th steps. □

Moreover, we put

$$A(s)=\sum_{i,j=1}^3\big(|[\sigma_{ij}-M(s)]^+|^2_{V(s)}+|[-\sigma_{ij}-M(s)]^+|^2_{V(s)}\big)\quad\text{for }0\le s\le T.$$

12th step. *For some positive number K_6 it holds that*

$$A(s) \le K_6\big(sA(s) + s^{1+\lambda}M(s)^2\big) \quad \textit{for } 0 \le s \le T. \tag{4.11}$$

PROOF. Let $0 \le s \le T$. By integrating (4.10) over $[0,\tau]$, $0 \le \tau \le s$, we see that

$$\begin{aligned}
&\sup_{0\le t\le s} E_0(t) + \int_0^s E_1(t)dt \\
&\le sK_5 \sum_{i,j=1}^{3} \big(|[\sigma_{ij} - M(s)]^+|^2_{L^{q_0}(Q(s))} + |[-\sigma_{ij} - M(s)]^+|^2_{L^{q_0}(Q(s))}\big) + s^{1+\lambda}K_5 M(s)^2 \\
&\le sC_0K_5 \sum_{i,j=1}^{3} \big(|[\sigma_{ij} - M(s)]^+|^2_{V(s)} + |[-\sigma_{ij} - M(s)]^+|^2_{V(s)}\big) + s^{1+\lambda}K_5 M(s)^2 \\
&\le sC_0K_5A(s) + K_5 s^{1+\lambda}M(s)^2.
\end{aligned}$$

Here, we have applied (1.9) because $q_0 = \frac{10}{3}$.

On the other hand, we have

$$\sup_{0\le t\le s} E_0(t) + \int_0^s E_1(t)dt \ge \frac{1}{18}\min\left\{\frac{1}{2}, \nu\right\}A(s).$$

Therefore, putting $K_5(C_0+1)/\min\left\{\frac{1}{2},\nu\right\} = K_6$, we get (4.11). □

13TH STEP. *There exists a positive constant K_7 such that*

$$M(s)^2 \le K_7(A(s) + s^\lambda M(s)^2) \quad \textit{for } 0 \le s \le T. \tag{4.12}$$

PROOF. From the proof of 8th step we observe that

$$\begin{aligned}
M(s)^2 \le K_3\Big\{&|\nabla \boldsymbol{u}_t : \boldsymbol{\varepsilon}_{1t}|^2_{L^q(Q(s))} + |\nabla \boldsymbol{u}_{2t} : \boldsymbol{\varepsilon}_t|^2_{L^q(Q(s))} + |\boldsymbol{\sigma} : \boldsymbol{\varepsilon}_{1t}|^2_{L^q(Q(s))} \\
&+ |\boldsymbol{\sigma}_2 : \boldsymbol{\varepsilon}_t|^2_{L^q(Q(s))} + \sum_{i,j=1}^{3} |\sigma_{ij}|^2_{L^{q_0}(Q(s))}\Big\} \quad \text{for } 0 \le s \le T.
\end{aligned}$$

From 10th–12th steps it follows the conclusion of this step. □

Now, we arrive at just before the point to accomplish the proof of the uniqueness.

PROOF OF THE UNIQUENESS. Taking a small positive number s_1 satisfying $s_1K_6 \le \frac{1}{2}$, (4.12) implies

$$A(s) \le 2s^{1+\lambda}K_6M(s)^2 \quad \text{for } 0 \le s \le s_1.$$

By substituting the above inequality into (4.12) we observe

$$\begin{aligned}M(s)^2 &\le K_7\big(2s^{1+\lambda}K_6M(s)^2 + s^\lambda M(s)^2\big)\\ &\le K_7(2K_6T+1)s^\lambda M(s)^2 \quad \text{for } 0 \le s \le s_1.\end{aligned}$$

Here, we choose a positive number $s_2 \le s_1$ with $K_7(2K_6T+1)s_2^\lambda \le \frac{1}{2}$. Then we have $M(s)^2 \le \frac{1}{2}M(s)^2$ for $0 \le s \le s_2$. Therefore, $M(s_2) = 0$ and $A(s_2) = 0$. It yields that $\boldsymbol{\sigma} = 0$, $\boldsymbol{u} = 0$ and $\theta = 0$ on $Q(s_2)$. In the above argument the choice of s_2 is independent of initial values. Thus we have proved the uniqueness on the whole interval $[0, T]$. □

5. Approximate solutions.

First, we approximate the indicator function I by using the Yosida approximation. For $\lambda > 0$ let I_λ be the Yosida-approximation of I. Easily, we have:

Lemma 5.1 (cf. [**19**, Section 4]) and [**4**, Theorem 2.1]). *Let $\lambda > 0$. If $\theta \in L^2(\Omega)$ and $\boldsymbol{\varepsilon} \in L^2(\Omega)^9$, then for $\sigma \in L^2(\Omega)$ it holds*

$$I_\lambda(\theta, \boldsymbol{\varepsilon}; \sigma) = \frac{1}{2\lambda}\big\{|[\sigma - f^*(\theta,\boldsymbol{\varepsilon})]^+|^2_{L^2(\Omega)} + |[f_*(\theta,\boldsymbol{\varepsilon}) - \sigma]^+|^2_{L^2(\Omega)}\big\},$$

$$\partial I_\lambda(\theta, \boldsymbol{\varepsilon}; \sigma) = \frac{1}{\lambda}\{[\sigma - f^*(\theta,\boldsymbol{\varepsilon})]^+ - [f_*(\theta,\boldsymbol{\varepsilon}) - \sigma]^+\} \text{ a.e. on } \Omega.$$

Next, let M be a positive number satisfying $M \ge \frac{4(\nu^2+4)}{\nu}$. Here, we consider the approximate problem (SMAP)(λ, M). The following lemma is concerned with the well-posedness of the approximate problem.

Lemma 5.2. *Let $\lambda > 0$. If $\boldsymbol{u}_0$, $\boldsymbol{v}_0$, θ_0 and $\boldsymbol{\sigma}_0$ satisfy* (A4)*, then there exist $T_\lambda \in (0, T]$ and a solution $\{\boldsymbol{u}, \theta, \boldsymbol{\sigma}\}$ of (SMAP)(λ, M) on $[0, T_\lambda]$, that is,* (2.2)–(2.6) *and* (S1)–(S4) *hold.*

By using the Banach fixed point theorem we can prove Lemma 5.2, because ∂I is Lipschitz continuous. From now on, we write T as T_λ in order to avoid surplus notations. The purpose of this section is to give uniform estimates for approximate solutions $\{\boldsymbol{u}_\lambda, \theta_\lambda, \boldsymbol{\sigma}_\lambda\}$ with respect to λ. Here, we put $\boldsymbol{u}_\lambda = (u_{\lambda 1}, u_{\lambda 2}, u_{\lambda 3})$ and $\boldsymbol{\sigma}_\lambda = (\sigma_{\lambda ij})$.

Lemma 5.3. *There exists a positive constant R_1 independent of λ such that*

$$|[\sigma_{\lambda ij}(t) - L_0]^+|_{L^2(\Omega)} + |[-\sigma_{\lambda ij}(t) - L_0]^+|_{L^2(\Omega)} \le R_1 \quad \text{for } 0 \le t \le T \text{ and } i, j;$$

$$\int_0^T |[\sigma_{\lambda ij}(t) - L_0]^+|^2_{H^1(\Omega)}dt + \int_0^T |[-\sigma_{\lambda ij}(t) - L_0]^+|^2_{H^1(\Omega)}dt \le R_1 \quad \text{for } i, j;$$

$$|\boldsymbol{u}_{\lambda t}(t)|^2_{L^2(\Omega)} + |\Delta \boldsymbol{u}_\lambda(t)|^2_{L^2(\Omega)} \le R_1 \quad \text{for } 0 \le t \le T;$$

$$\int_0^T |\nabla \boldsymbol{u}_{\lambda t}(t)|^2_{L^2(\Omega)}dt \le R_1 \quad \text{for } \lambda \in (0, 1].$$

PROOF. We multiply (2.4) by $[\sigma_{\lambda ij}(t)-L_0]^+$ and integrate it over Ω. Then we have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_\Omega |[\sigma_{\lambda ij}(t)-L_0]^+|^2dx+\nu\int_\Omega |\nabla[\sigma_{\lambda ij}(t)-L_0]^+|^2dx\\ &\quad\le c\int_\Omega \varepsilon_{\lambda ijt}(t)[\sigma_{\lambda ij}(t)-L_0]^+dx \quad \text{for a.e. } t\in[0,T] \text{ and } i,j,\end{aligned}\tag{5.1}$$

since $\partial I_\lambda(\theta_\lambda,\boldsymbol{\varepsilon}_\lambda;\sigma_{\lambda ij})[\sigma_{\lambda ij}-L_0]^+\ge 0$ a.e. on $Q(T)$. Similarly, we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_\Omega |[-\sigma_{\lambda ij}(t)-L_0]^+|^2dx+\nu\int_\Omega |\nabla[-\sigma_{\lambda ij}(t)-L_0]^+|^2dx\\ &\quad\le -c\int_\Omega \varepsilon_{\lambda ijt}(t)[-\sigma_{\lambda ij}(t)-L_0]^+dx \quad \text{for a.e. } t\in[0,T] \text{ and } i,j.\end{aligned}\tag{5.2}$$

It is obvious that

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_\Omega |\boldsymbol{u}_{\lambda t}(t)|^2dx+\frac{\gamma}{2}\frac{d}{dt}\int_\Omega |\Delta\boldsymbol{u}_\lambda(t)|^2dx+\mu\int_\Omega |\nabla\boldsymbol{u}_{\lambda t}(t)|^2dx\\ &\quad=-\sum_{i=1}^3\int_\Omega \boldsymbol{\sigma}_{\lambda i}(t)\cdot\nabla u_{\lambda it}(t)dx \quad \text{for a.e. } t\in[0,T].\end{aligned}\tag{5.3}$$

By adding (5.1)–(5.3) we see that

$$\begin{aligned}&\sum_{i,j=1}^3\left(\frac{1}{2}\frac{d}{dt}\int_\Omega |[\sigma_{\lambda ij}(t)-L_0]^+|^2dx+\nu\int_\Omega |\nabla[\sigma_{\lambda ij}(t)-L_0]^+|^2dx\right)\\ &\quad+\sum_{i,j=1}^3\left(\frac{1}{2}\frac{d}{dt}\int_\Omega |[-\sigma_{\lambda ij}(t)-L_0]^+|^2dx+\nu\int_\Omega |\nabla[-\sigma_{\lambda ij}(t)-L_0]^+|^2dx\right)\\ &\quad+\frac{c}{2}\frac{d}{dt}\int_\Omega |\boldsymbol{u}_{\lambda t}(t)|^2dx+\frac{c\gamma}{2}\frac{d}{dt}\int_\Omega |\Delta\boldsymbol{u}_\lambda(t)|^2dx+c\mu\int_\Omega |\nabla\boldsymbol{u}_{\lambda t}(t)|^2dx\\ &\le c\sum_{i,j=1}^3\int_\Omega \varepsilon_{\lambda ijt}(t)[\sigma_{\lambda ij}(t)-L_0]^+dx-c\sum_{i,j=1}^3\int_\Omega \varepsilon_{\lambda ijt}(t)[-\sigma_{\lambda ij}(t)-L_0]^+dx\\ &\quad-c\sum_{i,j=1}^3\int_\Omega \sigma_{\lambda ij}(t)\frac{\partial u_{\lambda it}(t)}{\partial x_j}dx \quad \text{for a.e. } t\in[0,T].\end{aligned}\tag{5.4}$$

Here, similarly to 3rd step of the proof of the uniqueness we can show that $\sigma_{\lambda ij}=\sigma_{\lambda ji}$ for i and j. Accordingly, we infer that

$$
\begin{aligned}
& c \sum_{i,j=1}^{3} \int_\Omega \varepsilon_{\lambda ijt}(t)[\sigma_{\lambda ij}(t)-L_0]^+ dx - c \sum_{i,j=1}^{3} \int_\Omega \varepsilon_{\lambda ijt}(t)[-\sigma_{\lambda ij}(t)-L_0]^+ dx \\
& \quad - c \sum_{i,j=1}^{3} \int_\Omega \sigma_{\lambda ij}(t) \frac{\partial u_{\lambda it}}{\partial x_j} dx \\
& \quad = c \sum_{i,j=1}^{3} \int_\Omega \frac{\partial u_{\lambda it}(t)}{\partial x_j} ([\sigma_{\lambda ij}(t)-L_0]^+ - [-\sigma_{\lambda ij}(t)-L_0]^+ - \sigma_{\lambda ij}(t)) dx \\
& \quad \le c L_0 \sum_{i,j=1}^{3} \int_\Omega \left| \frac{\partial u_{\lambda it}(t)}{\partial x_j} \right| dx \quad \text{for a.e. } t \in [0,T].
\end{aligned}
$$

Here, we use $|[\sigma_{\lambda ij}(t)-L_0]^+ - [-\sigma_{\lambda ij}(t)-L_0]^+ - \sigma_{\lambda ij}(t)| \le L_0$. Hence, it follows

$$
\begin{aligned}
& \frac{1}{2}\frac{d}{dt}\int_\Omega |[\sigma_{\lambda ij}(t)-L_0]^+|^2 dx + \nu \int_\Omega |\nabla[\sigma_{\lambda ij}(t)-L_0]^+|^2 dx \\
& \quad + \frac{1}{2}\frac{d}{dt}\int_\Omega |[-\sigma_{\lambda ij}(t)-L_0]^+|^2 dx + \nu \int_\Omega |\nabla[-\sigma_{\lambda ij}(t)-L_0]^+|^2 dx \\
& \quad + \frac{c}{2}\frac{d}{dt}\int_\Omega |\boldsymbol{u}_{\lambda t}(t)|^2 dx + \frac{c\gamma}{2}\frac{d}{dt}\int_\Omega |\Delta \boldsymbol{u}_\lambda(t)|^2 dx + \frac{c\mu}{2}\int_\Omega |\nabla \boldsymbol{u}_{\lambda t}(t)|^2 dx \\
& \quad \le \frac{9c}{2\mu} L_0^2 |\Omega| \quad \text{for a.e. } t \in [0,T].
\end{aligned} \tag{5.5}
$$

We integrate (5.5) over $[0,\tau]$, $0 \le \tau \le T$, and get the assertion of this lemma. □

Lemma 5.4. *Put $p = 10/3$. Then there exists a positive constant R_2 independent of λ such that*

$$
\begin{aligned}
& |\nabla \boldsymbol{u}_{\lambda t}|_{L^p(Q(T))} \le R_2(|\boldsymbol{\sigma}_\lambda|_{L^p(Q(T))} + |\boldsymbol{v}_0|_{H^2(\Omega)} + |\Delta \boldsymbol{u}_0|_{H^2(\Omega)}) \quad \text{for } \lambda \in (0,1], \qquad (5.6) \\
& \quad |\boldsymbol{\varepsilon}_{\lambda t}|_{L^p(Q(T))} \le R_2 \quad \text{for } \lambda \in (0,1].
\end{aligned}
$$

Proof. (5.6) is due to Lemma 3.5(i) since the embedding relation $H^2(\Omega) \subset W^{2-2/p,p}(\Omega)$ holds with $p = 10/3$ (cf. [**33**, Theorem 9.2.1]). Clearly, it holds that

$$
|\sigma_{\lambda ij}| \le |[\sigma_{\lambda ij} - L_0]^+| + |[-\sigma_{\lambda ij} - L_0]^+| + L_0 \quad \text{a.e. on } Q(T).
$$

Hence, Lemma 5.3 and (1.9) imply that $\{\boldsymbol{\sigma}_\lambda\}$ is the bounded set in $L^{10/3}(Q(T))^9$. □

Lemma 5.5. *There exists a positive constant R_3 such that*

$$
|\sigma_{\lambda ij}(t,x)| \le R_3 \quad \textit{for a.e. } (t,x) \in Q(T), \textit{ and } \lambda \in (0,1] \textit{ and } i,j.
$$

Proof. We shall prove this lemma in a similar way to those of [**21**, Theorem 7.1, Chapter 3] and [**3**, Lemma 4.3]. In this proof we fix i and j. First for $\ell \ge \ell_0 := \max\{L_0, |\sigma_{0ij}|_{L^\infty(\Omega)} + 1\}$ we put

$$A_\ell(t) = \{x \in \Omega : \sigma_{\lambda ij}(t,x) \geq \ell\}.$$

The inequality (5.1) still holds with ℓ instead of L_0. On account of $\ell \geq 1$ we observe that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{d\tau}\int_\Omega |[\sigma_{\lambda ij}(\tau)-\ell]^+|^2 dx + \nu\int_\Omega |\nabla[\sigma_{\lambda ij}(\tau)-\ell]^+|^2 dx \\
&\quad = c\int_\Omega \varepsilon_{\lambda ij\tau}(\tau)[\sigma_{\lambda ij}(\tau)-\ell]^+ dx \\
&\quad = c\int_{A_\ell(\tau)} \varepsilon_{\lambda ij\tau}(\tau)(\sigma_{\lambda ij}(\tau)-\ell) dx \\
&\quad \leq c\int_{A_\ell(\tau)} |\varepsilon_{\lambda ij\tau}(\tau)|(|\sigma_{\lambda ij}(\tau)-\ell|^2+\ell^2) dx \\
&\quad \leq c\Big(\int_{A_\ell(\tau)} |\varepsilon_{\lambda ij\tau}(\tau)|^{10/3} dx\Big)^{3/10}\Big(\int_{A_\ell(\tau)} (|\sigma_{\lambda ij}(\tau)-\ell|^2+\ell^2)^{10/7} dx\Big)^{7/10} \\
&\qquad\qquad \text{for a.e. } \tau \in [0,T].
\end{aligned}$$

Integrating the above inequality over $[0,t]$, $0 < t < T$, we have

$$\begin{aligned}
&|[\sigma_{\lambda ij}-\ell]^+|^2_{V(t)} \\
&\quad \leq N_1\Big(\int_0^t\int_{A_\ell(\tau)} |\varepsilon_{\lambda ij\tau}(\tau)|^{10/3} dxd\tau\Big)^{3/10}\Big(\int_0^t\int_{A_\ell(\tau)} (|\sigma_{\lambda ij}(\tau)-\ell|^2+\ell^2)^{10/7} dxd\tau\Big)^{7/10} \\
&\quad \leq N_1R_2\Big(\int_0^t\int_{A_\ell(\tau)} (|\sigma_{\lambda ij}(\tau)-\ell|^2+\ell^2)^{10/7} dxd\tau\Big)^{7/10} \qquad \text{for } t \in [0,T],
\end{aligned}$$

where N_1 is a positive constant depending only on ν and c. By applying Hölder's inequality we obtain

$$\Big(\int_0^t\int_{A_\ell(\tau)} (|\sigma_{\lambda ij}(\tau)-\ell|^2)^{10/7} dxd\tau\Big)^{7/10} \leq t^{1/10}|\Omega|^{1/10}|[\sigma_{\lambda ij}-\ell]^+|^2_{L^{10/3}(Q(t))}$$

and

$$\Big(\int_0^t\int_{A_\ell(\tau)} (\ell^2)^{10/7} dxd\tau\Big)^{7/10} \leq \ell^2\Big(\int_0^t |A_\ell(\tau)|d\tau\Big)^{7/10} \qquad \text{for } t \in [0,T].$$

Therefore, (1.9) implies

$$|[\sigma_{\lambda ij}-\ell]^+|_{V(t)} \leq \sqrt{N_1R_2}\Big\{C_0(t|\Omega|)^{1/20}|[\sigma_{\lambda ij}-\ell]^+|_{V(t)} + \ell\Big(\int_0^t |A_\ell(\tau)|d\tau\Big)^{7/20}\Big\}$$
$$\text{for } t \in [0,T].$$

Now, we choose $T_1 \in (0, T]$ such that $\sqrt{N_1 R_2} C_0 T_1^{1/20} |\Omega|^{1/20} \leq \frac{1}{2}$. Thus we infer that

$$|[\sigma_{\lambda ij} - \ell]^+|_{V(t)} \leq 2\ell \sqrt{N_1 R_2} \Big(\int_0^t |A_\ell(\tau)| d\tau \Big)^{7/20} \quad \text{for } 0 \leq t \leq T_1 \text{ and } \ell \geq \ell_0. \tag{5.7}$$

Putting $k_q = (2 - 2^{-q})\ell_1$ for $\ell_1 \geq \ell_0$ and $q = 0, 1, 2, \cdots$, (5.7) and (1.9) guarantee that

$$\begin{aligned}(k_{q+1} - k_q) \Big(\int_0^{T_1} |A_{k_{q+1}}(\tau)| d\tau \Big)^{3/10} &\leq \Big(\int_0^{T_1} \int_\Omega |[\sigma_{\lambda ij}(\tau) - k_q]^+|^{10/3} dx d\tau \Big)^{3/10} \\ &\leq C_0 |[\sigma_{\lambda ij}(\tau) - k_q]^+|_{V(T_1)} \\ &\leq N_2 k_q \Big(\int_0^{T_1} |A_{k_q}(\tau)| d\tau \Big)^{7/20} \\ &\qquad \text{for each } q = 0, 1, 2, \cdots, \end{aligned} \tag{5.8}$$

where $N_2 = 2C_0 \sqrt{N_1 R_2}$. Here, we put $a_q = \big(\int_0^{T_1} |A_{k_q}(\tau)| d\tau \big)^{3/10}$ for each $q = 0, 1, 2, \cdots$. Immediately, we have

$$a_{q+1} \leq N_3 2^q a_q^{7/6} \quad \text{for } q = 0, 1, 2, \cdots,$$

where $N_3 = 4N_2$. Also, we set $\ell_1 = N\ell_0$ for $N \geq 1$ and obtain the following inequality in a similar way to that of (5.8):

$$(\ell_1 - \ell_0) \Big(\int_0^{T_1} |A_{\ell_1}(t)| dt \Big)^{3/10} \leq N_2 \ell_0 \Big(\int_0^{T_1} |A_{\ell_0}(t)| dt \Big)^{7/20}$$

so that

$$a_0 = \Big(\int_0^{T_1} |A_{\ell_1}(\tau)| d\tau \Big)^{3/10} \leq \frac{N_2}{N-1} T_1^{7/20} |\Omega|^{7/20}.$$

Thus we can take N satisfying

$$N_2 T_1^{7/20} |\Omega|^{7/20} N_3^6 2^{36} + 1 \leq N.$$

Clearly, $a_0 \leq \frac{1}{N_3^6 2^{36}}$. Then by applying [**21**, Lemma 5.6 in Chapter 2] it holds that $a_q \to 0$ as $q \to \infty$ so that $\sigma_{\lambda ij} \leq 2N\ell_0$ on $Q(T_1)$.

Analogous arguments are valid on the cylinder $(T_1, 2T_1) \times \Omega$. Thus after a finite number of steps we get the required estimate. Of course the lower bound can be shown, similarly. □

The following lemma is useful in order to get uniform estimate and can be proved easily.

LEMMA 5.6. *$\{\boldsymbol{\varepsilon}_{\lambda t}\}$ and $\{\nabla \boldsymbol{u}_{\lambda t}\}$ are bounded sets in $L^4(Q(T))^9$.*

PROOF. For each i and $\lambda \in (0,1]$ $u_{\lambda i}$ satisfies (3.5) with $\boldsymbol{\sigma}_{\lambda i}$ instead of $\boldsymbol{f}$. Therefore, this lemma is a direct consequence of Lemmas 3.5(iii) and 5.5. □

LEMMA 5.7. *The set $\{\theta_\lambda\}$ is bounded in $W^{1,2}(0,T;L^2(\Omega))$ and $L^\infty(0,T;H^1(\Omega))$.*

PROOF. By Lemmas 5.5 and 5.6 the right hand side in (2.3) belongs to $L^2(Q(T))$ for each $\lambda \in (0,1]$, that is, $\{\boldsymbol{\sigma}_\lambda : \boldsymbol{\varepsilon}_{\lambda t} + \mu \nabla \boldsymbol{u}_{\lambda t} : \boldsymbol{\varepsilon}_{\lambda t}\}$ is the bounded set in $L^2(Q(T))$. Hence, this lemma is trivial. □

LEMMA 5.8. *There exists a positive constant R_4 such that*

$$\left.\begin{aligned}
&|I_\lambda(\theta_\lambda, \boldsymbol{\varepsilon}_\lambda; \sigma_{\lambda ij})|_{L^\infty(0,T)} \le R_4 \quad \textit{for } i,j,\\
&|\partial I_\lambda(\theta_\lambda, \boldsymbol{\varepsilon}_\lambda; \sigma_{\lambda ij})|_{L^2(0,T;L^2(\Omega))} \le R_4 \quad \textit{for } i,j,\\
&|\nabla \boldsymbol{\sigma}_\lambda(t)|_{L^2(\Omega)} \le R_4 \quad \textit{for } 0 \le t \le T,\\
&|\boldsymbol{\sigma}_\lambda|_{L^2(0,T;H^2(\Omega))} \le R_4,\\
&|\boldsymbol{\sigma}_{\lambda t}|_{L^2(Q(T))} \le R_4,
\end{aligned}\right\} \textit{for } \lambda \in (0,1].$$

PROOF. In this proof we fix i and j and out $\partial I_\lambda(\theta_\lambda, \boldsymbol{\varepsilon}_\lambda; \sigma_{\lambda ij}) = \xi_{\lambda ij}$ for $\lambda \in (0,1]$. Multiplying (2.4) by $\sigma_{\lambda ijt}$, we obtain

$$\begin{aligned}
&|\sigma_{\lambda ijt}(t)|^2_{L^2(\Omega)} + \frac{\nu}{2}\frac{d}{dt}|\nabla \sigma_{\lambda ij}(t)|^2_{L^2(\Omega)} + M\int_\Omega \xi_{\lambda ij}(t)\sigma_{\lambda ijt}(t)dx\\
&\quad \le \frac{1}{2}\int_\Omega |\sigma_{\lambda ijt}(t)|^2 dx + \frac{c^2}{2}\int_\Omega |\varepsilon_{\lambda ijt}(t)|^2 dx \quad \text{for a.e. } t \in [0,T].
\end{aligned}$$

Here, we calculate the time derivative of I_λ as follows (see Lemma 5.1):

$$\begin{aligned}
&\frac{d}{dt} I_\lambda(\theta_\lambda(t), \boldsymbol{\varepsilon}_\lambda(t); \sigma_{\lambda ij}(t))\\
&\quad = \frac{1}{\lambda}\int_\Omega \left(\sigma_{\lambda ijt}(t) - \frac{\partial}{\partial t} f^*(\theta_\lambda(t), \boldsymbol{\varepsilon}_\lambda(t))\right)[\sigma_{\lambda ij}(t) - f^*(\theta_\lambda(t), \boldsymbol{\varepsilon}_\lambda(t))]^+ dx\\
&\qquad + \frac{1}{\lambda}\int_\Omega \left(\frac{\partial}{\partial t} f_*(\theta_\lambda(t), \boldsymbol{\varepsilon}_\lambda(t)) - \sigma_{\lambda ijt}(t)\right)[f_*(\theta_\lambda(t), \boldsymbol{\varepsilon}_\lambda(t)) - \sigma_{\lambda ij}(t)]^+ dx\\
&\quad = \int_\Omega \xi_{\lambda ij}(t)\sigma_{\lambda ijt}(t)dx - \frac{1}{\lambda}\int_\Omega \frac{\partial}{\partial t} f^*(\theta_\lambda(t), \boldsymbol{\varepsilon}_\lambda(t))[\sigma_{\lambda ij}(t) - f^*(\theta_\lambda(t), \boldsymbol{\varepsilon}_\lambda(t))]^+ dx\\
&\qquad + \frac{1}{\lambda}\int_\Omega \frac{\partial}{\partial t} f_*(\theta_\lambda(t), \boldsymbol{\varepsilon}_\lambda(t))[f_*(\theta_\lambda(t), \boldsymbol{\varepsilon}_\lambda(t)) - \sigma_{\lambda ij}(t)]^+ dx\\
&\quad \le \int_\Omega \xi_{\lambda ij}(t)\sigma_{\lambda ijt}(t)dx + \int_\Omega F_\lambda(t)|\xi_{\lambda ij}(t)|dx \quad \text{for a.e. } t \in [0,T],
\end{aligned} \tag{5.9}$$

where $F_\lambda = \left|\frac{\partial}{\partial t} f^*(\theta_\lambda, \boldsymbol{\varepsilon}_\lambda)\right| + \left|\frac{\partial}{\partial t} f_*(\theta_\lambda, \boldsymbol{\varepsilon}_\lambda)\right|$. We note that

$$|[f_*(\theta_\lambda, \boldsymbol{\varepsilon}_\lambda) - \sigma_{\lambda ij}]^+| \le \lambda|\xi_{\lambda ij}|, \ |[\sigma_{\lambda ij} - f^*(\theta_\lambda, \boldsymbol{\varepsilon}_\lambda)]^+| \le \lambda|\xi_{\lambda ij}| \quad \text{a.e. on } Q(T).$$

From the above two inequalities it follows

$$\begin{aligned} &\frac{1}{2}|\sigma_{\lambda ijt}(t)|^2_{L^2(\Omega)} + \frac{\nu}{2}\frac{d}{dt}|\nabla\sigma_{\lambda ij}(t)|^2_{L^2(\Omega)} + M\frac{d}{dt}I_\lambda(\theta_\lambda(t), \boldsymbol{\varepsilon}_\lambda(t); \sigma_{\lambda ij}(t)) \\ &\quad \le \frac{c^2}{2}\int_\Omega |\varepsilon_{\lambda ijt}(t)|^2 dx + M\int_\Omega F_\lambda(t)\xi_{\lambda ij}(t)dx \quad \text{for a.e. } t \in [0,T]. \end{aligned} \tag{5.10}$$

Next, multiplying (2.4) by $\xi_{\lambda ij}$, we see that

$$\begin{aligned} &\int_\Omega \sigma_{\lambda ijt}(t)\xi_{\lambda ij}(t)dx - \nu\int_\Omega \Delta\sigma_{\lambda ij}(t)\xi_{\lambda ij}(t)dx + M\int_\Omega |\xi_{\lambda ij}(t)|^2 dx \\ &\quad = c\int_\Omega \varepsilon_{\lambda ijt}(t)\xi_{\lambda ij}(t)dx \quad \text{for a.e. } t \in [0,T]. \end{aligned}$$

By substituting (5.9) into the above inequality we get

$$\begin{aligned} &\frac{d}{dt}I_\lambda(\theta_\lambda(t), \boldsymbol{\varepsilon}_\lambda(t); \sigma_{\lambda ij}(t)) + M\int_\Omega |\xi_{\lambda ij}(t)|^2 dx \\ &\le \nu\int_\Omega |\Delta\sigma_{\lambda ij}(t)||\xi_{\lambda ij}(t)|dx + c\int_\Omega |\varepsilon_{\lambda ijt}(t)||\xi_{\lambda ij}(t)|dx + \int_\Omega |F_\lambda(t)||\xi_{\lambda ij}(t)|dx \\ &\le \frac{c^2}{2M}\int_\Omega |\varepsilon_{\lambda ijt}(t)|^2 dx + \frac{7M}{8}\int_\Omega |\xi_{\lambda ij}(t)|^2 dx + \frac{\nu^2}{M}\int_\Omega |\Delta\sigma_{\lambda ij}(t)|^2 dx + \frac{2}{M}\int_\Omega |F_\lambda(t)|^2 dx \end{aligned}$$

so that

$$\begin{aligned} &\frac{d}{dt}I_\lambda(\theta_\lambda(t), \boldsymbol{\varepsilon}_\lambda(t); \sigma_{\lambda ij}(t)) + \frac{M}{8}|\xi_{\lambda ij}(t)|^2_{L^2(\Omega)} \\ &\quad \le \frac{c^2}{2M}|\varepsilon_{\lambda ijt}(t)|^2_{L^2(\Omega)} + \frac{\nu^2}{M}|\Delta\sigma_{\lambda ij}(t)|^2_{L^2(\Omega)} + \frac{2}{M}|F_\lambda(t)|^2_{L^2(\Omega)} \quad \text{for a.e. } t \in [0,T]. \end{aligned} \tag{5.11}$$

Similarly to (5.10) and (5.11), by multiplying (2.4) by $-\Delta\sigma_{\lambda ij}$ we can show

$$\begin{aligned} &\frac{d}{dt}|\nabla\sigma_{\lambda ij}(t)|^2_{L^2(\Omega)} + \frac{\nu}{2}|\Delta\sigma_{\lambda ij}(t)|^2_{L^2(\Omega)} \\ &\quad \le \frac{M}{16}|\xi_{\lambda ij}(t)|^2_{L^2(\Omega)} + \frac{4}{M}|\Delta\sigma_{\lambda ij}(t)|^2_{L^2(\Omega)} + \frac{c^2}{2\nu}|\varepsilon_{\lambda ijt}(t)|^2_{L^2(\Omega)} \quad \text{for a.e. } t \in [0,T]. \end{aligned} \tag{5.12}$$

(5.11) and (5.12) imply that

$$\frac{d}{dt}I_\lambda(\theta_\lambda(t),\boldsymbol{\varepsilon}_\lambda(t);\sigma_{\lambda ij}(t))+\frac{M}{8}|\xi_{\lambda ij}(t)|^2_{L^2(\Omega)}+\frac{d}{dt}|\nabla\sigma_{\lambda ij}(t)|^2_{L^2(\Omega)}+\frac{\nu}{2}|\Delta\sigma_{\lambda ij}(t)|^2_{L^2(\Omega)}$$
$$\le\left(\frac{c^2}{2M}+\frac{c^2}{2\nu}\right)|\varepsilon_{\lambda ijt}(t)|^2_{L^2(\Omega)}+\frac{\nu^2+4}{M}|\Delta\sigma_{\lambda ij}(t)|^2_{L^2(\Omega)}+\frac{2}{M}|F_\lambda(t)|^2_{L^2(\Omega)}$$

for a.e. $t\in[0,T]$.

Now, it satisfies $\frac{\nu^2+4}{M}\le\frac{\mu}{4}$. Thus we have

$$\frac{d}{dt}I_\lambda(\theta_\lambda(t),\boldsymbol{\varepsilon}_\lambda(t);\sigma_{\lambda ij}(t))+\frac{M}{8}|\xi_{\lambda ij}(t)|^2_{L^2(\Omega)}+\frac{d}{dt}|\nabla\sigma_{\lambda ij}(t)|^2_{L^2(\Omega)}+\frac{\nu}{4}|\Delta\sigma_{\lambda ij}(t)|^2_{L^2(\Omega)}$$
$$\le\left(\frac{c^2}{2M}+\frac{c^2}{2\nu}\right)|\varepsilon_{\lambda ijt}(t)|^2_{L^2(\Omega)}+\frac{2}{M}|F_\lambda(t)|^2_{L^2(\Omega)}\quad\text{for a.e. } t\in[0,T]. \tag{5.13}$$

Integrating (5.13), we get the required estimates except for $|\boldsymbol{\sigma}_{\lambda t}|_{L^2(Q(T))}\le R_4$. In fact, it holds that

$$|F_\lambda(t)|_{L^2(\Omega)}\le 2L\left(|\theta_{\lambda t}(t)|_{L^2(\Omega)}+\sum_{i=1}^{3}|\varepsilon_{\lambda ijt}(t)|_{L^2(\Omega)}\right)\quad\text{for } t\in[0,T].$$

The rest estimate of this lemma is easily obtained from (5.10). □

Lemma 5.9. *The set $\{\boldsymbol{u}_\lambda\}$ is bounded in $W^{1,\infty}(0,T;H^2(\Omega)^3)$, $L^\infty(0,T;H^4(\Omega)^3)$ and $W^{1,2}(0,T;H^3(\Omega)^3)$. Therefore, $\{\boldsymbol{u}_{\lambda tt}\}$ is bounded in $L^\infty(0,T;L^2(\Omega)^3)$.*

Proof. We multiply (2.2) by $\Delta(\Delta\boldsymbol{u}_{\lambda t})$. Then by elementary calculation we see that

$$\frac{1}{2}|\Delta\boldsymbol{u}_{\lambda t}(t)|^2_{L^2(\Omega)}+\frac{\gamma}{2}|\Delta(\Delta\boldsymbol{u}_\lambda)(t)|^2_{L^2(\Omega)}+\frac{\mu}{2}\int_0^t|\nabla(\Delta\boldsymbol{u}_\tau)(\tau)|^2_{L^2(\Omega)}d\tau$$
$$\le\frac{1}{2\mu}\int_0^t|\nabla(\mathrm{div}\boldsymbol{\sigma}_\lambda)(\tau)|^2_{L^2(\Omega)}d\tau+\frac{1}{2}|\Delta\boldsymbol{v}_0|^2_{L^2(\Omega)}+\frac{\gamma}{2}|\Delta(\Delta\boldsymbol{u}_0)|^2_{L^2(\Omega)}\quad\text{for } t\in[0,T].$$

Also, (1.2) implies that $\{u_{\lambda tt}\}$ is bounded in $L^\infty(0,T;L^2(\Omega)^3)$. Thus we have proved this lemma. □

6. Proof of the existence.

The purpose of this section is give the proof of the existence of solutions. Before the proof we show the global existence of approximate solutions in time.

Proposition 6.1. *Let $T>0$ and assume* (A1)–(A4) *and $\mu^2>4\gamma$. Then for $\lambda\in(0,1]$ and $M>0$ with $M\ge\frac{4(\nu^2+4)}{\nu}$ (SMAP)(M,λ) has a solution on $[0,T]$.*

Proof. Let $[0, T_\lambda)$ be the maximal interval of existence of a solution to (SMAP)(M, λ). Suppose that $T_\lambda < T$. Then uniform estimates obtained in the previous section show that we can extend the solution beyond T_λ. Thus this proposition has been proved. $\square$

Proof of the existence. The uniform estimates shown in the previous section guarantee that we can take a subsequence $\{\lambda_k\}$ of $\{\lambda\}$ and functions $\boldsymbol{u}$, θ and $\boldsymbol{\sigma}$ satisfying (S1)–(S4),

$$
\begin{aligned}
\theta_k := \theta_{\lambda_k} \to \theta \quad & \text{weakly in } W^{1,2}(0,T;L^2(\Omega)),\\
& \text{weakly* in } L^\infty(0,T;H^1(\Omega)),\\
& \text{in } C([0,T];L^2(\Omega)),\\
\boldsymbol{\sigma}_k := \boldsymbol{\sigma}_{\lambda_k} \to \boldsymbol{\sigma} \quad & \text{weakly in } W^{1,2}(0,T;L^2(\Omega)^9),\\
& \text{weakly* in } L^\infty(0,T;H^1(\Omega)^9),\\
& \text{in } C([0,T];L^2(\Omega)^9),\\
\boldsymbol{u}_k := \boldsymbol{u}_{\lambda_k} \to \boldsymbol{u} \quad & \text{weakly* in } W^{1,\infty}(0,T;H^2(\Omega)^3) \text{ and } L^\infty(0,T;H^4(\Omega)^3),\\
& \text{weakly in } W^{1,2}(0,T;H^3(\Omega)^3) \text{ and } W^{2,2}(0,T;L^2(\Omega)^3),\\
I_{\lambda_k}(\theta_k, \boldsymbol{\varepsilon}_k; \sigma_{kij}) \to \hat{I}_{ij} \quad & \text{weakly* in } L^\infty(0,T) \text{ for } i,j,\\
\partial I_{\lambda_k}(\theta_k, \boldsymbol{\varepsilon}_k; \sigma_{kij}) \to \xi_{ij} \quad & \text{weakly in } L^2(Q(T)) \text{ for } i,j,
\end{aligned}
$$

as $k \to \infty$, where $\boldsymbol{\sigma}_k = (\sigma_{kji})$ and $\boldsymbol{\varepsilon}_k = \boldsymbol{\varepsilon}_{\lambda_k}$.

Hence, for each i we have

$$
\int_{Q(T)} (u_{itt} + \gamma \Delta(\Delta u_i) - \mu \Delta u_{it}) \eta dxdt = \int_{Q(T)} \eta \mathrm{div} \boldsymbol{\sigma}_i dxdt \quad \text{for } \eta \in L^2(Q(T)).
$$

where $\boldsymbol{u} = (u_1, u_2, u_3)$. Thus we know that (S5) is valid. Clearly, the initial conditions for $\boldsymbol{u}$ hold.

Next, by putting $\boldsymbol{\varepsilon} = \frac{1}{2}(\nabla \boldsymbol{u} + {}^t\nabla \boldsymbol{u})$ it is obvious that $\boldsymbol{\varepsilon}_{kt} \to \boldsymbol{\varepsilon}_t$ weakly in $L^2(0,T;L^2(\Omega)^9)$. Then we obtain

$$
\begin{aligned}
&\int_{Q(T)} \sigma_{ijt} \eta dxdt + \nu \int_{Q(T)} \nabla \sigma_{ij} \cdot \nabla \eta dxdt + M \int_{Q(T)} \xi_{ij} \eta dxdt\\
&= c \int_{Q(T)} \varepsilon_{ijt} \eta dxdt \quad \text{for } \eta \in L^2(0,T;H^1(\Omega)) \text{ and } i,j,
\end{aligned}
$$

where $\boldsymbol{\varepsilon} = (\varepsilon_{ij})$. Accordingly, in order to prove (S7) it is sufficient to show that

$$
M\xi_{ij} \in \partial I(\theta, \boldsymbol{\varepsilon}; \sigma_{ij}) \quad \text{a.e. on } Q(T) \text{ for each } i \text{ and } j. \tag{6.1}
$$

To do so, for i and j it holds that

$$|[\sigma_{\lambda ij} - f^*(\theta_\lambda, \boldsymbol{\varepsilon}_\lambda)]^+ - [f_*(\theta_\lambda, \boldsymbol{\varepsilon}_\lambda) - \sigma_{\lambda ij}]^+|_{L^2(Q(T))} = \lambda|\xi_{\lambda ij}|_{L^2(Q(T))} \to 0 \quad \text{as } \lambda \downarrow 0,$$

and $\sigma_{kij} \to \sigma_{ij}$ in $C([0,T]; L^2(\Omega))$ as $k \to \infty$. Also, the above convergences imply $f^*(\theta_k, \boldsymbol{\varepsilon}_k) \to f^*(\theta, \boldsymbol{\varepsilon})$ and $f_*(\theta_k, \boldsymbol{\varepsilon}_k) \to f_*(\theta, \boldsymbol{\varepsilon})$ in $L^2(Q(T))$ as $k \to \infty$, since $\boldsymbol{u}_k \to \boldsymbol{u}$ in $L^2(0,T; X^3)$. Hence, we have

$$|[\sigma_{ij} - f^*(\theta, \boldsymbol{\varepsilon})]^+ - [f_*(\theta, \boldsymbol{\varepsilon}) - \sigma_{ij}]^+|_{L^2(Q(T))} = 0$$

so that $f_*(\theta, \boldsymbol{\varepsilon}) \leq \sigma_{ij} \leq f^*(\theta, \boldsymbol{\varepsilon})$ a.e. on $Q(T)$. As the next step let $z \in L^2(Q(T))$ with $f_*(\theta, \boldsymbol{\varepsilon}) \leq z \leq f^*(\theta, \boldsymbol{\varepsilon})$ a.e. on $Q(T)$ and put

$$z_k = \max\{\min\{f^*(\theta_k, \boldsymbol{\varepsilon}_k), z\}, f_*(\theta_k, \boldsymbol{\varepsilon}_k)\}.$$

It is easy to see that

$$f_*(\theta_k, \boldsymbol{\varepsilon}_k) \leq z_k \leq f^*(\theta_k, \boldsymbol{\varepsilon}_k) \text{ a.e. on } Q(T) \text{ and } z_k \to z \text{ in } L^2(Q(T)) \text{ as } k \to \infty.$$

Consequently, we observe that

$$\int_{Q(T)} \xi_{kij}(z_k - \sigma_{kij}) dxdt \to \int_{Q(T)} \xi_{ij}(z - \sigma_{ij}) dxdt \text{ as } k \to \infty,$$
$$\int_{Q(T)} \xi_{kij}(z_k - \sigma_{kij}) dxdt \leq 0 \text{ for } k.$$

Hence, $\int_{Q(T)} M\xi_{ij}(z - \sigma_{ij}) dxdt \leq 0$. This means (6.1), that is, (S7) holds.

In order to prove (S6) it is sufficient to show that

$$\boldsymbol{\sigma}_k : \boldsymbol{\varepsilon}_{kt} + \mu \nabla \boldsymbol{u}_{kt} : \boldsymbol{\varepsilon}_{kt} \to \boldsymbol{\sigma} : \boldsymbol{\varepsilon}_t + \mu \nabla \boldsymbol{u}_t : \boldsymbol{\varepsilon}_t \text{weakly in } L^2(Q(T)). \tag{6.2}$$

Now, we know that the set $\{\boldsymbol{u}_{kt}\}$ is bounded in $L^\infty(0,T; H^2(\Omega)^3)$, $L^2(0,T; H^3(\Omega)^3)$ and $W^{1,\infty}(0,T; L^2(\Omega)^3)$. By applying Aubin's compact theorem (see [**22**, Theorem 5.1]) we know that $\boldsymbol{u}_{kt} \to \boldsymbol{u}_t$ in $L^2(0,T; H^2(\Omega)^3)$ as $k \to \infty$. It yields that $\nabla \boldsymbol{u}_{kt} \to \nabla \boldsymbol{u}_t$ and $\boldsymbol{\varepsilon}_{kt} \to \boldsymbol{\varepsilon}_t$ in $L^2(0,T; L^2(\Omega)^9)$ as $k \to \infty$. Hence, we have

$$\int_{Q(T)} (\boldsymbol{\sigma}_k : \boldsymbol{\varepsilon}_{kt} + \mu \nabla \boldsymbol{u}_{kt} : \boldsymbol{\varepsilon}_{kt}) \eta dxdt \to \int_{Q(T)} (\boldsymbol{\sigma} : \boldsymbol{\varepsilon}_t + \mu \nabla \boldsymbol{u}_t : \boldsymbol{\varepsilon}_t) \eta dxdt$$
$$\text{for } \eta \in C_0^\infty(Q(T)).$$

Obviously, (6.2) holds, because $\{\nabla u_{kt}\}$ and $\{\boldsymbol{\varepsilon}_{kt}\}$ are bounded in $L^\infty(0,T; L^6(\Omega))$. From the above argument we conclude that $\{\boldsymbol{u}, \theta, \boldsymbol{\sigma}\}$ is the solution of (SMAP) on $[0,T]$. $\square$

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