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A remark on the stability of saturated generic graphs

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Abstract. We show that any saturated generic graph satisfying some property (*) is strictly stable or ω-stable. As a corollary, we obtain that any saturated generic pseudoplane is strictly stable or ω-stable.

In 1988, Hrushovski [**6**] constructed a stable ω-categorical pseudoplane to refute Lachlan's conjecture: A *Hrushovski class* $\mathbf{K}_\alpha$ is derived from a dimension function δ_α. He constructed a $\mathbf{K}$-*generic* graph $G = (G, R)$ for some subclass $\mathbf{K}$ of $\mathbf{K}_\alpha$ such that G is ω-categorical and stable, and moreover satisfies the following property:

(†) For each $a \in G$ there are infinitely many $b \in G$ with $R(a, b)$, and for each distinct $a, b \in G$ there are finitely many $c \in G$ with $R(a, c)$ and $R(b, c)$.

Then the structure (G, G, R) is naturally regarded as a pseudoplane. (A *pseudoplane* is a structure with two sorts, points and lines, thogether with an incidence relation satisfying (i) on any line there are infinitely many points, and through any point there are infinitely many lines; (ii) any two lines intersect in finitely many points, and through any two points there are finitely many lines.) Applying Hrushovski's method, Baldwin [**2**] obtained an almost strongly minimal non-Desarguesian projective plane from a saturated $\mathbf{K}$-generic graph H for some subclass $\mathbf{K}$ of $\mathbf{K}_{\frac{1}{2}}$. In particular, this H satisfies the above property (†) with "finitely many" replaced by "exactly one". Hrushovski's example is strictly stable, and Baldwin's is ω-stable. In [**1**], Baldwin asked whether there is a 'generic' structure that is superstable but not ω-stable. In this paper, we study his question for the *ab initio* case, however we do not know whether the question holds or not, even if the structure is assumed to be saturated. For each $\mathbf{K} \subset \mathbf{K}_\alpha$ we consider the following property:

(*) Any finite graph with no cycles belongs to $\mathbf{K}$.

It is seen that the above two examples satisfy (*), and moreover that if a $\mathbf{K}$-generic graph satisfies (†) then $\mathbf{K}$ satisfies (*) (see 3.2). Our objective is to show that for *any* subclass $\mathbf{K}$ of $\mathbf{K}_\alpha$ with (*) a saturated $\mathbf{K}$-generic graph is strictly stable or ω-stable. This give a partial solution for Baldwin's question. As a corollary, we obtain that a saturated generic pseudoplane is strictly stable or ω-stable.

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1. Generic graphs.

Many papers [**3**], [**4**], [**5**], [**9**] have laid out the basics of generic structures. So we do not explain all of those detail here.

Let $R(*,*)$ be a binary relation of an undirected graph: It satisfies $\models \forall x(\neg R(x,x))$ and $\models \forall x \forall y(R(x,y) \to R(y,x))$. Let α be a positive real number. For a finite graph A, we write a dimension function $\delta_\alpha(A) = |A| - \alpha|R^A|$, where $R^A = \{\{a,b\} : A \models R(a,b)\}$. We denote $K_\alpha = \{A : A \text{ is a finite graph, } \delta_\alpha(B) \geq 0 \text{ for any } B \subset A\}$.

For finite $A \subset G$, A is said to be *closed in* G (in symbol, $A \leq G$), if $\delta_\alpha(XA) \geq \delta_\alpha(A)$ for any finite $X \subset G - A$. The *closure* of A in G is defined by $\mathrm{cl}_G(A) = \bigcap\{B : A \subset B \leq G, |B| < \omega\}$. Let $\mathbf{K} = (K, \leq)$ be a subclass of $\mathbf{K}_\alpha = (K_\alpha, \leq)$ that is closed under substructures.

Definition 1.1. A countable graph G is said to be $\mathbf{K}$-*generic*, if it satisfies the following: (i) for finite $A \subset G$, $A \in \mathbf{K}$; (ii) if $A \leq B \in \mathbf{K}$ and $A \leq G$, then there exists a copy B' of B over A with $B' \leq G$.

$\mathbf{K}$ is said to have *finite closures*, if there are no chains $A_0 \subset A_1 \subset \cdots$ of elements of $\mathbf{K}$ with $\delta_\alpha(A_{i+1}) < \delta_\alpha(A_i)$ for each $i < \omega$. If $\mathbf{K}$ has finite closures, then we can see that there exists a unique $\mathbf{K}$-generic graph G, and moreover that any finite set of G has finite closures. Note that if α is rational then $\mathbf{K}$ always has finite closures. We summarize our situation.

Assumption. $\mathbf{K} = (K, \leq)$ is derived from a dimension function δ_α for a positive real number α such that $\mathbf{K}$ is closed under substructures and has finite closures.

Our assumption is sufficiently natural: Indeed, for suitable irrational α, Hrushovski [**6**] defined a subclass $\mathbf{K}$ of $\mathbf{K}_\alpha$ satisfying our assumption, and constructed a $\mathbf{K}$-generic graph, which is stable and ω-categorical. Applying Hrushovski's construction, Baldwin [**2**] construct an almost strongly minimal $\mathbf{K}$-generic projective plane for some subclass $\mathbf{K}$ of $\mathbf{K}_{\frac{1}{2}}$ under our assumption, which gave a counter example of Zilber's conjecture.

For finite $A \subset G$, we define $d_G(A) = \delta(\mathrm{cl}_G(A))$. For finite A, B, we write $d_G(A/B) = d_G(AB) - d_G(B)$. For possibly infinite $B \subset G$, we define $d_G(A/B) = \inf\{d_G(A/B') : B' \subset B, B' \text{ is finite}\}$. The following result can be found, for instance, in [**8**].

Fact 1.2. Suppose that G is a saturated $\mathbf{K}$-generic graph. Then

(i) G is stable, and G is ω-stable if α is rational.
(ii) For any $A \leq B \leq G$ and $\bar{a} \in G$, $\mathrm{tp}(\bar{a}/B)$ does not fork over A if and only if $d_G(\bar{a}/B) = d_G(\bar{a}/A)$ and $\mathrm{cl}_G(\bar{a}A) \cap B = A$.

2. Construction of graphs.

The following remark may be elementary, but is necessary to construct an infinite graph such as in Lemma 2.3.

Remark 2.1. Let $\alpha > 0$ be irrational. It is known that for any $\epsilon > 0$ there are $a, b \in \boldsymbol{Z}$ such that $|a - b\alpha| < \epsilon$. From this it follows that $\{a - b\alpha : a, b \in \boldsymbol{N}\}$ is dense in $\boldsymbol{R}$.

To simplify our notation, we write $\delta(*)$ in place of $\delta_\alpha(*)$. For finite A, B, we write $\delta(A/B) = \delta(AB) - \delta(B)$.

For a finite graph AB with $A \cap B = \varnothing$, we say that a pair (B, A) is *biminimal*, if it satisfies the following: (i) $\delta(B/A) < 0$; (ii) $\delta(X/A) \geq 0$ for any nonempty proper subset X of B; (iii) $\delta(B/Y) \geq 0$ for any nonempty proper subset X of A.

A graph A is said to have *no cycles*, if for each $n > 2$ there do not exist distinct $a_1, a_2, \dots, a_n \in A$ with $R(a_1, a_2)$, $R(a_2, a_3), \dots, R(a_{n-1}, a_n)$ and $R(a_n, a_1)$. A graph A is said to be *connected*, if for any distinct $a, b \in A$ there exist $b_1, b_2, \dots, b_n (= b) \in A$ with $R(a, b_1), R(b_1, b_2), \dots, R(b_{n-1}, b_n)$.

LEMMA 2.2. *If α is irrational with $0 < \alpha < 1$, then for any ϵ with $0 < \epsilon < \alpha$ there is a finite graph eBC such that*

(1) *(C, eB) is biminimal*;
(2) *$\delta(C/eB) > -\epsilon$*;
(3) *eBC has no cycles*;
(4) *eB has no relations.*

PROOF. Take any $\epsilon > 0$. By Remark 2.1, there exists $p = \min\{a \in \boldsymbol{N} : 0 > a - b\alpha > -\epsilon\}$. Let $q_p \in \boldsymbol{N}$ be such that $0 > p - q_p\alpha > -\epsilon$. For each $n \in \boldsymbol{N}$ with $1 \leq n < p$, define $q_n = \max\{k \in \boldsymbol{N} : n - k\alpha > 0\}$. Then we denote $d_n = n - q_n\alpha$ for each n with $1 \leq n \leq p$. We can see that $0 < d_n < \alpha - \epsilon$ for any $n < p$. Hence $\{d_n\}_{1 \leq n \leq p}$ satisfies the following:

(a) $0 > d_p > -\epsilon$;
(b) If $1 \leq n < p$ then $d_n > 0$;
(c) If $1 \leq n < m \leq p$ then $d_n - d_m < \alpha$.

For the convenience, let $q_0 = -1$. Note that $q_i - q_{i-1} - 1 \geq 0$ for each i with $1 \leq i \leq p$. (In fact, since $0 < \alpha < 1$, we have $q_i - q_{i-1} > (\frac{i}{\alpha} - 1) - \frac{i-1}{\alpha} = \frac{1-\alpha}{\alpha} > 0$.) Then let $\{c_i : 1 \leq i \leq p\} \cup \{b_i^j : 1 \leq i \leq p, 1 \leq j \leq q_i - q_{i-1} - 1\}$ be a graph with the relations:

(i) $R(c_1, c_2), \dots, R(c_{p-1}, c_p)$;
(ii) $R(c_i, b_i^j)$ for each i, j with $1 \leq i \leq p$ and $1 \leq j \leq q_i - q_{i-1} - 1$.

Let $e = b_1^1$, $C = \{c_i : 1 \leq i \leq p\}$ and $B = \{b_i^j : 1 \leq i \leq p, 1 \leq j \leq q_i - q_{i-1} - 1\} - \{b_1^1\}$. Clearly eBC satisfies (3) and (4). By the definition of eBC, we have

$$\delta(C/eB) = p - \left\{(p-1) + \sum_{i=1}^{p}(q_i - q_{i-1} - 1)\right\}\alpha = p - q_p\alpha = d_p.$$

By (a), we have $0 > \delta(C/eB) > -\epsilon$, so (2) holds.

CLAIM. If $X (\subset C)$ is connected with $X \neq C$, then $\delta(X/eB) > 0$.

PROOF. By connectedness, we can denote $X = \{c_i\}_{n < i \leq m}$ for some n, m. If $n = 0$, then $\delta(X/eB) = m - q_m\alpha = d_m > 0$ by (b). If $n > 0$, then $\delta(X/eB) = (m - n) - (q_m - q_n - 1)\alpha = d_m - d_n + \alpha > 0$ by (c). (End of Proof of Claim)

We show (1). First take any $X \subset C$ with $X \neq C$. Let $X = \bigcup X_i$ where each X_i is connected component of X. Then $\delta(X/eB) = \sum \delta(X_i/eB) > 0$ by the claim. Next take any $Y \subset eB$ with $Y \neq eB$. Then $\delta(C/Y) \geq \delta(C/eB) + \alpha > -\epsilon + \alpha > 0$. Hence (1) holds. $\square$

LEMMA 2.3. *If α is irrational with $0 < \alpha < 1$, then there is a set $\{eB_iC_i\}_{i<\omega}$ of finite graphs such that*

(1) *D has no cycles*;
(2) *$B_n^* \leq eB_n^*C_n^* \leq D$ for each $n < \omega$*;
(3) *(C_n, eB_n) is biminimal for each $n < \omega$*;
(4) *eB^* has no relations*;
(5) *For each $i, j < \omega$ there are no relations between B_iC_i and B_jC_j,*

where $B_n^* = \bigcup_{i \leq n} B_i, C_n^* = \bigcup_{i \leq n} C_i, B^* = \bigcup_{i<\omega} B_i, C^* = \bigcup_{i<\omega} C_i$ and $D = eB^*C^*$.

PROOF. Take $i_0 < \omega$ with $\frac{1}{2^{i_0}} < \alpha$. By Lemma 2.2, for each $i < \omega$, there is eC_iB_i that satisfies (1)–(4) of Lemma 2.2 and moreover that satisfies $\delta(C_i/eB_i) > -\frac{1}{2^{i+i_0}}$. Clearly (3) holds. Denote $B_n^* = \bigcup_{i \leq n} B_i, C_n^* = \bigcup_{i \leq n} C_i, B^* = \bigcup_{i<\omega} B_i, C^* = \bigcup_{i<\omega} C_i$ and $D = eB^*C^*$. Then we can assume that (1), (4) and (5) hold. So it is enough to see (2). Let X_E denote $X \cap E$ for each X and E.

CLAIM 1. $eB_n^*C_n^* \leq D$.

PROOF. Take any finite $X \subset D - eB_n^*C_n^*$. We divide into two cases. First suppose $X_{B^*} \neq \varnothing$. Then $\delta(X/eB_n^*C_n^*) = \delta(X/e) = \delta(X_{C^*}/eX_{B^*}) + \delta(X_{B^*}/e) = \delta(X_{C^*}/eX_{B^*}) + |X_{B^*}| \geq \delta(X_{C^*}/eX_{B^*}) + 1 = \sum_i \delta(X_{C_i}/eX_{B_i}) + 1 \geq \sum_i \delta(C_i/eB_i) + 1 \geq -\sum_i \frac{1}{2^{i+i_0}} + 1 > 0$. Next suppose $X_{B^*} = \varnothing$. Then $\delta(X/eB_n^*C_n^*) = \delta(X_{C^*}/e) \geq |X_{C^*}| - |X_{C^*}|\alpha = |X_{C^*}|(1-\alpha) > 0$. In any case, claim 1 holds. $\square$

CLAIM 2. $B_n^* \leq eB_n^*C_n^*$.

PROOF. Take any $X \subset eC_n^*$. We divide into two cases. First suppose $e \in X$. Then $\delta(X/B_n^*) = \delta(X/B_n^*e) + \delta(e/B_n^*) = \sum_i \delta(X_{C_i}/B_ie) + 1 \geq \sum_i \delta(C_i/B_ie) + 1 = -\sum_i \frac{1}{2^{i+i_0}} + 1 \geq 0$. Next suppose $e \notin X$. Note that $\delta(Y/B_i) > 0$ for any $Y \subset C_i$. (Proof: If $Y = C_i$ then $\delta(Y/B_i) = \delta(C_i/B_i) > 0$. If $Y \neq C_i$ then $\delta(Y/B_i) \geq \delta(Y/B_ie) > 0$.) So $\delta(X/B_n^*) = \sum_i \delta(X_{C_i}/B_i) > 0$. In any case, claim 2 holds. $\square$

3. Proof of Theorem.

THEOREM 3.1. *For a positive real number α let $\mathbf{K} = (K, \leq)$ be a subclass of $\mathbf{K}_\alpha$ that is closed under substructures, and satisfy the following property*:

(*) *Any finite graph with no cycles belongs to $\mathbf{K}$.*

Then a saturated $\mathbf{K}$-generic graph G is ω-stable if α is rational and G is strictly stable if α is irrational.

PROOF. By Fact 1.2(i), G is stable, and G is ω-stable if α is rational. So it is enough to show that, if α is irrational then G is not superstable.

Claim 1. $\alpha < 1$.

Proof. Otherwise, we have $\alpha > 1$. Take some $m < \omega$ with $m(1-\alpha)+1 < 0$. Let $E = ab_1 \dots b_m$ be a graph with the relations $R(a, b_i)$ for each $i \leq m$. Since E has no cycles, by (*) we have $E \in \mathbf{K}$. On the other hand, $\delta(E) = (m+1) - m\alpha < 0$. This contradicts $\mathbf{K} \subset \mathbf{K}_\alpha$. (End of Proof of Claim 1) □

By claim 1 we can take $\{eB_iC_i\}_{i<\omega}$ as in Lemma 2.3. Let $D_j = \bigcup_{i \leq j} eB_iC_i$ for each $j < \omega$, and $D = \bigcup_{i<\omega} eB_iC_i$. Note that $D_1 \leq D_2 \leq \dots \leq D$. Since the D_j have no cycles, by (*) we have $D_j \in \mathbf{K}$. So we can assume that $D \leq G$.

Claim 2. $d_G(e/B_n^*) = \sum_{i \leq n} \delta(C_i/eB_i) + 1$ for each $n < \omega$.

Proof. By (2)–(5) of Lemma 2.3, we have $d_G(e/B_n^*) = \mathrm{d}_G(eB_n^*) - d_G(B_n^*) = \delta(eC_n^*B_n^*) - \delta(B_n^*) = \delta(eC_n^*/B_n^*) = \delta(C_n^*/eB_n^*) + 1 = \sum_{i \leq n} \delta(C_i/eB_i) + 1$. (End of Proof of Claim 2) □

By claim 2, we have $d_G(e/B_{n+1}^*) = \sum_{i \leq n+1} \delta(C_i/eB_i) + 1 = d_G(e/B_n^*) + \delta(C_{n+1}/eB_{n+1}) < d_G(e/B_n^*)$. By Fact 1.2, we obtain that $\mathrm{tp}(e/B_{n+1}^*)$ is a forking extension of $\mathrm{tp}(e/B_n^*)$ for each $n < \omega$. Hence G is not superstable.

Remark 3.2. Each of Hrushovski's and Baldwin's examples satisfies the property (*) in our theorem. In general, if a $\mathbf{K}$-generic graph G satisfies (†), then $\mathbf{K}$ satisfies (*). It can be shown as follows: Let A be a finite graph with no cycles. Take any $a_0 \in A$. Let C_0 be a connected component of a_0 in A. As A has no cycles, C_0 can be regarded as a tree so that height of a_0 is 0. Since G satisfies (†), we can inductively construct $C_0^* \subset G$ with $C_0^* \cong C_0$. Take any $a_1 \in A - C_0$. Let C_1 be a connected component of a_1. In the same way, we have $C_1^* \subset G$ with $C_0^*C_1^* \cong C_0C_1$. Iterating this process, we have $A^* \subset G$ with $A^* \cong A$. Hence $A \in \mathbf{K}$.

Remark 3.3. A $\mathbf{K}$-generic graph G is said to have *amalgamation over closed sets*, if for any models G_1, G_2 of $\mathrm{Th}(G)$, any set $A \in \mathbf{K}$, and any strong embeddings $f_1 : A \to G_1$ and $f_2 : A \to G_2$, there exist a model G_3 of $\mathrm{Th}(G)$ and elementary embeddings $g_1 : G_1 \to G_3$ and $g_2 : G_2 \to G_3$ with $g_1f_1 = g_2f_2$. It is seen that if G is saturated, then G has amalgamation over closed sets and $\mathbf{K}$ has finite closures. However, in our argument, we do not use the fact that $\mathbf{K}$ has finite closures. Note that, in [**8**], Fact 1.2 was proven under the conditon that G has amalgamation over closed sets. It follows that, in our theorem, the condition that G is saturated can be replaced by the weaker condition, amalgamation over closed sets.

When a structure $M = (G, G, R)$ is a pseudoplane (or projective plane) for some $\mathbf{K}$-generic graph G, we call M a *generic pseudoplane* (or *generic projective plane*). Hrushovski proved that there exists an ω-categorical generic pseudoplane [**6**]. On the other hand, it was shown that there are no ω-categorical generic projective planes [**7**]. As a corollary of Theorem 3.1, we obtain the following:

Corollary 3.4. *Any saturated generic pseudoplane is strictly stable or ω-stable.*

Question 3.5. Is any saturated $\mathbf{K}$-generic graph strictly stable or ω-stable?

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