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Convolution of Riemann zeta-values

Dedicated to Professor Isao Wakabayashi with great respect

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Abstract. In this note we are going to generalize Prudnikov's method of using a double integral to deduce relations between the Riemann zeta-values, so as to prove intriguing relations between double zeta-values of depth 2. Prior to this, we shall deduce the most well-known relation that expresses the sum $\sum_{j=1}^{m-2}\zeta(j+1)\zeta(m-j)$ in terms of $\zeta_2(1,m)$.

1. Introduction and statement of results.

If the Mellin transform

$$\mathscr{M}W(n)=\int_0^\infty x^{n-1}W(x)dx, \tag{1.1}$$

which is to yield (or at least, to be related to) zeta values, is computable in finite form for a suitable weight function W, then we may express the double integral

$$I(s)=\int_0^\infty\int_0^\infty (x+y)^{s-1}W(x)W(y)dxdy, \tag{1.2}$$

for positive integer values of $s=m$, as a convolution of zeta-values:

$$I(m)=\sum_{j=0}^{m-1}\binom{m-1}{j}\mathscr{M}W(j+1)\mathscr{M}W(m-j). \tag{1.3}$$

If, further, it so happens that, when we express (1.2) as a repeated integral

$$I(s)=\int_0^\infty z^{s-1}B(z)dz, \tag{1.4}$$

where $B(z)$ is the beta-type integral

$$B(z)=\int_0^z W(z-y)W(y)dy, \tag{1.5}$$

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we may express $B(z)$ in tractable form, then we may expect to obtain a relation among zeta-values.

We consider the case where $W(x)$ can be expressed as a series (finite or infinite). Then the implementation of the above method is a well-known one in number theory, i.e. expanding the product $W(x)W(y)$ into a series, extracting the diagonal terms (which is of the form (1.1)), being thereby left with non-diagonal terms to be summed accordingly.

A. P. Prudnikov [**6**] (cf. also Srivastava-Choi [**7**]) was the first who put the above idea into practice, using the (elliptic) theta series as $W(x)$.

In this note we shall use the weight functions

$$w_N(x) = \sum_{k=1}^{N} e^{-kx}$$

and

$$W_n(x) = x^n \lim_{N\to\infty} w_N(x) = x^n \sum_{k=1}^{\infty} e^{-kx} \left(= \frac{x^n}{e^x - 1} \right), \quad (x > 0)$$

to deduce the well-known fundamental relation among Riemann zeta-values anew (Theorem 1) and an intriguing relation between Riemann zeta-values and the multiple zeta-values of depth 2 (Theorem 2), respectively.

We use the following notation:

For a positive integer N let $\zeta_N(s)$ denote the N-th partial sum $\sum_{n=1}^{N} n^{-s}$ of the Riemann zeta-function $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$ $(\sigma = \Re s > 1)$, let H_N denote the N-th harmonic number $\sum_{n=1}^{N} \frac{1}{n}$, and let $\zeta_2(s_1, s_2)$ denote the double zeta-function defined for $\Re s_2 > 1, \Re(s_1 + s_2) > 2$ by

$$\zeta_2(s_1, s_2) = \sum_{m<n} \frac{1}{m^{s_1} n^{s_2}}, \tag{1.6}$$

so that $\zeta_2(1, s) = \sum_{n=1}^{\infty} H_{n-1} n^{-s}$. For double and multiple zeta-functions, cf. [**2**], [**5**], [**10**], [**11**].

We are now in a position to state the results.

THEOREM 1. *For each positive integer $m \geq 2$ and $N \to \infty$, we have the intermediate formula*

$$m\zeta_N(m+1) - 2 \sum_{h \leq N} \frac{H_{h-1}}{h^m} + o(1) = \sum_{j=1}^{m-2} \zeta_N(j+1)\zeta_N(m-j)$$

and in the limit

$$2\zeta_2(1, m) = m\zeta(m+1) - \sum_{j=1}^{m-2} \zeta(j+1)\zeta(m-j). \tag{1.7}$$

REMARK 1. The identity (1.7) is equivalent to Euler's sum formula

$$\zeta(k) = \sum_{j=2}^{k-1} \zeta_2(k-j, j) \quad (k \geq 2),$$

which is derived easily from the relation

$$\zeta(p)\zeta(q) = \zeta(p+q) + \zeta_2(p,q) + \zeta_2(q,p).$$

THEOREM 2. (1) *For integers* $1 \leq n_1 < n_2$, *we have*

$$\begin{aligned}
&\sum_{j=0}^{n_2-n_1} \binom{n_1+j}{n_1}\binom{n_2-j}{n_1} \zeta(n_1+1+j)\zeta(n_2+1-j) \\
&\quad - 2\sum_{j=0}^{n_1} (-1)^j \binom{n_1+j}{n_1}\binom{n_2-j}{n_2-n_1} \zeta(n_1+1+j)\zeta(n_2+1-j) \\
&\quad = \binom{n_1+n_2+1}{n_2-n_1} \zeta(n_1+n_2+2) \\
&\qquad - 2(-1)^{n_1} \sum_{j=0}^{n_1} \binom{n_1+j}{n_1}\binom{n_2-j}{n_2-n_1} \zeta_2(n_1+1+j, n_2+1-j).
\end{aligned}$$

(2) *For integers* $n \geq 1$, *we have*

$$\begin{aligned}
&\sum_{j=0}^{n-1} (-1)^j \binom{n+j}{n} \zeta(n+1+j)\zeta(n+1-j) \\
&\quad = (-1)^{n-1}\binom{2n}{n}\bigl\{\zeta(2n+2) + \zeta_2(1, 2n+1)\bigr\} + \zeta_2(n+1, n+1) \\
&\qquad + (-1)^n \sum_{j=0}^{n-1} \binom{n+j}{n} \zeta_2(n+1+j, n+1-j).
\end{aligned}$$

2. Proofs.

In this section we shall carry out the proof of our results in the lines indicated in §1.

PROOF OF THEOREM 1. We choose

$$w_N(x) = \sum_{k=1}^{N} e^{-kx} \quad (x > 0) \tag{2.1}$$

as a weight function. Then its Mellin transform (1.1) is

$$\mathscr{M} w_N(s) = \Gamma(s)\zeta_N(s), \tag{2.2}$$

so that the convolution formula (1.3) becomes

$$\begin{aligned} I(m) &= I_N(m) \\ &= 2\Gamma(m)H_N\zeta_N(m) + \sum_{j=1}^{m-2}\binom{m-1}{j}\Gamma(j+1)\Gamma(m-j)\zeta_N(j+1)\zeta_N(m-j). \end{aligned} \tag{2.3}$$

On the other hand, since

$$w_N(z-y)w_N(y) = w_N(z) + \sum_{\substack{h,k\le N \\ h\ne k}}\sum e^{-kz}e^{(k-h)y},$$

it follows that

$$B(z) = \int_0^z w_N(z-y)w_N(y)dy = zw_N(z) + 2\sum_{\substack{h,k\le N \\ h\ne k}}\sum \frac{e^{-hz}}{k-h}. \tag{2.4}$$

Hence its Mellin transform $I_N(s)$ is given by

$$\begin{aligned} I_N(s) &= \mathscr{M} w_N(s+1) + 2\sum_{\substack{h,k\le N \\ h\ne k}}\sum \frac{1}{k-h}\int_0^\infty z^{s-1}e^{-hz}dz \\ &= \Gamma(s+1)\zeta_N(s+1) + 2\Gamma(s)J_N(s), \end{aligned} \tag{2.5}$$

say, where

$$J_N(s) = \sum_{\substack{h,k\le N \\ h\ne k}}\sum \frac{1}{h^s(k-h)}. \tag{2.6}$$

Expressing the sum over k in (2.6) concretely, we obtain

$$\begin{aligned} J_N(s) &= \sum_{h=1}^N \frac{1}{h^s}\bigl(-H_{h-1} + H_{N-h}\bigr) \\ &= \sum_{h=1}^N \frac{1}{h^s}\Bigl\{-H_{h-1} + H_N - \Bigl(\frac{1}{N-h+1} + \cdots + \frac{1}{N}\Bigr)\Bigr\}, \end{aligned}$$

or

$$J_N(s) = H_N \zeta_N(s) - \sum_{h=1}^{N} \frac{H_{h-1}}{h^s} - J_N^{(1)}(s), \tag{2.7}$$

where

$$J_N^{(1)}(s) = \sum_{h=1}^{N} \frac{1}{h^s} \left(\frac{1}{N-h+1} + \cdots + \frac{1}{N} \right).$$

Since

$$\left| J_N^{(1)}(s) \right| \le \sum_{h=1}^{N} \frac{1}{h^\sigma} \frac{h}{N-h+1},$$

we derive, on dividing the sum into two at $h = \left[\frac{N}{2}\right]$, that

$$\left| J_N^{(1)}(s) \right| \ll N^{1-\sigma} \log N = o(1), \quad N \to \infty$$

for $\sigma > 1$, and therefore that

$$J_N(s) = H_N \zeta_N(s) - \sum_{h=1}^{N} \frac{H_{h-1}}{h^s} + o(1), \tag{2.8}$$

as $N \to \infty$ for $\sigma > 1$.

Substituting (2.8) into (2.5) and combining it with (2.3), thereby $s = m \ge 2$ an integer, we complete the proof of the intermediate formula. Letting then $N \to \infty$, (1.7) follows, and the proof is complete. □

Proof of Theorem 2. First we consider the case $\sigma > 1$. Corresponding to (2.1), we choose

$$W(x) = W_n(x) = x^n w(x), \tag{2.9}$$

where n is a positive integer and

$$w(x) = \sum_{k=1}^{\infty} e^{-kx} = \frac{1}{e^x - 1}. \tag{2.10}$$

Then its Mellin transform is

$$\mathscr{M} W_n(s) = \Gamma(n+s) \zeta(n+s) \tag{2.11}$$

corresponding to (2.2), and the convolution formula takes the form

$$\begin{aligned} I(m) &= \int_0^\infty \int_0^\infty (x+y)^{m-1} W_n(x) W_n(y) dx dy \\ &= \sum_{j=0}^{m-1} \binom{m-1}{j} \Gamma(n+j+1)\Gamma(m+n-j)\zeta(n+j+1)\zeta(m+n-j), \end{aligned} \tag{2.12}$$

corresponding to (2.3).

On the other hand, the same reasoning that led to (2.4) gives

$$B(z) = \int_0^z W_n(z-y) W_n(y) dy = B_1(z) + B_2(z), \tag{2.13}$$

where

$$B_1(z) = w(z) \int_0^z (z-y)^n y^n dy,$$

and

$$B_2(z) = \sum\sum_{h \neq k} e^{-kz} \int_0^z (z-y)^n y^n e^{(k-h)y} dy. \tag{2.14}$$

The integral in $B_1(z)$ is

$$z^{2n+1} B(n+1, n+1) = \frac{\Gamma(n+1)^2}{\Gamma(2n+2)} z^{2n+1},$$

so that

$$B_1(z) = \frac{\Gamma(n+1)^2}{\Gamma(2n+2)} z^{2n+1} w(z). \tag{2.15}$$

The integral in $B_2(z)$ can be treated by the well-known formulas involving the modified Bessel function I_μ (cf. e.g. Erdélyi [**3**]):

$$\int_0^u (x(u-x))^{\mu-1} e^{\beta x} dx = \sqrt{\pi} \left(\frac{u}{\beta}\right)^{\mu-1/2} \exp\left(\frac{\beta u}{2}\right) \Gamma(\mu) I_{\mu-1/2}\left(\frac{\beta u}{2}\right) \tag{2.16}$$

($\beta > 0$), and

$$I_{n+1/2}(x) = \frac{e^{-x}}{\sqrt{2\pi x}} \sum_{j=0}^{n} \frac{(n+j)!}{j!\,(n-j)!} \left(\frac{1}{2x}\right)^j \left((-1)^j e^{2x} + (-1)^{n+1}\right) \tag{2.17}$$

$(x > 0, n \in \boldsymbol{N})$.

Combining these, we deduce that

$$\int_0^z (z-y)^n y^n e^{\beta y} dy = \Gamma(n+1) \frac{z^n}{\beta^{n+1}} \sum_{j=0}^{n} \frac{(n+j)!}{j!\,(n-j)!} \left(\frac{1}{\beta z}\right)^j \bigl((-1)^j e^{\beta z} + (-1)^{n+1}\bigr),$$

whence further that

$$\int_0^z (z-y)^n y^n e^{\beta y} dy = \Gamma(n+1)(-1)^{n+1} \sum_{j=0}^{n} \frac{(n+j)!}{j!\,(n-j)!} \frac{z^{n-j}}{\beta^{n+j+1}} \bigl((-1)^{n+j+1} e^{\beta z} + 1\bigr). \tag{2.18}$$

Since (2.18) remains valid with β replaced by $-\beta$, we may substitute (2.18) with $\beta = k - h$ in $B_2(z)$ (in (2.14)), whereby we distinguish two cases: $n + j + 1$ is odd or even:

$$B_2(z) = \Gamma(n+1)(-1)^{n+1} \sum_{j=0}^{n} \frac{(n+j)!}{j!\,(n-j)!} z^{n-j} \Bigl(\sum\nolimits_{\mathrm{o}} + \sum\nolimits_{\mathrm{e}}\Bigr), \tag{2.19}$$

where

$$\sum\nolimits_{\mathrm{o}} = \sum_{\substack{h \neq k \\ n+j+1:\mathrm{odd}}}\sum e^{-kz} \frac{-e^{(k-h)z} + 1}{(k-h)^{n+j+1}} \tag{2.20}$$

and

$$\sum\nolimits_{\mathrm{e}} = \sum_{\substack{h \neq k \\ n+j+1:\mathrm{even}}}\sum e^{-kz} \frac{e^{(k-h)z} + 1}{(k-h)^{n+j+1}}. \tag{2.21}$$

We transform $\sum_{\mathrm{o}}$ as follows,

$$\begin{aligned}
\sum\nolimits_{\mathrm{o}} &= \sum_{h \neq k}\sum \frac{-e^{-hz} + e^{-kz}}{(k-h)^{n+j+1}} \\
&= -2 \sum_{h \neq k}\sum \frac{e^{-hz}}{(k-h)^{n+j+1}} \\
&= -2 \sum_{h} e^{-hz} \sum_{\substack{k \\ k \neq h}} \frac{1}{(k-h)^{n+j+1}} \\
&= -2 \sum_{h} e^{-hz} \sum_{l \geq h} \frac{1}{l^{n+j+1}},
\end{aligned}$$

which we record as

$$\sum\nolimits_{\mathrm{o}} = 2\sum_h e^{-hz}\biggl(\sum_{l<h}\frac{1}{l^{n+j+1}} - \zeta(n+j+1)\biggr). \tag{2.22}$$

The sum $\sum_{\mathrm{e}}$ is more readily transformed in the form of (2.22):

$$\sum\nolimits_{\mathrm{e}} = 2\sum_h e^{-hz}\biggl(\sum_{l<h}\frac{1}{l^{n+j+1}} + \zeta(n+j+1)\biggr). \tag{2.23}$$

Now (2.22) and (2.23) give

$$\sum\nolimits_{\mathrm{o}} + \sum\nolimits_{\mathrm{e}} = 2\sum_h e^{-hz}\biggl(\sum_{l<h}\frac{1}{l^{n+j+1}} + (-1)^{n+j+1}\zeta(n+j+1)\biggr).$$

Thus we conclude that

$$\begin{aligned} B_2(z) &= 2(-1)^{n+1}\Gamma(n+1)\sum_{j=0}^{n}\frac{(n+j)!}{j!\,(n-j)!}z^{n-j} \\ &\quad\times\sum_h e^{-hz}\biggl(\sum_{l<h}\frac{1}{l^{n+j+1}} + (-1)^{n+j+1}\zeta(n+j+1)\biggr). \end{aligned} \tag{2.24}$$

We substitute (2.15) and (2.24) in (2.13), obtaining the counterpart of (2.3), which we substitute in (1.4), to conclude that

$$\begin{aligned} I(s) = {}& \frac{\Gamma(n+1)^2}{\Gamma(2n+2)}\mathscr{M}W_{2n+1}(s) + 2(-1)^{n+1}\Gamma(n+1)\sum_{j=0}^{n}\frac{(n+j)!}{j!\,(n-j)!} \\ &\times\sum_h\biggl(\sum_{l<h}\frac{1}{l^{n+j+1}} + (-1)^{n+j+1}\zeta(n+j+1)\biggr)\int_0^\infty e^{-hz}z^{s+n-j-1}dz. \end{aligned}$$

Now using (2.11) for the first term and noting that the integral in the second term is $\Gamma(s+n-j)/h^{s+n-j}$ in the above formula, we finally deduce that

$$\begin{aligned} I(s) = {}& \frac{\Gamma(n+1)^2}{\Gamma(2n+2)}\Gamma(s+2n+1)\zeta(s+2n+1) \\ &+ 2(-1)^{n+1}\Gamma(n+1)\sum_{j=0}^{n}\frac{(n+j)!}{j!\,(n-j)!}\Gamma(s+n-j) \\ &\times\bigl\{\zeta_2(n+j+1, s+n-j) + (-1)^{n+j+1}\zeta(s+n-j)\zeta(n+j+1)\bigr\}. \end{aligned} \tag{2.25}$$

Putting $s = m \geq 2$, we have, from (2.12) and (2.15),

$$\sum_{j=0}^{m-1} \binom{m-1}{j} \Gamma(n+1+j)\Gamma(n+m-j)\zeta(n+1+j)\zeta(n+m-j)$$
$$= \frac{\Gamma(n+1)^2}{\Gamma(2n+2)}\Gamma(2n+m+1)\zeta(2n+m+1)$$
$$+ 2(-1)^{n+1}\Gamma(n+1)\sum_{r=0}^{n} \frac{(n+r)!}{r!\,(n-r)!}\Gamma(n+m-r)$$
$$\times \big\{\zeta_2(n+r+1, n-r+m) + (-1)^{n+r+1}\zeta(n+r+1)\zeta(n-r+m)\big\}.$$

Now rewriting $n_1 = n$ and $n_2 = n+m-1$ ($> n_1$), and dividing the both sides by $(n_2-n_1)!\,(n_1!)^2$, we get the formula (1) of Theorem 2.

To obtain the formula valid for $s = 1$, we rewrite the sum in $B_2(z)$ over h, k corresponding to $j = n$ and get

$$B_2(z) = 2(-1)^{n+1}\Gamma(n+1)$$
$$\times \left\{ \sum_{j=0}^{n-1} \frac{(n+j)!}{j!\,(n-j)!} z^{n-j} \sum_h e^{-hz} \left(\sum_{l<h} \frac{1}{l^{n+j+1}} + (-1)^{n+j+1}\zeta(n+j+1) \right) \right.$$
$$\left. - \frac{(2n)!}{n!} \sum_h e^{-hz} \left(\frac{1}{h^{2n+1}} + \sum_{l>h} \frac{1}{l^{2n+1}} \right) \right\}, \tag{2.26}$$

from which we get

$$\Gamma(n+1)^2\zeta(n+1)^2$$
$$= \Gamma(n+1)^2\zeta(2n+2) + (-1)^{n+1}\Gamma(n+1)\sum_{j=0}^{n-1} \frac{(n+j)!}{j!(n-j)!}\Gamma(n+1-j)$$
$$\times \big\{\zeta_2(n+1+j, n+1-j) + (-1)^{n+j+1}\zeta(n+1+j)\zeta(n+1-j)\big\}$$
$$- 2(-1)^{n+1}(2n)!\big\{\zeta(2n+2) + \zeta_2(1, 2n+1)\big\}.$$

Dividing both side by $(n!)^2$, we obtain the formula (2) of Theorem 2. □

We now illustrate by some examples. From Theorem 2 (1), we have

$$(n_1, n_2) = (2,3): \quad \zeta(7) = 3\zeta_2(3,4) + 4\zeta_2(4,3) - 2\zeta_2(2,5)$$
$$(n_1, n_2) = (3,4): \quad 2\zeta_2(4,5) + 2\zeta_2(5,4) + 10\zeta_2(6,3) + 5\zeta_2(3,6) - 5\zeta_2(2,7) = 0.$$

From Theorem 2 (2), we have

$$n = 1: \quad \zeta(4) = 2\zeta_2(2,2) - 2\zeta_2(1,3)$$
$$n = 2: \quad 4\zeta(6) = 3\zeta_2(2,4) + 6\zeta_2(4,2) - 6\zeta_2(1,5)$$
$$n = 3: \quad 13\zeta(8) = -20\zeta_2(1,7) + 10\zeta_2(2,6) - 4\zeta_2(3,5) + 2\zeta_2(4,4) + 20\zeta_2(6,2).$$

REMARK 2. We may consider more general integral

$$I_{n_1,n_2}(s) = \int_0^\infty \int_0^\infty (x+y)^{s-1} x^{n_1} y^{n_2} w(x) w(y) dx dy.$$

Since $I_{n_1,n_2}(s) = I_{n_2,n_1}(s)$, there is no harm to assume that $n_1 \le n_2$. From the trivial identity

$$x^{n_1} y^{n_2+1} = x^{n_1} y^{n_2}(x+y-x),$$

we have

$$I_{n_1,n_2+1}(s) = I_{n_1,n_2}(s+1) - I_{n_1+1,n_2}(s).$$

This means that $I_{n_1,n_2}(s)$ can be written as a linear combination of $I_{m,m}(s+j)$, e.g.

$$I_{n,n+1}(s) = \frac{1}{2} I_{n,n}(s+1)$$
$$I_{n,n+2}(s) = \frac{1}{2} I_{n,n}(s+2) - I_{n+1,n+1}(s)$$
$$I_{n,n+3}(s) = \frac{1}{2} I_{n,n}(s+3) - \frac{3}{2} I_{n+1,n+1}(s+1).$$

REMARK 3. We have not exhausted out the method and hope to return to the study of other possible relations among zeta and L-values by similar methods, elsewhere.

For harmonic numbers and their generalizations we refer e.g. to P. Flajolet and B. Salvy [**4**] and reference therein.

Tornheim's double series $T(r,s,t)$ [**8**] defined by

$$T(r,s,t) = \sum_{m,n=1}^{\infty} \frac{1}{m^r n^s (m+n)^t},$$

where r, s, t are non-negative integers subject to some conditions has received considerable attention by several authors including T. M. Apsotal and T. H. Vu [**1**] H. Tsumura [**9**] et al. We hope to turn to the study of $T(r,s,t)$ subsequently. We remark that Apostal and Vu considered $T(r,s,1)$ by denoting it by $T(r,s)$, but without referring to Tornheim's paper [**8**].

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