

## Martin boundary points of a John domain and unions of convex sets

By Hiroaki AIKAWA, Kentaro HIRATA and Torbjörn LUNDH

(Received Sep. 15, 2004)  
(Revised Mar. 9, 2005)

**Abstract.** We show that a John domain has finitely many minimal Martin boundary points at each Euclidean boundary point. The number of minimal Martin boundary points is estimated in terms of the John constant. In particular, if the John constant is bigger than  $\sqrt{3}/2$ , then there are at most two minimal Martin boundary points at each Euclidean boundary point. For a class of John domains represented as the union of convex sets we give a sufficient condition for the Martin boundary and the Euclidean boundary to coincide.

### 1. Introduction.

Let  $D$  be a bounded domain in  $\mathbf{R}^n$  with  $n \geq 2$ . Let  $\delta_D(x) = \text{dist}(x, \partial D)$  and  $x_0 \in D$ . We say that  $D$  is a John domain with John constant  $c_J > 0$  and John center at  $x_0$  if each  $x \in D$  can be joined to  $x_0$  by a rectifiable curve  $\gamma$  such that

$$\delta_D(y) \geq c_J \ell(\gamma(x, y)) \quad \text{for all } y \in \gamma, \quad (1.1)$$

where  $\gamma(x, y)$  is the subarc of  $\gamma$  from  $x$  to  $y$  and  $\ell(\gamma(x, y))$  is the length of  $\gamma(x, y)$ . In general  $0 < c_J \leq 1$ . It is easy to see that an open ball with center at  $x_0$  is a John domain with John constant  $c_J = 1$  and John center at  $x_0$ . We may say that the bigger  $c_J$  is, the smoother  $D$  is.

Since the main concern of this paper is the boundary behavior of functions in  $D$ , we may replace  $x_0$  by a compact subset  $K_0$  of  $D$ . We call such a domain a *general John domain with general John center  $K_0$  and general John constant  $c_J$* . Obviously, a John domain is a general John domain and vice versa. Note that a general John constant is improved, i.e., a John domain with John center at  $x_0$  and John constant  $c_J$  can be regarded as a general John domain with general John constant  $c'_J \geq c_J$  by replacing  $x_0$  by a larger compact set  $K_0$ . In fact, a smooth domain is a general John domain with general John constant  $c_J = 1$ ; whereas it cannot be a John domain with John constant 1 unless it is an open ball. Several general John domains have been studied in connection with the Martin boundary, e.g. Denjoy domains (Benedicks [10]), Lipschitz Denjoy domains (Ancona [6], [7] and Chevallier [11]), sectorial domains (Cranston-Salisbury [12]), quasi-sectorial domains (Lömker [18]), the connected union of a family of open balls with the

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2000 *Mathematics Subject Classification.* 31B05, 31B25, 31C35.

*Key Words and Phrases.* John domain, convex set, Martin boundary, quasihyperbolic metric, Carleson estimate, Domar's theorem, tract, weak boundary Harnack principle.

This work was supported in part by Grant-in-Aid for Scientific Research (B) (2) (No.15340046) Japan Society for the Promotion of Science.

same radius (Ancona [5]) and so on. The general John constants for these domains can be estimated by the geometrical assumption on the domains. For example, the general John constant  $c_J = 1$  for a Denjoy domain.

Let  $G(x, y)$  be the Green kernel for  $D$ . A Martin kernel at  $\xi \in \partial D$  (with reference point  $x_0$ ) is a limit of the ratio  $G(x, y_j)/G(x_0, y_j)$  with  $y_j \rightarrow \xi$ . The totality of Martin kernels gives an ideal boundary of  $D$ , referred to as the Martin boundary of  $D$ . We identify a Martin kernel and an ideal boundary point; a limit of the ratio  $G(x, y_j)/G(x_0, y_j)$  with  $y_j \rightarrow \xi$  is called a Martin boundary point at  $\xi$  as well. We say that a positive harmonic function  $h$  is *minimal* if every positive harmonic function less than or equal to  $h$  coincides with a constant multiple of  $h$ . If a Martin kernel is a minimal harmonic function, then we call it a minimal Martin kernel or a minimal Martin boundary point. In general, the Martin boundary need not be homeomorphic to the Euclidean boundary. There may be even infinitely many minimal Martin boundary points at a Euclidean boundary point (Martin [19]).

The purpose of this paper is to show that every John domain has finitely many minimal Martin boundary points at each Euclidean boundary point. Moreover, the number of minimal Martin boundary points is estimated in terms of the John constant.

**THEOREM 1.1.** *Let  $D$  be a general John domain with general John constant  $c_J$ .*

- (i) *The number of minimal Martin boundary points at every Euclidean boundary point  $\xi \in \partial D$  is bounded by a constant depending only on the general John constant  $c_J$ .*
- (ii) *If  $c_J > \sqrt{3}/2$ , then there are at most two minimal Martin boundary points at every Euclidean boundary point  $\xi \in \partial D$ .*

**REMARK 1.1.** Let  $D$  be a sectorial domain whose boundary near the origin lies on three equally distributed rays leaving the origin. Then  $D$  is a general John domain with John constant  $\sin(\pi/3) = \sqrt{3}/2$ . There may be three different minimal Martin boundary points at the origin. See Figure 1.1. This simple example shows that the bound  $c_J > \sqrt{3}/2$  in Theorem 1.1 is sharp. Note that the same bound  $c_J > \sqrt{3}/2$  also applies to the higher dimensional case.

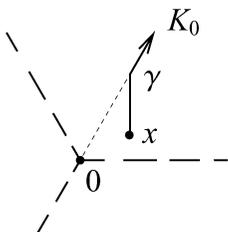


Figure 1.1. The bound  $c_J > \sqrt{3}/2$  in Theorem 1.1 is sharp.

**REMARK 1.2.** Theorem 1.1 generalizes some parts of [10], [6], [7], [11], [12] and [18]. One of the main interests of these papers was to give a criterion for the number of minimal Martin boundary points at a fixed Euclidean boundary point (via Kelvin transform for [10]). Such a criterion seems to be very difficult for a general John domain, since the boundary may disperse at every point (See e.g. [3, Figure 3b]).

One might think that the number of minimal Martin boundary points at a Euclidean boundary point would be equal to 1 provided the John constant  $c_J$  is sufficiently close to 1. This is not the case in view of Benedicks' work on a Denjoy domain ([10]). The best upper bound obtained from the John constant  $c_J$  is at least two as given in Theorem 1.1. Our second purpose is to find a certain class of John domains for which each boundary point has one minimal Martin boundary point.

We shall need some other information different from the John constant  $c_J$ . Ancona [5, Théorème] gave a condition for the union of a family of open balls with the same radius to have one minimal Martin boundary point at each Euclidean boundary point. By  $B(x, r)$  we denote the open ball with center at  $x$  and radius  $r$ . For a pair of distinct points  $x$  and  $y$  let  $[x, y]$  be the (open) line segment connecting  $x$  and  $y$ . For  $0 < \theta < \pi$  we denote by  $\Gamma_\theta(x, y)$  the open circular cone  $\{z \in \mathbf{R}^n : \angle zxy < \theta\}$  with vertex at  $x$ , axis  $[x, y]$  and aperture  $\theta$ . Ancona says that a domain  $D$  is *admissible* if

- (A1)  $D$  is the union of a family of open balls with the same radius  $\rho_0$ .
- (A2) Let  $\xi \in \partial D$ . If  $D$  includes two open balls  $B_1$  and  $B_2$  with radius  $\rho_0$  tangential to each other at  $\xi$ , then  $D$  includes a truncated circular cone  $\Gamma_\theta(\xi, y) \cap B(\xi, r)$  for some  $\theta > 0, r > 0$  and  $y$  in the hyperplane tangent to  $B_i$  at  $\xi$ . See Figure 1.2.

**THEOREM A (Ancona).** *Let  $D$  be a bounded admissible domain. Then every Euclidean boundary point of  $D$  has one Martin boundary point and it is minimal. Moreover, the Martin boundary of  $D$  is homeomorphic to the Euclidean boundary.*

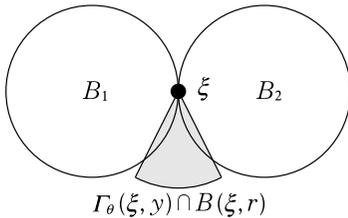


Figure 1.2. Condition (A2).

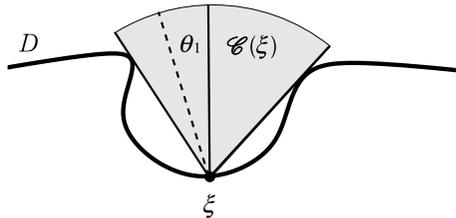


Figure 1.3. Condition (II).

Let us generalize both (A1) and (A2). Clearly, (A1) implies that  $D$  is a general John domain with general John constant 1. We would like to consider general convex sets rather than balls with the same radius. They need not be congruent. Observe that Ancona's condition (A2) implies that two balls  $B_1$  and  $B_2$  are *connected* by a truncated cone  $\Gamma_\theta(\xi, y) \cap B(\xi, r)$ . If  $0 < \theta' \leq \theta$ , then we have

$$\bigcup_{\substack{y \in D \\ \Gamma_{\theta'}(\xi, y) \cap B(\xi, r') \subset D}} \Gamma_{\theta'}(\xi, y) \cap B(\xi, r')$$

provided  $r' > 0$  is sufficiently small. In view of this observation, we generalize (A1) and (A2) as follows. Let  $A_0 \geq 1$  and  $\rho_0 > 0$ . We consider a bounded domain  $D$  such that

- (I)  $D$  is the union of a family of open convex sets  $\{C_\lambda\}_{\lambda \in A}$  such that  $B(z_\lambda, \rho_0) \subset$

$$C_\lambda \subset B(z_\lambda, A_0\rho_0).$$

(II) For each  $\xi \in \partial D$ , there are positive constants  $\theta_1 \leq \sin^{-1}(1/A_0)$  and  $\rho_1 \leq \rho_0 \cos \theta_1$  such that

$$\mathcal{C}(\xi) = \bigcup_{\substack{y \in D \\ \Gamma_{\theta_1}(\xi, y) \cap B(\xi, 2\rho_1) \subset D}} \Gamma_{\theta_1}(\xi, y) \cap B(\xi, 2\rho_1) \text{ is connected and non-empty.} \tag{1.2}$$

See Figure 1.3.

We observe that a bounded domain satisfying (I) becomes a general John domain. In fact, let  $K_0$  be the closure of  $\{z_\lambda\}_{\lambda \in A}$ . Then  $K_0$  is compact. For each  $x \in D$  there is  $\lambda_x \in A$  such that  $x \in C_{\lambda_x}$ . Since  $[x, z_{\lambda_x}] \subset C_{\lambda_x} \subset B(z_{\lambda_x}, A_0\rho_0)$  and  $\delta_{C_{\lambda_x}}(z_{\lambda_x}) \geq \rho_0$ , it follows from (2.4) in Section 2 that

$$\delta_D(w) \geq \delta_{C_{\lambda_x}}(w) \geq \frac{|x-w|}{|x-z_{\lambda_x}|} \delta_{C_{\lambda_x}}(z_{\lambda_x}) \geq A_0^{-1}|x-w| \quad \text{for all } w \in [x, z_{\lambda_x}].$$

Hence  $D$  is a general John domain with general John center  $K_0$  and general John constant  $A_0^{-1}$ . Thus Theorem 1.1 is applicable to such a domain, so that the number of minimal Martin boundary points at every Euclidean boundary point  $\xi \in \partial D$  is bounded by a constant  $N$ . Condition (II) implies that  $N = 1$ .

**THEOREM 1.2.** *Let  $D$  be a bounded domain satisfying (I) and (II). Then every Euclidean boundary point of  $D$  has only one Martin boundary point and it is minimal. Moreover the Martin boundary of  $D$  is homeomorphic to the Euclidean boundary.*

**REMARK 1.3.** Ancona’s admissible domains satisfy (I) and (II) of Theorem 1.2. The argument of Ancona depends on the special properties of a ball. His crucial lemma ([5, Lemme 1]) relies on the reflection with respect to a hyperplane, and is applied to a ball by the Kelvin transform ([5, Corollarie 2]). This approach is not applicable to our domains.

**REMARK 1.4.** A Denjoy domain can be represented as the union of a family of open balls with the same radius. A Lipschitz Denjoy domain, a sectorial domain and a quasi-sectorial domain can be represented as the union of a family of open convex sets  $C_\lambda$  satisfying (I). However, they cannot be represented as the union of a family of open balls with the same radius. Our Theorem 1.2 is applicable to these domains.

**REMARK 1.5.** Condition (II) is local in the following sense: Suppose  $D$  is the union of a family of open convex sets  $\{C_\lambda\}_{\lambda \in A}$  satisfying (I). If a particular point  $\xi \in \partial D$  satisfies (II), then there is only one Martin boundary point at  $\xi$  and it is minimal.

**REMARK 1.6.** Note that  $0 < \theta_1 < \pi/2$  by  $0 < \rho_1 \leq \rho_0 \cos \theta_1$ . The bounds  $\theta_1 \leq \sin^{-1}(1/A_0)$  and  $\rho_1 \leq \rho_0 \cos \theta_1$  are sharp. If one of the inequalities fails to hold, then there is a domain  $D$  satisfying both (I) and (1.2) and yet having a Euclidean boundary point  $\xi \in \partial D$  such that there are multiple Martin boundary points at  $\xi$ . See Examples 8.1 and 8.2 in Section 8.

Both Theorems 1.1 and 1.2 are based on a common geometrical notion, a *system of local reference points*. In Section 2, we shall introduce a quasihyperbolic metric and define a system of local reference points. Then we shall observe that Theorems 1.1 and 1.2 are decomposed into three propositions, namely, Propositions 2.1, 2.2 and 2.3. The first two propositions are purely geometric and will be proved in the same section. Proposition 2.3 involves many potential theoretic arguments. Among them, a Carleson type estimate (Lemma 5.1 in Section 5) for bounded positive harmonic functions vanishing on a portion of the boundary will be crucial. This estimate will be deduced from a Domar’s type theorem (Domar [13]) for nonnegative subharmonic functions, as was employed by Benedicks [10] and Chevallier [11]. Domar’s argument is applicable also to nonlinear equations in a metric measure space ([4]).

By the symbol  $A$  we denote an absolute positive constant whose value is unimportant and may change from line to line. If necessary, we use  $A_0, A_1, \dots$ , to specify them. We shall say that two positive functions  $f_1$  and  $f_2$  are comparable, written  $f_1 \approx f_2$ , if and only if there exists a constant  $A \geq 1$  such that  $A^{-1}f_1 \leq f_2 \leq Af_1$ . The constant  $A$  will be called the constant of comparison. We write  $B(x, r)$  and  $S(x, r)$  for the open ball and the sphere of center at  $x$  and radius  $r$ , respectively.

ACKNOWLEDGEMENT. The authors are very grateful to the referee for helpful comments and suggestions.

## 2. Local reference points.

### 2.1. Restatements of Theorems 1.1 and 1.2.

We define the quasihyperbolic metric  $k_D(x, y)$  by

$$k_D(x, y) = \inf_{\gamma} \int_{\gamma} \frac{ds(z)}{\delta_D(z)},$$

where the infimum is taken over all rectifiable curves  $\gamma$  connecting  $x$  to  $y$  in  $D$  and  $ds(z)$  stands for the line element on  $\gamma$ . We say that  $D$  satisfies a quasihyperbolic boundary condition if

$$k_D(x, x_0) \leq A \log \frac{\delta_D(x_0)}{\delta_D(x)} + A' \quad \text{for all } x \in D. \tag{2.1}$$

A domain satisfying the quasihyperbolic boundary condition is called a Hölder domain by Smith-Stegenga [20], [21]. It is easy to see that a John domain satisfies the quasihyperbolic boundary condition (see [16, Lemma 3.11]). We need more precise estimates.

DEFINITION 2.1. Let  $N$  be a positive integer and  $0 < \eta < 1$ . We say that  $\xi \in \partial D$  has a system of local reference points of order  $N$  with factor  $\eta$  if there exist  $R_{\xi} > 0$  and  $A_{\xi} > 1$  with the following property: for each positive  $R < R_{\xi}$  there are  $N$  points  $y_1 = y_1(R), \dots, y_N = y_N(R) \in D \cap S(\xi, R)$  such that  $A_{\xi}^{-1}R \leq \delta_D(y_i) \leq R$  for  $i = 1, \dots, N$  and

$$\min_{i=1,\dots,N} \{k_{D_R}(x, y_i)\} \leq A_\xi \log \frac{R}{\delta_D(x)} + A_\xi \quad \text{for } x \in D \cap \overline{B(\xi, \eta R)},$$

where  $D_R = D \cap B(\xi, \eta^{-3}R)$ . If  $\eta$  is not so important, we simply say that  $\xi \in \partial D$  has a system of local reference points of order  $N$ .

The proofs of Theorems 1.1 and 1.2 can be decomposed into the following three propositions. The first and the second are purely geometric; the third is potential theoretic.

**PROPOSITION 2.1.** *Let  $D$  be a general John domain with John constant  $c_J$ . Then every  $\xi \in \partial D$  has a system of local reference points of order  $N$  with  $N \leq N(c_J, n) < \infty$ . Moreover, if the John constant  $c_J > \sqrt{3}/2$ , then we can let  $N \leq 2$  by choosing a suitable factor  $0 < \eta < 1$ .*

**PROPOSITION 2.2.** *Let  $D$  be a bounded domain satisfying (I) and (II). Then every  $\xi \in \partial D$  has a system of local reference points of order 1.*

**REMARK 2.1.** In Proposition 2.1, the constants  $R_\xi$  and  $A_\xi$  in Definition 2.1 can be taken uniformly for  $\xi \in \partial D$ , whereas they may depend on  $\xi$  in Proposition 2.2.

By  $\mathcal{H}_\xi$  we denote the family of all kernel functions at  $\xi$  normalized at the John center  $x_0$ , i.e., the set of all positive harmonic functions  $h$  on  $D$  such that  $h(x_0) = 1$ ,  $h = 0$  q.e. (quasi everywhere) on  $\partial D$  and  $h$  is bounded on  $D \setminus B(\xi, r)$  for each  $r > 0$ . Here we say that a property holds q.e. if it holds outside a polar set. A Martin kernel at  $\xi$  (with reference point  $x_0$ ) is a limit of the ratio  $G(x, y_j)/G(x_0, y_j)$  of Green functions with  $y_j \rightarrow \xi$ . Suppose  $y_j \subset D \cap B(\xi, r/2)$ . Then the (global) boundary Harnack principle for a John domain (Bass and Burdzy [9]) implies that the ratio  $G(\cdot, y_j)/G(x_0, y_j)$  is bounded on  $D \setminus B(\xi, r)$ , and so is a Martin kernel at  $\xi$ . Obviously, a Martin kernel at  $\xi$  is a positive harmonic function vanishing q.e. on  $\partial D$  with value 1 at  $x_0$ , so that it belongs to  $\mathcal{H}_\xi$ . Thus Theorems 1.1 and 1.2 will follow from Propositions 2.1, 2.2 and the following:

**PROPOSITION 2.3.** *Let  $D$  be a general John domain. Suppose  $\xi \in \partial D$  has a system of local reference points of order  $N$ .*

- (i) *The number of minimal functions in  $\mathcal{H}_\xi$  is bounded by a constant depending only on  $N$ .*
- (ii) *If  $N \leq 2$ , then there are at most  $N$  minimal functions in  $\mathcal{H}_\xi$ . Moreover, if  $N = 1$ , then  $\mathcal{H}_\xi$  is a singleton and consists of a minimal function.*

**REMARK 2.2.** It is plausible that there are at most  $N$  minimal functions in  $\mathcal{H}_\xi$  even for  $N \geq 3$ . Unfortunately, our proof of Proposition 2.3 is based on (6.5), which is proved only for  $N \leq 2$ .

## 2.2. Proof of Proposition 2.1.

For the proof of the second assertion in Proposition 2.1, we prepare an elementary geometrical observation.

**LEMMA 2.1.** *Let  $e_1, e_2$  and  $e_3$  be points on the unit sphere  $S(0, 1)$ . Then*

$$\max \min_{i \neq j} |e_i - e_j| = \sqrt{3},$$

where the maximum is taken over all positions of  $e_1, e_2$  and  $e_3$ .

PROOF. This is a well-known fact (Fejes [14]). For the convenience sake of the reader we provide a proof. We can easily prove the lemma for  $n = 2$ . Let  $n \geq 3$ . We observe from the compactness of  $S(0, 1)$  that the maximum  $d$  is taken by some points  $e_1, e_2$  and  $e_3$  on  $S(0, 1)$ . There is a unique 2-dimensional plane  $\Pi$  containing  $e_1, e_2$  and  $e_3$ , since three distinct points on  $S(0, 1)$  cannot be collinear. Observe that  $S(0, 1) \cap \Pi$  is a circle with radius at most 1. Since  $e_1, e_2$  and  $e_3$  are points on this circle, it follows from the case  $n = 2$  that  $d \leq \sqrt{3}$ . The lemma follows.  $\square$

PROOF OF PROPOSITION 2.1. We prove the proposition with  $R_\xi = \delta_D(K_0)$ . Let  $\xi \in \partial D$  and  $0 < R < \delta_D(K_0)$ . Let us prove the first assertion with  $\eta = 1/2$ . Take  $x \in D \cap \overline{B(\xi, R/2)}$ . By definition there is a rectifiable curve  $\gamma$  starting from  $x$  and terminating at  $K_0$  such that (1.1) holds. Then the first hit  $y(x)$  of  $S(\xi, R)$  along  $\gamma$  satisfies  $2^{-1}c_J R \leq \delta_D(y(x)) \leq R$  and  $k_{D_R}(x, y(x)) \leq A \log \frac{R}{\delta_D(x)}$ . We associate  $y(x)$  with  $x$ , although it may not be unique.

Consider, in general, the family of balls  $B(y, 4^{-1}c_J R)$  with  $y \in S(\xi, R)$ . These balls are included in  $B(\xi, (4^{-1}c_J + 1)R)$ , so that at most  $N(c_J, n)$  balls among them can be mutually disjoint. Hence we find  $N$  points  $x_1, \dots, x_N \in D \cap \overline{B(\xi, R/2)}$  with  $N \leq N(c_J, n)$  such that  $\{B(y_1, 4^{-1}c_J R), \dots, B(y_N, 4^{-1}c_J R)\}$  is maximal, where  $y_j = y(x_j) \in D \cap S(\xi, R)$  is the point associated with  $x_j$  as above. This means that if  $x \in D \cap \overline{B(\xi, R/2)}$ , then  $B(y(x), 4^{-1}c_J R)$  intersects some of  $B(y_1, 4^{-1}c_J R), \dots, B(y_N, 4^{-1}c_J R)$ , say  $B(y_i, 4^{-1}c_J R)$ . Since  $B(y(x), 4^{-1}c_J R) \cap B(y_i, 4^{-1}c_J R) \neq \emptyset$  and  $B(y(x), 2^{-1}c_J R) \cup B(y_i, 2^{-1}c_J R) \subset D_R$ , it follows that  $k_{D_R}(y(x), y_i) \leq A'$ . Hence

$$k_{D_R}(x, y_i) \leq k_{D_R}(x, y(x)) + k_{D_R}(y(x), y_i) \leq A \log \frac{R}{\delta_D(x)} + A'.$$

Repeating some points, say  $y_1 = y(x_1)$ , if necessary, we may assume that this property holds with  $N$  independent of  $R$  and  $N \leq N(c_J, n)$ . Thus the first assertion follows.

For the proof of the second assertion, let  $\sqrt{3}/2 < b' < b < c_J$  and  $\eta = 1 - b/c_J > 0$ . Let us prove that  $\xi$  has a system of local reference points of order at most 2 with factor  $\eta$ . Let  $0 < R < \delta_D(K_0)$ . Suppose  $x \in D \cap \overline{B(\xi, \eta R)}$ . In the same way as in the proof of the first assertion, we find  $y(x) \in S(\xi, R)$  such that  $k_{D_R}(x, y(x)) \leq A \log \frac{R}{\delta_D(x)}$  and

$$\delta_D(y(x)) \geq c_J(1 - \eta)R = bR > b'R > \frac{\sqrt{3}}{2}R.$$

Lemma 2.1 says that at most two disjoint balls of radius  $b'R$  can be placed so that their centers lie on the sphere  $S(\xi, R)$ . Hence we can choose  $x_1, x_2 \in D \cap \overline{B(\xi, \eta R)}$  such that  $B(y(x), b'R)$  intersects  $B(y_i, b'R)$  for some  $i = 1, 2$ , where  $y_i = y(x_i)$ . Since  $B(y(x), b'R) \cap B(y_i, b'R) \neq \emptyset$  and  $B(y(x), bR) \cup B(y_i, bR) \subset D_R$ , it follows that  $k_{D_R}(y(x), y_i) \leq A$ .

Hence the proposition follows. □

REMARK 2.3. In case  $c_J \leq \sqrt{3}/2$ , we may have an estimate of  $N$  better than the above proof, by considering a lemma similar to Lemma 2.1.

**2.3. Proof of Proposition 2.2.**

In this subsection, we assume, by translation and dilation, that  $\xi = 0$  and  $\rho_1 = 1$  for simplicity. The aperture  $\theta_1 \leq \sin^{-1}(1/A_0)$  is fixed and we write  $\Gamma(x, y)$  for  $\Gamma_{\theta_1}(x, y)$ . Note that  $1 = \rho_1 \leq \rho_0 \cos \theta_1$ , so that  $0 < \theta_1 < \pi/2$  and  $\rho_0 \geq \sec \theta_1$ . Let  $C_\lambda$  be a convex set appearing in (I) and let  $B(z_\lambda, \rho_0) \subset C_\lambda \subset B(z_\lambda, A_0\rho_0)$ . If  $x \in \overline{C_\lambda} \setminus B(z_\lambda, \rho_0)$ , then

$$\Gamma(x, z_\lambda) \cap B(x, 2) \subset \text{co}(\{x\} \cup B(z_\lambda, \rho_0)) \subset C_\lambda, \tag{2.2}$$

where  $\text{co}(\{x\} \cup B(z_\lambda, \rho_0))$  is the interior of the convex hull of  $\{x\} \cup B(z_\lambda, \rho_0)$ . Let

$$\mathcal{Y} = \{y \in S(0, 1) : \Gamma(0, y) \cap B(0, 2) \subset D\}.$$

We first show that  $\mathcal{Y} \neq \emptyset$  and that the point 0 can be accessible along a ray issuing from the origin toward a point in  $\mathcal{Y}$ .

LEMMA 2.2. *There is a positive constant  $R_0 < 1$  such that if  $C_\lambda \cap B(0, R_0) \neq \emptyset$ , then  $C_\lambda \cap \mathcal{Y} \neq \emptyset$ . In particular,  $\mathcal{Y} \neq \emptyset$ .*

PROOF. Suppose to the contrary, there is a sequence  $C_{\lambda_j}$  with  $\text{dist}(0, C_{\lambda_j}) \rightarrow 0$  and  $C_{\lambda_j} \cap \mathcal{Y} = \emptyset$ . Let  $z_{\lambda_j}$  be such that  $B(z_{\lambda_j}, \rho_0) \subset C_{\lambda_j} \subset B(z_{\lambda_j}, A_0\rho_0)$ . Taking a subsequence, if necessary, we may assume that  $z_{\lambda_j}$  converges, say to  $z_0$ . We claim

$$\Gamma(0, z_0) \cap B(0, 2) \subset \bigcup_j C_{\lambda_j}. \tag{2.3}$$

We find  $x_{\lambda_j} \in \partial C_{\lambda_j}$  with  $x_{\lambda_j} \rightarrow 0$ . Take  $x \in \Gamma(0, z_0) \cap B(0, 2)$ . Then  $\angle x_0 z_0 < \theta_1$  and  $|x| < 2$  by definition. If  $j$  is sufficiently large, then  $\angle x x_{\lambda_j} z_{\lambda_j} < \theta_1$  and  $|x - x_{\lambda_j}| < 2$  by continuity, so that

$$x \in \Gamma(x_{\lambda_j}, z_{\lambda_j}) \cap B(x_{\lambda_j}, 2) \subset \text{co}(\{x_{\lambda_j}\} \cup B(z_{\lambda_j}, \rho_0)) \subset C_{\lambda_j},$$

by (2.2). Thus (2.3) follows. Now, by definition,  $y_0 = z_0/|z_0| \in \mathcal{Y}$  and  $y_0 \in \Gamma(0, z_0) \cap B(0, 2) \subset \bigcup_j C_{\lambda_j}$ . This contradicts  $C_{\lambda_j} \cap \mathcal{Y} = \emptyset$ . The lemma follows. □

Observe that if  $C$  is a convex set, then the distance function  $\delta_C(x) = \text{dist}(x, \partial C)$  is a concave function on  $\overline{C}$ , i.e.,

$$\delta_C(z) \geq \frac{|z - y|}{|x - y|} \delta_C(x) + \frac{|x - z|}{|x - y|} \delta_C(y) \quad \text{for } z \in [x, y], \tag{2.4}$$

whenever  $x$  and  $y$  are distinct points in  $\overline{C}$ . This fact will be used in the following lemma.

LEMMA 2.3. *Let  $0 < R_0 < 1$  be as in Lemma 2.2. Suppose  $0 < R < \min\{R_0, 3^{-1} \sin \theta_1\}$ . If  $C_\lambda \cap B(0, R) \neq \emptyset$  and  $y \in C_\lambda \cap \mathcal{Y}$ , then there exists a point  $w \in C_\lambda \cap \Gamma(0, y) \cap B(0, 3R/\sin \theta_1)$  such that*

$$\delta_{C_\lambda \cap \Gamma(0, y)}(w) \geq \frac{\sin \theta_1}{4} R.$$

PROOF. Take  $x \in C_\lambda \cap B(0, R)$ . Then  $[x, y] \subset C_\lambda$ . Observe that there is a point  $w_1 \in [x, y] \cap \overline{\Gamma(0, y)}$  with  $|w_1| \leq R/\sin \theta_1$ . In fact, if  $x \in \overline{\Gamma(0, y)}$ , then  $w_1 = x$  satisfies the condition. Otherwise, let  $w_1$  be the intersection of  $[x, y]$  and  $\partial\Gamma(0, y)$ . By elementary geometry

$$R > \text{dist}(x, [0, y]) \geq \text{dist}(w_1, [0, y]) = |w_1| \sin \theta_1,$$

so that  $|w_1| \leq R/\sin \theta_1$ . Since  $|w_1 - y| \geq 1 - R/\sin \theta_1$  and  $3R/\sin \theta_1 < 1$ , we find a point  $w_2 \in [w_1, y] \subset C_\lambda \cap \Gamma(0, y)$  with  $|w_1 - w_2| = R/\sin \theta_1$ . See Figure 2.1.

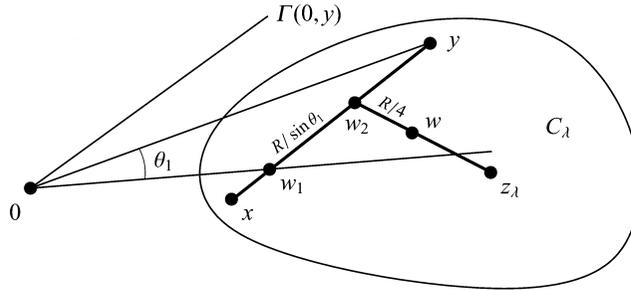


Figure 2.1.  $\delta_{C_\lambda \cap \Gamma(0, y)}(w) \geq 4^{-1} \sin \theta_1 R$ .

By (2.4) with  $C = \Gamma(0, y)$  we obtain

$$\delta_{\Gamma(0, y)}(w_2) \geq \frac{|w_1 - w_2|}{|w_1 - y|} \delta_{\Gamma(0, y)}(y) \geq \frac{R/\sin \theta_1}{R/\sin \theta_1 + 1} \sin \theta_1 > \frac{R}{2}.$$

Moreover  $|w_2| \leq 2R/\sin \theta_1$ . Since  $|w_2 - z_\lambda| \geq \rho_0 - 2R/\sin \theta_1 > R$  by  $3R/\sin \theta_1 < 1 \leq \rho_0$ , we can take a point  $w \in [w_2, z_\lambda] \subset C_\lambda$  such that  $|w - w_2| = R/4$ . Then it follows from (2.4) with  $C = C_\lambda$  that

$$\delta_{C_\lambda}(w) \geq \frac{|w - w_2|}{|z_\lambda - w_2|} \delta_{C_\lambda}(z_\lambda) \geq \frac{R/4}{A_0 \rho_0} \rho_0 \geq \frac{\sin \theta_1}{4} R.$$

Hence

$$\delta_{\Gamma(0, y) \cap C_\lambda}(w) \geq \min \left\{ \frac{R}{2} - \frac{R}{4}, \frac{\sin \theta_1}{4} R \right\} = \frac{\sin \theta_1}{4} R.$$

Moreover,

$$|w| \leq |w - w_2| + |w_2 - w_1| + |w_1| \leq \frac{R}{4} + \frac{R}{\sin \theta_1} + \frac{R}{\sin \theta_1} < \frac{3R}{\sin \theta_1}.$$

Thus the lemma is proved. □

PROOF OF PROPOSITION 2.2. Let  $0 < R_0 < 1$  be as in Lemma 2.2 and let  $0 < \eta^3 < 6^{-1} \sin \theta_1$ . Suppose  $0 < R < \min\{R_0, 3^{-1} \sin \theta_1\}$ . By Lemma 2.2 we fix  $y_0 \in \mathcal{C}$  and write  $y_R = Ry_0$ . It is sufficient to show that

$$k_{D_R}(x, y_R) \leq A \log \frac{R}{\delta_D(x)} + A \quad \text{for } x \in D \cap \overline{B(0, \eta R)}, \tag{2.5}$$

where  $A$  is independent of  $x$  and  $R$ . Take  $x \in D \cap \overline{B(0, \eta R)}$ . Then there is a convex set  $C_\lambda$  containing  $x$  and there is  $y \in C_\lambda \cap \mathcal{C}$  by Lemma 2.2. By Lemma 2.3 we find a point  $w \in C_\lambda \cap \Gamma(0, y) \cap B(0, 3R/\sin \theta_1)$  such that  $\delta_{C_\lambda \cap \Gamma(0, y)}(w) \geq 4^{-1}R \sin \theta_1$ . Then  $[x, w] \subset B(0, 2^{-1}\eta^{-3}R) \cap C_\lambda$ , and therefore  $\delta_{D_R}(z) = \delta_D(z) \geq \delta_{C_\lambda}(z)$  for  $z \in [x, w]$ . Since

$$\delta_{D_R}(z) \geq \delta_{C_\lambda}(z) \geq \frac{|x - z|}{|x - w|} \delta_{C_\lambda}(w) \geq \frac{\sin^2 \theta_1}{16} |x - z| \quad \text{for } z \in [x, w]$$

by (2.4), it follows that

$$k_{D_R}(x, w) \leq \int_{[x, w]} \frac{ds(z)}{\delta_{D_R}(z)} \leq A \log \frac{R}{\delta_D(x)} + A.$$

Since

$$\delta_{D_R}(z) \geq \delta_{\Gamma(0, y)}(z) \geq \frac{|w - z|}{|w - Ry|} \delta_{\Gamma(0, y)}(Ry) \geq \frac{\sin^2 \theta_1}{4} |w - z| \quad \text{for } z \in [w, Ry],$$

it also follows that

$$k_{D_R}(w, Ry) \leq \int_{[w, Ry]} \frac{ds(z)}{\delta_{D_R}(z)} \leq A \log \frac{R}{\delta_D(x)} + A.$$

Note that  $\mathcal{C}(0) \cap S(0, 1)$  is connected by the assumption (II). In view of  $\text{dist}(\mathcal{C}, S(0, 1) \setminus \mathcal{C}(0)) \geq \sin \theta_1$  and  $\mathcal{C}(0) \subset D$ , we see that  $k_{D_R}(Ry, y_R) \leq A$  with  $A$  independent of  $R, y$  and  $y_R$ . Thus (2.5) follows from the triangle inequality. □

### 3. Refinement of Domar’s theorem.

Domar [13, Theorem 2] gave a criterion for the boundedness of a subharmonic function majorized by a positive function. We need its quantitative refinement, i.e., the

dependency of the bound is given explicitly.

LEMMA 3.1. *Let  $u$  be a nonnegative subharmonic function on a bounded domain  $\Omega$ . Suppose there is  $\varepsilon > 0$  such that*

$$I = \int_{\Omega} (\log^+ u)^{n-1+\varepsilon} dx < \infty.$$

Then

$$u(x) \leq \exp(2 + AI^{1/\varepsilon} \delta_{\Omega}(x)^{-n/\varepsilon}), \tag{3.1}$$

where  $A$  is a positive constant depending only on  $\varepsilon$  and the dimension  $n$ .

For the proof we prepare the following.

LEMMA 3.2. *Let  $u$  be a nonnegative subharmonic function on  $B(x, R)$ . Suppose  $u(x) \geq t > 0$  and*

$$R \geq L_n |\{y \in B(x, R) : e^{-1}t < u(y) \leq et\}|^{1/n}, \tag{3.2}$$

where  $L_n = (e^2/v_n)^{1/n}$  and  $v_n$  is the volume of the unit ball. Then there exists a point  $x' \in B(x, R)$  with  $u(x') > et$ .

PROOF. Observe that (3.2) is equivalent to

$$\frac{|\{y \in B(x, R) : e^{-1}t < u(y) \leq et\}|}{|B(x, R)|} \leq \frac{1}{e^2}.$$

Suppose  $u \leq et$  on  $B(x, R)$ . Then the mean value property of subharmonic functions yields

$$\begin{aligned} t \leq u(x) &\leq \frac{1}{|B(x, R)|} \int_{B(x, R)} u(y) dy \\ &= \frac{1}{|B(x, R)|} \left( \int_{B(x, R) \cap \{u \leq e^{-1}t\}} u dy + \int_{B(x, R) \cap \{u > e^{-1}t\}} u dy \right) \\ &\leq e^{-1}t + \frac{1}{e^2} et < t. \end{aligned}$$

This is a contradiction. □

PROOF OF LEMMA 3.1. Since the right hand side of (3.1) is not less than  $e^2$ , it is sufficient to show that

$$\delta_{\Omega}(x) \leq AI^{1/n} (\log u(x))^{-\varepsilon/n}, \quad \text{whenever } u(x) > e^2. \tag{3.3}$$

Fix  $x_1 \in \Omega$  with  $u(x_1) > e^2$  and let us prove (3.3) with  $x = x_1$ . Let

$$R_j = L_n |\{y \in \Omega : e^{j-2}u(x_1) < u(y) \leq e^j u(x_1)\}|^{1/n} \quad \text{for } j \geq 1.$$

We choose a sequence  $\{x_j\}$  as follows: If  $\delta_\Omega(x_1) < R_1$ , then we stop. If  $\delta_\Omega(x_1) \geq R_1$ , then  $B(x_1, R_1) \subset \Omega$ , so that there exists  $x_2 \in B(x_1, R_1)$  such that  $u(x_2) > eu(x_1)$  by Lemma 3.2. Next we consider  $\delta_\Omega(x_2)$ . If  $\delta_\Omega(x_2) < R_2$ , then we stop. If  $\delta_\Omega(x_2) \geq R_2$ , then  $B(x_2, R_2) \subset \Omega$ , so that there exists  $x_3 \in B(x_2, R_2)$  such that  $u(x_3) > e^2u(x_1)$  by Lemma 3.2. Repeat this procedure to obtain a finite or infinite sequence  $\{x_j\}$ . We claim

$$\delta_\Omega(x_1) \leq 2 \sum_{j=1}^{\infty} R_j. \tag{3.4}$$

Suppose first  $\{x_j\}$  is finite. If  $\delta_\Omega(x_1) < R_1$ , then (3.4) trivially holds. If  $\delta_\Omega(x_1) \geq R_1$ , then we have an integer  $J \geq 2$  such that

$$\begin{aligned} \delta_\Omega(x_1) \geq R_1, \dots, \delta_\Omega(x_{J-1}) \geq R_{J-1}, \delta_\Omega(x_J) < R_J, \\ x_2 \in B(x_1, R_1), x_3 \in B(x_2, R_2), \dots, x_J \in B(x_{J-1}, R_{J-1}). \end{aligned}$$

Hence we have

$$\delta_\Omega(x_1) \leq |x_1 - x_2| + \dots + |x_{J-1} - x_J| + \delta_\Omega(x_J) < R_1 + \dots + R_{J-1} + R_J,$$

so that (3.4) follows. Suppose next  $\{x_j\}$  is infinite. Since  $u(x_j) > e^j u(x_1) \rightarrow \infty$ , it follows from the local boundedness of a subharmonic function that  $x_j$  goes to the boundary. Hence, there is an integer  $J \geq 2$  such that  $\delta_\Omega(x_J) \leq \frac{1}{2} \delta_\Omega(x_1)$ . Then

$$\delta_\Omega(x_1) \leq |x_1 - x_2| + \dots + |x_{J-1} - x_J| + \delta_\Omega(x_J) \leq R_1 + \dots + R_{J-1} + \frac{1}{2} \delta_\Omega(x_1),$$

so that (3.4) follows. In view of (3.4) we observe that (3.3) follows from

$$\sum_{j=1}^{\infty} R_j \leq AI^{1/n} (\log u(x_1))^{-\varepsilon/n}. \tag{3.5}$$

To show (3.5), let  $j_1$  be the integer such that  $e^{j_1} < u(x_1) \leq e^{j_1+1}$ . Then  $j_1 \geq 2$  and

$$R_j \leq L_n |\{y \in \Omega : e^{j_1+j-2} < u(y) \leq e^{j_1+j+1}\}|^{1/n}.$$

Since the family of intervals  $\{(e^{j_1+j-2}, e^{j_1+j+1})\}_j$  overlaps at most 3 times, it follows from Hölder’s inequality that

$$\begin{aligned}
 \sum_{j=1}^{\infty} R_j &\leq 3L_n \sum_{j=j_1}^{\infty} |\{y \in \Omega : e^{j-1} < u(y) \leq e^j\}|^{1/n} \\
 &\leq 3L_n \left( \sum_{j=j_1}^{\infty} \frac{1}{j^{(n-1+\varepsilon)/(n-1)}} \right)^{(n-1)/n} \left( \sum_{j=j_1}^{\infty} j^{n-1+\varepsilon} |\{y \in \Omega : e^{j-1} < u(y) \leq e^j\}| \right)^{1/n} \\
 &\leq A j_1^{-\varepsilon/n} \left( \int_{\Omega} (\log^+ u)^{n-1+\varepsilon} dy \right)^{1/n} \\
 &\leq A (\log u(x_1))^{-\varepsilon/n} I^{1/n}.
 \end{aligned}$$

Thus (3.5) follows. The lemma is proved. □

#### 4. Integrability of negative power of the distance function.

Inspired by Smith and Stegenga [20, Theorem 4] we have proved that for a bounded John domain there is a positive constant  $\tau$  such that

$$\int_D \delta_D(x)^{-\tau} dx < \infty$$

([1, Lemma 5]). We need its local version.

LEMMA 4.1. *Let  $D$  be a general John domain with general John constant  $c_J$  and general John center  $K_0$ . Then there are positive constants  $\tau$  and  $A$  depending on  $c_J$  such that*

$$\int_{D \cap B(\xi, R)} \left( \frac{R}{\delta_D(x)} \right)^{\tau} dx \leq AR^n$$

for each  $\xi \in \partial D$  and  $0 < R < \delta_D(K_0)$ .

PROOF. Let

$$V_j = \{x \in D \cap B(\xi, R + (1 + c_J^{-1})2^{1-j}R) : 2^{-j-1}R \leq \delta_D(x) < 2^{-j}R\}$$

for  $j \geq 0$ . For a moment we fix  $x \in \bigcup_{i=j+1}^{\infty} V_i$ . By definition there is a rectifiable curve  $\gamma$  connecting  $x$  and  $K_0$  with (1.1). Hence we find  $y \in \gamma$  such that  $\delta_D(y) = 2^{-j}R \geq c_J|x - y|$ . In other words  $x \in B(y, c_J^{-1}2^{-j}R)$ . We observe

$$|B(y, 5c_J^{-1}2^{-j}R)| \leq A|V_j \cap B(y, c_J^{-1}2^{-j}R)|. \tag{4.1}$$

In fact, take  $y^* \in \partial D$  such that  $|y - y^*| = 2^{-j}R$ , and then take  $y' \in [y, y^*]$  with  $\delta_D(y') = \frac{1}{2}(2^{-j}R + 2^{-j-1}R)$ . An elementary geometrical observation and  $c_J \leq 1$  give  $B(y', 2^{-j-2}R) \subset V_j \cap B(y, c_J^{-1}2^{-j}R)$ , so that (4.1) follows.

Now the covering lemma yields a sequence  $\{y_k\}$  such that

$$\bigcup_{i=j+1}^{\infty} V_i \subset \bigcup_k \overline{B(y_k, 5c_J^{-1}2^{-j}R)}$$

and  $\{\overline{B(y_k, c_J^{-1}2^{-j}R)}\}_k$  are disjoint. Hence

$$\sum_{i=j+1}^{\infty} |V_i| = \left| \bigcup_{i=j+1}^{\infty} V_i \right| \leq \sum_k |B(y_k, 5c_J^{-1}2^{-j}R)| \leq A_1 \sum_k |V_j \cap B(y_k, c_J^{-1}2^{-j}R)| \leq A_1 |V_j|$$

by (4.1). Let  $1 < t < 1 + A_1^{-1}$ . In the same way as in the proof of [1, Lemma 5] we have

$$\sum_{j=0}^{\infty} t^j |V_j| \leq \frac{1}{1 - (t-1)A_1} \sum_{j=0}^{\infty} |V_j| \leq A |B(\xi, R + (1 + c_J^{-1})2R)| \leq AR^n.$$

Since  $t^j < (R/\delta_D(x))^\tau \leq t^{j+1}$  on  $V_j$  with  $\tau = \log t / \log 2 > 0$ , it follows that

$$\int_{D \cap B(\xi, R)} \left( \frac{R}{\delta_D(x)} \right)^\tau dx \leq \sum_{j=0}^{\infty} t^{j+1} |V_j| \leq AR^n.$$

Thus the lemma follows. □

### 5. Growth of positive harmonic functions.

In this section we shall show Proposition 2.3 (i) by investigating the growth of  $h \in \mathcal{H}_\xi$ . Throughout the section we let  $D$  be a general John domain and let  $\xi \in \partial D$  be fixed. We say that  $x, y \in D$  are connected by a Harnack chain  $\{B(x_j, \frac{1}{2}\delta_D(x_j))\}_{j=1}^k$  if  $x \in B(x_1, \frac{1}{2}\delta_D(x_1))$ ,  $y \in B(x_k, \frac{1}{2}\delta_D(x_k))$ , and  $B(x_j, \frac{1}{2}\delta_D(x_j)) \cap B(x_{j+1}, \frac{1}{2}\delta_D(x_{j+1})) \neq \emptyset$  for  $j = 1, \dots, k-1$ . The number  $k$  is called the length of the Harnack chain. We observe that the shortest length of the Harnack chain connecting  $x$  and  $y$  is comparable to  $k_D(x, y) + 1$ . Therefore, the Harnack inequality yields that there is a constant  $A_2 > 1$  depending only on  $n$  such that

$$\exp(-A_2(k_D(x, y) + 1)) \leq \frac{h(x)}{h(y)} \leq \exp(A_2(k_D(x, y) + 1)) \tag{5.1}$$

for every positive harmonic function  $h$  on  $D$ . If  $D$  is a John domain with John constant  $c_J$  and John center  $x_0$ , then we have from (2.1)

$$\frac{h(x)}{h(x_0)} \leq A_3 \left( \frac{\delta_D(x_0)}{\delta_D(x)} \right)^\lambda \tag{5.2}$$

with  $\lambda$  and  $A_3 > 0$  depending only on the John constant  $c_J$ . If  $D$  is a general John domain with John constant  $c_J$  and John center  $K_0$ , then (5.2) holds with the same  $\lambda$

and another  $A_3$  depending only on  $c_J$ ,  $x_0$  and  $K_0$ .

Let  $\Omega$  be an open set intersecting  $\partial D$ . Let  $h$  be a bounded positive harmonic function in  $D \cap \Omega$  vanishing q.e. on  $\partial D \cap \Omega$ . We extend  $h$  to  $\Omega \setminus D$  by 0 outside  $D$  and denote by  $h^*$  its upper regularization. Then we observe that  $h^*$  is a nonnegative subharmonic function on  $\Omega$  ([8, Theorem 5.2.1]). We shall apply the refinement of Domar's theorem (Lemma 3.1) to the subharmonic function  $h^*$  to obtain a Carleson type estimate.

LEMMA 5.1. *Let  $\xi \in \partial D$  have a system of local reference points  $y_1, \dots, y_N \in D \cap S(\xi, R)$  of order  $N$  with factor  $\eta$  for  $0 < R < R_\xi$ . Suppose  $h$  is a positive harmonic function in  $D \cap B(\xi, \eta^{-3}R)$  vanishing q.e. on  $\partial D \cap B(\xi, \eta^{-3}R)$ . If  $h$  is bounded in  $D \cap B(\xi, \eta R) \setminus \overline{B(\xi, \eta^3 R)}$ , then*

$$h \leq A \sum_{i=1}^N h(y_i) \quad \text{on } D \cap S(\xi, \eta^2 R), \tag{5.3}$$

where  $A$  is independent of  $h$  and  $R$ .

PROOF. Let  $0 < R < R_\xi$ . Then we find  $y_1, \dots, y_N \in D \cap S(\xi, R)$  with  $\delta_D(y_i) \approx R$  such that

$$\min_{i=1, \dots, N} \{k_{D_R}(x, y_i)\} \leq A \log \frac{R}{\delta_D(x)} + A \quad \text{for } x \in D \cap \overline{B(\xi, \eta R)}.$$

By (5.1) we find a constant  $A_4 > 1$  such that

$$h(x) \leq A_4 \left( \frac{R}{\delta_D(x)} \right)^\lambda \sum_{i=1}^N h(y_i) \quad \text{for } x \in D \cap \overline{B(\xi, \eta R)}. \tag{5.4}$$

Let us apply Lemma 3.1 to  $\varepsilon = 1$ ,  $u = h^*/(A_4 \sum_{i=1}^N h(y_i))$  and  $\Omega = B(\xi, \eta R) \setminus \overline{B(\xi, \eta^3 R)}$ . Let  $\tau > 0$  be as in Lemma 4.1. Apply the elementary inequality:

$$(\log t)^n \leq \left( \frac{n}{\tau} \right)^n t^\tau \quad \text{for } t \geq 1$$

to  $t = R/\delta_D(x) \geq 1$  for  $x \in \Omega$ . Then

$$\left[ \log^+ \left( \frac{R}{\delta_D(x)} \right) \right]^n \leq A \left( \frac{R}{\delta_D(x)} \right)^\tau,$$

so that it follows from (5.4) and Lemma 4.1 that

$$I = \int_\Omega (\log^+ u)^n dx \leq A \int_{D \cap B(\xi, R)} \left( \frac{R}{\delta_D(x)} \right)^\tau dx \leq AR^n.$$

Hence, Lemma 3.1 yields that  $u \leq \exp(2 + AIR^{-n}) \leq A$  on  $S(\xi, \eta^2R)$ , i.e., (5.3) holds. □

Let us apply Lemma 5.1 to a kernel function  $h \in \mathcal{H}_\xi$  to obtain the following growth estimate.

LEMMA 5.2. *Let  $\xi \in \partial D$  have a system of local reference points  $y_1, \dots, y_N \in D \cap S(\xi, R)$  of order  $N$  with factor  $\eta$  for  $0 < R < R_\xi$ . Let  $h \in \mathcal{H}_\xi$ . Then*

$$h(x) \leq A|x - \xi|^{-\lambda} \quad \text{for } x \in D,$$

where  $\lambda > 0$  is as in (5.2) and  $A$  is independent of  $R$ ,  $x$  and  $h$ .

PROOF. By Lemma 5.1 we have (5.3). Since  $h$  is bounded apart from a neighborhood of  $\xi$ , the maximum principle gives

$$h(x) \leq A \sum_{i=1}^N h(y_i) \quad \text{for } x \in D \setminus B(\xi, \eta^2R).$$

Apply (5.2) to each  $y_i \in D \cap S(\xi, R)$  with  $\delta_D(y_i) \approx R$ . Then obtain  $h(y_i) \leq AR^{-\lambda}$ . This, together with the above estimate, yields  $h(x) \leq A|x - \xi|^{-\lambda}$  for  $x \in D$ . The lemma is proved. □

Here we record another application of Lemma 5.1, as this will be useful later.

LEMMA 5.3. *Let  $\xi \in \partial D$  have a system of local reference points  $y_1, \dots, y_N \in D \cap S(\xi, R)$  of order  $N$  with factor  $\eta$  for  $0 < R < R_\xi$ . Let  $h$  be a bounded positive harmonic function on  $D \cap B(\xi, \eta^{-3}R)$  vanishing q.e. on  $\partial D \cap B(\xi, \eta^{-3}R)$ . Then*

$$h \leq A \sum_{i=1}^N h(y_i) \quad \text{on } D \cap \overline{B(\xi, \eta^2R)},$$

where  $A$  is independent of  $R$  and  $h$ .

PROOF. We have (5.3). Apply the maximum principle to  $D \cap B(\xi, \eta^2R)$ . □

The following lemma is well-known. For the sake of the reader's convenience, we state it with a proof.

LEMMA 5.4. *Suppose there exist a positive integer  $M$  and a positive constant  $A$  with the following property: if  $h_0, \dots, h_M \in \mathcal{H}_\xi$ , then there is  $j$  such that*

$$h_j \leq A \sum_{i \neq j} h_i \quad \text{on } D.$$

Then  $\mathcal{H}_\xi$  has at most  $M$  minimal harmonic functions.

PROOF. Suppose there are  $M+1$  different minimal harmonic functions  $h_0, \dots, h_M \in \mathcal{H}_\xi$ . If necessary relabeling, we may assume that

$$h_0 \leq A \sum_{i=1}^M h_i \quad \text{on } D.$$

We may also assume that  $A > 1$ . Then  $(A \sum_{i=1}^M h_i - h_0)/(AM - 1) \in \mathcal{H}_\xi$ . Let  $h$  be this function. We have

$$\frac{1}{AM} h_0 + \left(1 - \frac{1}{AM}\right) h = \frac{1}{M} \sum_{i=1}^M h_i.$$

Compare the Martin representation measures for the both sides. The measure for the left hand side has at least  $\frac{1}{AM}$  mass at  $h_0$ , whereas the measure for the right hand side has 0 mass at  $h_0$ . This contradicts the uniqueness of the Martin representation.  $\square$

Let  $u$  be an unbounded subharmonic function on  $\mathbf{R}^n$ . For  $k \in \mathbf{R}$  and  $x_0 \in \mathbf{R}^n$  we define the limit component containing  $x_0$  by

$$C(x_0, k) = \bigcup_{R > |x_0|} C_R(x_0, k),$$

where  $C_R(x_0, k)$  is the connected component of  $\{x : u(x) \geq k, |x| \leq R\}$  containing  $x_0$ . By  $N(k)$  we denote the total number of limit components. It is known that  $N(k)$  is an increasing function of  $k$  and the limit  $\lim_{k \rightarrow \infty} N(k)$  is referred to as the number of tracts of  $u$ . There is a close connection between the growth of  $u$  and the number of tracts. See [15, Section 1] and [17, Section 4.6] for details.

PROOF OF PROPOSITION 2.3 FOR  $N \geq 3$ . Let  $h_j \in \mathcal{H}_\xi$  for  $j = 0, \dots, M$ . Let  $h_j^*$  be the upper regularization of the extension of  $h_j$  to  $\mathbf{R}^n \setminus \{\xi\}$  as before Lemma 5.1 and let  $H_j$  be the Kelvin transform of  $h_j^*$  with respect to  $S(\xi, 1)$ , i.e.,

$$H_j(x) = |x - \xi|^{2-n} h_j^*(\xi + |x - \xi|^{-2}(x - \xi)).$$

Observe that  $H_j$  is a nonnegative subharmonic function on  $\mathbf{R}^n$  which is positive and harmonic on the Kelvin image  $D^*$  of  $D$  and is equal to 0 q.e. outside  $D^*$ . Moreover, Lemma 5.2 shows

$$H_j(x) \leq A|x - \xi|^{2-n+\lambda}.$$

Thus  $H_j$  is of order at most  $2 - n + \lambda$ . As in Benedicks [10, Theorem 2], we let

$$w = \max_{j=0, \dots, M} \left\{ H_j - \sum_{i \neq j} H_i \right\}$$

and let  $w^+$  be the upper regularization of  $\max\{w, 0\}$ . Then  $w^+$  is a nonnegative subharmonic function on  $\mathbf{R}^n$  of order at most  $2 - n + \lambda$ . If none of  $\{x : H_j(x) > \sum_{i \neq j} H_i(x)\}$  is empty, then  $w^+$  has  $M + 1$  tracts. Hence, [15, Theorem 3] yields

$$2 - n + \lambda \geq \frac{1}{2} \log \left( \frac{M + 1}{4} \right) + \frac{3}{2} \quad \text{if } M \geq 3.$$

Hence, if  $M > 4 \exp(1 - 2n + 2\lambda) - 1$ , then  $\{x : H_j(x) > \sum_{i \neq j} H_i(x)\} = \emptyset$  for some  $j = 0, \dots, M$ . This means that  $H_j \leq \sum_{i \neq j} H_i$  on  $D^*$ , so that

$$h_j \leq \sum_{i \neq j} h_i \quad \text{on } D.$$

Hence Lemma 5.4 implies that  $\mathcal{H}_\xi$  has at most  $M$  minimal harmonic functions, or equivalently there are at most  $M$  minimal Martin boundary points at  $\xi$ . Thus the number of minimal Martin boundary points at  $\xi$  is bounded by  $4 \exp(1 - 2n + 2\lambda)$ .  $\square$

REMARK 5.1. The above proof gives a coarse estimate of the number of minimal harmonic functions of  $\mathcal{H}_\xi$  in terms of  $\lambda$  depending on the John constant  $c_J$ . More delicate arguments will be needed for a sharp estimate.

### 6. Weak boundary Harnack principle.

In this section we shall prove Proposition 2.3 for  $N \leq 2$ . Throughout the section we let  $D$  be a general John domain and fix  $\xi \in \partial D$ . Since most arguments are valid for any  $N \geq 1$ , except for (6.5), we shall state the results for general  $N$ . Proposition 2.3 will be derived from a certain estimate of the Green function. There is a difference of the behavior of the Green function  $G$  for  $D$  between the cases  $n = 2$  and  $n \geq 3$ , i.e., if  $n \geq 3$  and  $R > 0$  is small, then

$$G(x, y) \approx R^{2-n} \quad \text{for } x \in S\left(y, \frac{1}{2}\delta_D(y)\right) \text{ with } \delta_D(y) \approx R;$$

if  $n = 2$ , then this estimate does not necessarily hold. To avoid this difficulty we consider the Green function  $G_R$  for the intersection  $\tilde{D}_R = D \cap B(\xi, A_5 R)$  with sufficiently large  $A_5 > \eta^{-3}$ . Then we have for any  $n \geq 2$ ,

$$G_R(x, y) \approx R^{2-n} \quad \text{for } x \in S\left(y, \frac{1}{2}\delta_D(y)\right) \text{ with } \delta_D(y) \approx R, \tag{6.1}$$

where the constant of comparison depends only on  $D$  and  $A_5$ .

By  $\omega(x, E, U)$  we denote the harmonic measure of  $E$  for an open set  $U$  evaluated at  $x$ . The box argument in [2, Lemma 2] (see [9] for the original form) gives the following estimate of the harmonic measure.

LEMMA 6.1. *Let  $\xi \in \partial D$  have a system of local reference points  $y_1, \dots, y_N \in D \cap S(\xi, R)$  of order  $N$  with factor  $\eta$  for  $0 < R < R_\xi$ . If  $x \in D \cap \overline{B(\xi, \eta^3 R)}$ , then*

$$\omega(x, D \cap S(\xi, \eta^2 R), D \cap B(\xi, \eta^2 R)) \leq AR^{n-2} \sum_{i=1}^N G_R(x, y_i), \tag{6.2}$$

where  $A$  depends only on  $n, c_J, R_\xi$  and  $A_\xi$ .

PROOF. Let us begin with an estimate of harmonic measure in a John domain. For  $0 < r < \delta_D(K_0)$  let  $U(r) = \{x \in D : \delta_D(x) < r\}$ . Then each point  $x \in U(r)$  can be connected to  $K_0$  by a curve such that (1.1) holds. Hence,  $B(x, A_6 r) \setminus U(r)$  includes a ball with radius  $r$ , provided  $A_6$  is large. This implies that

$$\omega(x, U(r) \cap S(x, A_6 r), U(r) \cap B(x, A_6 r)) \leq 1 - \varepsilon_0 \quad \text{for } x \in U(r)$$

with  $0 < \varepsilon_0 < 1$  depending only on  $A_6$  and the dimension. Let  $R \geq r$  and repeat this argument with the maximum principle. Then there exist positive constants  $A_7$  and  $A_8$  such that

$$\omega(x, U(r) \cap S(x, R), U(r) \cap B(x, R)) \leq \exp(A_7 - A_8 R/r). \tag{6.3}$$

See [2, Lemma 1] for details.

Let  $0 < R < R_\xi$ . For each  $x \in D \cap \overline{B(\xi, \eta R)}$  there is a local reference point  $y(x) \in \{y_1, \dots, y_N\}$  such that

$$k_{D_R}(x, y(x)) \leq A_\xi \log \frac{R}{\delta_D(x)} + A_\xi$$

by definition. Let  $y'(x) \in S(y(x), \frac{1}{2}\delta_D(y(x)))$ . Then we observe that  $k_{D_R \setminus \{y(x)\}}(x, y'(x)) \leq A_\xi \log(R/\delta_D(x)) + A_\xi$ . Letting  $u(x) = R^{n-2} \sum_{i=1}^N G_R(x, y_i)$ , we obtain from (5.1) and (6.1) that

$$u(x) \geq A \left( \frac{\delta_D(x)}{R} \right)^\lambda \quad \text{for } x \in D \cap \overline{B(\xi, \eta R)}$$

with some  $\lambda > 0$  depending only on  $n, c_J, R_\xi$  and  $A_\xi$ . Let  $D_j = \{x \in \tilde{D}_R : \exp(-2^{j+1}) \leq u(x) < \exp(-2^j)\}$  and  $U_j = \{x \in \tilde{D}_R : u(x) < \exp(-2^j)\}$ . Then we see that

$$U_j \cap B(\xi, \eta R) \subset \left\{ x \in D : \delta_D(x) < AR \exp\left(-\frac{2^j}{\lambda}\right) \right\}.$$

Define a decreasing sequence  $R_j$  by  $R_0 = \eta^2 R$  and

$$R_j = \left( \eta^2 - \frac{6(\eta^2 - \eta^3)}{\pi^2} \sum_{k=1}^j \frac{1}{k^2} \right) R \quad \text{for } j \geq 1.$$

Let  $\omega_0 = \omega(\cdot, D \cap S(\xi, \eta^2 R), D \cap B(\xi, \eta^2 R))$  and put

$$d_j = \begin{cases} \sup_{x \in D_j \cap B(\xi, R_j)} \frac{\omega_0(x)}{u(x)} & \text{if } D_j \cap B(\xi, R_j) \neq \emptyset, \\ 0 & \text{if } D_j \cap B(\xi, R_j) = \emptyset. \end{cases}$$

It is sufficient to show that  $d_j$  is bounded by a constant independent of  $R$  and  $j$ , since  $R_j > \eta^3 R$  for all  $j \geq 0$ . Apply the maximum principle to  $U_j \cap B(\xi, R_{j-1})$  to obtain

$$\omega_0(x) \leq \omega(x, U_j \cap S(\xi, R_{j-1}), U_j \cap B(\xi, R_{j-1})) + d_{j-1}u(x) \quad \text{for } x \in U_j \cap B(\xi, R_{j-1}).$$

Divide the both sides by  $u(x)$  and take the supremum over  $D_j \cap B(\xi, R_j)$ . Then (6.3) yields

$$d_j \leq \exp \left( 2^{j+1} + A_7 - A_8 \frac{R_{j-1} - R_j}{AR \exp(-2^j/\lambda)} \right) + d_{j-1},$$

provided  $j$  is so large, say  $j \geq j_0$ , that

$$\frac{R_{j-1} - R_j}{AR \exp(-2^j/\lambda)} = \frac{6(\eta^2 - \eta^3) \exp(2^j/\lambda)}{\pi^2 A j^2} \geq 1.$$

Hence, for  $j \geq j_0$ ,

$$d_j \leq d_{j_0-1} + \sum_{j=j_0}^{\infty} \exp \left( 2^{j+1} + A_7 - A_8 \frac{6(\eta^2 - \eta^3) \exp(2^j/\lambda)}{\pi^2 A j^2} \right) < \infty.$$

For  $j \leq j_0$  we have  $d_j \leq \exp(2^{j+1}) \leq \exp(2^{j_0+1})$ . Hence we obtain  $\sup_{j \geq 0} d_j < \infty$ . Thus (6.2) follows. □

LEMMA 6.2. *Let  $\xi \in \partial D$  have a system of local reference points  $y_1, \dots, y_N \in D \cap S(\xi, R)$  of order  $N$  with factor  $\eta$  for  $0 < R < R_\xi$ . If  $x \in D \cap B(\xi, \eta^3 R)$  and  $y \in D \cap S(\xi, \eta^{-3} R)$ , then*

$$G_R(x, y) \leq AR^{n-2} \sum_{i=1}^N G_R(x, y_i) \sum_{j=1}^N G_R(y_j, y), \tag{6.4}$$

where  $A$  depends only on  $n, c_J, R_\xi$  and  $A_\xi$ .

PROOF. Apply Lemma 5.3 to  $h(x) = G_R(x, y)$  with  $y \in D \cap S(\xi, \eta^{-3} R)$ . Then

$$G_R(x, y) \leq A \sum_{j=1}^N h(y_j) \quad \text{for } x \in D \cap S(\xi, \eta^2 R).$$

Hence (6.2) yields

$$G_R(x, y) \leq AR^{n-2} \sum_{i=1}^N G_R(x, y_i) \sum_{j=1}^N h(y_j) \quad \text{for } x \in D \cap \overline{B(\xi, \eta^3 R)}$$

by the maximum principle. The lemma follows.  $\square$

For further arguments we need the following improvement of (6.4): If  $x \in D \cap S(\xi, \eta^9 R)$  and  $y \in D \cap S(\xi, \eta^{-3} R)$ , then

$$G_R(x, y) \leq AR^{n-2} \sum_{i=1}^N G_R(x, y_i) G_R(y_i, y) \tag{6.5}$$

where  $A$  depends only on  $n, c_J, R_\xi$  and  $A_\xi$ . Note that the cross terms  $G_R(x, y_i) G_R(y_j, y)$  ( $i \neq j$ ) disappear from the right hand side of (6.4).

If  $N = 1$ , then (6.5) is nothing but (6.4). If  $N \leq 2$ , then Ancona's ingenious trick [6, Théorème 7.3] gives (6.5) from (6.4). However, the proof is rather complicated and we postpone the proof to the next section. The remaining arguments are rather easy and hold for arbitrary  $N \geq 1$ , provided (6.5) holds. Let us show the weak boundary Harnack principle defined by Ancona [6, Définition 2.3].

LEMMA 6.3 (Weak Boundary Harnack Principle). *Let  $\xi \in \partial D$  have a system of local reference points  $y_1, \dots, y_N \in D \cap S(\xi, R)$  of order  $N$  with factor  $\eta$  for  $0 < R < R_\xi$ . Moreover, suppose (6.5) holds. Let  $h_0, h_1, \dots, h_N \in \mathcal{H}_\xi$ . Then*

$$h_0(x) \leq A \sum_{i=1}^N \frac{h_0(y_i)}{h_i(y_i)} h_i(x) \quad \text{for } x \in D \setminus B(\xi, \eta^9 R) \tag{6.6}$$

where  $A$  depends only on  $n, c_J, R_\xi$  and  $A_\xi$ .

PROOF. In (6.5) we replace the roles of  $x$  and  $y$  and write  $z$  for  $y$ . By dilation and changing  $A_5$  we obtain from the symmetry of the Green function that if  $x \in D \cap S(\xi, \eta^9 R)$  and  $z \in D \cap S(\xi, \eta^{21} R)$ , then

$$G_R(x, z) \leq AR^{n-2} \sum_{i=1}^N G_R(x, z_i) G_R(z_i, z),$$

where  $z_1, \dots, z_N \in D \cap S(\xi, \eta^{12} R)$  are local reference points. Moreover, for each  $z_i$  we find a local reference point  $y_{j(i)} \in D \cap S(\xi, R)$  such that  $k_{\tilde{D}_R \setminus \{x, z\}}(z_i, y_{j(i)}) \leq A$ . In view of (5.1), we have  $G_R(x, z_i) \approx G_R(x, y_{j(i)})$  and  $G_R(z_i, z) \approx G_R(y_{j(i)}, z)$ , whenever

$x \in D \cap S(\xi, \eta^9 R)$  and  $z \in D \cap S(\xi, \eta^{21} R)$ . Hence we obtain that if  $x \in D \cap S(\xi, \eta^9 R)$  and  $z \in D \cap S(\xi, \eta^{21} R)$ , then

$$G_R(x, z) \leq AR^{n-2} \sum_{i=1}^N G_R(x, y_i) G_R(y_i, z). \tag{6.7}$$

Let  $r = \eta^{-3} R$  and  $\rho = \eta^{21} R$ . Observe that the regularized reduced function  $\widehat{R}_{h_0}^{D \cap (S(\xi, r) \cup S(\xi, \rho))}$  with respect to  $\widetilde{D}_R$  is a Green potential of measures  $\mu$  concentrated on  $D \cap S(\xi, r)$  and  $\nu$  on  $D \cap S(\xi, \rho)$  such that  $\widehat{R}_{h_0}^{D \cap (S(\xi, r) \cup S(\xi, \rho))} = h_0$  on  $D \cap B(\xi, r) \setminus \overline{B(\xi, \rho)}$ . It follows from (6.5) and (6.7) that for  $x \in D \cap S(\xi, \eta^9 R)$ ,

$$\begin{aligned} h_0(x) &= \int_{D \cap S(\xi, r)} G_R(x, y) d\mu(y) + \int_{D \cap S(\xi, \rho)} G_R(x, z) d\nu(z) \\ &\leq AR^{n-2} \sum_{i=1}^N \left( \int_{D \cap S(\xi, r)} G_R(x, y_i) G_R(y_i, y) d\mu(y) \right. \\ &\quad \left. + \int_{D \cap S(\xi, \rho)} G_R(x, y_i) G_R(y_i, z) d\nu(z) \right) \\ &= AR^{n-2} \sum_{i=1}^N G_R(x, y_i) h_0(y_i). \end{aligned}$$

Let  $\varepsilon = 1 - \eta^9$ . Observe from (6.1) and the Harnack inequality that  $h_i(y_i) R^{n-2} G_R(x, y_i) \approx h_i(x)$  for  $x \in S(y_i, \varepsilon \delta_D(y_i))$ , and so  $h_i(y_i) R^{n-2} G_R(x, y_i) \leq Ah_i(x)$  for  $x \in D \cap S(\xi, \eta^9 R) \subset \widetilde{D}_R \setminus B(y_i, \varepsilon \delta_D(y_i))$  by the maximum principle. Hence (6.6) follows for  $x \in D \setminus B(\xi, \eta^9 R)$  by the maximum principle.  $\square$

PROOF OF PROPOSITION 2.3 (ii) FOR  $N \leq 2$ . Obviously (6.5) holds for  $N = 1$ . As we shall show in the next section, (6.5) holds for  $N = 2$ . Hence Lemma 6.3 is applicable. Varying  $R$  in Lemma 6.3, we obtain relationships among kernel functions in  $\mathcal{H}_\xi$  (cf. Lemma 5.4), which yield Proposition 2.3. This procedure is the same as in Ancona [6, Théorème 2.5] and we omit the details.  $\square$

REMARK 6.1. We do not know whether the weak boundary Harnack principle holds for  $N \geq 3$ . In special cases, such as a sectorial domain whose boundary lies on  $N$  rays leaving  $\xi$ , we can apply the weak boundary Harnack principle repeatedly to subdomains containing just one ray and conclude the weak boundary Harnack principle for the sectorial domain itself (cf. Cranston and Salisbury [12, p. 36]).

### 7. Proof of (6.5).

In this section we shall prove the following:

LEMMA 7.1. *Let  $\xi \in \partial D$  have a system of local reference points  $y_1, y_2 \in D \cap S(\xi, R)$*

of order 2 with factor  $\eta$  for  $0 < R < R_\xi$ . If  $x \in D \cap S(\xi, \eta^9 R)$  and  $y \in D \cap S(\xi, \eta^{-3} R)$ , then (6.5) holds.

In order to apply (5.1) to the Green function, we need the following elementary lemma.

LEMMA 7.2. *Let  $\Omega$  be a subdomain of  $\mathbf{R}^n$  and let  $z \in \Omega$ . Then*

$$k_{\Omega \setminus \{z\}}(x, y) \leq \pi + 3k_\Omega(x, y) \quad \text{for } x, y \in \Omega \setminus B\left(z, \frac{1}{2}\delta_D(z)\right).$$

PROOF. We first claim that

$$\delta_\Omega(w) \leq 3\delta_{\Omega \setminus \{z\}}(w) \quad \text{for } w \in \Omega \setminus B\left(z, \frac{1}{2}\delta_\Omega(z)\right).$$

Indeed, let  $w \in \Omega \setminus B(z, 2^{-1}\delta_\Omega(z))$ . Observe that  $\delta_{\Omega \setminus \{z\}}(w) = \min\{\delta_\Omega(w), |z - w|\}$ . If  $\delta_{\Omega \setminus \{z\}}(w) = \delta_\Omega(w)$ , then there is nothing to prove. Otherwise,  $\delta_{\Omega \setminus \{z\}}(w) = |z - w| \geq 2^{-1}\delta_\Omega(z)$ , so that

$$\delta_\Omega(w) \leq \delta_\Omega(z) + |z - w| \leq 3\delta_{\Omega \setminus \{z\}}(w).$$

Now let  $\gamma$  be an arbitrary rectifiable curve in  $\Omega$  connecting  $x$  to  $y$ . If  $\gamma \subset \Omega \setminus B(z, 2^{-1}\delta_\Omega(z))$ , then the claim shows that

$$\int_\gamma \frac{ds(w)}{\delta_{\Omega \setminus \{z\}}(w)} \leq 3 \int_\gamma \frac{ds(w)}{\delta_\Omega(w)}.$$

Suppose  $\gamma \setminus B(z, 2^{-1}\delta_\Omega(z)) \neq \emptyset$ . Let  $w_1$  and  $w_2$  be the first hit of  $\gamma$  to  $S(z, 2^{-1}\delta_\Omega(z))$  and the last hit, respectively. Observe that  $w_1$  and  $w_2$  can be connected by the circle  $\gamma_1$  in  $S(z, 2^{-1}\delta_\Omega(z))$  whose length is at most  $\pi 2^{-1}\delta_\Omega(z)$ . Let  $\gamma' = \gamma(x, w_1) \cup \gamma_1 \cup \gamma(w_2, y)$ , where  $\gamma(x, w_1)$  (resp.  $\gamma(w_2, y)$ ) is the subcurve of  $\gamma$  connecting  $x$  and  $w_1$  (resp.  $w_2$  and  $y$ ). It follows from the above claim that

$$\int_{\gamma'} \frac{ds(w)}{\delta_{\Omega \setminus \{z\}}(w)} \leq \int_{\gamma_1} \frac{ds(w)}{|z - w|} + 3 \int_{\gamma(x, w_1) \cup \gamma(w_2, y)} \frac{ds(w)}{\delta_\Omega(w)} \leq \pi + 3 \int_\gamma \frac{ds(w)}{\delta_\Omega(w)}.$$

Taking the infimum with respect to  $\gamma$ , we obtain the lemma. □

We employ Ancona's trick [6, Théorème 7.3]. Since our setting is slightly different from Ancona's, we provide a proof for the sake of the reader's convenience.

PROOF OF LEMMA 7.1. Besides the local reference points  $y_1, y_2 \in D \cap S(\xi, R)$ , we take local reference points  $y_1^*, y_2^* \in D \cap S(\xi, \eta^6 R)$  with

$$\min_{i=1,2} \{k_{D \cap B(\xi, \eta^3 R)}(x, y_i^*)\} \leq A_\xi \log \frac{\eta^6 R}{\delta_D(x)} + A_\xi \quad \text{for } x \in D \cap \overline{B(\xi, \eta^7 R)}.$$

Then

$$\min_{j=1,2} \{k_{D_R}(y_i^*, y_j)\} \leq A_\xi \log \frac{R}{\delta_D(y_i^*)} + A_\xi \leq A_\xi.$$

So, we may assume either

$$k_{D_R}(y_1^*, y_1) \leq A \quad \text{and} \quad k_{D_R}(y_2^*, y_1) \leq A, \tag{7.1}$$

or

$$k_{D_R}(y_1^*, y_1) \leq A \quad \text{and} \quad k_{D_R}(y_2^*, y_2) \leq A, \tag{7.2}$$

by replacing the roles of  $y_1$  and  $y_2$ , if necessary.

First consider the case when (7.1) holds. Let  $x \in D \cap S(\xi, \eta^9 R)$  and suppose  $y \in D \cap S(\xi, \eta^3 R)$ . Then  $y_1^*, y_2^*, y_1 \in D_R \setminus (B(x, 2^{-1}\delta_{D_R}(x)) \cup B(y, 2^{-1}\delta_{D_R}(y)))$ . By (7.1) and Lemma 7.2 we have  $k_{D_R \setminus \{x\}}(y_i^*, y_1) \leq A$  and  $k_{D_R \setminus \{y\}}(y_i^*, y_1) \leq A$  for  $i = 1, 2$ . Hence (5.1) and (6.4) for  $y_1^*$  and  $y_2^*$  yield

$$G_R(x, y) \leq AR^{n-2} \sum_{i,j} G_R(x, y_i^*) G_R(y_j^*, y) \leq AR^{n-2} G_R(x, y_1) G_R(y_1, y).$$

By the maximum principle the same inequality holds for  $y \in D_R \setminus B(\xi, \eta^3 R)$ , and in particular for  $y \in D \cap S(\xi, \eta^{-3} R)$ . Hence the lemma follows in this case.

Next consider the case when (7.2) holds. Let  $\Phi = \{z \in \tilde{D}_R : G_R(z, y_1) \geq G_R(z, y_2)\}$ . If either  $x, y \in \Phi$  or  $x, y \in \tilde{D}_R \setminus \Phi$ , then (6.5) follows from (6.4). Let us consider the remaining cases. If necessary, exchanging the roles of  $y_1$  and  $y_2$ , we may assume that  $x \in \Phi \cap S(\xi, \eta^9 R)$  and  $y \in (\tilde{D}_R \setminus \Phi) \cap S(\xi, \eta^{-3} R)$ . Let  $E = \Phi \setminus B(\xi, \eta^3 R)$  and consider the regularized reduced function  $\widehat{R}_{G_R(\cdot, y)}^E$  with respect to  $\tilde{D}_R$ . This function is represented as the Green potential of a measure  $\mu$  concentrated on  $\partial E$ . For a moment let  $z \in E$ . Then we have from (6.4) for  $y_1^*, y_2^*$  and the maximum principle

$$G_R(x, z) \leq AR^{n-2} \sum_{i,j} G_R(x, y_i^*) G_R(y_j^*, z). \tag{7.3}$$

It is easy to see from (7.2) that  $k_{D_R \setminus \{x\}}(y_i^*, y_i) \leq A$ , so that  $G_R(x, y_i^*) \leq AG_R(x, y_i)$  for  $i = 1, 2$  by (5.1). We also have  $G_R(y_j^*, z) \leq AG_R(y_j, z)$  for  $j = 1, 2$ . In fact, if  $z \in B\left(y_j, \frac{1-\eta^6}{2}\delta_D(y_j)\right)$ , then  $G_R(y_j, z) \approx |y_j - z|^{2-n} \geq AR^{2-n} \geq AG_R(y_j^*, z)$ ; if  $z \in \tilde{D}_R \setminus B\left(y_j, \frac{1-\eta^6}{2}\delta_D(y_j)\right)$ , then (7.2) gives  $k_{D_R \setminus \{z\}}(y_j^*, y_j) \leq A$ , and hence  $G_R(y_j^*, z) \approx G_R(y_j, z)$  by (5.1). Hence (7.3) becomes

$$G_R(x, z) \leq AR^{n-2} \sum_{i,j} G_R(x, y_i) G_R(y_j, z) \leq AR^{n-2} G_R(x, y_1) G_R(y_1, z)$$

by the definition of  $\Phi$ . Therefore

$$\begin{aligned} \widehat{R}_{G_R(\cdot, y)}^E(x) &\leq AR^{n-2}G_R(x, y_1) \int_E G_R(y_1, z) d\mu(z) \\ &= AR^{n-2}G_R(x, y_1) \widehat{R}_{G_R(\cdot, y)}^E(y_1) \leq AR^{n-2}G_R(x, y_1)G_R(y_1, y). \end{aligned} \quad (7.4)$$

Let  $v_y = G_R(\cdot, y) - \widehat{R}_{G_R(\cdot, y)}^E$ . Then

$$v_y = 0 \quad \text{q.e. on } E = \Phi \setminus B(\xi, \eta^3 R). \quad (7.5)$$

By (6.4) we have

$$v_y(z) \leq G_R(z, y) \leq AR^{n-2}G_R(z, y_2)G_R(y_2, y) \quad \text{for } z \in D \cap \partial\Phi \cap \overline{B(\xi, \eta^3 R)}. \quad (7.6)$$

Observe that

$$D \cap \partial(\Phi \cap B(\xi, \eta^3 R)) \subset (\Phi \setminus B(\xi, \eta^3 R)) \cup (D \cap \partial\Phi \cap \overline{B(\xi, \eta^3 R)}).$$

Hence (7.5), (7.6) and the maximum principle yield

$$v_y \leq AR^{n-2}G_R(\cdot, y_2)G_R(y_2, y) \quad \text{on } \Phi \cap \overline{B(\xi, \eta^3 R)}.$$

This, together with (7.4), implies

$$G_R(x, y) \leq AR^{n-2}(G_R(x, y_1)G_R(y_1, y) + G_R(x, y_2)G_R(y_2, y)).$$

The proof is complete. □

### 8. Sharpness of Theorem 1.2.

In this section we give two examples to demonstrate the sharpness of the bounds  $\theta_1 \leq \sin^{-1}(1/A_0)$  and  $\rho_1 \leq \rho_0 \cos \theta_1$ . Each example satisfies (I) in Section 1 and (1.2) with  $\rho_1$  and  $\theta_1$  violating the bounds; and yet there are two minimal Martin boundary points at the origin. For simplicity we let  $n = 2$ ,  $\rho_0 = 1$  and  $\theta_0 = \sin^{-1}(1/A_0)$  with  $A_0 > 1$ . Write  $\mathbf{R}_+^2 = \{(x_1, x_2) \in \mathbf{R}^2 : x_2 > 0\}$  and  $\mathbf{R}_-^2 = \{(x_1, x_2) \in \mathbf{R}^2 : x_2 < 0\}$ . For  $z, w \in \mathbf{R}^2$  with  $|z - w| = A_0$  we let  $V(z, w) = \text{co}(\{z\} \cup B(w, 1))$ , where we recall that  $\text{co}(\{z\} \cup B(w, 1))$  is the interior of the convex hull of  $\{z\} \cup B(w, 1)$ . Obviously,  $B(w, 1) \subset V(z, w) \subset B(w, A_0)$ . Our domains  $D$  in the following examples will be given as unions of open balls of radius 1 and  $V(z, w)$  with suitable  $z$  and  $w$ . Hence (I) will be satisfied. Let us recall

$$\mathcal{C}(0) = \bigcup_{\substack{y \in D \\ \Gamma_{\theta_1}(0, y) \cap B(0, 2\rho_1) \subset D}} \Gamma_{\theta_1}(0, y) \cap B(0, 2\rho_1).$$

For both examples, we shall show  $\mathcal{C}(0) = B(0, 2\rho_1) \cap \mathbf{R}_-^2$ , a connected set. Thus (1.2) will hold.

EXAMPLE 8.1. The case when  $\theta_1 > \theta_0$  and  $\rho_1 > 0$ . We may assume that  $0 < \rho_1 < 1$ . Let  $z^0 = (0, A_0)$  and

$$D = V(0, z^0) \cup \left( B(0, A_0 + 1) \setminus \overline{(B(0, A_0 - 1) \cap \mathbf{R}_+^2)} \right).$$

See Figure 8.1. Obviously, there are two minimal Martin boundary points at 0. Since  $D = V(0, z^0) \cup (\bigcup_{z \in E_1} B(z, 1))$  with  $E_1 = S(0, A_0) \cup (B(0, A_0) \cap \{(x_1, x_2) \in \mathbf{R}^2 : x_2 \leq -1\})$ , it follows that  $D$  satisfies (I). It is easy to see that  $B(0, 2\rho_1) \cap \mathbf{R}_-^2 \subset \mathcal{C}(0)$ . By an elementary geometrical observation  $V(0, z^0) \subset \Gamma_{\theta_0}(0, z^0)$ , so that  $\Gamma_{\theta_1}(0, y) \cap B(0, 2\rho_1)$  is not included in  $V(0, z^0)$  for  $\theta_1 > \theta_0$ . Hence  $\mathcal{C}(0) = B(0, 2\rho_1) \cap \mathbf{R}_-^2$ , so that (1.2) holds.

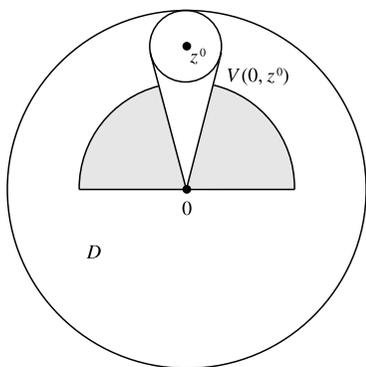


Figure 8.1. Example 8.1:  
 $\theta_1 > \theta_0$  and  $\rho_1 > 0$ .

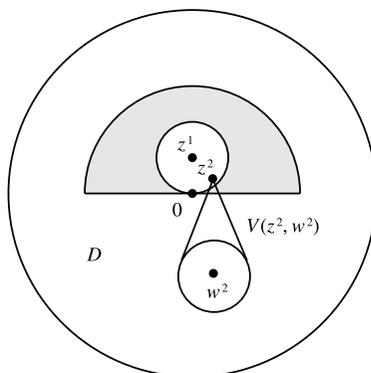


Figure 8.2. Example 8.2:  
 $0 < \theta_1 \leq \theta_0$  and  $\rho_1 > \cos \theta_1$ .

EXAMPLE 8.2. The case when  $0 < \theta_1 \leq \theta_0$  and  $\cos \theta_1 < \rho_1 < 1$ . Let  $z^1 = (0, 1)$  and we choose a point  $z^2 \in B(z^1, 1)$  such that

$$V(z^2, w^2) \cap \mathbf{R}_+^2 \subset B(0, 2 \cos \theta_0) \tag{8.1}$$

and  $0 \notin \overline{V(z^2, w^2)}$ , where  $w^2 = (z_1^2, z_2^2 - A_0)$ . Define

$$D = V(z^2, w^2) \cup B(z^1, 1) \cup \left( B(0, 5) \setminus \overline{(B(0, 3) \cap \mathbf{R}_+^2)} \right).$$

See Figure 8.2. Since  $D = V(z^2, w^2) \cup (\bigcup_{z \in E_2} B(z, 1))$  with  $E_2 = \{z^1\} \cup S(0, 4) \cup (B(0, 4) \cap \{(x_1, x_2) \in \mathbf{R}^2 : x_2 \leq -1\})$ , it follows that  $D$  satisfies (I). There are two minimal Martin boundary points at 0 since  $0 \notin \overline{V(z^2, w^2)}$ . It is easy to see that  $B(0, 2\rho_1) \cap \mathbf{R}_-^2 \subset \mathcal{C}(0)$ . Observe that  $\Gamma_{\theta_1}(0, y) \cap B(0, 2\rho_1)$  consists of rays of length  $2\rho_1$  issuing from the origin; while  $\Gamma_{\theta_1}(0, z^1) \cap B(z^1, 1)$  consists of rays of length in between 2 and  $2 \cos \theta_1$

and  $B(z^1, 1) \setminus \Gamma_{\theta_1}(0, z^1)$  consists of rays of length not greater than  $2 \cos \theta_1$ . Since  $\rho_1 > \cos \theta_1 \geq \cos \theta_0$ , we infer from (8.1) that if  $y \in \mathbf{R}_+^2$ , then  $\Gamma_{\theta_1}(0, y) \cap B(0, 2\rho_1)$  cannot be included in  $D$ . Hence  $\mathcal{C}(0) = B(0, 2\rho_1) \cap \mathbf{R}_-^2$ , so that (1.2) holds.

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Hiroaki AIKAWA

Department of Mathematics

Hokkaido University

Sapporo 060-0810

Japan

E-mail: aik@math.sci.hokudai.ac.jp

Kentaro HIRATA

Department of Mathematics

Hokkaido University

Sapporo 060-0810

Japan

E-mail: hirata@math.sci.hokudai.ac.jp

Torbjörn LUNDH

Department of Mathematics

Chalmers University of Technology

SE-412 96 Göteborg

Sweden

E-mail: torbjrn@math.chalmers.se