Japan J. Indust. Appl. Math., 24 (2007), 79–104 Area ⟨1⟩

Propagation Speeds of Traveling Fronts for Higher Order Autocatalytic Reaction-Diffusion Systems

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Received July 21, 2006
Revised October 13, 2006

This paper investigates the existence of traveling fronts and their propagation speeds for the two component higher order autocatalytic reaction-diffusion systems with any diffusion coefficients. Our elementary analysis of the vector fields in the phase space gives the estimate of the minimal propagation speeds in terms of the order of autocatalysis and the diffusion coefficients.

Key words: traveling fronts, propagation speed, autocatalytic reaction, phase space

1. Introduction

Autocatalytic reaction-diffusion systems including the Brusselator [20], the Field–Noyes model [7] and the Gray–Scott model [11], have stimulated an extensive amount of theoretical studies on waves and patterns produced by chemical reactions (see for example, [15]). One of the basic elements responsible for chemical pattern formation is traveling waves which describe the development of chemical processes. The papers by Needham et al. ([2]–[5], [16]–[19]) studied extensively the traveling waves in autocatalytic reactions. Focant and Gallay [8] and Hosono and Kawahara [14] also discussed the traveling waves for the mixed order autocatalytic two component systems and their minimal propagation speeds. The similar type of traveling waves appears in the combustion problem and the speeds of combustion waves were discussed by the several authors ([9], [23] and the references therein). Our concern is traveling fronts and their speeds for the higher order autocatalytic reaction-diffusion system of the form:

$$\begin{cases} u_t = d_1 u_{xx} - k_1 u v^m, \\ v_t = d_2 v_{xx} + k_2 u v^m, \end{cases} \tag{1}$$

where u and v are concentrations of the reactant and the autocatalyst respectively, d_1 and d_2 are diffusion coefficients, and k_1 and k_2 are any positive constants.

Then, traveling front solutions for (1) are defined as follows. The nonnegative bounded functions of the form $(u(x,t), v(x,t)) = (U(z), V(z))$ with $z = x - ct$ are

*This work was in part supported by Grant-in-Aid for Scientific Research No. 14540143 and No. 18540144 Japan Society for the Promotion of Science.

said to be traveling front solutions for (1) when they satisfy the equations

$$\begin{cases} d_1U'' + cU' - k_1UV^m = 0, \\ d_2V'' + cV' + k_2UV^m = 0, \end{cases} \tag{2}$$

with the boundary conditions

$$P_- \equiv (U(-\infty), V(-\infty)) = (0, v_0), \quad P_+ \equiv (U(+\infty), V(+\infty)) = (u_0, 0). \tag{3}$$

Here u_0 and v_0 are positive, and $'$ denotes d/dz. This condition is imposed so that on the far right there is only reactant and on the far left there is only autocatalyst, and these two states should be the critical states of the system (2).

By applying the comparison argument, Takase and Sleeman [22] proved that there exists the minimal wave speed c^* such that traveling front solutions for (1) exist for each $c \geq c^*$ assuming that u_0 is sufficiently small for $m > 1$. The purpose of this paper is to discuss the properties of the minimal wave speed c^*, especially the dependence of c^* on the parameters m, d_1 and d_2 for arbitrarily fixed $u_0 > 0$. The method of the proofs employed here is the shooting argument which looks for the connection orbits of (2) and (3) in the 3-dim phase space. Throughout this paper, we always assume that $m > 1$ without notice.

In the next section, we present the preliminary results required for the later discussions. In Section 3, we investigate the existence of traveling fronts for (1) and their minimal propagation speeds when $0 < d_1 < d_2$. In Section 4, we study the same problem when $d_1 > d_2 > 0$.

2. The preliminary results

We first write the equations in the dimensionless form. Introducing the dimensionless dependent variables $\tilde{u} = u/u_0$, $\tilde{v} = v/(ru_0)$ with $r = k_2/k_1$ and scaling the independent variables by $\tilde{t} = k_1(ru_0)^m t$, $\tilde{x} = \sqrt{k_1(ru_0)^m}x$, we write (1) as

$$\begin{cases} u_t = d_1u_{xx} - uv^m, \\ v_t = d_2v_{xx} + uv^m, \end{cases} \tag{4}$$

where $\tilde{}$ is omitted for the simplicity of notations. We may also suppose that either of d_1 and d_2 is 1 by scaling the independent variable x. The corresponding traveling equations are

$$\begin{cases} d_1U'' + cU' - UV^m = 0, \\ d_2V'' + cV' + UV^m = 0, \end{cases} \tag{5}$$

with the boundary conditions

$$P_- \equiv (U(-\infty), V(-\infty)) = (0, \beta), \quad P_+ \equiv (U(+\infty), V(+\infty)) = (1, 0). \tag{6}$$

Here β is a constant to be determined.

We already know the following properties of the traveling front solutions for (4), that is, the solutions of (5) and (6).

Proposition 1 ([3]). *Assume that there exists a traveling front solution* $(U(z), V(z))$ *for* (4)*. Then it satisfies the followings for all* $z \in \mathbb{R}$*.*
(i) $0 < U < 1$, $0 < V < 1$.
(ii) $0 < U' < +\infty$, $-\infty < V' < 0$.
(iii) $U + V - 1 \begin{cases} > 0, & \text{for } d_2 > d_1 \geq 0, \\ = 0, & \text{for } d_1 = d_2, \\ < 0, & \text{for } d_1 > d_2 \geq 0. \end{cases}$

Furthermore, $\lim_{z \to -\infty}(U(z), V(z)) = (0, 1)$*, that is* $\beta = 1$*, and* $c > 0$*.*

For the equal diffusion case: $d_1 = d_2 = 1$, we know the following theorem.

Theorem 2 ([21]). *Assume that* $d_1 = d_2 = 1$*. Then, there exists some positive* c_1^* *such that only for each* $c \geq c_1^*$*,* (1) *has a unique traveling front solution. Furthermore, the minimal wave speed* c_1^* *satisfies that*

$$\frac{2}{m(m+1)} \leq {c_1^*}^2 \leq \frac{2}{(m-1)m}. \tag{7}$$

For the extreme case: $d_1 = 0$, we may assume $d_2 = 1$ without loss of generality, and have the following result.

Theorem 3 ([13]). *Assume that* $d_1 = 0$*,* $d_2 = 1$*. Then, there exists* c_0^* *such that only for each* $c \geq c_0^*$*,* (1) *has a unique traveling front solution. Furthermore, the minimal wave speed* c_0^* *satisfies*

$$\frac{1}{m} < {c_0^*}^2 \leq \frac{1}{m-1}. \tag{8}$$

For another extreme case: $d_2 = 0$, we may assume $d_1 = 1$ without loss of generality, and easily have the following result.

Theorem 4. *Assume that* $d_1 = 1$*,* $d_2 = 0$*. Then, there exists a unique traveling front solution for* (1) *for each positive* c*.*

Proof. See Appendix.

In the next two sections, on the basis of these results, we discuss the general case that the both diffusion coefficients are not zero. To do so, we require some definitions and the Wazewski theorem which was formulated by Dunbar [6]. Let $\mathbf{y}(z; \mathbf{y}_0)$ be a solution of the initial value problem

$$\mathbf{y}' = \mathbf{f}(\mathbf{y}), \quad \mathbf{y}(0) = \mathbf{y}_0, \tag{9}$$

where $' = d/dz$, $\mathbf{y}, \mathbf{y}_0 \in \mathbb{R}^n$, and $\mathbf{f}(\mathbf{y})$ is Lipschitz continuous. Set $\mathbf{y}(z; \mathbf{y}_0) = \mathbf{y}_0 \cdot z$ and $V \cdot Z = \{\mathbf{y}_0 \cdot z \colon \mathbf{y}_0 \in V,\, z \in Z\}$. Given $W \subseteq \mathbb{R}^n$, let $W^- = \{\mathbf{y}_0 \in W \colon$ any $z > 0$, $\mathbf{y}_0 \cdot [0, z) \not\subset W\}$. W^- is called the immediate exit set. Given $\Sigma \subseteq W$, let

$\Sigma^0 = \{\mathbf{y}_0 \in \Sigma$: there is an $z_0 = z_0(\mathbf{y}_0)$ such that $\mathbf{y}_0 \cdot z_0 \notin W\}$. For $\mathbf{y}_0 \in \Sigma^0$, define $T(\mathbf{y}_0) = \sup\{z\colon \mathbf{y}_0 \cdot [0,z] \subseteq W\}$, which is called an exit time. We denote the closure of a set W by $\mathrm{cl}(W)$.

Proposition 5 ([6]). *Suppose*

1) *If* $\mathbf{y}_0 \in \Sigma$ *and* $\mathbf{y}_0 \cdot [0,z] \subseteq \mathrm{cl}(W)$, *then* $\mathbf{y}_0 \cdot [0,z] \subseteq W$.
2) *If* $\mathbf{y}_0 \in \Sigma$, $\mathbf{y}_0 \cdot z \in W$ *and* $\mathbf{y}_0 \cdot z \notin W^-$, *then there is an open set* V_z *about* $\mathbf{y}_0 \cdot z$ *disjoint from* W^-.
3) $\Sigma = \Sigma^0$, Σ *is compact, and* Σ *intersect an orbit of* (9) *only once in* W.

Then the mapping $F(\mathbf{y}_0) = \mathbf{y}_0 \cdot T(\mathbf{y}_0)$ *is homeomorphism from* Σ *to its image on* W^-.

A set $W \subseteq \mathbb{R}^n$ satisfying (1) and (2) in Proposition 5 is called a Wazewski set.

3. The case $0 < d_1 < d_2$

For $d_1 \geq 0$ and $d_2 > 0$, the system (4) can be written as

$$\begin{cases} u_t = du_{xx} - uv^m, \\ v_t = v_{xx} + uv^m. \end{cases} \tag{10}$$

by the change of the independent variable x, where $d = d_1/d_2$. Then the traveling equations for (10) are

$$\begin{cases} dU'' + cU' - UV^m = 0, \\ V'' + cV' + UV^m = 0. \end{cases} \tag{11}$$

The boundary conditions are specified by (6) with $\beta = 1$.

Adding the above two equations and integrating the resulting equation, we have the relation $dU' + V' + c(U + V - 1) = 0$ with the aid of the boundary condition (6). Set $X = U + V - 1$. Proposition 1 assures that X is positive when $0 < d < 1$. Then (11) is reduced to the first order system

$$\begin{cases} X' = -\dfrac{c}{d}X - \left(\dfrac{1}{d} - 1\right)W, \\ V' = W, \\ W' = -cW - (1 - V + X)V^m. \end{cases} \tag{12}$$

The boundary conditions are

$$\begin{aligned} &(X(-\infty), V(-\infty), W(-\infty)) = (0,1,0), \\ &(X(+\infty), V(+\infty), W(+\infty)) = (0,0,0). \end{aligned} \tag{13}$$

Introducing the new dependent variables by

$$q = V, \quad p = \frac{q'}{q}, \tag{14}$$

we can write (12) as

$$\begin{cases} X' = -\frac{c}{d}X - \left(\frac{1}{d} - 1\right)pq, \\ q' = pq, \\ p' = -p(p+c) - (1-q+X)q^{m-1}. \end{cases} \tag{15}$$

For later use, we denote (15) in the vector form as $\mathbf{u}' = \mathbf{f}_d(\mathbf{u})$ with $\mathbf{u} = (X, q, p)$, and $\mathbf{u}(z; \mathbf{u}_0)$ denotes a solution of (15) satisfying $\mathbf{u}(0; \mathbf{u}_0) = \mathbf{u}_0$.

We should note that the singularity $(0,0,0)$ of (12) is split into two critical points $P_0 = (0,0,0)$ and $P_c = (0,0,-c)$ in (15). Thus, (15) has three critical points $P_0 = (0,0,0)$, $P_c = (0,0,-c)$ and $P_1 = (0,1,0)$. The properties of these critical points are as follows. $P_1 = (0,1,0)$ has the 1-dim unstable manifold and the 2-dim stable manifold. $P_0 = (0,0,0)$ is a topologically stable node. $P_c = (0,0,-c)$ has the 2-dim stable manifold and the 1-dim unstable manifold.

Now our problem is to find an orbit of (15) connecting $P_1 = (0,1,0)$ with $P_0 = (0,0,0)$ or with $P_c = (0,0,-c)$, which lies entirely in $\Omega^+ = \{(X,q,p)\colon X > 0,\ 0 < q < 1,\ p < 0\}$. Here, let $\mathcal{U}_d(c)$ be the part of the unstable manifold of P_1 lying in Ω^+, whose existence will be assured in Section 3.2. Our main goal of this section is to find the values of c for which $\mathcal{U}_d(c)$ approaches $P_0 = (0,0,0)$ or $P_c = (0,0,-c)$.

3.1. The property of the vector field

We first discuss the property of the vector field of (15) and the behavior of the unstable manifold $\mathcal{U}_d(c)$. To do this, we require the information on the case of $d = 0$. For $d = 0$, the first equation of (15) gives $X = -pq/c$ and (15) is reduced to

$$\begin{cases} q' = pq \\ p' = -p(p+c) - \left(1 - q - \frac{pq}{c}\right)q^{m-1}, \end{cases} \tag{16}$$

which has three critical points $(0,0), (1,0)$ and $(0,-c)$. The proof of Theorem 3 assures the followings. There exists a unique orbit of (16) connecting $(1,0)$ to $(0,-c)$ if $c = c_0^*$ and to $(0,0)$ if $c > c_0^*$. This orbit can be represented by $p = \psi_c(q) < 0$ for $0 < q < 1$, and it satisfies that $\psi_c(1) = 0$ for $c \geq c_0^*$, $\psi_{c_0^*}(0) = c_0^*$, and $\psi_c(0) = 0$ for $c > c_0^*$. For c_0^*, let Ω_0^* be the set $\{(q,p)\colon 0 < q < 1,\ \psi_{c_0^*}(q) < p < 0\}$. Then any orbit of (16) starting from a point in Ω_0^* stays in Ω_0^* for $z \geq 0$ and converges to the critical point $(0,0)$ as $z \to \infty$. Also, it holds that $\psi_c'(1) = 1/c$, which is easily derived from the linearized analysis of the critical point $(1,0)$. By the use of the orbit $p = \psi_c(q)$, we introduce the region Ω_1 defined by

$$\Omega_1 \equiv \left\{(X,q,p)\colon 0 < X < -\frac{1}{c}\psi_c(q)q,\ 0 < q < 1,\ \psi_c(q) < p < 0\right\}.$$

Then, the boundary of Ω_1, denoted by $\partial\Omega_1$, consists of the followings:

$$S_1 = \Big\{(X,q,p)\colon 0<q<1,\ X=-\psi_c(q)\frac{q}{c},\ \psi_c(q)<p<0\Big\},$$
$$S_2 = \Big\{(X,q,p)\colon 0<q<1,\ 0<X<-\psi_c(q)\frac{q}{c},\ p=\psi_c(q)\Big\},$$
$$S_3 = \{(X,q,p)\colon 0<q<1,\ X=0,\ \psi_c(q)q<p<0\},$$
$$S_4 = \Big\{(X,q,p)\colon 0<q<1,\ 0<X<-\psi_c(q)\frac{q}{c},\ p=0\Big\},$$
$$J_1 = \{(X,q,p)\colon 0<q<1,\ X=0,\ p=0\},$$
$$J_2 = \{(X,q,p)\colon 0<q<1,\ X=0,\ p=\psi_c(q)\},$$
$$J_3 = \Big\{(X,q,p)\colon 0<q<1,\ X=-\psi_c(q)\frac{q}{c},\ p=0\Big\},$$
$$J_4 = \Big\{(X,q,p)\colon 0<q<1,\ X=-\psi_c(q)\frac{q}{c},\ p=\psi_c(q)\Big\},$$
$$I_0 = \{(X,q,p)\colon X=0,\ q=0,\ \psi_c(0)<p<0\},$$

P_0, P_1 and P_c.

That is,

$$\partial\Omega_1 = \left(\bigcup_{i=1}^{4} S_i\right) \cup \left(\bigcup_{i=1}^{4} J_i\right) \cup I_0 \cup P_0 \cup P_1 \cup P_c.$$

Here, note that $I_0 = \varnothing$ for any $c > c_0^*$.

Proposition 6. *Let d be fixed in $(0,1)$ and $c \geq c_0^*$. Any orbit of (15) starting from a point $\mathbf{u}_0 \in \Omega_1$, denoted by $\mathbf{u}(z;\mathbf{u}_0)$, stays in Ω_1 for all $z \geq 0$.*

Proof. P_0, P_1 and P_c are critical points and I_0 is an invariant manifold, so that $\mathbf{u}(z;\mathbf{u}_0)$ cannot reach any point of the set $I_0 \cup P_0 \cup P_1 \cup P_c$ in a finite time. To prove that $\mathbf{u}(z;\mathbf{u}_0)$ cannot leave Ω_1 through S_i $(i=1,\dots,4)$, it suffices to show that the inner products of the outward normal $\mathbf{n}_i$ of S_i and the vector field $\mathbf{f}_d$ are all negative.

We begin with S_1. On S_1, $\mathbf{n}_1 = (1,(1/c)(\psi_c(q)+q\psi_c'(q)),0)$. Hence we have

$$\begin{aligned}\mathbf{n}_1\cdot\mathbf{f}_d &= -\frac{c}{d}X - \left(\frac{1}{d}-1\right)pq + pq\frac{1}{c}(\psi_c + q\psi_c')\\ &= \frac{1}{d}\psi q - \frac{1}{d}pq + \frac{pq}{c}(c+\psi_c+q\psi_c') \leq \frac{pq}{c}(c+\psi_c+q\psi_c').\end{aligned}$$

It follows from (16) that

$$q\psi_c'(q) = -(\psi_c + c) - \frac{1}{\psi_c}\left(1-q-\frac{\psi_c q}{c}\right)q^{m-1},$$

which gives

$$\mathbf{n}_1\cdot\mathbf{f}_d \leq -\frac{pq}{c\psi_c}\left(1-q-\frac{\psi_c q}{c}\right)q^{m-1} < 0 \quad \text{for any} \quad \mathbf{u}\in S_1.$$

On S_2, $\mathbf{n}_2 = (0, \psi_c'(q), -1)$. Hence we have

$$\begin{aligned}\mathbf{n}_2 \cdot \mathbf{f}_d &= \psi_c' pq + p(p+c) + (1-q+X)q^{m-1} \\ &= -\psi_c(\psi_c + c) - \left(1 - q - \frac{\psi_c q}{c}\right) q^{m-1} \\ &\quad + \psi_c(\psi_c + c) + (1-q+X)q^{m-1} \\ &= \left(X + \frac{\psi_c q}{c}\right) q^{m-1} < 0 \quad \text{for any} \quad \mathbf{u} \in S_2.\end{aligned}$$

On S_3, $\mathbf{n}_3 = (-1, 0, 0)$, so that

$$\mathbf{n}_3 \cdot \mathbf{f}_d = \frac{c}{d} X + \left(\frac{1}{d} - 1\right) pq = \left(\frac{1}{d} - 1\right) pq < 0, \quad \text{for any} \quad \mathbf{u} \in S_3.$$

Similarly, on S_4, $\mathbf{n}_4 = (0, 0, 1)$, so that

$$\mathbf{n}_4 \cdot \mathbf{f}_d = -p(p+c) - (1-q+X)q^{m-1} = -(1-q+X)q^{m-1} < 0, \quad \text{for any} \quad \mathbf{u} \in S_4.$$

For the remaining part of the boundary J_i $(i = 1, \ldots, 4)$, we have to examine the orbit $\mathbf{u}(z; \mathbf{u}_0)$ for $\mathbf{u}_0 \in J_i$. For $\mathbf{u}_0 = (0, q_0, 0) \in J_1$, we see that

$$\begin{aligned}\mathbf{u}(z; \mathbf{u}_0) &= \mathbf{u}_0 + \mathbf{u}'(0; \mathbf{u}_0) z + \frac{1}{2} \mathbf{u}''(0; \mathbf{u}_0) z^2 + (\text{h.o.t}) \\ &= \mathbf{u}_0 + \mathbf{f}_d(\mathbf{u}_0) z + \frac{1}{2} \mathrm{D}\mathbf{f}_d(\mathbf{u}_0) \mathbf{u}'(0; \mathbf{u}_0) z^2 + (\text{h.o.t}) \\ &= (0, q_0, 0) + (0, 0, -(1-q_0)q_0^{m-1}) z \\ &\quad + \frac{1}{2}(1-q_0)q_0^{m-1} \left(\left(\frac{1}{d} - 1\right) q_0, -q_0, c \right) z^2 + (\text{h.o.t}),\end{aligned}$$

where $\mathrm{D}\mathbf{f}_d(\mathbf{u})$ is the Jacobi matrix of $\mathbf{f}_d(\mathbf{u})$. This shows that $X(z; \mathbf{u}_0) > 0$ and $p(z; \mathbf{u}_0) < 0$ for sufficiently samll positive z, so that any orbit $\mathbf{u}(z; \mathbf{u}_0)$ for $\mathbf{u}_0 \in \Omega_1$ cannot traverse J_1. Similar argument assures that the same result holds for J_2, J_3 and J_4. This completes the proof. □

3.2. The local property of $\mathcal{U}_d(c)$

We next discuss the local behavior of $\mathcal{U}_d(c)$ near the critical point P_1. The linearization of (15) at P_1 becomes

$$\begin{pmatrix} -\dfrac{c}{d} & 0 & 1 - \dfrac{1}{d} \\ 0 & 0 & 1 \\ -1 & 1 & -c \end{pmatrix}. \tag{17}$$

Its characteristic equation becomes

$$g(\lambda, c, d) \equiv \left(\lambda + \frac{c}{d}\right) f^c(\lambda) - \left(\frac{1}{d} - 1\right) \lambda = 0, \tag{18}$$

with $f^c(\lambda) = (\lambda^2 + c\lambda - 1)$. This can be factorized as

$$g(\lambda, c, d) = (\lambda + c)\left(\lambda^2 + \frac{c}{d}\lambda - \frac{1}{d}\right)$$
$$= (\lambda + c)(\lambda - \lambda_-^d(c))(\lambda - \lambda_+^d(c)),$$

where $\lambda_-^d(c) = -(1/(2d))(c + \sqrt{c^2 + 4d}) < 0 < \lambda_+^d(c) = -(1/(2d))(c - \sqrt{c^2 + 4d})$. Therefore, (17) has two negative eigenvalues $-c$ and $\lambda_-^d(c)$, and one positive eigenvalue $\lambda_+^d(c)$. For $d = 1$, we see that $\lambda_\pm^1 = (1/2)(-c \pm \sqrt{c^2 + 4})$ are zeros of $f^c(\lambda)$. Moreover, it is obvious that $\lambda_+^d(c)$ is strictly monotone decreasing with respect to $d > 0$, since $\lambda_+^d(c) = -(1/(2d))(c - \sqrt{c^2 + 4d}) = 2/(\sqrt{c^2 + 4d} + c)$. The eigenvector $\mathbf{p}_+^d(c)$ corresponding to the eigenvalue $\lambda_+^d(c)$ is given by $\mathbf{p}_+^d(c) = (-f^c(\lambda_+^d(c)), 1, \lambda_+^d(c))$ for $d > 0$.

Thus we have the following two propositions.

Proposition 7. *Let c be fixed positive. For $d > 0$, $\mathcal{U}_d(c)$ has the tangential direction $\mathbf{p}_+^d(c) = (-f^c(\lambda_+^d(c)), 1, \lambda_+^d(c))$ at P_1, and $\lambda_+^d(c)$ and $f^c(\lambda_+^d(c))$ are both strictly monotone decreasing with respect to d. Therefore, it holds that if $0 < d < 1$,*

$$0 = f^c(\lambda_+^1(c)) < f^c(\lambda_+^d(c)) < f^c(\lambda_+^0(c)) = \frac{1}{c}\lambda_+^0(c). \tag{19}$$

and if $d > 1$,

$$-1 = f^c(0) < f^c(\lambda_+^d(c)) < f^c(\lambda_+^1(c)) = 0. \tag{20}$$

Proof. We already see that $\lambda_+^d(c)$ is strictly monotone decreasing with respect to d, so that the monotonicity of $f^c(\lambda_+^d)$ with respect to d are also obvious because $f^{c\prime}(\lambda) = 2\lambda + c > 0$ for any $\lambda > 0$. This completes the proof. □

Proposition 8. *Let d be fixed positive. For $c > 0$, $\lambda_+^d(c)$ is strictly monotone decreasing with respect to c, and*

$$\frac{\partial}{\partial c} f^c(\lambda_+^d(c)) \begin{cases} < 0, & \text{for } 0 < d < 1, \\ > 0, & \text{for } d > 1. \end{cases} \tag{21}$$

Proof. The expression $\lambda_+^d = -(1/(2d))(c - \sqrt{c^2 + 4d})$ directly shows that $(\partial/\partial c)\lambda_+^d = -\lambda_+^d/\sqrt{c^2+4d} < 0$. Since $(\partial/\partial c) f^c(\lambda_+^d(c)) = 2(d-1)\left(2\lambda_+^d(c)^2\right)/\sqrt{c^2+4d}$, (21) is obvious, so that the proof is completed. □

We next show that $\mathcal{U}_d(c)$ enters Ω_1. Since $\mathcal{U}_d(c)$ has the tangential direction $\mathbf{p}_+^d$ at P_1, the projection of $\mathcal{U}_d(c)$ to the plane $X = 0$ has a slope λ_+^d and the projection of $\mathcal{U}_d(c)$ to the plane $p = 0$ has a slope $-f(\lambda_+^d)$. On the other hand, the boundary surfaces $p = \psi_c(q)$ and $X = -\psi_c(q)q/c$ have slopes $\lambda_+^0 = 1/c$ and $-\lambda_+^0/c$, respectively at $q = 1$. Therefore, the inequalities (19) assures that $\mathcal{U}_d(c)$ enters Ω_1. Thus, from Proposition 6 we can conclude that $\mathcal{U}_d(c)$ stays in Ω_1 for all

$z \in \mathbb{R}$. There exists no critical point in Ω_1 and $q' = pq < 0$, so that $q \to 0$. This implies that $X(q) \to 0$ as $q \to 0$, and hence $\mathcal{U}_d(c)$ approaches $\mathrm{cl}(I_0)$, which consists of the ordinary points I_0 and the critical points P_0 and P_c. Therefore $\mathcal{U}_d(c)$ has to approach P_0 or P_c. In fact, the following lemma holds.

LEMMA 9. *Assume that* $0 < d < 1$. *Then, for each* $c \geq c_0^*$, *there exists an orbit of* (15) *which connects* P_1 *with* P_0.

Proof. It suffices to show that $\mathcal{U}_d(c)$ cannot approach to P_c as $z \to \infty$ for $c = c_0^*$ because Ω_1 does not contain P_c for any $c > c_0^*$. Assume that $c = c_0^*$ in the following. We first choose $\mathbf{u}_0 = (X_0, q_0, p_0) \in \mathcal{U}_d$ such that $p_0/(q_0 - 1) < \psi_c(q_0)/(q_0 - 1)$, which is possible since

$$\lim_{q_0 \to 1} \frac{p_0}{q_0 - 1} = \lambda_+^d < \lambda_+^0 = \frac{1}{c} = \lim_{q_0 \to 1} \frac{\psi_c(q_0)}{q_0 - 1}.$$

Let $\mathbf{q}_0$ be (q_0, p_0) and $(q(z; \mathbf{q}_0), p(z; \mathbf{q}_0))$ be the solution of (16) satisfying $(q(0; \mathbf{q}_0), p(0; \mathbf{q}_0)) = \mathbf{q}_0$. Then, $\mathbf{q}_0 \in \Omega_0^*$, so that $(q(z; \mathbf{q}_0), p(z; \mathbf{q}_0))$ stays in Ω_0^* for all $z \geq 0$ and converges to $(0, 0)$ as $z \to \infty$. We can represent this orbit by $p = \tilde{\psi}_c(q)$, $(0 \leq q \leq q_0)$, where $\tilde{\psi}_c(0) = 0$ and $\tilde{\psi}_c(q_0) = p_0$. Now we define the region $\tilde{\Omega}_1$ by $\big\{(X, q, p)\colon 0 < X < -(1/c)\tilde{\psi}_c(q)q,\ 0 < q < q_0,\ \tilde{\psi}_c(q) < p < 0\big\}$. Repeating the same arguments in the proof of Proposition 6, we conclude that $\mathcal{U}_d(c)$ stays in $\tilde{\Omega}_1$ for all $z \geq 0$ and converges to $(0, 0, 0)$ as $z \to \infty$ because $\tilde{\psi}_c(q) \to 0$ as $q \to 0$. This completes the proof. □

3.3. The existence of the orbit connecting P_1 and P_c

In this subsection, we shall prove the existence of the orbit connecting P_1 and P_c with the aid of the Wazewski theorem. In order to apply the Wazewski theorem, we rewrite the system (15) as

$$\begin{cases} X' = -\dfrac{c}{d}X - \left(\dfrac{1}{d} - 1\right)pq, \\ q' = pq, \\ p' = -p(p + c) - (1 - q + X)q^{m-1}, \\ c' = 0, \end{cases} \tag{22}$$

which is simply denoted by the vector notation: $\mathbf{U}' = \mathbf{F}(\mathbf{U})$ with $\mathbf{U} = (\mathbf{u}, c) = (X, q, p, c)$, and $\mathbf{U}(z; \mathbf{U}_0)$ denotes a solution of (22) which satisfies $\mathbf{U}(0; \mathbf{U}_0) = \mathbf{U}_0$. This system has three critical manifolds $\tilde{P}_0(c) = (0, 0, 0, c)$, $\tilde{P}_1(c) = (0, 1, 0, c)$ and $\tilde{P}_2(c) = (0, 0, -c, c)$. In $\mathbb{R}^4$, we shall construct the Wazewski set for (22).

Let W_c be

$$W_c = \{\mathbf{u} = (X, q, p)\colon 0 \leq X \leq 1,\ l_c(p) \leq q \leq 1,\ -c \leq p \leq 0\},$$

where $l_c(p)$ is defined by

$$l_c(p) = \begin{cases} \left(\dfrac{1}{k}(p+c)\right)^{1/(m-1)}, & -c \le p \le -\dfrac{c}{2}, \\ q_k, & -\dfrac{c}{2} \le p \le 0, \end{cases}$$

with $q_k = (c/(2k))^{1/(m-1)}$. We also define $\tilde{W}$ by $\tilde{W} = \{\mathbf{U} = (\mathbf{u}, c) \colon \mathbf{u} \in W_c,\ c_1 \le c \le c_2\}$, where c_1 and c_2 are any constants satisfying $0 < c_1 < c_2$ and will be specified later. The boundary of $\tilde{W}$ is consists of $\partial\tilde{W}_1 = \{\mathbf{U} = (\mathbf{u}, c) \colon \mathbf{u} \in \partial W_c,\ c_1 \le c \le c_2\}$ and $\partial\tilde{W}_2 = \bigcup_{i=1}^{2}\{\mathbf{U} = (\mathbf{u}, c) \colon \mathbf{u} \in \mathrm{int}(W_c),\ c = c_i\}$, that is, $\partial\tilde{W} = \partial\tilde{W}_1 \cup \partial\tilde{W}_2$. Here, for each fixed c, $\mathrm{int}(W_c)$ and ∂W_c denote the interior and the boundary of W_c in $\mathbb{R}^3$, respectively. ∂W_c consists of two critical points P_1 and P_c, and the following surfaces H_c^i $(i = 1, 2, \ldots, 6)$ and the one dimensional invariant manifold J_c.

$$\begin{aligned}
H_c^1 &= \{(X, q, p) \colon 0 \le X \le 1,\ q = l_c(p),\ -c < p \le 0\}, \\
H_c^2 &= \{(X, q, p) \colon 0 \le X \le 1,\ 0 < q \le 1,\ p = -c\}, \\
H_c^3 &= \{(X, q, p) \colon 0 \le X \le 1,\ q = 1,\ -c < p < 0\}, \\
H_c^4 &= \left\{(X, q, p) \colon 0 \le X \le 1,\ \frac{c}{2k} < q \le 1,\ p = 0\right\} \setminus P_1, \\
H_c^5 &= \{(X, q, p) \colon X = 0,\ l_c(p) < q < 1,\ -c < p < 0\}, \\
H_c^6 &= \{(X, q, p) \colon X = 1,\ l_c(p) < q < 1,\ -c < p < 0\}, \\
J_c &= \{(X, q, p) \colon 0 < X \le 1,\ q = 0,\ p = -c\}.
\end{aligned}$$

That is, ∂W_c is the disjoint union of these sets, and expressed as

$$\partial W_c = \left(\bigcup_{i=1}^{6} H_c^i\right) \cup J_c \cup P_1 \cup P_c.$$

Set $\tilde{H}^i = \{(\mathbf{u}, c) \colon \mathbf{u} \in H_c^i,\ c_1 \le c \le c_2\}$ for $i = 1, 2$. Then, we have the proposition.

Proposition 10. *For sufficiently large positive k, the immediate exit set $\tilde{W}^-$ of $\tilde{W}$ is the disjoint union of $\tilde{H}^1$ and $\tilde{H}^2$, that is,*

$$\tilde{W}^- = \tilde{H}^1 \cup \tilde{H}^2, \quad \tilde{H}^1 \cap \tilde{H}^2 = \varnothing,$$

for any c_1 and c_2 satisfying $0 < c_1 < c_2$.

Proof. It is obvious that $\partial\tilde{W}_2$ is not an immediate exit set because $\mathrm{int}(W_c)$ is open in $\mathbb{R}^3$ and $\{(\mathbf{u}, c) \colon c = c_i\}$ $(i = 1, 2)$ are invariant manifolds. Therefore, it suffices to discuss $\partial\tilde{W}_1$. We denote the outward normal of $\partial\tilde{W}_1$ by $\tilde{\mathbf{n}}$.

Let us first consider $\tilde{H}^1$. For $-c < p \le -c/2$, $q = l_c(p)$, which implies $p = kq^{m-1} - c$ $(0 < q \le q_k)$. This gives $\tilde{\mathbf{n}} = (0, -k(m-1)q^{m-2}, 1, 1)$ on $\tilde{H}^1$ for $-c < p \le -c/2$. Thus, noting that $0 \le X \le 1$ in $\mathrm{cl}(\tilde{W})$, we have

$$\begin{aligned}
\tilde{\mathbf{n}} \cdot \mathbf{F} &= -k(m-1)pq^{m-1} - p(p+c) - (1-q+X)q^{m-1} \\
&= -k(m-1)pq^{m-1} - kpq^{m-1} - (1-q+X)q^{m-1} \\
&= -mkq^{m-1}\left(p + \frac{1}{mk}(1-q+X)\right) \\
&\ge mkq^{m-1}\left(\frac{c}{2} - \frac{2}{mk}\right) > 0,
\end{aligned}$$

if we choose $k > 4/(mc_1)$. For $-c/2 < p < 0$, $q = q_k > 0$, so that on H_c^1 we see that $\tilde{\mathbf{n}} \cdot \mathbf{F} = -pq_k > 0$. On $E_1 = \{(X, q, p, c)\colon 0 \le X \le 1,\ q = q_k,\ p = 0,\ c_1 \le c \le c_2\}$, we have $\tilde{\mathbf{U}}' = \tilde{\mathbf{F}} = (-(c/d)X, 0, -(1-q_k+X)q_k^{m-1}, 0)$. Hence for $\tilde{\mathbf{U}}_0 = (X_0, q_k, 0, c_0) \in E_1$

$$\begin{aligned}
q\big(z : \tilde{\mathbf{U}}_0\big) &= q\big(0 : \tilde{\mathbf{U}}_0\big) + q'\big(0 : \tilde{\mathbf{U}}_0\big)z + \frac{1}{2}q''\big(0 : \tilde{\mathbf{U}}_0\big)z^2 + O(z^3) \\
&= q_k + \frac{1}{2}p'\big(0 : \tilde{\mathbf{U}}_0\big)q\big(0 : \tilde{\mathbf{U}}_0\big)z^2 + O(z^3) \\
&= q_k - \frac{1}{2}(1 - q_k + X_0)q_k^m z^2 + O(z^3) < q_k,
\end{aligned}$$

for any sufficiently small positive z, since we can choose k so large as $q_k = (c_0/(2k))^{1/(m-1)} < 1/2$. Therefore, we know that $\tilde{H}^1$ is the immediate exit set.

Next, let us consider $\tilde{H}^2$. Since $\tilde{\mathbf{n}} = (0, 0, -1, -1)$ on $\tilde{H}^2$, we see that

$$\tilde{\mathbf{n}} \cdot \mathbf{F} = p(p+c) + (1-q+X)q^{m-1} = (1-q+X)q^{m-1} > 0$$

for $q \ne 1$ or $X \ne 0$. On $E_2 = \{(X, q, p, c)\colon X = 0,\ q = 1,\ p = -c,\ c_1 \le c \le c_2\}$, $\tilde{\mathbf{F}} = (((1/d) - 1)c, -c, 0, 0)$, so that

$$\begin{aligned}
p'' &= -(2p+c)p' - (m-1)(1+X)q^{m-2}q' + mq^{m-1}q' - X'q^{m-1} \\
&= -(m-1)q' + mq' - X' = q' - X' = -c - \left(\frac{1}{d} - 1\right)c = -\frac{c}{d} < 0.
\end{aligned}$$

Hence, for $\tilde{\mathbf{U}}_0 = (0, 1, -c, c) \in E_2$ we have

$$\begin{aligned}
p\big(z : \tilde{\mathbf{U}}_0\big) &= p\big(0 : \tilde{\mathbf{U}}_0\big) + p'\big(0 : \tilde{\mathbf{U}}_0\big)z + \frac{1}{2}p''\big(0 : \tilde{\mathbf{U}}_0\big)z^2 + O(z^3) \\
&= -c - \frac{c}{2d}z^2 + O(z^3) < -c,
\end{aligned}$$

for any sufficiently small positive z. Thus we also know that $\tilde{H}^2$ is the immediate exit set. The disjointness of $\tilde{H}^1$ and $\tilde{H}^2$ is obvious.

Similarly, we can show that any orbit starting from

$$\tilde{W}^{+} = \left\{ \mathbf{U} = (\mathbf{u}, c) \colon \mathbf{u} \in \bigcup_{i=3}^{6} H_c^i,\ c_1 \le c \le c_2 \right\},$$

enters immediately into $\mathrm{int}\big(\tilde{W}\big)$ for any c_1 and c_2 satisfying $0 < c_1 < c_2$, which completes the proof. $\square$

Now, we shall prove the following lemma.

Lemma 11. *Assume that* $0 < d < 1$*. Then, there exists some positive constant* c_d^* $(< c_0^*)$ *such that, for* $c = c_d^*$, (15) *has an orbit connecting* P_1 *with* P_c*.*

Proof. We first consider the case that $m \ge 2$, which assures that $\mathbf{F}(\mathbf{U})$ of the system (22) is Lipschitz continuous. In order to apply the Wazewski theorem, we first identify Σ as follows. Let $\mathbf{u}_c = (X_c, q_c, p_c)$ be a point on the unstable manifold $\mathcal{U}_d(c)$ for each c satisfying $c_1 \le c \le c_2$, where q_c is chosen sufficiently close to 1 and satisfies $0 < q_c < 1$. Set $\Sigma = \{\mathbf{U}_c = (\mathbf{u}_c, c) \colon c_1 + \epsilon \le c \le c_2 - \epsilon\} \subset \tilde{W}$ with sufficiently small $\epsilon > 0$. Then Σ is compact and intersects a trajectory of (22) only onces. $\tilde{W}$ is seen to be a Wazewski set. In fact, (1) in Proposition 5 is trivially satisfied since $\tilde{W}$ is closed. If $\mathbf{U}_c \in \Sigma$, $\mathbf{U}_c \cdot z \in \tilde{W}$ and $\mathbf{U}_c \cdot z \notin \tilde{W}^{-}$, then $\mathbf{U}_c \cdot z \in \mathrm{int}(\tilde{W}) \cup \partial\tilde{W}_2 \cup \big(\partial\tilde{W}_1 \setminus \tilde{W}^{-}\big)$. The definition of Σ assures that $\mathbf{U}_c \cdot z \notin \partial\tilde{W}_2$. Here we note that $\partial\tilde{W}_1 \setminus \tilde{W}^{-} = \tilde{W}^{+} \cup \tilde{W}^{0}$ with $\tilde{W}^{0} = \{(\mathbf{u}, c) \colon \mathbf{u} \in (J_c \cup P_1 \cup P_c),\ c_1 \le c \le c_2\}$. In the proof of Proposition 10 we already see that the direction of the vector field is inward on $\tilde{W}^{+}$, so that $\mathbf{U}_c \cdot z \notin \tilde{W}^{+}$. Since $\tilde{W}^{0}$ consists of the invariant manifold and the critical points, it is also obvious that $\mathbf{U}_c \cdot z \notin \tilde{W}^{0}$. Therefore, it holds that $\mathbf{U}_c \cdot z \in \mathrm{int}(\tilde{W})$, which assures (2).

Now, we consider the behavior of $\mathcal{U}_d(c)$ for large c and for small $c > 0$. For any $c \ge c_0^*$, Lemma 9 asserts that $\mathcal{U}_d(c)$ approaches to P_0 as $z \to \infty$. This assures that $\mathcal{U}_d(c)$ has to leave W_c through H_c^1 for $c \ge c_0^*$. Of course, it cannot traverse H_c^2 before it hits H_c^1 because H_c^1 is the immediate exit set of W_c. Thus we see that any orbit starting from $\mathbf{U}_c \in \Sigma$ hits $\tilde{H}^1$ with an exit time $T(\mathbf{U}_c)$ for $c \ge c_0^*$. Since the slope of the projection of $\mathcal{U}_d(c)$ to the p-q plane at P_1 is $\lambda_{+}^{d}(c)$ which approaches to $1/\sqrt{d}$ as $c \to 0$, there exists some small positive $c_1{}'$ such that $\mathcal{U}_d(c)$ intersects the plane $p = -c$ for $c \in (0, c_1{}']$, which implies that $\mathcal{U}_d(c)$ goes out from W_c through H_c^2. Hence we see that an orbit starting from $\mathbf{U}_c \in \Sigma$ hits $\tilde{H}^2$ with an exit time $T(\mathbf{U}_c)$ for any $c \in (0, c_1{}']$. Choose $c_1 + \epsilon = c_1{}'$ and $c_2 - \epsilon = c_0^*$. Then, if $\Sigma = \Sigma^0$, Proposition 5 says that $F(\Sigma)$ is the continuous image of the connected set Σ on $\tilde{W}^{-} = \tilde{H}^1 \cup \tilde{H}^2$. On the other hand, we have shown in the above that $F(\mathbf{U}_c) \equiv \mathbf{U}_c{\cdot}T(\mathbf{U}_c) \in \tilde{H}^1$ for $c = c_1 + \epsilon$, $F(\mathbf{U}_c) \in \tilde{H}^2$ for $c = c_2 - \epsilon$ and $\tilde{H}^1 \cap \tilde{H}^2 \ne \varnothing$, which contradicts the connectedness of $F(\Sigma)$. Therefore, $\Sigma \ne \Sigma^0$, so that there exists a point $\mathbf{U}_{c^*} = (\mathbf{u}_{c^*}, c^*) \in \Sigma$ such that the solution $\mathbf{U}(z; \mathbf{U}_{c^*}) = (\mathbf{u}(z; \mathbf{u}_{c^*}), c^*)$ of (22) stays in $\mathrm{int}(\tilde{W})$ for all $z > 0$.

Next, we show that $\mathbf{u}(z; \mathbf{u}_{c^*})$ gives an orbit connecting P_1 with P_c. Note that $\mathbf{u}(z; \mathbf{u}_{c^*})$ stays in $\mathrm{int}(W_{c^*})$ for all $z > 0$ and there is no critical point in

$\mathrm{int}(W_{c^*})$. In $\mathrm{int}(W_{c^*})$, $q' = pq < 0$ so that $q(z;\mathbf{u}_{c^*}) \to 0$ as $z \to \infty$. This assures that $p(z;\mathbf{u}_{c^*}) \to -c^*$ as $z \to \infty$ since $0 \le l_{c^*}(p) < q$ in $\mathrm{int}(W_{c^*})$ and $l_{c^*}(p) = 0$ if and only if $p = -c^*$. We choose now sufficiently large positive z_1 such that $-c^* \le p(z;\mathbf{u}_{c^*}) \le -c^*/2$ for all $z \ge z_1$. It easily follows from the second equation of (22) that $q(z;\mathbf{u}_{c^*}) \le q(z_1;\mathbf{u}_{c^*})e^{-(c^*/2)(z-z_1)}$. The first equation of (22) is integrated as

$$X(z) = e^{-(c^*/d)(z-z_1)}\left(X(z_1) - \left(\frac{1}{d}-1\right)\int_{z_1}^{z} e^{(c^*/d)(\zeta-z_1)}p(\zeta)q(\zeta)\,d\zeta\right), \tag{23}$$

where we omit the dependency of the initial value $\mathbf{u}_{c^*}$ for simplicity. By using the above estimates of p and q, we have the following inequality.

$$\begin{aligned} X(z) &\le e^{-(c^*/d)(z-z_1)}\left(X(z_1) + \left(\frac{1}{d}-1\right)c^*q(z_1)\int_{z_1}^{z} e^{(c^*/d)(\zeta-z_1)}e^{-(c^*/2)(\zeta-z_1)}\,d\zeta\right) \\ &\le e^{-(c^*/d)(z-z_1)}\left(X(z_1) + \left(\frac{1}{d}-1\right)c^*q(z_1)\int_{z_1}^{z} e^{(1/d-1/2)c^*(\zeta-z_1)}\,d\zeta\right) \\ &\le e^{-(c^*/d)(z-z_1)}\left(X(z_1) + \frac{2(1-d)}{2-d}q(z_1)\left[e^{(1/d-1/2)c^*(z-z_1)}-1\right]\right) \\ &\le e^{-(c^*/d)(z-z_1)}X(z_1) + e^{-(1/2)c^*(z-z_1)}q(z_1), \end{aligned}$$

which asserts that $X(z) \to 0$ as $z \to \infty$. Thus we conclude that $\lim_{z\to\infty}\mathbf{u}(z;\mathbf{u}_{c^*}) = P_{c^*}$. It is obvious that $\lim_{z\to-\infty}\mathbf{u}(z;\mathbf{u}_{c^*}) = P_1$ since $\mathbf{u}_{c^*} \in \mathcal{U}_d(c^*)$.

For the case that $1 < m < 2$, in order to assure the Lipschitz continuity of $\mathbf{F}(\mathbf{U})$, we introduce another change of the dependent variables by

$$\hat{q} = V^{m-1}, \quad \hat{p} = \frac{\hat{q}'}{\hat{q}}, \tag{24}$$

which rewrite the system (12) as follows.

$$\begin{cases} X' = -\dfrac{c}{d}X - \dfrac{1}{m-1}\left(\dfrac{1}{d}-1\right)\hat{p}\hat{q}^{1/(m-1)}, \\ \hat{q}' = \hat{p}\hat{q}, \\ \hat{p}' = -\dfrac{1}{m-1}\hat{p}(\hat{p}+(m-1)c) - (m-1)\big(1-\hat{q}^{1/(m-1)}+X\big)\hat{q}. \end{cases} \tag{25}$$

Here, the critical points of this system are $\hat{P}_0 = (0,0,0)$, $\hat{P}_1 = (0,1,0)$, and $\hat{P}_0 = (0,0,-(m-1)c)$. We write (25) in the vector form as $\hat{\mathbf{u}}' = \hat{\mathbf{f}}_d(\hat{\mathbf{u}})$ with $\hat{\mathbf{u}} = (X,\hat{q},\hat{p})$, and $\hat{\mathbf{u}}(z;\hat{\mathbf{u}}_0)$ denotes a solution of (25) satisfying $\hat{\mathbf{u}}(0;\hat{\mathbf{u}}_0) = \hat{\mathbf{u}}_0$. Now $\hat{\mathbf{f}}_d(\hat{\mathbf{u}})$ is Lipshitz continuous. It follows from (14) and (24) that

$$q = \hat{q}^{1/(m-1)}, \quad p = \frac{1}{m-1}\hat{p}. \tag{26}$$

This relation assures the one to one correspondence between orbits of (15) in the phase space Ω^+ and orbits of (25) in $\hat{\Omega}^+ = \{(X,\hat{q},\hat{p})\colon X > 0,\ 0 < \hat{q} < 1,\ \hat{p} < 0\}$.

Through this correspondence, the whole of the above arguments for $m \geq 2$ is valid also for $0 < m < 1$ since Propositions 6, 7, 8, 10 and Lemma 9 hold for $m > 1$, and hence we know the existence of some c^* such that for $c = c^*$ (25) has a solution $\hat{\mathbf{u}}(z; \hat{\mathbf{u}}_0)$ satisfying $\lim_{z\to-\infty} \hat{\mathbf{u}}(z; \hat{\mathbf{u}}_0) = \hat{P}_1$ and $\lim_{z\to\infty} \hat{\mathbf{u}}(z; \hat{\mathbf{u}}_0) = \hat{P}_{c^*}$. Again, through the correspondence (26), for this c^* we have obtained the orbit of (12) connecting P_1 with P_{c^*}. Choosing $c_d^* = c^*$, we complete the proof. □

REMARK 1. It follows from the proof of Lemma 11 that the traveling front solution of (11) obtained above, denoted by $(U_d(z; c^*), V_d(z; c^*))$, decays to $(1, 0)$ exponentially as $z \to \infty$.

Thus, we have established the existence of the traveling front solution of (11) and (6) for $c \geq c_0^*$ and $c = c_d^*$. In the next subsection, we shall prove that c_d^* is the minimal wave speed of traveling front solutions and state the main theorem.

3.4. The properties of the connection orbit

We first discuss the existence of connection orbits of (15) for any $c > c_d^*$. We can express the orbit of (15) obtained in Lemma 11 as $(X, q, p) = (X^*(q), q, p^*(q))$ $(0 \leq q \leq 1)$ in the phase space, since $q' = pq < 0$ in Ω^+. Here we note that $p^*(0) = -c_d^*$,

$$p^{*\prime}(q) = -\frac{p^*(p^* + c_d^*) + (1 - q + X^*)q^{m-1}}{p^*q}, \tag{27}$$

and

$$X^{*\prime}(q) = -\frac{c_d^*}{d}\frac{X^*}{p^*q} - \left(\frac{1}{d} - 1\right), \tag{28}$$

By the use of this orbit, the region Ω^* is defined, similarly as Ω_1, by

$$\Omega^* \equiv \{(X, q, p) \colon 0 < X < X^*(q),\ 0 < q < 1,\ p^*(q) < p < 0\}.$$

The boundary of Ω^* is given by

$$\partial\Omega^* = \left(\bigcup_{i=1}^{4} S_i^*\right) \cup \left(\bigcup_{i=1}^{4} J_i^*\right) \cup I_0^* \cup P_0 \cup P_1 \cup P_{c_d^*},$$

where

$$\begin{aligned}
S_1^* &= \{(X, q, p) \colon 0 < q < 1,\ X = X^*(q),\ p^*(q) < p < 0\},\\
S_2^* &= \{(X, q, p) \colon 0 < q < 1,\ 0 < X < X^*(q),\ p = p^*(q)\},\\
S_3^* &= \{(X, q, p) \colon 0 < q < 1,\ X = 0,\ p^*(q) < p < 0\},\\
S_4^* &= \{(X, q, p) \colon 0 < q < 1,\ 0 < X < X^*(q),\ p = 0\},\\
J_1^* &= \{(X, q, p) \colon 0 < q < 1,\ X = 0,\ p = 0\},
\end{aligned}$$

$$J_2^* = \{(X,q,p)\colon 0<q<1,\ X=0,\ p=p^*(q)\},$$
$$J_3^* = \{(X,q,p)\colon 0<q<1,\ X=X^*(q),\ p=0\},$$
$$J_4^* = \{(X,q,p)\colon 0<q<1,\ X=X^*(q),\ p=p^*(q)\},$$
$$I_0^* = \{(X,q,p)\colon X=0,\ q=0,\ -c_d^*<p<0\}.$$

PROPOSITION 12. *Let d be fixed in $(0,1)$. For each $c>c_d^*$, any orbit of (15) starting from a point $\mathbf{u}_0 \in \Omega^*$, denoted by $\mathbf{u}_c(z;\mathbf{u}_0)$, stays in Ω^* for all $z \geq 0$.*

Proof. The proof of Proposition 6 can be applied to this proposition with the minor changes by noting that $c>c_d^*$. In fact, on S_1^*, $X=X^*(q)$, so that $\mathbf{n}_1^* = (1,-X^{*\prime},0)$. Here, $\mathbf{n}_i^*$ are the outward normal of the surfaces S_i^* $(i=1,2,3,4)$. Hence, using (28) we have

$$\begin{aligned}
\mathbf{n}_1^*\cdot\mathbf{f}_d &= -\frac{c}{d}X - \left(\frac{1}{d}-1\right)pq - X^{*\prime}pq \\
&= -\frac{c}{d}X^* - \left(\frac{1}{d}-1\right)pq + \left\{\frac{c_d^*}{d}\frac{X^*}{p^*q} + \left(\frac{1}{d}-1\right)\right\}pq \\
&= \frac{1}{d}\left(\frac{c_d^*p}{p^*} - c\right)X^* < \frac{1}{d}(c_d^*-c)X^* < 0,
\end{aligned}$$

for any $\mathbf{u}=(X,q,p)\in S_1^*$.

On S_2^*, $p=p^*(q)$, so that $\mathbf{n}_2^*=(0,p^{*\prime}(q),-1)$. We then have

$$\begin{aligned}
\mathbf{n}_2^*\cdot\mathbf{f}_d &= p^{*\prime}pq + p(p+c) + (1-q+X)q^{m-1} \\
&= -p^*(p^*+c_d^*) - (1-q+X^*)q^{m-1} \\
&\quad + p^*(p^*+c) + (1-q+X)q^{m-1} \\
&= p^*(c-c_d^*) + (X-X^*)q^{m-1} < 0,
\end{aligned}$$

for any $\mathbf{u}\in S_2^*$. The remaining part of the proof is the same as that of Proposition 6. This completes the proof. □

LEMMA 13. *Let d be fixed. Then, for each $c>c_d^*$, there exists an orbit of (15) connecting P_1 with P_0 lying in Ω^+. For each positive $c<c_d^*$, there exists no orbit of (15) connecting P_1 with P_0 or P_c.*

Proof. It follows from Proposition 8 that for each $c>c_d^*$, $\mathcal{U}_d(c)$ enters Ω^*. Then Proposition 12 assures that the orbit corresponding to $\mathcal{U}_d(c)$ cannot leave Ω^*. Hence, it must approach to P_0 as $z\to\infty$ since $q'=pq<0$ in Ω^* and the critical point P_c does not belong to $\mathrm{cl}(\Omega^*)$, which proves the first half of this lemma.

Assume that for some $c^*<c_d^*$, there exists an orbit connecting P_1 with P_c for $c=c^*$. Then Proposition 8 asserts the existence of an orbit connecting P_1 with P_0 for $c=c_d^*$ because of $c_d^*>c^*$. This contradicts the uniqueness of the orbit corresponding to $\mathcal{U}_d(c)$. Next, assume that for some $c^*<c_d^*$, there exists an orbit connecting P_1 with P_0 for $c=c^*$. Then, we can choose $c_2-\epsilon=c^*$ in place of

c_0^* in the proof of Lemma 11, so that from Lemma 11, we see the existence of an orbit connecting P_1 with P_c for $c = c_d^{*\prime}$ ($< c^* < c_d^*$). Thus, the first part of this lemma again asserts that the existence of an orbit connecting P_1 with P_0 for $c = c_d^*$, which leads us to the contradiction. This proves the last half of the assertion, which completes the proof. $\square$

Lemmas 11 and 13 allow us to call c_d^* the minimal wave speed of traveling front solutions.

Remark 2. It follows from the proof of Lemma 13 that c_d^* is determined uniquely. Also, for each $c \geq c_d^*$, the traveling front solution of (11) exists uniquely except translation since $\mathcal{U}_d(c)$ is the unique orbit entering P_1 as $z \to -\infty$.

Next, we consider the d-dependence of connection orbits.

Lemma 14. *Let $d \in (0,1)$ be fixed. Then, for each $\bar{d} \in (d,1)$, there exists an orbit of* (15) *connecting P_1 with P_0 in Ω^+.*

Proof. We first show that for each fixed $c > 0$, $\mathcal{U}_d(c)$ lies strictly below the surface $S_0 = \{(X,q,p)\colon 0 < q < 1,\ X = -pq/c,\ p < 0\}$ for any $d \in (0,1)$. The normal vector $\mathbf{n}_0$ of S_0 is (c,p,q), so that

$$\begin{aligned}\mathbf{n}_0 \cdot \mathbf{f}_d &= c\Big\{-\frac{c}{d}X - \Big(\frac{1}{d} - 1\Big)pq\Big\} + p^2 q + q\Big\{-p(p+c) - (1-q+X)q^{m-1}\Big\}\\ &= -(1-q+X)q^m < 0,\end{aligned}$$

for any $\mathbf{u} = (X,q,p) \in S_0$. This implies that any orbit lying in Ω^+ cannot traverse S_0 from the region $X < -pq/c$ to $X > -pq/c$. On the other hand, $\mathcal{U}_d(c)$ is represented near P_1 by

$$(X(q), q, p(q)) = (0,1,0) - (-f^c(\lambda_+^d(c)), 1, \lambda_+^d(c))h + o(h))$$

for sufficiently small positive h. Therefore,

$$\begin{aligned}X(q) + p(q)\frac{q}{c} &= f^c(\lambda_+^d(c))h - (1-h)\lambda_+^d(c)\frac{h}{c} + o(h)\\ &= \Big(f^c(\lambda_+^d(c)) - \frac{\lambda_+^d(c)}{c}\Big)h + o(h) < 0,\end{aligned}$$

since Proposition 7 assures that $f^c(\lambda_+^d(c))/\lambda_+^d(c) < f^c(\lambda_+^0(c))/\lambda_+^0(c) = 1/c$. Thus we know that $\mathcal{U}_d(c)$ lies strictly below S_0 near P_1 and hence cannot traverse S_0 in Ω^+.

We again consider the region Ω^* and prove that $\mathcal{U}_{\bar{d}}(c_d^*)$ stays in Ω^* for all z for $\bar{d} \in (d,1)$. For simplicity, we denote c_d^* by c^* in the following. Proposition 7 assures that $\mathcal{U}_{\bar{d}}(c^*)$ enters Ω^* for $\bar{d} > d$. On the surface S_1^*, using (28) with $c_d^* = c^*$, we have

$$\begin{aligned}
\mathbf{n}_1^* \cdot \mathbf{f}_{\bar{d}} &= -\frac{c^*}{\bar{d}}X - \left(\frac{1}{\bar{d}} - 1\right)pq - X^{*\prime}pq \\
&= -\frac{c^*}{\bar{d}}X^* - \left(\frac{1}{\bar{d}} - 1\right)pq + \left\{\frac{c^*}{d}\frac{X^*}{p^*q} + \left(\frac{1}{d} - 1\right)\right\}pq \\
&= -c^*\left(\frac{1}{\bar{d}} - \frac{1}{d}\frac{p}{p^*}\right)X^* - \left(\frac{1}{\bar{d}} - \frac{1}{d}\right)pq \\
&\leq -\left(\frac{1}{\bar{d}} - \frac{1}{d}\right)(c^*X^* + pq) < 0,
\end{aligned}$$

for any $\mathbf{u} = (X, q, p) \in S_1^*$, since $\mathcal{U}_d(c^*)$ stays in the region below S_0 where $X^* < -pq/c^*$.

On S_2^*, $p = p^*(q)$, so that $\mathbf{n}_2^* = (0, p^{*\prime}(q), -1)$. We then have

$$\begin{aligned}
\mathbf{n}_2^* \cdot \mathbf{f}_d &= p^{*\prime}pq + p(p + c^*) + (1 - q + X)q^{m-1} \\
&= -p^*(p^* + c^*) - (1 - q + X^*)q^{m-1} \\
&\quad + p^*(p^* + c^*) + (1 - q + X)q^{m-1} \\
&= (X - X^*)q^{m-1} < 0,
\end{aligned}$$

for any $\mathbf{u} \in S_2^*$. By repeating the same argument as in Proposition 12, we can conclude that $\mathcal{U}_{\bar{d}}(c^*)$ stays in Ω^* for all z for $\bar{d} \in (d, 1)$.

Next, assume that $\mathcal{U}_{\bar{d}}(c^*)$ connects P_1 with P_{c^*}. We can express this orbit by $(X_{\bar{d}}(q), q, p_{\bar{d}}(q))$ $(0 \leq q \leq 1)$. Since $\mathcal{U}_{\bar{d}}(c^*)$ stays in Ω^* for $0 < q < 1$, it holds that $0 < X_{\bar{d}}(q) < X^*(q)$ and $p^*(q) < p_{\bar{d}}(q) < 0$ for $0 < q < 1$. We should note that

$$\begin{aligned}
X_{\bar{d}}(0) = X^*(0) = X_{\bar{d}}(1) = X^*(1) = 0, \\
p_{\bar{d}}(0) = p^*(0) = -c^*, \quad p_{\bar{d}}(1) = p^*(1) = 0,
\end{aligned} \tag{29}$$

and

$$\begin{aligned}
X_{\bar{d}}{}'(1) = -f^{c^*}(\lambda_+^{\bar{d}}(c^*)), \quad X^{*\prime}(1) = -f^{c^*}(\lambda_+^{d}(c^*)), \\
p_{\bar{d}}{}'(1) = \lambda_+^{\bar{d}}(c^*), \quad p^{*\prime}(1) = \lambda_+^{d}(c^*).
\end{aligned} \tag{30}$$

Since $p^*(q)$ and $p_d(q)$ satisfy

$$\frac{d}{dq}p^* = -\frac{p^*(p^* + c^*) + (1 - q + X^*)q^{m-1}}{p^*q},$$

and

$$\frac{d}{dq}p_{\bar{d}} = -\frac{p_{\bar{d}}(p_{\bar{d}} + c^*) + (1 - q + X_d)q^{m-1}}{p_{\bar{d}}q},$$

respectively, we have

$$\begin{aligned}\frac{d}{dq}(p^* - p_{\bar{d}}) &= -\frac{1}{q}(p^* - p_{\bar{d}}) - \left(\frac{1-q+X^*}{p^*} - \frac{1-q+X_{\bar{d}}}{p_{\bar{d}}}\right)q^{m-2} \\ &= -\frac{a(q)}{q}(p_{\bar{d}} - p^*) + b(q),\end{aligned}$$

where $a(q) = 1 - ((1-q+X^*)q^{m-2})/(p^* p_{\bar{d}})$ and $b(q) = -(q^{m-2}/p_{\bar{d}})(X^* - X_{\bar{d}})$. This can be solved as

$$p^*(q) - p_{\bar{d}}(q) = e^{A(q)}\left(p^*(q_1) - p_{\bar{d}}(q_1) + \int_{q_1}^{q} e^{-A(q')}b(q')\,dq'\right), \tag{31}$$

where $A(q) = -\int_{1/2}^{q} a(q')/q'\,dq'$. $A(q)$ is evaluated as follows. By using (29) and (30), we see that

$$\begin{aligned}X_{\bar{d}}(q) &= -f^{c^*}\big(\lambda_+^{\bar{d}}(c^*)\big)(q-1) + o(q-1), \\ X^*(q) &= -f^{c^*}\big(\lambda_+^{d}(c^*)\big)(q-1) + o(q-1), \\ p_{\bar{d}}(q) &= \lambda_+^{\bar{d}}(c^*)(q-1) + o(q-1), \\ p^*(q) &= \lambda_+^{d}(c^*)(q-1) + o(q-1).\end{aligned}$$

These expressions easily prove that there exists sufficiently small $\epsilon > 0$ such that

$$a(q) < 1 - \frac{C_1}{1-q} \quad \text{for} \quad q \in [1-\epsilon, 1),$$

with some positive constant C_1. Hence, we have

$$-\frac{a(q)}{q} > \frac{C_1 - (1-q)}{(1-q)q} > \frac{C_1 - \epsilon}{1-q} > 0,$$

for $q \in [1-\epsilon, 1)$, so that

$$\begin{aligned}A(q) &= -\int_{1/2}^{q} \frac{a(q')}{q'}\,dq' = -\int_{1/2}^{1-\epsilon} \frac{a(q')}{q'}\,dq' - \int_{1-\epsilon}^{q} \frac{a(q')}{q'}\,dq' \\ &> A(1-\epsilon) + \int_{1-\epsilon}^{q} \frac{C_1 - \epsilon}{1-q'}\,dq' = A(1-\epsilon) + (C_1 - \epsilon)\log\left(\frac{\epsilon}{1-q}\right).\end{aligned}$$

Thus, we have obtained the required inequality

$$e^{A(q)} > e^{A(1-\epsilon)} e^{\log(\epsilon/(1-q))^{C_1-\epsilon}} = e^{A(1-\epsilon)}\left(\frac{\epsilon}{1-q}\right)^{C_1-\epsilon}. \tag{32}$$

Now, take the limit $q \to 1$ in (31). Noting that (29) and (32), we obtain the relation

$$p^*(q_1) - p_{\bar{d}}(q_1) + \int_{q_1}^{1} e^{-A(q')}b(q')\,dq' = 0.$$

Then, take the limit $q_1 \to 0$ in the above relation. By (29), we finally conclude that

$$\int_0^1 e^{-A(q')} b(q')\, dq' = 0,$$

which is impossible because $b(q) > 0$ for $0 < q < 1$. This proves that $\mathcal{U}_{\bar{d}}(c^*)$ starting from P_1 cannot approach P_{c^*} as $z \to \infty$. Therefore, it must approach P_0 as $z \to \infty$. This completes the proof. □

Lemmas 11 and 13 assure that the existence of a unique connection orbit of (15) only for each $c \geq c_d^*$, that is, c_d^* is the minimal wave speed. Furthermore, with the aid of Lemma 14 we can show that $c_{\bar{d}}^* < c_d^*$ for $\bar{d} > d$. In fact, Lemma 14 asserts that for $c = c_d^*$ there exists an orbit connecting P_1 with P_0 for $\bar{d} > d$, so that $c_{\bar{d}}^* \leq c_d^*$ if $\bar{d} > d$. If $c_d^* = c_{\bar{d}}^*$, the definition of $c_{\bar{d}}^*$ implies that $\mathcal{U}_{\bar{d}}(c_{\bar{d}}^*)$ connects P_1 with $P_{c_{\bar{d}}^*}$. This is a contradiction. Combining these results, we obtain the following theorem.

THEOREM 15. *Assume that* $0 < d < 1$. *Then, there exists some* c_d^*, *such that a traveling front solution for* (10) *exists uniquely (except translation) only for each* $c \geq c_d^*$. *Furthermore, the minimal wave speed* c_d^* *is strictly monotone decreasing with respect to* d, *and it satisfies that* $c_1^* < c_d^* < c_0^*$.

Proof. It suffices for us to prove that $c_1^* < c_d^*$. For $d = 1$, the system (15) becomes

$$\begin{cases} X' = -\dfrac{c}{d} X, \\ q' = pq, \\ p' \ = -p(p+c) - (1 - q + X) q^{m-1}. \end{cases} \tag{33}$$

The first equation of (33) with the boundary condition (13) gives that $X \equiv 0$. Therefore, we consider the problem in the invariant manifold $X = 0$ in $\mathrm{cl}(\Omega^+)$, which is denoted by $\Omega_0 = \{(X, q, p) \colon X = 0,\ 0 < q < 1,\ p < 0\}$. In Ω_0, (33) is reduced to

$$\begin{cases} q' = pq, \\ p' = -p(p+c) - (1 - q) q^{m-1}. \end{cases} \tag{34}$$

The system (34) has three critical points $P_0^0 = (0,0)$, $P_1^0 = (1,0)$ and $P_c^0 = (0,-c)$ in the (q,p)-space. At $P_1^0 = (1,0)$, the eigenvalues are $\lambda_1^1(c) < 0$ and $\lambda_+^1(c) > 0$, and hence P_1^0 has the 1-dim unstable manifold $\mathcal{U}_1(c)$. The slope of $\mathcal{U}_1(c)$ at $q = 1$ is $\lambda_+^1(c)$.

Now, consider the region $\Omega_1^* = \{(q,p) \colon 0 < q < 1,\ p^*(q) < p < 0\}$, where $(X^*(q), q, p^*(q))$ is the same as before. We should note that $p^*(1) = 0$, $p^*(0) = -c_d^*$ and $p^{*\prime}(0) = \lambda_+^d(c_d^*)$. The boundary of Ω_1^* consists of $\mathcal{C} = \{(q,p) \colon 0 < q < 1,\ p = p^*(q)\}$, $l_1 = \{(q,p) \colon 0 < q < 1,\ p = 0\}$, $l_2 = \{(q,p) \colon q = 0,\ -c_d^* < p < 0\}$, P_0 P_1 and P_c. Let us examine the behavior of the orbit $\mathcal{U}_1(c_d^*)$. Since $\lambda_+^1(c_d^*) < \lambda_+^d(c_d^*)$ by (19), $\mathcal{U}_1(c_d^*)$ enters Ω_1^*. The argument in the proof of Lemma 14 with $X = 0$

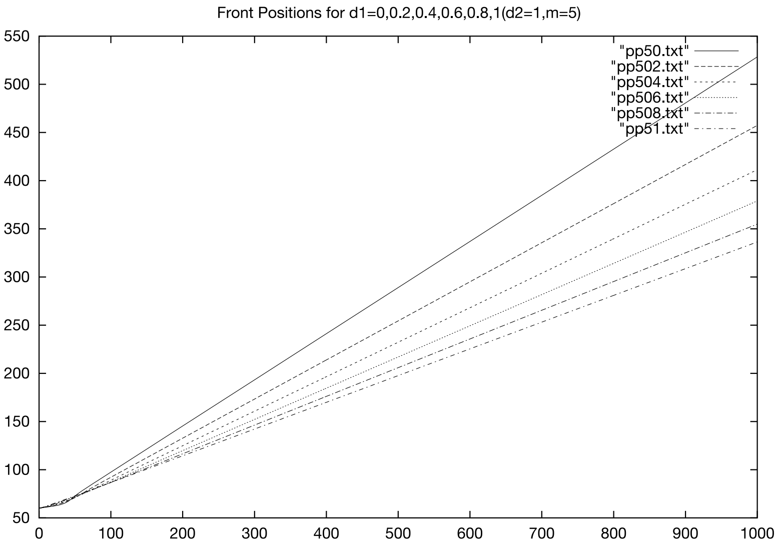

Fig. 1. The position of the front $x(t)$ of $u(x(t), t) = 1/2$ $(0 \le t \le 1000)$ for $d = 0$ (pp50), 0.2 (pp502), 0.4 (pp504), 0.6 (pp506), 0.8 (pp508), 1.0 (pp51), $m = 5$.

verifies that the inner products of the vector field (34) with $c = c_d^*$ and the outward normal on $\mathcal{C}$, l_1 and l_2 are all negative, so that that $\mathcal{U}_1(c_d^*)$ cannot leave Ω_1^* for all z. This assures the existence of the connection orbit of (34) approaching P_0 or P_c for $c = c_d^*$, which implies that $c_1^* \le c_d^*$ for any fixed $d \in (0, 1)$. If $c_1^* = c_d^*$, choose d' satisfying $d < d' < 1$. Then $c_1^* \le c_{d'}^*$ and $c_{d'}^* < c_d^*$. This is a contradiction. Hence we have $c_1^* < c_d^*$, which completes the proof. □

Fig. 1 shows the numerical result of the propagation speeds of the traveling fronts obtained by solving the evolutional system (10) with the appropriate initial data of the step function type. This result illustrates numerically the last assertion of Theorem 15.

4. The case $d_1 > d_2 > 0$

The condition $d_1 > d_2 > 0$ implies $d = d_1/d_2 > 1$. Then, it follows from Proposition 1 that $X = U + V - 1 < 0$. Hence, for $d > 1$, our problem is to find an orbit of (15) connecting $P_1 = (0, 1, 0)$ with $P_0 = (0, 0, 0)$ or with $P_c = (0, 0, -c)$, which lies entirely in $\Omega^- = \{(X, q, p)\colon X < 0,\ 0 < q < 1,\ p < 0\}$. Repeating the similar arguments in Section 3, we can prove the existence of the connection orbits. Therefore, we discuss only the different points from the case that $0 < d < 1$.

In order to examine the local behavior of the unstable manifold $\mathcal{U}_d(c)$ near P_1, it suffices to refer to the statements for $d > 1$ in Propositions 7 and 8. They assert that $\mathcal{U}_d(c)$ has the tangential direction $\mathbf{p}_+^d(c) = (-f^c(\lambda_+^d(c)), 1, \lambda_+^d(c))$ at P_1. Since $f^c(\lambda_+^d(c))$ is negative, $\mathcal{U}_d(c)$ enters Ω^-.

To discuss the global behavior of $\mathcal{U}_d(c)$, we require the connection orbit of (15) for $d = 1$, which was already considered in the proof of Theorem 15. For $d = 1$, we have the system (34), and we know that there exists a unique orbit connecting P_1^0 with P_0^0 for $c > c_1^*$ and with P_c^0 for $c = c_1^*$, which was assured in the proof of Theorem 2. Now, we denote this unique orbit by $(q, \phi_c(q))$ for $c \geq c_1^*$, which satisfies

$$\frac{d}{dq}\phi_c = -\frac{\phi_c(\phi_c + c) + (1-q)q^{m-1}}{\phi_c q}. \tag{35}$$

Let us define the region Ω_1^- by

$$\Omega_1^- \equiv \{(X, q, p)\colon q - 1 < X < 0,\ 0 < q < 1,\ \phi_c(q) < p < 0\}.$$

Then, the boundary $\partial\Omega_1^-$ of Ω_1^- consists of the followings:

$$\begin{aligned}
S_1^- &= \{(X, q, p)\colon 0 < q < 1,\ X = q - 1,\ \phi_c(q) < p < 0\},\\
S_2^- &= \{(X, q, p)\colon 0 < q < 1,\ q - 1 < X < 0,\ p = \phi_c(q)\},\\
S_3^- &= \{(X, q, p)\colon 0 < q < 1,\ X = 0,\ \phi_c(q) < p < 0\},\\
S_4^- &= \{(X, q, p)\colon 0 < q < 1,\ q - 1 < X < 0,\ p = 0\},\\
S_5^- &= \{(X, q, p)\colon q = 0,\ -1 \leq X < 0,\ p_1(0) \leq p \leq 0\},\\
J_1^- &= \{(X, q, p)\colon 0 < q < 1,\ X = 0,\ p = 0\},\\
J_2^- &= \{(X, q, p)\colon 0 < q < 1,\ X = 0,\ p = \phi_c(q)\},\\
J_3^- &= \{(X, q, p)\colon 0 < q < 1,\ X = q - 1,\ p = 0\},\\
J_4^- &= \{(X, q, p)\colon 0 < q < 1,\ X = q - 1,\ p = \phi_c(q)\},\\
I_0^- &= \{(X, q, p)\colon X = 0,\ q = 0,\ \phi_1(0) < p < 0\},\\
&P_0,\ P_1 \text{ and } P_c.
\end{aligned}$$

That is,

$$\partial\Omega_1 = \left(\bigcup_{i=1}^{5} S_i^-\right) \cup \left(\bigcup_{i=1}^{4} J_i^-\right) \cup I_0^- \cup P_0 \cup P_1 \cup P_c.$$

Here, note that $J_0 = \varnothing$ for any $c > c_0^*$. We have the followings which correspond to Proposition 6 and Lemma 9 respectively.

Proposition 16. *Let $d > 1$ and $c \geq c_1^*$. Any orbit of* (15) *starting from a point $\mathbf{u}_0 \in \Omega_1^-$, denoted by $\mathbf{u}(z; \mathbf{u}_0)$, stays in Ω_1^- for all $z \geq 0$.*

Proof. We only consider S_1^- and S_5^-. S_5^- is an invariant manifold, so that $\mathbf{u}(z; \mathbf{u}_0)$ cannot reach any point of this surface. On S_1^-, the outward normal $\mathbf{n}_1^-$ is $(-1, 1, 0)$, so that the inner product

$$\mathbf{n}_1^- \cdot \mathbf{f}_d = \frac{c}{d}X + \left(\frac{1}{d} - 1\right)pq + pq = \frac{c}{d}(q-1) + \frac{1}{d}pq < 0.$$

Hence, $\mathbf{u}(z;\mathbf{u}_0)$ cannot leave Ω_1^- through S_1^-. To prove the remaining part, it suffices for us to follow the argument in the proof of Proposition 6 by replacing ψ_c with ϕ_c. This completes the proof. □

Lemma 17. *Assume that $d > 1$. Then, for each $c \geq c_1^*$, there exists an orbit of* (15) *which connects P_1 with P_0.*

Proof. Since $0 < \lambda_+^d < \lambda_+^1$ and $-1 = f^c(0) < f^c(\lambda_+^d) < f^c(\lambda_+^1) = 0$, $\mathcal{U}_d(c)$ enters Ω_1^-. Then Proposition 16 assures that $q \to 0$ as $z \to \infty$, since $q' = pq < 0$ in Ω_1^-. The first equation of (15) gives (23) with c in place of c^*, so that we have

$$\begin{aligned}|X(z)| &\leq e^{-(c/d)(z-z_1)}\left(|X(z_1)| + \int_{z_1}^{z} e^{(c/d)(\zeta - z_1)}|p(\zeta)|q(\zeta)\,d\zeta\right)\\ &\leq e^{-(c/d)(z-z_1)}|X(z_1)| + \frac{d}{c}M_p q(z_1),\end{aligned}$$

for any z, z_1 satisfying $z > z_1$, where $M_p = \max_{0\leq q\leq 1}|\phi_c(q)|$. Taking the limit $z \to \infty$ in the above inequality, we have $\lim_{z\to\infty}|X(z)| \leq (d/c)M_p q(z_1)$. Again take the limit $z_1 \to \infty$. Then we see $\lim_{z\to\infty}|X(z)| = 0$. This implies that $\mathcal{U}_d(c)$ must approach $\mathrm{cl}(I_0^-)$, which assures that $\mathcal{U}_d(c)$ approach P_0 or P_c as $z \to \infty$. To show that $\mathcal{U}_d(c)$ cannot approach to P_c as $z \to \infty$ for any $c \geq c_1^*$, it suffices for us to repeat the corresponding part of the proof of Lemma 9. This completes the proof. □

We define W_c^- by

$$W_c^- = \{\mathbf{u} = (X, q, p)\colon q - 1 \leq X \leq 0,\ l_c(p) \leq q \leq 1,\ -c \leq p \leq 0\}.$$

This plays the same role as W_c in the Subsection 3.3, and we can construct the Wazewski set $\tilde{W}^- = \{\mathbf{U} = (\mathbf{u}, c)\colon \mathbf{u} \in W_c^-,\ c_1 \leq c \leq c_2\}$ for the extended system (22). By noting that $X < 0$ and $1 - q + X > 0$, the whole arguments in the Subsection 3.3 with c_1^* in place of c_0^* are valid for $d > 1$. Thus we have

Lemma 18. *Assume that $d > 1$. Then, there exists some positive constant c_d^* ($< c_1^*$) such that, for $c = c_d^*$,* (15) *has an orbit connecting P_1 with P_c.*

Remark 3. Remark 1 is also valid for $d > 1$.

In order to prove that c_d^* is the minimal wave speed, we introduce the region Ω_-^* by Ω^* with the reversed inequality with respect to X, that is,

$$\Omega_-^* \equiv \{(X, q, p)\colon X^*(q) < X < 0,\ 0 < q < 1,\ p^*(q) < p < 0\},$$

where $(X^*(q), q, p^*(q))$ denotes the orbit connecting P_1 with P_c for $c = c_d^*$. Then, the boundary of Ω_-^* is given by

$$\partial\Omega_-^* = \left(\bigcup_{i=1}^{4} S_i^{-*}\right) \cup \left(\bigcup_{i=1}^{4} J_i^{-*}\right) \cup I_0^{-*} \cup P_0 \cup P_1 \cup P_{c_d^*},$$

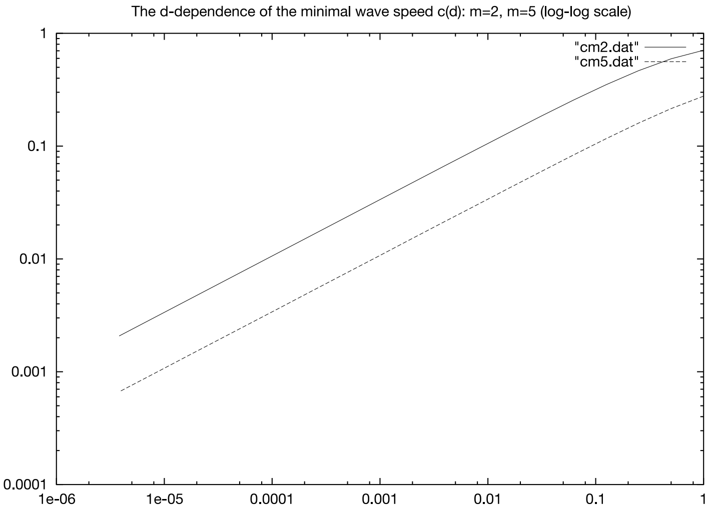

Fig. 2. The dependency of c_d^* on $\delta = 1/d$ (log-log scale).

where S_i^{-*}, J_i^{-*} $(i = 1, 2, 3, 4)$ and I_0^{-*} are S_i^-, J_i^- $(i = 1, 2, 3, 4)$ and I_0^- with $p^*(q)$ in place of $\phi_c(q)$. Proposition 8 assures that $\mathcal{U}_d(c)$ enters Ω_-^* for any $c > c_d^*$, and then applying the arguments of the proofs of Proposition 12 and Lemma 13 to Ω_-^*, we see

Lemma 19. *Let $d > 1$ be fixed. Then, for each $c > c_d^*$, there exists an orbit of* (15) *connecting P_1 with P_0 lying in Ω_-^*. For each positive $c < c_d^*$, there exists no orbit of* (15) *connecting P_1 with P_0 or P_c.*

As for the d-dependence, unfortunately, Proposition 7 asserts that $\mathcal{U}_{\bar{d}}(c)$ does not enter Ω_-^* for any $\bar{d} > d > 1$. Therefore, except the monotone decreasing property of c_d^* with respect to d, we have the similar result of Theorem 15 for $d > 1$.

Theorem 20. *Assume that $d > 1$. Then, there exists some c_d^*, such that a traveling front solution for* (10) *exists uniquely (except translation) for each $c \geq c_d^*$. Furthermore, the minimal wave speed c_d^* satisfies that $c_d^* < c_1^* \leq \sqrt{2/(m(m-1))}$.*

5. Concluding remarks

For all $d_1 \geq 0$ and $d_2 > 0$, we have shown the existence of the minimal wave speed $c_d^* > 0$ such that the traveling front solutions exist for any $c \geq c_d^*$.

As for the front profile, we have seen that the solutions of (5) with the minimal wave speed c_d^* decay to $(U, V) = (1, 0)$ exponentially as $z \to \infty$. We can also see that the the solutions of (5) with the speed $c > c_d^*$ decay to $(U, V) = (1, 0)$ algebraically as $z \to \infty$. This can be proved by the routine work of the local analysis of the (15) near P_0 with the aid of the center manifold theory (see, for example [12]). Of course, it is obvious that any traveling front solutions decay to $(U, V) = (0, 1)$ exponentially as $z \to -\infty$.

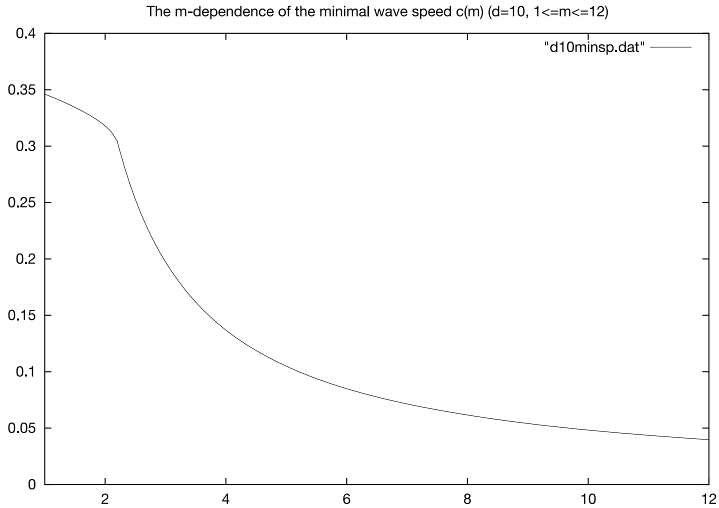

Fig. 3. The dependency of c_d^* on m ($d = 10$, $1 \leq m \leq 12$).

We have also discussed the dependence of c_d^* on the diffusion coefficients and the order of autocatalytic reactions. When $d > 1$, unfortunately we cannot establish the monotone dependence of the minimal wave speed on d. However, Fig. 2 shows that the minimal wave speed c_d^* is monotone decreasing with respect to the parameter d for $m = 2$ (cm2.dat) and $m = 5$ (cm5.dat). This is obtained numerically by the shooting method which follows the unstable manifold of the critical point P_1 of (15). Also, this numerical result shown in Fig. 2 suggests that $c_d^* = \sqrt{\delta}\,\{\sigma(m) + o(1)\}$, where $\delta = 1/d$. Fig. 3 shows that he minimal wave speed c_d^* is monotone decreasing with respect to the parameter m for $d = 10$. The justifications of these numerical results require the further study.

6. Appendix

We give the proof of Theorem 4. For $d_1 = 1$ and $d_2 = 0$, (5) becomes

$$\begin{cases} U'' + cU' - UV^m = 0, \\ cV' + UV^m = 0. \end{cases} \tag{36}$$

Adding these two equations and integrating the result with the aid of (6), we have

$$U' + c(U + V - 1) = 0,$$

so that (36) is reduced to the first order system

$$\begin{cases} U' = -c(U + V - 1), \\ V' = -\dfrac{1}{c}UV^m, \end{cases} \tag{37}$$

which has two critical points $P_+ = (1,0)$ and $P_- = (0,1)$. From Proposition 1, we see that $U + V < 1$ and $U' > 0$. This shows $c > 0$. Hence we look for an orbit connecting P_+ and P_- which lies entirely in $\Omega_0 = \{(U,V)\colon 0 < U < 1,\ 0 < V < 1,\ U + V < 1\}$ for $c > 0$.

The eigenvalues of the linearized equation of (37) about the critical point P_- are $\lambda_\pm = (1/2)[-c \pm \sqrt{c^2+4}]$. The corresponding eigenvectors are ${}^t(c\lambda_\pm, -1)$, respectvely. Since $c\lambda_+ = (c/2)(\sqrt{c^2+4} - c) = 2/(\sqrt{1+4/c^2} + 1) < 1$, the 1-dim unstable manifold $\mathcal{U}_0$ through P_- enters Ω_0. We easily see that this $\mathcal{U}_0$ must stay in Ω_0 for all $z \in \mathbb{R}$. In fact, on the boundary $l_1 = \{(0,V)\colon 0 < V < 1\}$, the vector field of (37) is $(c(1-V), 0)$, and it is $(0, U(1-U)^m)$ on $l_2 = \{(U,V)\colon 0 < U < 1, U + V = 1\}$. These imply that the vector fields on l_1 and l_2 are directed to the inside of Ω_0. The boundary $l_3 = \{(U,0)\colon 0 \le U \le 1\}$ is an invariant manifold. Thus we know that $\mathcal{U}_0$ cannot traverse the boundary of Ω_0 and stays in Ω_0. Noting that there is no critical point in Ω_0 and $U' > 0$, we can conclude that $\mathcal{U}_0$ must approach P_+ as $z \to +\infty$. This completes the proof. □

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