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NECESSARY AND SUFFICIENT CONDITIONS FOR THE BEDROSIAN IDENTITY

LIHUI TAN, LIHUA YANG AND DAREN HUANG

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ABSTRACT. This paper focuses on necessary and sufficient conditions for the Bedrosian identity to be valid. It extends the results of Brown (1986), Xu and Yan (2006) for $L^2(\mathbb{R})$ functions. Meanwhile, the Bedrosian identity for periodic functions is also discussed and some necessary and sufficient conditions are obtained.

1. Introduction.

The Hilbert transform, which is defined as the Cauchy principal value of the following singular integral

$$(Hf)(x) = \text{p.v}\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{f(y)}{x-y}dy, \tag{1.1}$$

has been widely used in physics, engineering and mathematics. It is a typical Calderon-Zygmund singular integral operator in mathematics and a basic mathematical tool for the study of analytic signals, with which the concept of instantaneous frequency and amplitude of non-stationary signals are defined precisely. A signal is said to be analytic if it has no negative frequencies, that is, $\hat{f}(\omega) = 0$ for $\omega < 0$ and dual-analytic if it has no positive frequencies, namely, $\hat{f}(\omega) = 0$ for $\omega > 0$, where $\hat{f}(\omega)$ is the Fourier transform of f defined by

$$\hat{f}(\omega) := \int_{-\infty}^{\infty} f(t)e^{-i\omega t}dt \tag{1.2}$$

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for $f \in L^1(\mathbb{R})$ and the L^2-limit of the Fourier transform of functions in $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ for $f \in L^2(\mathbb{R})$ (see [**5**]). For $1 \le p \le \infty$, $L^p(\mathbb{R})$ stands for the space of all the p-power integrable functions on $\mathbb{R}$ endowed with the following norm

$$\|f\|_p := \begin{cases} \left(\int_{\mathbb{R}} |f(t)|^p dt\right)^{1/p} & 1 \le p < \infty, \\ \operatorname{essup}_{x \in \mathbb{R}} |f(t)| & p = \infty. \end{cases} \tag{1.3}$$

It is well-known that the Hilbert transform is a bounded linear operator on $L^p(\mathbb{R})$ for $1 < p < \infty$ and the Fourier transform of Hf satisfies

$$(Hf)\hat{}(\omega) = -i\operatorname{sgn}(\omega)\hat{f}(\omega) \quad \text{a.e.} \quad \omega \in \mathbb{R} \tag{1.4}$$

for $f \in L^p(\mathbb{R})$ $(1 < p \le 2)$ or $f, Hf \in L^1(\mathbb{R})$, where $\operatorname{sgn}(\omega)$ is the signum function defined as $\operatorname{sgn}(0) := 0$ and $\operatorname{sgn}(\omega) := \omega/|\omega|$ for $\omega \ne 0$ (cf. [**5**]).

A (real) signal is usually represented by its amplitude and frequency according to its physical attributes. In signal analysis, one needs to compute the amplitude and frequency of a signal s, that is, to demodulate the signal. A classical method of demodulation is to add the Hilbert transform Hs of s as the imaginary part to the signal to produce an analytic signal $s(t) + iHs(t) = A(t)e^{i\theta(t)}$, $t \in \mathbb{R}$ (cf. [**7**]), and then extract $A(t)$ and $\theta(t)$ as its instantaneous amplitude and phase. This means that $A(t)$ and $\theta(t)$ should satisfy

$$H[A(t)\cos\theta(t)] = A(t)\sin\theta(t). \tag{1.5}$$

However, it is observed that under proper conditions the following equality

$$H[A(t)\cos\theta(t)] = A(t)H\cos\theta(t) \tag{1.6}$$

holds, which means that (1.5) is true if and only if $H\cos\theta(t) = \sin\theta(t)$. Therefore, demodulation (1.5) can be reduced to a question of frequency demodulation of a unitary amplitude signal [**9, 18**]. Equation (1.6) is a very important equality for demodulation of signals in the non-stationary signal processing, whose general form is

$$H(fg) = fHg. \tag{1.7}$$

Equation (1.7) was first studied by Bedrosian in 1963 and was called the Bedrosian identity in honor of him. In [**1**] it was proved that (1.7) holds if either of the following two conditions is satisfied:

(i) The Fourier transforms $\hat{f}, \hat{g}$ satisfy for some $a > 0$ that $\hat{f}(\omega) = 0,\ |\omega| > a$ and $\hat{g}(\omega) = 0,\ |\omega| < a$;

(ii) Both $f(t)$ and $g(t)$ are analytic or dual-analytic functions.

Later, Nuttall and Bedrosian (1966), and Brown (1974) obtained more general sufficient conditions by weakening the above condition (ii) [**3, 12**]. And in 1986, Brown established the first necessary and sufficient condition in the time domain and a parellel result in the frequency domain for the Bedrosian identity [**4**], which we state as follows:

(a) If $f, g \in L^2(\mathbb{R})$ are bounded on $\mathbb{R}$, then $H(fg) = fH(g)$ if and only if

$$H(f_+(t)g_-(t)) = if_+(t)g_-(t), \qquad H(f_-(t)g_+(t)) = -if_-(t)g_+(t), \tag{1.8}$$

or

$$\hat{f}_+ * \hat{g}_-(\omega) = 0 \quad \text{a.e. } \omega \in \mathbb{R}_+, \quad \hat{f}_- * \hat{g}_+(\omega) = 0 \quad \text{a.e. } \omega \in \mathbb{R}_-, \tag{1.9}$$

where '$*$' denotes the convolution operator, $g_+(t) = (\hat{g}(\omega)\chi_{\mathbb{R}_+})\check{}(t)$ and $g_-(t) = (\hat{g}(\omega)\chi_{\mathbb{R}_-})\check{}(t)$ with $\mathbb{R}_+ = (0, \infty)$ and $\mathbb{R}_- = (-\infty, 0)$. Hereafter, $\check{f}$ denotes the inverse Fourier transform of f and χ_E the characteristic function of set E.

Recently, as the advent of the empirical mode decomposition for the non-stationary signal processing and researches on relevant mathematics problems [**6, 8, 10, 14**], the Bedrosian identity received much attention again. A new necessary and sufficient condition was proposed by Xu and Yan in their recent paper [16]:

(b) If $f, f', g \in L^2(\mathbb{R})$, then $H(fg) = fH(g)$ if and only if

$$\int_0^1 dt \int_{\mathbb{R}} \frac{\omega}{t^2} e^{-2i\pi x\omega(t-1)/t} \hat{f}\left(\frac{\omega}{t}\right) \hat{g}(-\omega) d\omega = 0. \tag{1.10}$$

Moreover, Chen, Huang, Riemenschneider and Xu proved that the Bedrosian identity holds if the function g has vanishing moment of order k for some $k \in \mathbb{N}$ in the sense of Cauchy principal value and function f is a polynomial of order $n \le k$ [**6**].

Another up-to-date development was given by Yu and Zhang [**17**]. It was proved that (1.9) is still a necessary and sufficient condition for the Bedrosian identity even if the functions f and g in (b) are not bounded (see Theorem 2.4 of [**17**]), that is:

(c) If $f, g \in L^2(\mathbb{R})$, then $H(fg) = fH(g)$ if and only if (1.9) holds.

In Section 2 of this paper it is shown that (1.8) is also a necessary and sufficient condition for the Bedrosian identity to hold provided that $f, g \in L^2(\mathbb{R})$. We also weaken the conditions in (b). Considering the significance of periodic signals in reality, the Bedrosian identity for periodic functions is discussed in Section 3.

2. The Bedrosian Identity in $L^2(\mathbb{R})$.

Analytic signal is a fundamental concept in time-frequency analysis. It is well-known that $f \in L^2(\mathbb{R})$ is analytic if and only if $Hf = -if$ and $f \in L^2(\mathbb{R})$ is dual analytic if and only if $Hf = if$ [**3**]. The result remains true if $f \in L^p(\mathbb{R}), 1 < p \le 2$, or $f, Hf \in L^1(\mathbb{R})$ [**5**]. Based on it, it seems that (1.8) and (1.9) are equivalent for all $f, g \in L^2(\mathbb{R})$ since (1.9) means that $f_- g_+$ is analytic and $f_+ g_-$ is dual analytic. However, we should point out that the deduction is unreliable because even if $f, g \in L^2(\mathbb{R})$ are bounded we do not have $f_+ g_-, f_- g_+ \in L^p(\mathbb{R})$ for some $1 < p < \infty$. Thus, the proof in [**4**] for the equivalence of (1.8) and (1.9) is not precise. To clarify this conclusion, we need to know if $Hf = -if$ is still a characterization of Lebesgue integrable analytic signals f. Qian proved that f is analytic if it is a tempered upper-Hardy distribution represented by the boundary value of a function in $H^p(\mathbb{C}^+)$ with $1 \le p \le \infty$ (see [**13**]). In this paper, we extend the characterization of analytic and dual analytic signals to $L^1(\mathbb{R})$ by making use of the following lemma from [**17**].

Lemma 2.1. *Let $f, g \in L^1(\mathbb{R})$. Then $Hf = g$ if and only if*

$$-i\mathrm{sgn}(\omega)\hat{f}(\omega) = \hat{g}(\omega), \qquad \omega \in \mathbb{R}.$$

With the lemma, the characterization of analytic and dual analytic signals is extended to $L^1(\mathbb{R})$ as follows.

Lemma 2.2. *Let $f \in L^1(\mathbb{R})$. Then $Hf = -if$ if and only if $\hat{f}(\omega) = 0$ for $\omega \in \mathbb{R}_-$; $Hf = if$ if and only if $\hat{f}(\omega) = 0$ for $\omega \in \mathbb{R}_+$.*

Proof. Suppose $Hf = -if$, then we have $f, Hf \in L^1(\mathbb{R})$. Since $(Hf)\hat{}(\omega) = -i\mathrm{sgn}(\omega)\hat{f}(\omega)$ (see [5, Chapter 8]) we conclude that $-i\hat{f}(\omega) = -i\mathrm{sgn}(\omega)\hat{f}(\omega)$, which implies that $\hat{f}(\omega) = 0$ for $\omega \in \mathbb{R}_-$. Conversely, if $\hat{f}(\omega) = 0$ for $\omega \in \mathbb{R}_-$, we have $-i\mathrm{sgn}(\omega)\hat{f}(\omega) = -i\hat{f}(\omega)$ for $\omega \in \mathbb{R}$, which concludes that $Hf = -if$ by Lemma 2.1. The equivalence between $Hf = if$ and $\hat{f}(\omega) = 0$ for $\omega \in \mathbb{R}_+$ can be shown similarly. □

Based on Lemma 2.2 it is shown in the following theorem that (1.8) and (1.9) are equivalent, which shows that (1.8) is still a necessary and sufficient condition for the Bedrosian identity to be valid even if f, g are unbounded.

Theorem 2.3. *Let $f, g \in L^2(\mathbb{R})$. Then* (1.8) *and* (1.9) *are equivalent. Consequently, $H(fg) = fHg$ if and only if* (1.8) *holds.*

Proof. Since $f_+g_- \in L^1(\mathbb{R})$, by Lemma 2.2 it is deduced that $H(f_+g_-) = if_+g_-$ if and only if $(f_+g_-)\hat{}(\omega) = 0$ for $\omega \in \mathbb{R}_+$. Similarly, $H(f_-g_+) = -if_-g_+$ if and only if $(f_-g_+)\hat{}(\omega) = 0$ for $\omega \in \mathbb{R}_-$. Therefore (1.8) is equivalent to (1.9) for all $f, g \in L^2(\mathbb{R})$. □

Theorem 2.4. *Let $f, g \in L^2(\mathbb{R})$ and let $\rho(\omega)$ be a Lebesgue measurable function on $\mathbb{R}$ satisfying*

$$\text{(2.1)} \quad \rho(\omega) \neq 0 \ \text{ a.e. } \ \omega \in \mathbb{R} \quad \textit{and} \quad \int_A |\rho(\omega)\hat{g}(\lambda)\hat{f}(\omega - \lambda)| d\omega d\lambda < \infty,$$

where $A := (\mathbb{R}_- \times \mathbb{R}_+) \cup (\mathbb{R}_+ \times \mathbb{R}_-)$. Then $H(fg) = fHg$ if and only if

$$\text{(2.2)} \quad \int_{\mathbb{R}} d\omega \int_0^1 \rho\left(\frac{\omega}{t} - \omega\right) \frac{\omega}{t^2} e^{i(\frac{1}{t}-1)\omega x} \hat{g}(-\omega)\hat{f}\left(\frac{\omega}{t}\right) dt = 0 \ \text{ a.e. } \ x \in \mathbb{R},$$

Proof. By the result (c) in Section 1, it is sufficient to prove the equivalence between (2.2) and (1.9). Denote

$$(2.3)\qquad \begin{cases} \xi_\rho := \int_0^\infty \rho(u)e^{iux}du \int_{-\infty}^0 \hat{g}(\lambda)\hat{f}(u-\lambda)d\lambda, \\ \eta_\rho := \int_{-\infty}^0 \rho(u)e^{iux}du \int_0^\infty \hat{g}(\lambda)\hat{f}(u-\lambda)d\lambda. \end{cases}$$

With (2.1) and Fubini's theorem, it can be seen that

$$\xi_\rho = \int_{-\infty}^0 d\lambda \int_{-\lambda}^\infty \rho(\lambda+\nu)\hat{g}(\lambda)\hat{f}(\nu)e^{i(\nu+\lambda)x}d\nu.$$

Replacing ν by $-\frac{\lambda}{t}$ and then λ by $-\omega$ in the above equality, we conclude that

$$\xi_\rho = \int_0^\infty d\omega \int_0^1 \rho\left(\frac{\omega}{t}-\omega\right)\frac{\omega}{t^2}e^{i\left(\frac{1}{t}-1\right)\omega x}\hat{g}(-\omega)\hat{f}\left(\frac{\omega}{t}\right)dt.$$

Similarly, we have

$$\eta_\rho = -\int_{-\infty}^0 d\omega \int_0^1 \rho\left(\frac{\omega}{t}-\omega\right)\frac{\omega}{t^2}e^{i\left(\frac{1}{t}-1\right)\omega x}\hat{g}(-\omega)\hat{f}\left(\frac{\omega}{t}\right)dt.$$

It is easy to learn that (1.9) is equivalent to $\xi_\rho = \eta_\rho = 0$ a.e. $x \in \mathbb{R}$. Therefore (1.9) implies $\xi_\rho = \eta_\rho$ a.e. $x \in \mathbb{R}$, that is, (2.2) holds.

Conversely, suppose (2.2) holds. By (2.3) we conclude that

$$\int_{\mathbb{R}} \rho(u)e^{iux}\left[\chi_{\mathbb{R}_+}(u)\int_{-\infty}^0 \hat{g}(\lambda)\hat{f}(u-\lambda)d\lambda - \chi_{\mathbb{R}_-}(u)\int_0^\infty \hat{g}(\lambda)\hat{f}(u-\lambda)d\lambda\right]du = 0.$$

Hence

$$\chi_{\mathbb{R}_+}(u)\int_{-\infty}^0 \hat{g}(\lambda)\hat{f}(u-\lambda)d\lambda - \chi_{\mathbb{R}_-}(u)\int_0^\infty \hat{g}(\lambda)\hat{f}(u-\lambda)d\lambda = 0 \quad \text{a.e. } u \in \mathbb{R},$$

which implies (1.9) immediately. □

The function ρ in Theorem 2.4 is introduced to ensure the validity of Fubini's theorem. There are many such functions. For example, if

$\rho \in L^1(\mathbb{R})$, it is easy to verify by the Hölder inequality that (2.1) holds for any $f, g \in L^2(\mathbb{R})$.

As a special case, if $\hat{g}(\lambda)\hat{f}(\omega - \lambda) \in L^1(\mathbb{R}^2)$, we get the following corollary by choosing $\rho \equiv 1$ in Theorem 2.4.

Corollary 2.5. *Let $f, g \in L^2(\mathbb{R})$. If*

$$\int_A |\hat{g}(\lambda)\hat{f}(\omega - \lambda)| d\omega d\lambda < \infty, \tag{2.4}$$

where $A := [\mathbb{R}_- \times \mathbb{R}_+] \cup [\mathbb{R}_+ \times \mathbb{R}_-]$, then $H(fg) = fHg$ if and only if (1.10) *holds.*

Corollary 2.6. *Let $f, g \in L^2(\mathbb{R})$. If there exists $\alpha > \frac{1}{2}$ such that $|x|^\alpha \hat{f}(x), |x|^{1-\alpha}\hat{g}(x) \in L^2(\mathbb{R})$, Then $H(fg) = fHg$ if and only if* (1.10) *holds.*

Proof. Denote

$$I := \int_0^\infty d\lambda \int_{-\infty}^0 |\hat{g}(\lambda)\hat{f}(\omega - \lambda)| d\omega.$$

Then we have

$$I = \int_0^\infty |\hat{g}(\lambda)| d\lambda \int_{-\infty}^{-\lambda} |\hat{f}(\nu)| d\nu = \int_{-\infty}^0 |\hat{f}(\nu)| d\nu \int_0^{-\nu} |\hat{g}(\lambda)| d\lambda.$$

Substituting λ by $-\nu t$, we get

$$I = \int_{-\infty}^0 |\hat{f}(\nu)| d\nu \int_0^1 |\hat{g}(-\nu t)||\nu| dt = \int_0^1 dt \int_{-\infty}^0 |\hat{f}(\nu)\hat{g}(-\nu t)\nu| d\nu.$$

The Hölder inequality gives that

$$\int_{-\infty}^0 |\hat{f}(\nu)\hat{g}(-\nu t)\nu| d\nu \le t^{\alpha - \frac{3}{2}} \||\nu|^\alpha \hat{f}(\nu)\|_2 \cdot \||\nu|^{1-\alpha}\hat{g}(\nu)\|_2$$

for all $t > 0$, which yields

$$I \le \||\nu|^\alpha \hat{f}(\nu)\|_2 \cdot \||\nu|^{1-\alpha}\hat{g}(\nu)\|_2 \int_0^1 t^{\alpha - \frac{3}{2}} dt < \infty.$$

Similarly, we have

$$\int_{-\infty}^{0} d\lambda \int_{0}^{\infty} |\hat{g}(\lambda)\hat{f}(\omega-\lambda)| d\omega < \infty.$$

Hence (2.4) is satisfied. By Corollary 2.5, $H(fg) = fHg$ if and only if (1.10) holds. □

The above corollary shows that the result (b) of Section 1 established by Xu and Yan in [**16**] can be obtained by Corollary 2.6. In fact, if $f' \in L^2(\mathbb{R})$ we have $|\omega \hat{f}(\omega)| = |(f')\hat{}(\omega)| \in L^2(\mathbb{R})$, the condition of Corollary 2.6 is satisfied with $\alpha = 1$. It is easily seen that Corollary 2.6 includes many other cases such as those corresponding to $\frac{1}{2} < \alpha < 1$. Besides, (2.4) is also satisfied if $\hat{f}, \hat{g} \in L^1(\mathbb{R})$.

3. The Bedrosian Identity for Periodic Functions. In practice, people consider not only signals with finite energy signals but also periodic ones such as amplitude modulation signals with linear frequencies. We shall discuss in this section necessary and sufficient conditions for the Bedrosian identity (1.7) to be valid when g or both f and g are periodic.

In order to consider the Bedrosian identity for periodic signals, we need to deal with the Hilbert transform of periodic functions. A Lebesgue measurable function f with T-period ($T > 0$) is said to be in L_T^p if $\|f\|_{L_T^p} < \infty$, where the L_T^p-norm is defined by

$$\|f\|_{L_T^p} := \begin{cases} \left(\int_0^T |f(t)|^p dt\right)^{1/p} & 1 \le p < \infty, \\ \operatorname{essup}_{t\in\mathbf{R}} |f(t)| & p = \infty. \end{cases}$$

For $f \in L_{2\pi}^2$ the Hilbert transform Hf is defined independently of that for functions in $L^p(\mathbb{R})$ and called the circular Hilbert transform in many literatures, for example [**2, p. 15**]. The definition is as follows:

$$H\left(\sum_{k\in\mathbb{Z}} c_k e^{ikt}\right) := -i \sum_{k\in\mathbb{Z}} \operatorname{sgn}(k) c_k e^{ikt}$$

for all $\{c_k\}_{k\in\mathbb{Z}} \in l^2$, where

$$l^2 := \left\{ \{c_k\}_{k\in\mathbb{Z}} | \sum_{k\in\mathbb{Z}} |c_k|^2 < \infty \right\}.$$

For any $f \in L^1_T$ we call

$$c_k^T(f) := \frac{1}{T}\int_0^T f(t)e^{-i\frac{2\pi}{T}kt}dt$$

the k-th Fourier coefficient of f. It is also denoted as $c_k(f)$ for simplicity if no confusions occur. It is well-known that if $f \in L^2_T$ then it has the following Fourier expansion:

$$f(t) = \sum_{k\in\mathbb{Z}} c_k(f)e^{i\frac{2\pi}{T}kt},$$

in L^2_T. With this expansion, the Hilbert transform of $f \in L^2_T$ is defined as

$$H\left(\sum_{k\in\mathbb{Z}} c_k e^{i\frac{2\pi}{T}kt}\right) := -i\sum_{k\in\mathbb{Z}} \mathrm{sgn}(k)c_k e^{i\frac{2\pi}{T}kt} \ (\forall \{c_k\}_{k\in\mathbb{Z}} \in l^2).$$

A T-periodic function is obviously nT-periodic for $n \in \mathbb{N}$. It can be verified that

$$c_k^{nT}(f) = c_{k/n}^T(f)$$

with $c_{k/n}^T(f) := 0$ for $k/n \notin \mathbb{Z}$, and consequently the Hilbert transform obtained from viewing it as a nT-periodic function is the same as that from viewing it as a T-periodic function. This note will help us avoid possible confusions when a common multiple $T = nT_1 = mT_2$ of different periods T_1 and T_2 is considered in Section 3.2.

3.1. *The Bedrosian Identity for $f \in L^2(\mathbb{R})$ and $g \in L^2_T$.*

Lemma 3.1. *Let $f \in L^2(\mathbb{R})$, $g \in L^2_T$, and $\sum_{k\in\mathbb{Z}} |c_k(g)| < \infty$. Then $fg, fHg \in L^2(\mathbb{R})$ and*

$$(fg)\hat{}(\omega) = \sum_{k\in\mathbb{Z}} c_k(g)\hat{f}\left(\omega - \frac{2\pi}{T}k\right), \tag{3.1}$$

$$(fHg)\hat{}(\omega) = \sum_{k\in\mathbb{Z}} c_k(g)(-i\mathrm{sgn}(k))\hat{f}\left(\omega - \frac{2\pi}{T}k\right),$$

where the series converge in $L^2(\mathbb{R})$.

Proof. The condition $\sum_{k\in\mathbb{Z}} |c_k(g)| < \infty$ implies that $\sum_{k\in\mathbb{Z}} c_k(g)e^{i\frac{2\pi}{T}kt}$ converges uniformly to g and g is bounded on $\mathbb{R}$. It follows that $fg \in L^2(\mathbb{R})$. By the Lebesgue dominated convergence theorem, there holds

$$\begin{aligned}\left\|(fg)\hat{} - \sum_{k=-n}^{m} c_k(g)\hat{f}\left(\cdot - \frac{2\pi}{T}k\right)\right\|_2^2 \\ = \int_{\mathbb{R}} \left| f(t)\left[g(t) - \sum_{k=-n}^{m} c_k(g)e^{i\frac{2\pi}{T}kt}\right]\right|^2 dt \to 0 \ (n, m \to \infty).\end{aligned}$$

The first equality of (3.1) is proved.

By the definition of the Hilbert transform of periodic functions, we have

$$(Hg)(t) = \sum_{k\in\mathbb{Z}} c_k(g)(-i\mathrm{sgn}(k))e^{i\frac{2\pi}{T}kt},$$

which shows that $Hg \in L_T^2$ and $\sum_{k\in\mathbb{Z}} |c_k(Hg)| < \infty$. Similar arguments as those used in the proof of the first equality in (3.1) show that $fHg \in L^2(\mathbb{R})$ and the second equality of (3.1) holds. □

Theorem 3.2. *Let $f \in L^2(\mathbb{R})$, $g \in L_T^2$, and $\sum_{k\in\mathbb{Z}} |c_k(g)| < \infty$. Then $H(fg) = fHg$ if and only if*

$$\begin{cases} c_0(g)\hat{f}(\omega) + 2\sum_{k=-\infty}^{-1} c_k(g)\hat{f}\left(\omega - \frac{2\pi}{T}k\right) = 0 \quad \text{a.e. } \ \omega \in (0, \infty), \\ c_0(g)\hat{f}(\omega) + 2\sum_{k=1}^{\infty} c_k(g)\hat{f}\left(\omega - \frac{2\pi}{T}k\right) = 0 \quad \text{a.e. } \ \omega \in (-\infty, 0)\,. \end{cases} \tag{3.2}$$

Proof. By Lemma 3.1 we have

$$\begin{aligned}(H(fg) - fHg)\hat{}(\omega) &= -i\sum_{k\in\mathbb{Z}} c_k(g)[\mathrm{sgn}(\omega) - \mathrm{sgn}(k)]e^{i\frac{2\pi}{T}kt}f(t)\hat{}(\omega) \\ &= -i\sum_{k\in\mathbb{Z}} c_k(g)[\mathrm{sgn}(\omega) - \mathrm{sgn}(k)]\hat{f}\left(\omega - \frac{2\pi}{T}k\right),\end{aligned}$$

from which the theorem follows immediately. □

Equation (3.2) is essentially the counterpart of (1.9) as the continuous Fourier transform $\hat{g}$ is replaced by discrete Fourier spectrum $c_k(g)$ for periodic function g. Based on it, we immediately have the following sufficient condition for the Bedrosian identity to hold.

Corollary 3.3. *Let $f \in L^2(\mathbb{R})$, $g \in L^2_T$ and $\sum_{k\in\mathbb{Z}} |c_k(g)| < \infty$. If there exists an $N \in \mathbb{N}$ such that*

$$\operatorname{supp} \hat{f} \subset \left[-\frac{2\pi}{T}N, \frac{2\pi}{T}N\right], \ c_k(g) = 0 \ (\forall |k| < N), \tag{3.3}$$

then $H(fg) = fHg$.

Below is a special case of Theorem 3.2 from signal processing, in which a signal $f(t)$ is usually modulated with a carrier wave $e^{i\omega_0 t}$ of high carrier frequency for transmission.

Corollary 3.4. *Let $f \in L^2(\mathbb{R})$ and $\omega_0 \neq 0$. Then $H(f(t)e^{i\omega_0 t}) = f(t)He^{i\omega_0 t}$ if and only if*

$$\begin{cases} \hat{f}(\omega) = 0 \ \text{ a.e. } \ \omega \in (-\infty, -\omega_0) & \text{if } \omega_0 > 0, \\ \hat{f}(\omega) = 0 \ \text{ a.e. } \ \omega \in (-\omega_0, \infty,) & \text{if } \omega_0 < 0. \end{cases} \tag{3.4}$$

Proof. If $\omega_0 > 0$, then the period of $e^{i\omega_0 t}$ is $T = 2\pi/\omega_0$. By Theorem 3.2, $H(f(t)e^{i\omega_0 t}) = f(t)He^{i\omega_0 t}$ holds if and only if $\hat{f}(\omega - \frac{2\pi}{T}) = 0$ a.e. $\omega \in (-\infty, 0)$, that is, $\hat{f}(\omega) = 0$ a.e. $\omega \in (-\infty, -\omega_0)$.

If $\omega_0 < 0$, the result can be proved similarly. □

The condition "$\sum_{k\in\mathbb{Z}} |c_k(g)| < \infty$" in Lemma 3.1 and Theorem 3.2 is a constraint on g in the Fourier transform domain. The analogical condition $\hat{g} \in L^1(\mathbb{R})$ is not necessary for the case that $f, g \in L^2(\mathbb{R})$ (see Section 2). Thus it is natural to ask if the condition can be removed. We note that, in the case that $f, g \in L^2(\mathbb{R})$, there holds $fg, fHg \in L^1(\mathbb{R})$ and consequently the Fourier transforms of fg and fHg are well-defined. However, the assumption that $f \in L^2(\mathbb{R})$ and $g \in L^2_T$ does not imply $fg, fHg \in L^2(\mathbb{R})$ or $L^1(\mathbb{R})$, which makes the

Fourier transforms of fg and fHg undefined. Therefore, if we want to remove the condition "$\sum_{k\in\mathbb{Z}}|c_k(g)|<\infty$" we need to supplement other conditions such that $fg, fHg \in L^2(\mathbb{R})$ or $L^1(\mathbb{R})$. To do this, let us first introduce an important space $\mathcal{L}_T^2(\mathbb{R})$ for $T>0$, which is defined as the space of all the Lebesgue integrable functions on $\mathbb{R}$ satisfying

$$|f|_T := \left\|\sum_{k\in\mathbb{Z}}|f(\cdot+kT)|\right\|_{L_T^2} < \infty.$$

The space corresponding to $T=1$ was first introduced and used in wavelet construction by Jia and Micchelli in [**10**]. With the space $\mathcal{L}_T^2(\mathbb{R})$ we have the following lemma.

Lemma 3.5. *Let $f\in\mathcal{L}_T^2(\mathbb{R})$ and $g\in L_T^2$. Then $fg, fHg\in L^1(\mathbb{R})$ and there holds* (3.1) *in which the series converge in $L^\infty(\mathbb{R})$.*

Proof. By the Hölder inequality, we have

$$\int_{\mathbb{R}}|f(t)g(t)|dt=\int_0^T|g(t)|\sum_{k\in\mathbb{Z}}|f(t+kT)|dt \le |f|_T\left(\int_0^T|g(t)|^2dt\right)^{\frac{1}{2}}<\infty,$$

which implies that $fg\in L^1(\mathbb{R})$. Similarly, there holds $fHg\in L^1(\mathbb{R})$.

Using the convergence property for $g\in L_T^2$ and the Hölder inequality, we also deduce that

$$\begin{aligned}\left\|(fg)\hat{} - \sum_{k=-n}^{m} c_k(g)\hat{f}\left(\cdot-\frac{2\pi}{T}k\right)]\right\|_\infty &\le \int_{\mathbb{R}}|f(t)[g(t)-g_{n,m}(t)]|dt\\ &\le |f|_T\,\|g-g_{n,m}\|_{L_T^2}\to 0\ (n,m\to\infty),\end{aligned}$$

where $g_{n,m}(t) := \sum_{k=-n}^m c_k(g)e^{i\frac{2\pi}{T}kt}$. The first equality of (3.1) is proved. The second one can be shown similarly. ◻

With Lemma 3.5 a parallel result of Theorem 3.2 is established as follows.

Theorem 3.6. *Let $f\in\mathcal{L}_T^2(\mathbb{R})$, $g\in L_T^2$. Then $H(fg)=fHg$ if and only if* (3.2) *holds.*

Proof. Since $fg, fHg \in L^1(\mathbb{R})$, by Lemma 2.1, $H(fg) = fHg$ if and only if $-i\mathrm{sgn}(\omega)(fg)\hat{}(\omega) = (fHg)\hat{}(\omega)$, that is,

$$(3.5)\quad -i\mathrm{sgn}(\omega)\sum_{k\in\mathbb{Z}} c_k(g)\hat{f}\left(\omega - \frac{2\pi}{T}k\right) = -i\sum_{k\in\mathbb{Z}} c_k(g)\mathrm{sgn}(k)\hat{f}\left(\omega - \frac{2\pi}{T}k\right) \quad \omega\in\mathbb{R}$$

which, by Lemma 3.5, implies (3.2) immediately. □

Below is a counterpart of Corollary 3.3.

Corollary 3.7. *Let $f \in \mathcal{L}^2_T(\mathbb{R})$, $g \in L^2_T$. If there exists an $N \in \mathbb{N}$ such that* (3.3) *holds, then $H(fg) = fHg$.*

It should be pointed out that the condition $f \in \mathcal{L}^2_T(\mathbb{R})$, $g \in L^2_T$ in Theorem 3.6 cannot be replaced by $f \in L^2(\mathbb{R})$, $g \in L^2_T$ generally. It is because that for $f \in L^2(\mathbb{R})$ and $g \in L^2_T$ the fact $fg \in L^p(\mathbb{R})$ $(1 \le p < \infty)$ is not guaranteed and consequently $H(fg)$ is not well-defined according to (1.1). To understand it, let us consider the following example. Put

$$f(t) := \frac{2^{j/2}}{(1+|k|)(j+1)} \quad t \in \left(k + \frac{1}{2^{j+1}}, k + \frac{1}{2^j}\right] \quad (k \in \mathbb{Z};\ j = 0, 1, 2, \cdots)$$

and

$$g(t) := \frac{2^{j/2}}{j+1} \quad t \in \left(k + \frac{1}{2^{j+1}}, k + \frac{1}{2^j}\right] \quad (k \in \mathbb{Z},\ j = 0, 1, 2, \cdots).$$

One can see that g is a periodic function with period $T = 1$. A simple calculation gives that

$$\|f\|_2^2 = \frac{1}{2}\sum_{k\in\mathbb{Z}}\sum_{j=0}^{\infty}\frac{1}{(1+|k|)^2(j+1)^2} < \infty, \quad \|g\|^2_{L^2_T} = \frac{1}{2}\sum_{j=0}^{\infty}\frac{1}{(j+1)^2} < \infty$$

and

$$\int_{\mathbb{R}} |f(t)g(t)|^p dt = \frac{1}{2}\sum_{k\in\mathbb{Z}}\sum_{j=0}^{\infty}\left|\frac{2^j}{(1+|k|)(j+1)^2}\right|^p \frac{1}{2^j}$$
$$= \frac{1}{2}\sum_{k\in\mathbb{Z}}\frac{1}{(1+|k|)^p}\sum_{j=0}^{\infty}\frac{2^{j(p-1)}}{(j+1)^{2p}} = \infty$$

for any $1 \le p < \infty$, which shows that $f \in L^2(\mathbb{R})$ and $g \in L^2_T$, but $fg \notin L^p(\mathbb{R})$ for any $1 \le p < \infty$.

Because of the importance of the space $\mathcal{L}^2_T(\mathbb{R})$, let us give some discussion on it at the end of this subsection. It is easy to verify that $\mathcal{L}^2_T(\mathbb{R}) \subset L^2(\mathbb{R})$. Below are some characterizations of the space.

Proposition 3.8. *Let $f \in L^1(\mathbb{R})$. Then $f \in \mathcal{L}^2_T(\mathbb{R})$ if and only if $\sum_{k\in\mathbb{Z}} |\hat{f}(k\frac{2\pi}{T})|^2 < \infty$ and $f(x) \le g(x)$* a.e. *$x \in \mathbb{R}$ for some nonnegative function $g \in \mathcal{L}^2_T(\mathbb{R})$.*

Proof. For $f \in L^1(\mathbb{R})$, it is easily seen that $f \in \mathcal{L}^2_T(\mathbb{R})$ if and only if $F(t) := \sum_{k\in\mathbb{Z}} |f(t+kT)| \in L^2_T$. The Fourier coefficients of $F(t)$ is easily calculated as:

$$c_k(F) = \frac{1}{T}(|f|)^{\wedge}\left(k\frac{2\pi}{T}\right) \ (\forall k \in \mathbb{Z}).$$

By the Riesz-Fischer theorem [**15**] it is concluded that $f \in \mathcal{L}^2_T(\mathbb{R})$ if and only if

$$\sum_{k\in\mathbb{Z}}\left|(|f|)^{\wedge}\left(k\frac{2\pi}{T}\right)\right|^2 < \infty.$$

Necessity: Suppose $f \in \mathcal{L}^2_T(\mathbb{R})$, then both $|f|$ and $|f| - f$ are in $\mathcal{L}^2_T(\mathbb{R})$, which implies that

$$\sum_{k\in\mathbb{Z}}\left|(|f|)^{\wedge}\left(k\frac{2\pi}{T}\right)\right|^2 < \infty, \qquad \sum_{k\in\mathbb{Z}}\left|(|f|-f)^{\wedge}\left(k\frac{2\pi}{T}\right)\right|^2 < \infty$$

and consequently $\sum_{k\in\mathbb{Z}} |\hat{f}(k\frac{2\pi}{T})|^2 < \infty$. The function $g := |f|$ is a nonnegative function in $\mathcal{L}^2_T(\mathbb{R})$ satisfying $f(x) \le g(x)$ a.e. $x \in \mathbb{R}$.

Sufficiency: Suppose $f \in L^1(\mathbb{R})$ satisfies $\sum_{k\in\mathbb{Z}} |\hat{f}(k\frac{2\pi}{T})|^2 < \infty$ and $f(x) \le g(x)$ a.e. $x \in \mathbb{R}$ for some non-negative function $g \in \mathcal{L}_T^2(\mathbb{R})$. It follows that $\sum_{k\in\mathbb{Z}} |\hat{g}(k\frac{2\pi}{T})|^2 < \infty$ and consequently $\sum_{k\in\mathbb{Z}} |(g-f)\hat{\ }(k\frac{2\pi}{T})|^2 < \infty$, which concludes that $g - f \in \mathcal{L}_T^2(\mathbb{R})$. Hence $f \in \mathcal{L}_T^2(\mathbb{R})$. $\square$

Proposition 3.9. *Let f be a Lebesgue measurable function on $\mathbb{R}$ satisfying*

$$\sum_{k\in\mathbb{Z}} \left(\int_{kT}^{(k+1)T} |f(t)|^2 dt \right)^{1/2} < \infty.$$

Then $f \in \mathcal{L}_T^2(\mathbb{R})$. Consequently, if $(1+|\cdot|)^s f \in L^2(\mathbb{R})$ for some $s > \frac{1}{2}$, then $f \in \mathcal{L}_T^2(\mathbb{R})$.

Proof. By

$$\begin{aligned} |f|_T = \left\| \sum_{k\in\mathbb{Z}} |f(\cdot + kT)| \right\|_{L_T^2} &\le \sum_{k\in\mathbb{Z}} \|f(\cdot + kT)\|_{L_T^2} \\ &= \sum_{k\in\mathbb{Z}} \left(\int_{kT}^{(k+1)T} |f(t)|^2 dt \right)^{1/2} < \infty, \end{aligned}$$

we conclude $f \in \mathcal{L}_T^2(\mathbb{R})$.

Suppose that $g := (1+|\cdot|)^s f \in L^2(\mathbb{R})$ for some $s > \frac{1}{2}$. We have

$$\begin{aligned} \int_{kT}^{(k+1)T} |f(t)|^2 dt &= \int_{kT}^{(k+1)T} \left| \frac{g(t)}{(1+|t|)^s} \right|^2 dt \\ &\le \frac{1}{(1+|kT|)^{2s}} \int_{kT}^{(k+1)T} |g(t)|^2 dt. \end{aligned}$$

It follows that

$$\sum_{k\in\mathbb{Z}} \left(\int_{kT}^{(k+1)T} |f(t)|^2 dt \right)^{1/2} \le \left(\sum_{k\in\mathbb{Z}} \frac{1}{(1+|kT|)^{2s}} \right)^{1/2} \|g\|_2 < \infty.$$

Hence $f \in \mathcal{L}_T^2(\mathbb{R})$. $\square$

3.2. *The Bedrosian Identity for $f \in L^2_{T_1}$ and $g \in L^2_{T_2}$.* Another important case is that both f and g are periodic signals. Some researches, for example [**14, 17**], have been done to the circular Bedrosian identity $H(fg) = fHg$ for $f, g \in L^2_{2\pi}$. However, the case that f and g are periodic signals but with different periods T_1 and T_2 is more often encountered in engineering applications, such as the beat waves in signal processing [**11**]. In this subsection, we consider a special case that $f \in L^2_{T_1},\ g \in L^2_{T_2}$ and T_1, T_2 have a common multiple T, that is, $T = nT_1 = mT_2$ for some $n, m \in \mathbb{N}$. Before this, let us give some preliminaries. The following lemma is from [**17**].

Lemma 3.10. *Let $f, g \in L^1_T$. Then $Hf = g$ if and only if $c_k(g) = -i\mathrm{sgn}(k)c_k(f)$ for all $k \in \mathbb{Z}$.*

Based on Lemma 3.10, we have the following result similar to Lemma 3.1.

Lemma 3.11. *Let $f, g \in L^2_T$. Then $fg, fHg \in L^1_T$ and*

$$\begin{aligned} c_n(fg) &= \sum_{k\in\mathbb{Z}} c_k(g)c_{n-k}(f), \\ c_n(fHg) &= -i\sum_{k\in\mathbb{Z}} \mathrm{sgn}(k)c_k(g)c_{n-k}(f) \end{aligned} \qquad \text{(for all } n \in \mathbb{Z}\text{)}.$$

The main result of this subsection is the following theorem.

Theorem 3.12. *Let $T_1, T_2 > 0$ have a common multiple $T = nT_1 = mT_2$ and $f \in L^2_{T_1}, g \in L^2_{T_2}$. Then $H(fg) = fHg$ if and only if*

$$\sum_{k\in\mathbb{Z}} [sgn(l) - sgn(k)]c_k^{T_2}(g)c^{T_1}_{\frac{l-km}{n}}(f) = 0 \quad \textit{for all} \quad l \in \mathbb{Z}. \tag{3.6}$$

Proof. Since $fg, fHg \in L^1_T$, we learn from Lemma 3.10 that $H(fg) = fHg$ if and only if

$$c_l^T(fHg) = -i\mathrm{sgn}(l)c_l^T(fg) \quad \text{for all} \quad l \in \mathbb{Z}.$$

By Lemma 3.11 it follows that

$$\sum_{k\in\mathbb{Z}}[\mathrm{sgn}(l)-\mathrm{sgn}(k)]c_k^{mT_2}(g)c_{l-k}^{nT_1}(f)=0 \quad \text{for all} \quad l\in\mathbb{Z},$$

which can further be rewritten as (3.6). □

Similar to Corollary 3.4, a special case of Theorem 3.12 is given below.

Corollary 3.13. *Let* $T_1, T_2 > 0$ *have a common multiple* $T = nT_1 = mT_2$ *and* $f \in L^2_{T_1}$. *Then* $H\left(f(t)e^{i\frac{2\pi}{T_2}t}\right) = fHe^{i\frac{2\pi}{T_2}t}$ *if and only if* $c_j^{T_1}(f) = 0$ *for all* $j \in \mathbb{Z},\ j \le -m/n$; $H\left(f(t)e^{-i\frac{2\pi}{T_2}t}\right) = fHe^{-i\frac{2\pi}{T_2}t}$ *if and only if* $c_j^{T_1}(f) = 0$ *for all* $j \in \mathbb{Z},\ j \ge -m/n$.

Proof. For $g(t) = e^{i\frac{2\pi}{T_2}t}$ we have $c_1^{T_2}(g) = 1$ and $c_k^{T_2}(g) = 0$ for all $k \neq 1$. By Theorem 3.12 it is concluded that $H\left(f(t)e^{i\frac{2\pi}{T_2}t}\right) = fHe^{i\frac{2\pi}{T_2}t}$ holds if and only if $c^{T_1}_{\frac{l-m}{n}}(f)$ for all $l \le 0$, that is, $c_j(f) = 0$ for all $j \in \mathbb{Z}$ satisfying $j \le -m/n$. The second claim can be proved similarly. □

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School of Mathematics and Computational Science, Sun Yat-sen University, Guangzhou City, 510275, P. R. China
Email address: tanlihui198@tom.com

School of Mathematics and Computational Science, Sun Yat-sen University, Guangzhou City, 510275, P. R. China
Email address: mcsylh@mail.sysu.edu.cn

School of Mathematics and Computational Science, Sun Yat-sen University, Guangzhou City, 510275, P. R. China
Email address: hdren@mail.sysu.edu.cn