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BOUNDARY INTEGRAL EQUATIONS IN BENDING OF THERMOELASTIC PLATES WITH MIXED BOUNDARY CONDITIONS

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This article is dedicated to Professor Zuhair Nashed,
a good friend and colleague,
for his long and distinguished service to the mathematical profession.

ABSTRACT. Initial-boundary value problems for bending of a thermoelastic plate with transverse shear deformation are studied under the assumption that different parts of the boundary are subjected to different types of physical conditions. The solutions of these problems are represented as single-layer and double-layer thermoelastic potentials, which leads to time-dependent systems of boundary integral equations. The unique solvability of these systems is proved in spaces of distributions.

1. Introduction. Mathematical models of elastic plates aim to replace the study of full three-dimensional problems with that of simpler theories in only two dimensions, concentrating the computation on the phenomenon of bending and disregarding other, less significant, effects. Kirchhoff's classical theory reduces to the solution of a fourth-order equation with two boundary conditions. Later models (see, for example, [1]) yield more information by including the action of transverse shear forces into a system of three second-order equations accompanied by three boundary conditions. The model considered in [1] was later extended to bending motions and, more recently, to thermoelastic plates,

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where deformation caused by temperature variations is also taken into account [2]. In this paper we study the bending of a thin thermoelastic plate subject to mixed boundary conditions and homogenous initial conditions. The variational formulation of this problem and a statement on its unique solvability may be found in [3]. The corresponding results in the absence of thermal effects are detailed in [4-7].

2. Formulation of the Problem. We consider a thin elastic plate of thickness $h_0 = \text{const} > 0$, which occupies a region $\bar{S} \times [-h_0/2, h_0/2]$ in $\mathbb{R}^3$, where S is a domain in $\mathbb{R}^2$. The displacement vector at a point x' in this region at $t \geq 0$ is denoted by $v(x', t) = (v_1(x', t), v_2(x', t), v_3(x', t))^{\mathrm{T}}$, where the superscript T signifies matrix transposition. The temperature in the plate is denoted by $\theta(x', t)$. Let $x' = (x, x_3)$, $x = (x_1, x_2) \in \bar{S}$. In plate models with transverse shear deformation it is assumed [1] that

$$v(x', t) = (x_3 u_1(x, t), x_3 u_2(x, t), u_3(x, t))^{\mathrm{T}}.$$

If thermal effects are taken into account, we also consider the "averaged weighted temperature" across thickness defined by [2]

$$u_4(x, t) = \frac{1}{h^2 h_0} \int\limits_{-h_0/2}^{h_0/2} x_3 \theta(x, x_3, t)\, dx_3, \quad h^2 = \frac{h_0^2}{12}.$$

The factor $1/h^2$ has been introduced for reasons of convenience. Then the vector function $U(x, t) = (u(x, t)^{\mathrm{T}}, u_4(x, t))^{\mathrm{T}}$, where $u(x, t) = (u_1(x, t), u_2(x, t), u_3(x, t))^{\mathrm{T}}$, satisfies the equation

$$\text{(1)} \quad \mathsf{L}U(x,t) = \mathsf{B}_0 \partial_t^2 U(x,t) + \mathsf{B}_1 \partial_t U(x,t) + \mathsf{A}U(x,t) = \mathsf{Q}(x,t), \ (x,t) \in G,$$

where $G = S \times (0, \infty)$, $\mathsf{B}_0 = \operatorname{diag}\{\rho h^2, \rho h^2, \rho, 0\}$, $\partial_t = \partial/\partial t$, $\rho > 0$ is the constant density of the material,

$$\mathsf{B}_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \eta\partial_1 & \eta\partial_2 & 0 & \varkappa^{-1} \end{pmatrix}, \quad \mathsf{A} = \begin{pmatrix} & & & h^2\gamma\partial_1 \\ & A & & h^2\gamma\partial_2 \\ & & & 0 \\ 0 & 0 & 0 & -\Delta \end{pmatrix},$$

$$A = \begin{pmatrix} -h^2\mu\Delta - h^2(\lambda+\mu)\partial_1^2 + \mu & -h^2(\lambda+\mu)\partial_1\partial_2 & \mu\partial_1 \\ -h^2(\lambda+\mu)\partial_1\partial_2 & -h^2\mu\Delta - h^2(\lambda+\mu)\partial_2^2 + \mu & \mu\partial_2 \\ -\mu\partial_1 & -\mu\partial_2 & -\mu\Delta \end{pmatrix},$$

$\partial_\alpha = \partial/\partial x_\alpha$, $\alpha = 1, 2$, $\eta, \varkappa$, and γ are positive physical constants, λ and μ are the Lamé coefficients of the material satisfying $\lambda + \mu > 0$, $\mu > 0$, and $\mathsf{Q}(x,t) = (q(x,t)^{\mathrm{T}}, q_4(x,t))^{\mathrm{T}}$, where $q(x,t) = (q_1(x,t), q_2(x,t), q_3(x,t))^{\mathrm{T}}$ is a combination of the forces and moments acting on the plate and its faces and $q_4(x,t)$ is a combination of the averaged heat source density and the temperature and heat flux on the faces. Without loss of generality, in what follows we consider only the case of the homogeneous equation (1), that is, $\mathsf{Q}(x,t) \equiv 0$, and homogeneous initial conditions

$$U(x,0) = 0, \quad \partial_t u(x,0) = 0, \quad x \in S, \tag{2}$$

since, as shown in [8] and [9], the nonhomogeneity in both can easily be transferred to the boundary conditions.

We assume that the boundary ∂S consists of four open arcs ∂S_i, $i = 1, \ldots, 4$, counted counterclockwise, such that

$$\partial S = \bigcup_{i=1}^{4} \partial \bar{S}_i, \quad \partial S_i \cap \partial S_j = \varnothing, \quad i \neq j, \quad i, j = 1, \ldots, 4.$$

We write

$$\Gamma = \partial S \times (0,\infty), \ \ \Gamma_i = \partial S_i \times (0,\infty), \ \ \partial S_{ij} = \partial S_i \cup \partial S_j \cup (\overline{\partial S}_i \cap \overline{\partial S}_j),$$
$$\Gamma_{ij} = \partial S_{ij} \times (0,\infty), \ \ i, j = 1, \ldots, 4.$$

We further assume that the displacements and temperature are prescribed on Γ_1 by

$$u(x,t) = f(x,t), \quad u_4(x,t) = f_4(x,t), \quad (x,t) \in \Gamma_1, \tag{3}$$

or $U(x,t) = F(x,t)$, where

$$F(x,t) = (f(x,t)^{\mathrm{T}}, f_4(x,t))^{\mathrm{T}}, \quad f(x,t) = (f_1(x,t), f_2(x,t), f_3(x,t))^{\mathrm{T}},$$

and the displacements and heat flux on Γ_2 by

$$u(x,t) = f(x,t), \quad \partial_n u_4(x,t) = g_4(x,t), \quad (x,t) \in \Gamma_2. \tag{4}$$

In (4), $n = n(x) = (n_1(x), n_2(x))^{\mathrm{T}}$ is outward unit normal to ∂S and $\partial_n = \partial/\partial n$.

Let T be the boundary moment-force operator defined by

$$\begin{pmatrix} h^2\big[(\lambda+2\mu)n_1\partial_1+\mu n_2\partial_2\big], & h^2(\lambda n_1\partial_2 + \mu n_2\partial_1) & 0 \\ h^2(\mu n_1\partial_2 + \lambda n_2\partial_1) & h^2\big[(\lambda+2\mu)n_2\partial_2 + \mu n_1\partial_1\big] & 0 \\ \mu n_1 & \mu n_2 & \mu\partial_n \end{pmatrix}. \tag{5}$$

Tu is the vector of the averaged moments and shear force acting on the lateral part of the plate boundary. We assume that the boundary moments and force and the heat flux are prescribed on Γ_3 by

$$\begin{aligned} Tu(x,t) - h^2\gamma n(x)u_4(x,t) &= g(x,t), \\ \partial_n u_4(x,t) = g_4(x,t), &\quad (x,t) \in \Gamma_3, \end{aligned} \tag{6}$$

or

$$(\mathcal{T}U)(x,t) = G(x,t) = (g(x,t)^{\mathrm{T}}, g_4(x,t))^{\mathrm{T}},$$

where

$$g(x,t) = (g_1(x,t), g_2(x,t), g_3(x,t)^{\mathrm{T}}$$

and

$$(\mathcal{T}U)(x,t) = \begin{pmatrix} (Tu)(x,t) - h^2\gamma n(x)u_4(x,t) \\ \partial_n u_4(x,t) \end{pmatrix} = \begin{pmatrix} (\mathcal{T}_e U)(x,t) \\ (\mathcal{T}_\theta U)(x,t) \end{pmatrix}.$$

To keep the notation simple, in (6) and below we also denote by $n(x)$ the three-component vector $(n_1(x), n_2(x), 0)^{\mathrm{T}}$.

Finally, on Γ_4 we prescribe the boundary force and moments and the temperature:

$$(7)\quad Tu(x,t) - h^2\gamma n(x)u_4(x,t) = g(x,t),\quad u_4(x,t) = f_4(x,t),\quad (x,t) \in \Gamma_4.$$

The functions $f(x,t)$, $f_4(x,t)$, $g(x,t)$, and $g_4(x,t)$ in (3)–(6) are known.

Let S^+ and S^- be, respectively, the interior and exterior domains bounded by ∂S, and let $G^\pm = S^\pm \times (0,\infty)$. We consider simultaneously the interior and exterior initial-boundary value problems (TM$^\pm$), which consist in finding $U \in C^2(G^\pm) \cap C^1(\bar{G}^\pm)$ that satisfy the homogeneous equation (1) in $G^\pm$, (2) in $S^\pm$, and (3)–(7).

The solutions of (TM$^\pm$) will be represented in terms of thermoelastic single-layer and double-layer potentials. These representations lead to systems of time-dependent boundary integral equations for the unknown densities. The aim of the paper is to prove the unique solvability of these equations and to show that the potentials with densities obtained in this way are the solutions of (TM$^\pm$).

3. The Laplace-transformed Boundary Value Problems. First, we apply the Laplace transformation in (TM$^\pm$) and study the unique solvability of the transformed problems (TM$_p^\pm$). In what follows, we denote Laplace transforms of (vector-valued or scalar) functions by a "hat" above their symbols; thus,

$$\hat{U}(x,p) = (\mathcal{L}U)(x,p) = \int_0^\infty e^{-pt}U(x,t)dt.$$

The transformed problems (TM$_p^\pm$) depend on the complex parameter p and consist in finding $\hat{U}(x,p) \in \mathrm{C}^2(S^\pm) \cap \mathrm{C}^1(\bar{S}^\pm)$ that satisfy, respectively, the equations

$$(8)\qquad p^2\mathsf{B}_0\hat{U}(x,p) + p\,\mathsf{B}_1\hat{U}(x,p) + \mathsf{A}\hat{U}(x,p) = 0,\quad x \in S^\pm,$$

and the boundary conditions

$$\begin{aligned}
&\hat{u}(x,p) = \hat{f}(x,p), && \hat{u}_4(x,p) = \hat{f}_4(x,p), && x \in \partial S_1,\\
&\hat{u}(x,p) = \hat{f}(x,p), && \partial_n \hat{u}_4(x,p) = \hat{g}_4(x,p), && x \in \partial S_2,\\
&T\hat{u}(x,p) - h^2\gamma n(x)\hat{u}_4(x,p) = \hat{g}(x,p), && \partial_n \hat{u}_4(x,p) = \hat{g}_4(x,p), && x \in \partial S_3,\\
&T\hat{u}(x,p) - h^2\gamma n(x)\hat{u}_4(x,p) = \hat{g}(x,p), && \hat{u}_4(x,p) = \hat{f}_4(x,p), && x \in \partial S_4.
\end{aligned}$$

Let $H_m(\mathbb{R}^2)$, $m \in \mathbb{R}$, be the standard Sobolev space of functions $\hat{v}_4(x)$ with norm

$$\|\hat{v}_4\|_m = \left\{ \int\limits_{\mathbb{R}^2} (1+|\xi|^2)^m |\tilde{v}_4(\xi)|^2 d\xi \right\}^{1/2},$$

where $\tilde{v}_4(\xi)$ is the Fourier transform of $\hat{v}_4(x)$.

For every $m \in \mathbb{R}$ and $p \in \mathbb{C}$, $\mathbf{H}_{m,p}(\mathbb{R}^2)$ is the space of three-component vector functions $\hat{v}(x)$ that coincides with $[H_m(\mathbb{R}^2)]^3$ as a set but is endowed with the norm

$$\|\hat{v}\|_{m,p} = \left\{ \int\limits_{\mathbb{R}^2} (1+|\xi|^2+|p|^2)^m |\tilde{v}(\xi)|^2 d\xi \right\}^{1/2}.$$

The spaces $H_m(S^\pm)$ and $\mathbf{H}_{m,p}(S^\pm)$ consist of the restrictions to $S^\pm$ of the elements $\hat{v}_4 \in H_m(\mathbb{R}^2)$ and $\hat{v} \in \mathbf{H}_{m,p}(\mathbb{R}^2)$, respectively. Their norms are defined by

$$\begin{aligned}
\|\hat{u}_4\|_{m;S^\pm} &= \inf_{\hat{v}_4 \in H_m(\mathbb{R}^2):\hat{v}_4|_{S^\pm}=\hat{u}_4} \|\hat{v}_4\|_m,\\
\|\hat{u}\|_{m,p;S^\pm} &= \inf_{\hat{v} \in \mathbf{H}_{m,p}(\mathbb{R}^2):\hat{v}|_{S^\pm}=\hat{u}} \|\hat{v}\|_{m,p}.
\end{aligned}$$

Let $H_{1/2}(\partial S)$ and $\mathbf{H}_{1/2,p}(\partial S)$ be the spaces of the traces on ∂S of all $\hat{u}_4 \in H_1(S^+)$ and $\hat{u} \in \mathbf{H}_{1,p}(S^+)$, with norms

$$\begin{aligned}
\|\hat{f}_4\|_{1/2;\partial S} &= \inf_{\hat{u}_4 \in H_1(S^+):\hat{u}_4|_{\partial S}=\hat{f}_4} \|\hat{u}_4\|_{1;S^+},\\
\|\hat{f}\|_{1/2,p;\partial S} &= \inf_{\hat{u} \in \mathbf{H}_{1,p}(S^+):\hat{u}|_{\partial S}=\hat{f}} \|\hat{u}\|_{1,p;S^+},
\end{aligned}$$

respectively. The continuous (uniformly with respect to $p \in \mathbb{C}$) trace operators from $H_1(S^\pm)$ to $H_{1/2}(\partial S)$ and from $\mathbf{H}_{1,p}(S^\pm)$ to $\mathbf{H}_{1/2,p}(\partial S)$ are denoted by the same symbols $\gamma^\pm$.

$H_{-1/2}(\partial S)$ and $\mathbf{H}_{-1/2,p}(\partial S)$ are the duals of $H_{1/2}(\partial S)$ and $\mathbf{H}_{1/2,p}(\partial S)$ with respect to the duality generated by the inner products in $L^2(\partial S)$ and $[L^2(\partial S)]^3$. Their norms are denoted by $\|\hat{g}_4\|_{-1/2;\partial S}$ and $\|\hat{g}\|_{-1/2,p;\partial S}$.

Let $\partial\tilde{S} \subset \partial S$ be any open part of ∂S with $\operatorname{mes}\partial\tilde{S} > 0$ (in particular, $\partial\tilde{S}$ may coincide with ∂S), and let $\tilde{\pi}$ be the operator of restriction of functions from ∂S to $\partial\tilde{S}$. The spaces $H_{\pm 1/2}(\partial\tilde{S})$ and $\mathbf{H}_{\pm 1/2,p}(\partial\tilde{S})$ consist of the restrictions to $\partial\tilde{S}$ of all the elements of $H_{\pm 1/2}(\partial S)$ and $\mathbf{H}_{\pm 1/2,p}(\partial S)$, respectively. Their norms are defined by

$$\|\hat{e}_4\|_{\pm 1/2;\partial\tilde{S}} = \inf_{\hat{r}_4 \in H_{\pm 1/2}(\partial S):\, \tilde{\pi}\hat{r}_4 = \hat{e}_4} \|\hat{r}_4\|_{\pm 1/2;\partial S},$$

$$\|\hat{e}\|_{\pm 1/2,p;\partial\tilde{S}} = \inf_{\hat{r} \in \mathbf{H}_{\pm 1/2,p}(\partial S):\, \tilde{\pi}\hat{r} = \hat{e}} \|\hat{r}\|_{\pm 1/2,p;\partial S}.$$

$\mathring{H}_{\pm 1/2}(\partial\tilde{S})$ and $\mathring{\mathbf{H}}_{\pm 1/2,p}(\partial\tilde{S})$ are the subspaces of $H_{\pm 1/2}(\partial S)$ and $\mathbf{H}_{\pm 1/2,p}(\partial S)$ consisting of all the elements with support in $\overline{\partial\tilde{S}}$. The norms of $\hat{e}_4 \in \mathring{H}_{\pm 1/2}(\partial\tilde{S})$ and $\hat{e} \in \mathring{H}_{\pm 1/2}(\partial\tilde{S})$ are denoted by $\|\hat{e}_4\|_{\pm 1/2;\partial S}$ and $\|\hat{e}\|_{\pm 1/2,p;\partial S}$. We remark that $H_{\pm 1/2}(\partial\tilde{S})$ are the duals of $\mathring{H}_{\mp 1/2}(\partial\tilde{S})$ and $\mathbf{H}_{\pm 1/2,p}(\partial\tilde{S})$ the duals of $\mathring{\mathbf{H}}_{\mp 1/2,p}(\partial\tilde{S})$ with respect to the duality generated by the inner products in $L^2(\partial\tilde{S})$ and $[L^2(\partial\tilde{S})]^3$.

Let π_i and π_{ij}, $i, j = 1, \dots, 4$, be the operators of restriction from ∂S to ∂S_i and to ∂S_{ij}, respectively. $H_1(S^\pm; \partial S_{23})$ and $\mathbf{H}_{1,p}(S^\pm; \partial S_{34})$ are the subspaces of $H_1(S^\pm)$ and $\mathbf{H}_{1,p}(S^\pm)$ of all $\hat{u}_4 \in H_1(S^\pm)$ and $\hat{u} \in \mathbf{H}_{1,p}(S^\pm)$ such that $\pi_{41}\gamma^\pm \hat{u}_4 = 0$ and $\pi_{12}\gamma^\pm \hat{u} = 0$.

We consider the spaces $\mathcal{H}_{1,p}(S^\pm) = \mathbf{H}_{1,p}(S^\pm) \times H_1(S^\pm)$ of elements $\hat{U} = (\hat{u}^{\mathrm{T}}, \hat{u}_4)^{\mathrm{T}}$ with norm

$$|||\hat{U}|||_{1,p;S^\pm} = \|\hat{u}\|_{1,p;S^\pm} + \|\hat{u}_4\|_{1;S^\pm},$$

and their subspaces

$$\mathcal{H}_{1,p}(S^\pm; \partial S_{34}, \partial S_{23}) = \mathbf{H}_{1,p}(S^\pm; \partial S_{34}) \times H_1(S^\pm; \partial S_{23}).$$

Finally, the spaces $\mathcal{H}_{\pm 1/2,p}(\partial\tilde{S}) = \mathbf{H}_{\pm 1/2,p}(\partial\tilde{S}) \times H_{\pm 1/2}(\partial\tilde{S})$ are endowed with the norms

$$|||\hat{\mathcal{E}}|||_{\pm 1/2,p;\partial\tilde{S}} = \|\hat{e}\|_{\pm 1/2,p;\partial\tilde{S}} + \|\hat{e}_4\|_{\pm 1/2;\partial\tilde{S}},$$

where $\hat{\mathcal{E}} = (\hat{e}^{\mathrm{T}}, \hat{e}_4)^{\mathrm{T}}$. $\mathring{\mathcal{H}}_{\pm 1/2,p}(\partial\tilde{S}) = \mathring{\mathbf{H}}_{\pm 1/2,p}(\partial\tilde{S}) \times \mathring{H}_{\pm 1/2}(\partial\tilde{S})$ are the subspaces of $\mathcal{H}_{\pm 1/2,p}(\partial S)$ consisting of all $\hat{\mathcal{E}}$ with $\operatorname{supp}\hat{\mathcal{E}} \subset \overline{\partial\tilde{S}}$.

We now turn to the variational formulation of $(\mathrm{TM}_p^\pm)$. Let $\kappa > 0$, and let $\mathbb{C}_\kappa = \{p = \sigma + i\tau \in \mathbb{C} : \sigma > \kappa\}$. In what follows we denote by c all positive constants occurring in estimates, which are independent of the functions in those estimates and of $p \in \mathbb{C}_\kappa$, but may depend on κ. Also, $(\cdot,\cdot)_{0;S^\pm}$ and $(\cdot,\cdot)_{0;\partial\tilde{S}}$ are the inner products in $[L^2(S^\pm)]^m$ and $[L^2(\partial\tilde{S})]^m$ for all $m \in \mathbb{N}$. The norms on these spaces are $\|\cdot\|_{0;S^\pm}$ and $\|\cdot\|_{0;\partial\tilde{S}}$.

Let $\hat{U} = (\hat{u}^{\mathrm{T}}, \hat{u}_4)^{\mathrm{T}} \in \mathrm{C}^2(S^\pm) \cap \mathrm{C}^1(\bar{S}^\pm)$ be the classical solution of $(\mathrm{TM}_p^\pm)$, and let $\hat{W} = (\hat{w}^{\mathrm{T}}, \hat{w}_4)^{\mathrm{T}} \in \mathrm{C}_0^\infty(\bar{S}^\pm)$ be any function (with compact support in S^- in the case of S^-) such that $\hat{w}(x,p) = 0$, $x \in \partial S_{12}$, and $\hat{w}_4(x,p) = 0$, $x \in \partial S_{41}$. We multiply (8) by $\hat{W}$ in $[L^2(S^\pm)]^4$ and arrive at

$$\Upsilon_{\pm,p}(\hat{U},\hat{W}) = \pm L(\hat{W}), \tag{9}$$

where

$$\begin{gathered}
\Upsilon_{\pm,p}(\hat{U},\hat{W}) = a_\pm(\hat{u},\hat{w}) + (\nabla\hat{u}_4, \nabla\hat{w}_4)_{0;S^\pm} + p^2(B_0^{1/2}\hat{u}, B_0^{1/2}\hat{w})_{0;S^\pm} \\
+ \varkappa^{-1}p(\hat{u}_4,\hat{w}_4)_{0;S^\pm} - h^2\gamma(\hat{u}_4, \operatorname{div}\hat{w})_{0;S^\pm} + \eta p(\operatorname{div}\hat{u}, \hat{w}_4)_{0;S^\pm}, \\
a_\pm(\hat{u},\hat{w}) = 2\int\limits_{S^\pm} E(\hat{u},\hat{w})\,dx, \\
2E(\hat{u},\hat{w}) = h^2E_0(\hat{u},\hat{w}) + h^2\mu(\partial_2\hat{u}_1 + \partial_1\hat{u}_2)(\partial_2\bar{\hat{w}}_1 + \partial_1\bar{\hat{w}}_2) \\
+ \mu[(\hat{u}_1 + \partial_1\hat{u}_3)(\bar{\hat{w}}_1 + \partial_1\bar{\hat{w}}_3) + (\hat{u}_2 + \partial_2\hat{u}_3)(\bar{\hat{w}}_2 + \partial_2\bar{\hat{w}}_3)], \\
E_0(\hat{u},\hat{w}) = (\lambda + 2\mu)[(\partial_1\hat{u}_1)(\partial_1\bar{\hat{w}}_1) + (\partial_2\hat{u}_2)(\partial_2\bar{\hat{w}}_2)] \\
+\lambda[(\partial_1\hat{u}_1)(\partial_2\bar{\hat{w}}_2) + (\partial_2\hat{u}_2)(\partial_1\bar{\hat{w}}_1)], \\
B_0 = \operatorname{diag}\{\rho h^2, \rho h^2, \rho\}, \quad L(\hat{W}) = (\hat{g}_4, \hat{w}_4)_{0;\partial S_{23}} + (\hat{g}, \hat{w})_{0;\partial S_{34}}.
\end{gathered}$$

In view of (9), the variational problems $(\mathrm{TM}_p^\pm)$ consist in finding $\hat{U} \in \mathcal{H}_{1,p}(S^\pm)$ that satisfy

$$\pi_{12}\gamma^\pm\hat{u} = \hat{f}, \quad \pi_{41}\gamma^\pm\hat{u}_4 = \hat{f}_4$$

and (9) for any $\hat{W} \in \mathcal{H}_{1,p}(S^\pm; \partial S_{34}, \partial S_{23})$.

The following assertion is proved in [3].

Lemma 1. For all $\hat{f} \in \mathbf{H}_{1/2,p}(\partial S_{12})$, $\hat{f}_4 \in H_{1/2}(\partial S_{41})$, $\hat{g} \in \mathbf{H}_{-1/2,p}(\partial S_{34})$, and $\hat{g}_4 \in H_{-1/2}(\partial S_{23})$, $p \in \mathbb{C}_\kappa$, $\kappa > 0$, problems $(\mathrm{TM}_p^\pm)$ have unique solutions $\hat{U} \in \mathcal{H}_{1,p}(S^\pm)$, which satisfy the estimates

$$\|\hat{U}\|_{1,p;S^\pm} \le c\big\{|p|(\|\hat{f}\|_{1/2,p;\partial S_{12}} + \|\hat{f}_4\|_{1/2;\partial S_{41}}) \\ + |p|\,\|\hat{g}\|_{-1/2,p;\partial S_{34}} + \|\hat{g}_4\|_{-1/2;\partial S_{23}}\big\}. \tag{10}$$

For $\partial\tilde{S} \subseteq \partial S$, we write $\mathbf{H}_{\pm 1/2}(\partial\tilde{S}) = \mathbf{H}_{\pm 1/2,0}(\partial\tilde{S})$ and $\mathbf{H}_1(S^\pm) = \mathbf{H}_{1,0}(S^\pm)$, and denote the norms on these spaces by $\|\cdot\|_{\pm 1/2;\partial\tilde{S}}$ and $\|\cdot\|_{1;S^\pm}$, respectively.

For any $\kappa > 0$ and $k \in \mathbb{R}$, we introduce the spaces $\mathbf{H}^{\mathcal{L}}_{\pm 1/2,k,\kappa}(\partial\tilde{S})$ and $\mathbf{H}^{\mathcal{L}}_{1,k,\kappa}(S^\pm)$, which consist of all three-component vector-valued functions $\hat{e}(x,p)$ and $\hat{u}(x,p)$ that

(i) define holomorphic mappings

$$\hat{e}(x,p) : \mathbb{C}_\kappa \to \mathbf{H}_{\pm 1/2}(\partial\tilde{S}), \quad \hat{u}(x,p) : \mathbb{C}_\kappa \to \mathbf{H}_1(S^\pm);$$

(ii) have norms

$$\|\hat{e}\|^2_{\pm 1/2,k,\kappa;\partial\tilde{S}} = \sup_{\sigma>\kappa} \int_{-\infty}^{\infty} (1+|p|^2)^k \|\hat{e}(x,p)\|^2_{\pm 1/2,p;\partial\tilde{S}}\, d\tau < \infty,$$

$$\|\hat{u}\|^2_{1,k,\kappa;S^\pm} = \sup_{\sigma>\kappa} \int_{-\infty}^{\infty} (1+|p|^2)^k \|\hat{u}(x,p)\|^2_{1,p;S^\pm}\, d\tau < \infty.$$

$H^{\mathcal{L}}_{\pm 1/2,k,\kappa}(\partial\tilde{S})$ and $H^{\mathcal{L}}_{1,k,\kappa}(S^\pm)$ consist of all functions $\hat{e}_4(x,p)$ and $\hat{u}_4(x,p)$ that

(i) define holomorphic mappings

$$\hat{e}_4(x,p) : \mathbb{C}_\kappa \to H_{\pm 1/2}(\partial\tilde{S}), \quad \hat{u}_4(x,p) : \mathbb{C}_\kappa \to H_1(S^\pm);$$

(ii) have norms

$$\|\hat{e}_4\|^2_{\pm 1/2,k,\kappa;\partial\tilde{S}} = \sup_{\sigma>\kappa} \int_{-\infty}^{\infty} (1+|p|^2)^k \|\hat{e}_4(x,p)\|^2_{\pm 1/2;\partial\tilde{S}}\, d\tau < \infty,$$

$$\|\hat{u}_4\|^2_{1,k,\kappa;S^\pm} = \sup_{\sigma>\kappa} \int_{-\infty}^{\infty} (1+|p|^2)^k \|\hat{u}_4(x,p)\|^2_{1;S^\pm}\, d\tau < \infty.$$

The norms on the spaces $\mathcal{H}^{\mathcal{L}}_{1,k,l,\kappa}(S^\pm) = \mathbf{H}^{\mathcal{L}}_{1,k,\kappa}(S^\pm) \times H^{\mathcal{L}}_{1,l,\kappa}(S^\pm)$ and $\mathcal{H}^{\mathcal{L}}_{\pm 1/2,k,l,\kappa}(\partial\tilde{S}) = \mathbf{H}^{\mathcal{L}}_{\pm 1/2,k,\kappa}(\partial\tilde{S}) \times H^{\mathcal{L}}_{\pm 1/2,l,\kappa}(\partial\tilde{S})$ are

$$|||\hat{U}|||_{1,k,l,\kappa;S^\pm} = \|\hat{u}\|_{1,k,\kappa;S^\pm} + \|\hat{u}_4\|_{1,l,\kappa;S^\pm},$$

$$|||\hat{\mathcal{E}}|||_{\pm 1/2,k,l,\kappa;\partial\tilde{S}} = \|\hat{e}\|_{\pm 1/2,k,\kappa;\partial\tilde{S}} + \|\hat{e}_4\|_{\pm 1/2,l,\kappa;\partial\tilde{S}}.$$

$\mathring{\mathcal{H}}^{\mathcal{L}}_{\pm 1/2,k,l,\kappa}(\partial\tilde{S}) = \mathring{\mathbf{H}}^{\mathcal{L}}_{\pm 1/2,k,\kappa}(\partial\tilde{S}) \times \mathring{H}^{\mathcal{L}}_{\pm 1/2,l,\kappa}(\partial\tilde{S})$ are the subspaces of $\mathcal{H}^{\mathcal{L}}_{\pm 1/2,k,l,\kappa}(\partial S)$ consisting of all $\hat{\mathcal{E}}$ with $\operatorname{supp}\hat{\mathcal{E}} \subset \overline{\partial\tilde{S}}$.

Let $\tilde{\Gamma} = \partial\tilde{S} \times (0,\infty)$. For $\kappa > 0$ and $k, l \in \mathbb{R}$, the spaces

$$\begin{gathered}
\mathbf{H}^{\mathcal{L}^{-1}}_{1,k,\kappa}(G^\pm), \quad H^{\mathcal{L}^{-1}}_{1,l,\kappa}(G^\pm), \quad \mathcal{H}^{\mathcal{L}^{-1}}_{1,k,l,\kappa}(G^\pm),\\
\mathbf{H}^{\mathcal{L}^{-1}}_{\pm 1/2,k,\kappa}(\tilde{\Gamma}), \quad H^{\mathcal{L}^{-1}}_{\pm 1/2,l,\kappa}(\tilde{\Gamma}), \quad \mathcal{H}^{\mathcal{L}^{-1}}_{\pm 1/2,k,l,\kappa}(\tilde{\Gamma}),\\
\mathring{\mathbf{H}}^{\mathcal{L}^{-1}}_{\pm 1/2,k,\kappa}(\tilde{\Gamma}), \quad \mathring{H}^{\mathcal{L}^{-1}}_{\pm 1/2,l,\kappa}(\tilde{\Gamma}), \quad \mathring{\mathcal{H}}^{\mathcal{L}^{-1}}_{\pm 1/2,k,l,\kappa}(\tilde{\Gamma})
\end{gathered}$$

consist, respectively, of the inverse Laplace transforms of all the functions in

$$\begin{gathered}
\mathbf{H}^{\mathcal{L}}_{1,k,\kappa}(S^\pm), \quad H^{\mathcal{L}}_{1,l,\kappa}(S^\pm), \quad \mathcal{H}^{\mathcal{L}}_{1,k,l,\kappa}(S^\pm),\\
\mathbf{H}^{\mathcal{L}}_{\pm 1/2,k,\kappa}(\partial\tilde{S}), \quad H^{\mathcal{L}}_{\pm 1/2,l,\kappa}(\partial\tilde{S}), \quad \mathcal{H}^{\mathcal{L}}_{\pm 1/2,k,l,\kappa}(\partial\tilde{S}),\\
\mathring{\mathbf{H}}^{\mathcal{L}}_{\pm 1/2,k,\kappa}(\partial\tilde{S}), \quad \mathring{H}^{\mathcal{L}}_{\pm 1/2,l,\kappa}(\partial\tilde{S}), \quad \mathring{\mathcal{H}}^{\mathcal{L}}_{\pm 1/2,k,l,\kappa}(\partial\tilde{S}).
\end{gathered}$$

The norms on these spaces are

$$\begin{gathered}
\|u\|_{1,k,\kappa;G^\pm} = \|\hat{u}\|_{1,k,\kappa;S^\pm}, \quad \|u_4\|_{1,l,\kappa;G^\pm} = \|\hat{u}_4\|_{1,l,\kappa;S^\pm},\\
|||U|||_{1,k,l,\kappa;G^\pm} = |||\hat{U}|||_{1,k,l,\kappa;S^\pm},\\
\|e\|_{\pm 1/2,k,\kappa;\tilde{\Gamma}} = \|\hat{e}\|_{\pm 1/2,k,\kappa;\partial\tilde{S}}, \quad \|e_4\|_{\pm 1/2,l,\kappa;\tilde{\Gamma}} = \|\hat{e}_4\|_{\pm 1/2,l,\kappa;\partial\tilde{S}},\\
|||\mathcal{E}|||_{\pm 1/2,k,l,\kappa;\tilde{\Gamma}} = |||\hat{\mathcal{E}}|||_{\pm 1/2,k,l,\kappa;\partial\tilde{S}}.
\end{gathered}$$

The trace operators from $G^{\pm}$ to Γ and the operators of restriction from Γ to its parts Γ_{ij}, $i,j=1,\dots,4$, are again denoted by $\gamma^{\pm}$ and π_{ij}. We say that $U=(u^{\mathrm{T}},u_4)^{\mathrm{T}}\in\mathcal{H}^{\mathcal{L}^{-1}}_{1,0,0,\kappa}(G^{\pm})$ is a weak solution of $(\mathrm{TM}^{\pm})$ if

(i) $\gamma_0 u=0$, where γ_0 is the trace operator on $S^{\pm}\times\{t=0\}$;

(ii) $\pi_{12}\gamma^{\pm}u=f$ and $\pi_{41}\gamma^{\pm}u_4=f_4$;

(iii) for all $W=(w^{\mathrm{T}},w_4)^{\mathrm{T}}\in\mathrm{C}^{\infty}_0(\bar{G}^{\pm})$ such that $w(x,t)=0$, $(x,t)\in\Gamma_{12}$, and $w_4(x,t)=0$, $(x,t)\in\Gamma_{41}$,

$$\Upsilon_{\pm}(U,W)=\pm L(W),$$

where

$$\begin{aligned}\Upsilon_{\pm}(U,W)=\int_0^{\infty}\big\{&a_{\pm}(u,w)+(\nabla u_4,\nabla w_4)_{0;S^{\pm}}-(B_0^{1/2}\partial_t u,B_0^{1/2}\partial_t w)_{0;S^{\pm}}\\&-\varkappa^{-1}(u_4,\partial_t w_4)_{0;S^{\pm}}-h^2\gamma(u_4,\operatorname{div}w)_{0;S^{\pm}}-\eta(\operatorname{div}u,\partial_t w_4)_{0;S^{\pm}}\big\}\,dt,\end{aligned}$$

$$L(W)=\int_0^{\infty}\big\{(g,w)_{0;\partial S_{34}}+(g_4,w_4)_{0;\partial S_{23}}\big\}\,dt.$$

The next assertion was also proved in [3].

Theorem 1. Let $U(x,t)=\mathcal{L}^{-1}\hat{U}(x,p)$ be the inverse Laplace transform of the weak solution $\hat{U}(x,p)$ of problem $(\mathrm{TM}^{\pm}_p)$. If

$$\begin{aligned}&f\in\mathbf{H}^{\mathcal{L}^{-1}}_{1/2,l+1,\kappa}(\Gamma_{12}),\quad f_4\in H^{\mathcal{L}^{-1}}_{1/2,l+1,\kappa}(\Gamma_{41}),\\&g\in\mathbf{H}^{\mathcal{L}^{-1}}_{-1/2,l+1,\kappa}(\Gamma_{34}),\quad g_4\in H^{\mathcal{L}^{-1}}_{-1/2,l,\kappa}(\Gamma_{23}),\end{aligned}$$

$\kappa>0$, and $l\in\mathbb{R}$, then $U\in\mathcal{H}^{\mathcal{L}^{-1}}_{1,l,l,\kappa}(G^{\pm})$ and

$$\begin{aligned}|||U|||_{1,l,l,\kappa;G^{\pm}}\le c\big\{&\|f\|_{1/2,l+1,\kappa;\Gamma_{12}}+\|f_4\|_{1/2,l+1,\kappa;\Gamma_{41}}\\&+\|g\|_{-1/2,l+1,\kappa;\Gamma_{34}}+\|g_4\|_{-1/2,l,\kappa;\Gamma_{23}}\big\}.\end{aligned}$$

If $l\ge 0$, then U is the unique weak solution of $(\mathrm{TM}^{\pm})$.

We now consider three particular cases: problems $(\mathrm{TD}^{\pm})$ with Dirichlet boundary conditions, where $\partial S_1 = \partial S$ and $\partial S_i = \varnothing$, $i = 2, 3, 4$, therefore, $\Gamma_1 = \Gamma$; problems $(\mathrm{TN}^{\pm})$ with Neumann boundary conditions, where $\partial S_3 = \partial S$ and $\partial S_i = \varnothing$, $i = 1, 2, 4$, therefore, $\Gamma_3 = \Gamma$; and the "reduced" problems $(\mathrm{T'M}^{\pm})$ with mixed boundary conditions, where $\partial S_2 = \varnothing$ and $\partial S_4 = \varnothing$. The corresponding Laplace-transformed boundary value problems are $(\mathrm{TD}_p^{\pm})$, $(\mathrm{TN}_p^{\pm})$, and $(\mathrm{T'M}_p^{\pm})$, respectively.

Strictly speaking, the next assertions do not follow directly from Lemma 1 and Theorem 1, but their proofs are almost identical to those of the others.

Corollary 1. (i) *For any $\hat{F} \in \mathcal{H}_{1/2,p}(\partial S)$, $p \in \mathbb{C}_\kappa$, and $\kappa > 0$, problems $(\mathrm{TD}_p^{\pm})$ have unique solutions $\hat{U} \in \mathcal{H}_{1,p}(S^{\pm})$, which satisfy the estimates*

$$|||\hat{U}|||_{1,p;S^{\pm}} \leq c|p|\,|||\hat{F}|||_{1/2,p;\partial S}.$$

(ii) *For any $\hat{G} = (\hat{g}^{\mathrm{T}}, \hat{g}_4)^{\mathrm{T}} \in \mathcal{H}_{-1/2,p}(\partial S)$, $p \in \mathbb{C}_\kappa$, and $\kappa > 0$, problems $(\mathrm{TN}_p^{\pm})$ have unique solutions $\hat{U} \in \mathcal{H}_{1,p}(S^{\pm})$, which satisfy the estimates*

$$|||\hat{U}|||_{1,p;S^{\pm}} \leq c(|p|\|\hat{g}\|_{-1/2,p;\partial S} + \|\hat{g}_4\|_{-1/2;\partial S}).$$

(iii) *For any $\hat{F} \in \mathcal{H}_{1/2,p;\partial S_1}$, $\hat{G} = (\hat{g}^{\mathrm{T}}, \hat{g}_4)^{\mathrm{T}} \in \mathcal{H}_{-1/2,p}(\partial S_3)$, $p \in \mathbb{C}_\kappa$, and $\kappa > 0$, problems $(\mathrm{T'M}_p^{\pm})$ have unique solutions $\hat{U} \in \mathcal{H}_{1,p}(S^{\pm})$, which satisfy the estimates*

$$|||\hat{U}|||_{1,p;S^{\pm}} \leq c(|p|\,|||\hat{F}|||_{1/2,p;\partial S_1} + |p|\|\hat{g}\|_{-1/2,p;\partial S_3} + \|\hat{g}_4\|_{-1/2;\partial S_3}).$$

Corollary 2. (i) *Let $U(x,t) = \mathcal{L}^{-1}\hat{U}(x,p)$ be the inverse Laplace transform of the weak solution $\hat{U}(x,p)$ of problem $(\mathrm{TD}_p^{\pm})$. If $F \in \mathcal{H}^{\mathcal{L}^{-1}}_{1/2,l+1,l+1,\kappa}(\Gamma)$, $\kappa > 0$, and $l \in \mathbb{R}$, then $U \in \mathcal{H}^{\mathcal{L}^{-1}}_{1,l,l,\kappa}(G^{\pm})$ and*

$$|||U|||_{1,l,l,\kappa;G^{\pm}} \leq c|||F|||_{1/2,l+1,l+1,\kappa;\Gamma}.$$

If $l \geq 0$, then $U(x,t)$ is the unique weak solution of $(\mathrm{TD}^{\pm})$.

(ii) *Let $U(x,t) = \mathcal{L}^{-1}\hat{U}(x,p)$ be the inverse Laplace transform of the weak solution $\hat{U}(x,p)$ of problem $(\mathrm{TN}_p^{\pm})$. If $G \in \mathcal{H}^{\mathcal{L}^{-1}}_{-1/2,l+1,l,\kappa}(\Gamma)$, $\kappa > 0$, and $l \in \mathbb{R}$, then $U \in \mathcal{H}^{\mathcal{L}^{-1}}_{1,l,l,\kappa}(G^{\pm})$ and*

$$|||U|||_{1,l,l,\kappa;G^\pm} \le c|||G|||_{-1/2,l+1,l,\kappa;\Gamma}.$$

If $l \ge 0$, then $U(x,t)$ is the unique weak solution of ($\mathrm{TN}^\pm$).

(iii) *Let $U(x,t) = \mathcal{L}^{-1}\hat{U}(x,p)$ be the inverse Laplace transform of the weak solution $\hat{U}(x,p)$ of problem* ($\mathrm{T'M}_p^\pm$). *If $F \in \mathcal{H}^{\mathcal{L}^{-1}}_{1/2,l+1,l+1,\kappa}(\Gamma_1)$, $G \in \mathcal{H}^{\mathcal{L}^{-1}}_{-1/2,l+1,l,\kappa}(\Gamma_3)$, $\kappa > 0$, and $l \in \mathbb{R}$, then $U \in \mathcal{H}^{\mathcal{L}^{-1}}_{1,l,l,\kappa}(G^\pm)$ and*

$$|||U|||_{1,l,l,\kappa;G^\pm} \le c(|||F|||_{1/2,l+1,l+1,\kappa;\Gamma_1} + |||G|||_{-1/2,l+1,l,\kappa;\Gamma_3}).$$

If $l \ge 0$, then $U(x,t)$ is the unique weak solution of ($\mathrm{T'M}^\pm$).

4. Properties of the Boundary Operators. We introduce the Poincaré-Steklov operators $\mathcal{T}_p^\pm$ acting on elements $\hat{F} \in \mathcal{H}_{1/2,p}(\partial S)$ by

$$(\mathcal{T}_p^\pm \hat{F}, \hat{\Phi})_{0;\partial S} = \pm \Upsilon_{\pm,p}(\hat{U}, \hat{W}),$$

where $\hat{U} \in \mathcal{H}_{1,p}(S^\pm)$ is the solution of ($\mathrm{TD}_p^\pm$) with boundary data $\hat{F}$, $\hat{\Phi}$ is an arbitrary element of $\mathcal{H}_{1/2,p}(\partial S)$, and $\hat{W} \in \mathcal{H}_{1,p}(S^\pm)$ is any extension of $\hat{F}$ to $\bar{S}^\pm$. Also, we write

$$\mathcal{T}_p^\pm \hat{F} = \begin{pmatrix} \mathcal{T}_{p,e}^\pm \hat{F} \\ \mathcal{T}_{p,\theta}^\pm \hat{F} \end{pmatrix}$$

and remark that for sufficiently smooth vector-valued functions $\hat{U}(x,p)$ (and, respectively, $\hat{F}(x,p)$), $(\mathcal{T}_p^\pm \hat{F})(x,p)$ coincide with the Laplace transforms of $(\mathcal{T}U)(x,t)$ introduced in Section 2.

Later on we use another norm on $\mathcal{H}_{-1/2,p}(\partial\tilde{S})$, namely,

$$\langle \hat{G} \rangle_{-1/2,p;\partial\tilde{S}} = |p|\|\hat{g}\|_{-1/2,p;\partial\tilde{S}} + \|g_4\|_{-1/2;\partial\tilde{S}}.$$

The next assertions were proved in [10].

Lemma 2. For any $p \in \mathbb{C}_0$, the operators $\mathcal{T}_p^\pm$ are homeomorphisms from $\mathcal{H}_{1/2,p}(\partial S)$ to $\mathcal{H}_{-1/2,p}(\partial S)$. If $\hat{U} \in \mathcal{H}_{1,p}(S^\pm)$ are the solutions of ($\mathrm{TD}_p^\pm$) with boundary data $\hat{F} \in \mathcal{H}_{1/2,p}(\partial S)$, then for any $p \in \mathbb{C}_\kappa$, $\kappa > 0$,

$$\langle \mathcal{T}_p^{\pm}\hat{F}\rangle_{-1/2,p;\partial S} \le c|p|^2 |||\hat{F}|||_{1/2,p;\partial S}, \tag{11}$$

$$|||\hat{F}|||_{1/2,p;\partial S} \le c\langle \mathcal{T}_p^{\pm}\hat{F}\rangle_{-1/2,p;\partial S}, \tag{12}$$

$$\begin{aligned} \|\mathcal{T}_{p,e}^{\pm}\hat{F}\|_{-1/2,p;\partial S} &\le c|||\hat{U}|||_{1,p;S^{\pm}}, \\ \|\mathcal{T}_{p,\theta}^{\pm}\hat{F}\|_{-1/2,p;\partial S} &\le c|p|\, |||\hat{U}|||_{1,p;S^{\pm}}. \end{aligned} \tag{13}$$

Let operators $\mathcal{T}_{\mathcal{L}}^{\pm}$ on $\mathcal{H}_{1/2,l,l,\kappa}^{\mathcal{L}}(\partial S)$ and $(\mathcal{T}_{\mathcal{L}}^{\pm})^{-1}$ on $\mathcal{H}_{1/2,l+1,l,\kappa}^{\mathcal{L}}(\partial S)$ be defined by $(\mathcal{T}_{\mathcal{L}}^{\pm}\hat{F})(x,p) = (\mathcal{T}_p^{\pm}\hat{F})(x,p)$ and $((\mathcal{T}_{\mathcal{L}}^{\pm})^{-1}\hat{G})(x,p) = ((\mathcal{T}_p^{\pm})^{-1}\hat{G})(x,p)$, $x \in \partial S$, $p \in \mathbb{C}_0$. Operators $\mathcal{T}^{\pm}$ on $\mathcal{H}_{1/2,l,l,\kappa}^{\mathcal{L}^{-1}}(\Gamma)$ and $(\mathcal{T}^{\pm})^{-1}$ on $\mathcal{H}_{1/2,l+1,l,\kappa}^{\mathcal{L}^{-1}}(\Gamma)$ are defined by $\mathcal{T}^{\pm} = \mathcal{L}^{-1}\mathcal{T}_{\mathcal{L}}^{\pm}\mathcal{L}$ and $(\mathcal{T}^{\pm})^{-1} = \mathcal{L}^{-1}(\mathcal{T}_{\mathcal{L}}^{\pm})^{-1}\mathcal{L}$.

Lemma 3. *For any $\lambda > 0$ and $k \in \mathbb{R}$, the operators*

$$\mathcal{T}^{\pm} : \mathcal{H}_{1/2,l,l,\kappa}^{\mathcal{L}^{-1}}(\Gamma) \to \mathcal{H}_{-1/2,l-1,l-2,\kappa}^{\mathcal{L}^{-1}}(\Gamma)$$

are continuous and injective, and their ranges are dense in $\mathcal{H}_{-1/2,l-1,l-2,\kappa}^{\mathcal{L}^{-1}}(\Gamma)$. Their inverses, extended by continuity from their ranges, define continuous and injective mappings

$$(\mathcal{T}^{\pm})^{-1} : \mathcal{H}_{-1/2,l-1,l-2,\kappa}^{\mathcal{L}^{-1}}(\Gamma) \to \mathcal{H}_{1/2,l-2,l-2,\kappa}^{\mathcal{L}^{-1}}(\Gamma),$$

whose ranges are dense in the spaces $\mathcal{H}_{1/2,l-2,l-2,\kappa}^{\mathcal{L}^{-1}}(\Gamma)$.

Proof. The continuity of $\mathcal{T}^{\pm}$ and $(\mathcal{T}^{\pm})^{-1}$ follows from (11) and (12), respectively. The statement about the ranges of $\mathcal{T}^{\pm}$ being dense in $\mathcal{H}_{-1/2,l-1,l-2,\kappa}^{\mathcal{L}^{-1}}(\Gamma)$ follows from the fact that $\mathcal{H}_{-1/2,l+1,l,\kappa}^{\mathcal{L}^{-1}}(\Gamma)$ is dense in $\mathcal{H}_{-1/2,l-1,l-2,\kappa}^{\mathcal{L}^{-1}}(\Gamma)$, and that about the ranges of $(\mathcal{T}^{\pm})^{-1}$ from the fact that $\mathcal{H}_{1/2,l,l,\kappa}^{\mathcal{L}^{-1}}(\Gamma)$ is dense in $\mathcal{H}_{1/2,l-2,l-2,\kappa}^{\mathcal{L}^{-1}}(\Gamma)$. $\square$

Suppose that $\partial\tilde{S}_i \subset \partial S$, $i = 1,2$, are two open arcs such that $\partial\tilde{S}_1 \cap \partial\tilde{S}_2 = \varnothing$ and $\overline{\partial\tilde{S}_1} \cup \overline{\partial\tilde{S}_2} = \partial S$, and let $\tilde{\pi}_i$, $i = 1,2$, be the operators of restriction from Γ to $\tilde{\Gamma}_i = \partial\tilde{S}_i \times (0,\infty)$. Also, let $F \in \mathcal{H}_{1/2,l,l,\kappa}(\Gamma)$. We define boundary operators π_{ij}, $i,j = 1,2$, $i \ne j$, by

$$\pi^{\pm}_{ij}F = \{\tilde{\pi}_i\hat{F}, \tilde{\pi}_j\mathcal{T}^{\pm}F\} = \{\hat{F}_i, \hat{G}_j\}.$$

Lemma 4. *For any $\kappa > 0$ and $l \in \mathbb{R}$, the operators*

$$\pi^{\pm}_{ij} : \mathcal{H}^{\mathcal{L}^{-1}}_{1/2,l,l,\kappa}(\Gamma) \to \mathcal{H}^{\mathcal{L}^{-1}}_{1/2,l,l,\kappa}(\tilde{\Gamma}_i) \times \mathcal{H}^{\mathcal{L}^{-1}}_{-1/2,l-1,l-2,\kappa}(\tilde{\Gamma}_j)$$

are continuous and injective, and their ranges are dense in the spaces $\mathcal{H}^{\mathcal{L}^{-1}}_{1/2,l,l,\kappa}(\tilde{\Gamma}_i) \times \mathcal{H}^{\mathcal{L}^{-1}}_{-1/2,l-1,l-2,\kappa}(\tilde{\Gamma}_j)$. Their inverses, extended by continuity from their ranges, define continuous and injective mappings

$$(\pi^{\pm}_{ij})^{-1} : \mathcal{H}^{\mathcal{L}^{-1}}_{1/2,l,l,\kappa}(\tilde{\Gamma}_i) \times \mathcal{H}^{\mathcal{L}^{-1}}_{-1/2,l,l-1,\kappa}(\tilde{\Gamma}_j) \to \mathcal{H}^{\mathcal{L}^{-1}}_{1/2,l-1,l-1,\kappa}(\Gamma),$$

whose ranges are dense in $\mathcal{H}^{\mathcal{L}^{-1}}_{1/2,l-1,l-1,\kappa}(\Gamma)$.

Proof. The continuity of $\pi^{\pm}_{ij}$ follows from the statement in Lemma 3 about the continuity of $\mathcal{T}^{\pm}$. Let $\{F_i, G_j\} \in \mathcal{H}^{\mathcal{L}^{-1}}_{1/2,l,l,\kappa}(\tilde{\Gamma}_i) \times \mathcal{H}^{\mathcal{L}^{-1}}_{-1/2,l,l-1,\kappa}(\tilde{\Gamma}_j)$, and let $U \in \mathcal{H}^{\mathcal{L}^{-1}}_{1,l-1,l-1,\kappa}(G^{\pm})$ be the solutions of (T$'$M$^{\pm}$) with boundary data $\{F_i, G_j\}$. Since $(\pi^{\pm}_{ij})^{-1}\{F_i, G_j\} = \gamma^{\pm}U$, the continuity of $(\pi^{\pm}_{ij})^{-1}$ follows from the trace theorem. The assertion about the ranges of $\pi^{\pm}_{ij}$ and $(\pi^{\pm}_{ij})^{-1}$ being dense in the corresponding spaces can be verified by the method used in the proof of Lemma 3. □

Let $G \in \mathcal{H}_{-1/2,l+1,l,\kappa}(\Gamma)$. We define boundary operators ε_{ij}, $i, j = 1, 2$, $i \neq j$, by

$$\varepsilon^{\pm}_{ij}G = \{\tilde{\pi}_i(\mathcal{T}^{\pm})^{-1}G, \tilde{\pi}_jG\} = \{\hat{F}_i, \hat{G}_j\}.$$

Lemma 5. *For any $\kappa > 0$ and $l \in \mathbb{R}$, the operators*

$$\varepsilon^{\pm}_{ij} : \mathcal{H}^{\mathcal{L}^{-1}}_{-1/2,l+1,l,\kappa}(\Gamma) \to \mathcal{H}^{\mathcal{L}^{-1}}_{1/2,l,l,\kappa}(\tilde{\Gamma}_i) \times \mathcal{H}^{\mathcal{L}^{-1}}_{-1/2,l+1,l,\kappa}(\tilde{\Gamma}_j)$$

are continuous and injective, and their ranges are dense in the spaces $\mathcal{H}^{\mathcal{L}^{-1}}_{1/2,l,l,\kappa}(\tilde{\Gamma}_i) \times \mathcal{H}^{\mathcal{L}^{-1}}_{-1/2,l+1,l,\kappa}(\tilde{\Gamma}_j)$. Their inverses, extended by continuity from their ranges, define continuous and injective mappings

$$(\varepsilon^{\pm}_{ij})^{-1} : \mathcal{H}^{\mathcal{L}^{-1}}_{1/2,l+1,l+1,\kappa}(\tilde{\Gamma}_i) \times \mathcal{H}^{\mathcal{L}^{-1}}_{-1/2,l+1,l,\kappa}(\tilde{\Gamma}_j) \to \mathcal{H}^{\mathcal{L}^{-1}}_{-1/2,l,l-1,\kappa}(\Gamma),$$

whose ranges are dense in $\mathcal{H}^{\mathcal{L}^{-1}}_{-1/2,l,l-1,\kappa}(\Gamma)$.

Proof. The continuity of $\varepsilon^{\pm}_{ij}$ follows from the statement in Lemma 3 about the continuity of $(\mathcal{T}^{\pm})^{-1}$. Let $\{F_i, G_j\} \in \mathcal{H}^{\mathcal{L}^{-1}}_{1/2,l+1,l+1,\kappa}(\tilde{\Gamma}_i) \times \mathcal{H}^{\mathcal{L}^{-1}}_{-1/2,l+1,l,\kappa}(\tilde{\Gamma}_j)$, and let $U \in \mathcal{H}^{\mathcal{L}^{-1}}_{1,l,l,\kappa}(G^{\pm})$ be the solutions of (T′M$^{\pm}$) with boundary data $\{F_i, G_j\}$. The continuity of $(\varepsilon^{\pm}_{ij})^{-1}$ now follows from (13), while the assertion about the ranges of $\varepsilon^{\pm}_{ij}$ and $(\varepsilon^{\pm}_{ij})^{-1}$ being dense in the corresponding spaces is established as in the proof of Lemma 3. □

For any $\kappa > 0$ and $l \in \mathbb{R}$, we introduce the space

$$\begin{aligned}\mathcal{H}^{\mathcal{L}^{-1}}_{\mathrm{mixed},l,\kappa}(\Gamma) = \mathbf{H}^{\mathcal{L}^{-1}}_{1/2,l+1,\kappa}(\Gamma_{12}) \times H^{\mathcal{L}^{-1}}_{1/2,l+1,\kappa}(\Gamma_{41})\\ \times \mathbf{H}^{\mathcal{L}^{-1}}_{-1/2,l+1,\kappa}(\Gamma_{34}) \times H^{\mathcal{L}^{-1}}_{-1/2,l,\kappa}(\Gamma_{23}).\end{aligned}$$

Naturally, the norm of

$$\Phi(x,t) = \{f_{12}(x,t), f_{4,41}(x,t), g_{34}(x,t), g_{4,23}(x,t)\} \in \mathcal{H}^{\mathcal{L}^{-1}}_{\mathrm{mixed},l,\kappa}(\Gamma)$$

is

$$\begin{aligned}|||\Phi|||_{\mathrm{mixed},l,\kappa;\Gamma} = \|f_{12}\|_{1/2,l+1,\kappa;\Gamma_{12}} + \|f_{4,41}\|_{1/2,l+1,\kappa;\Gamma_{41}}\\ + \|g_{34}\|_{-1/2,l+1,\kappa;\Gamma_{34}} + \|g_{4,23}\|_{-1/2,l,\kappa;\Gamma_{23}}.\end{aligned}$$

Let $\Phi \in \mathcal{H}^{\mathcal{L}^{-1}}_{\mathrm{mixed},l,\kappa}(\Gamma)$. By Theorem 1, problems (TM$^{\pm}$) have unique solutions $U \in \mathcal{H}^{\mathcal{L}^{-1}}_{1,l,l,\kappa}(G^{\pm})$, which satisfy the estimates

$$|||U|||_{1,l,l,\kappa;G^{\pm}} \leq c|||\Phi|||_{\mathrm{mixed},l,\kappa;\Gamma}. \tag{14}$$

Let $F = \gamma^{\pm}U \in \mathcal{H}^{\mathcal{L}^{-1}}_{1/2,l,l,\kappa}(\Gamma)$, and let $G = \mathcal{T}^{\pm}F$. We define boundary operators $\rho^{\pm}$ and $\eta^{\pm}$ by $\rho^{\pm}F = \Phi$ and $\eta^{\pm}G = \Phi$.

Lemma 6. *For any $\kappa > 0$ and $l \in \mathbb{R}$, the operators*

$$\rho^{\pm} : \mathcal{H}^{\mathcal{L}^{-1}}_{1/2,l+2,l+2,\kappa}(\Gamma) \to \mathcal{H}^{\mathcal{L}^{-1}}_{\mathrm{mixed},l,\kappa}(\Gamma),$$
$$\eta^{\pm} : \mathcal{H}^{\mathcal{L}^{-1}}_{-1/2,l+2,l+1,\kappa}(\Gamma) \to \mathcal{H}^{\mathcal{L}^{-1}}_{\mathrm{mixed},l,\kappa}(\Gamma)$$

are continuous and injective, and their ranges are dense in $\mathcal{H}^{\mathcal{L}^{-1}}_{\mathrm{mixed},l,\kappa}(\Gamma)$. *Their inverses, extended by continuity from their ranges, define continuous and injective mappings*

$$(\rho^{\pm})^{-1} : \mathcal{H}^{\mathcal{L}^{-1}}_{\mathrm{mixed},l,\kappa}(\tilde{\Gamma}) \to \mathcal{H}^{\mathcal{L}^{-1}}_{1/2,l,l,\kappa}(\Gamma),$$
$$(\eta^{\pm})^{-1} : \mathcal{H}^{\mathcal{L}^{-1}}_{\mathrm{mixed},l,\kappa}(\tilde{\Gamma}) \to \mathcal{H}^{\mathcal{L}^{-1}}_{-1/2,l,l-1,\kappa}(\Gamma),$$

whose ranges are dense,respectively, in $\mathcal{H}^{\mathcal{L}^{-1}}_{1/2,l,l,\kappa}(\Gamma)$ *and* $\mathcal{H}^{\mathcal{L}^{-1}}_{-1/2,l,l-1,\kappa}(\Gamma)$.

Proof. Let $F = (f^{\mathrm{T}}, f_4)^{\mathrm{T}} \in \mathcal{H}^{\mathcal{L}^{-1}}_{1/2,l+2,l+2,\kappa}(\Gamma)$. By Lemma 3, $G = (g^{\mathrm{T}}, g_4)^{\mathrm{T}} = \mathcal{T}^{\pm} F \in \mathcal{H}^{\mathcal{L}^{-1}}_{-1/2,l+1,l,\kappa}(\Gamma)$. We write

$$\Phi = \rho^{\pm} F = \{\pi_{12} f, \pi_{41} f_4, \pi_{34} g, \pi_{23} g_4\}.$$

Obviously, $\rho^{\pm}$ define continuous mappings

$$\rho^{\pm} : \mathcal{H}^{\mathcal{L}^{-1}}_{1/2,l+2,l+2,\kappa}(\Gamma) \to \mathbf{H}^{\mathcal{L}^{-1}}_{1/2,l+2,\kappa}(\Gamma_{12}) \times H^{\mathcal{L}^{-1}}_{1/2,l+2,\kappa}(\Gamma_{41})$$
$$\times \mathbf{H}^{\mathcal{L}^{-1}}_{-1/2,l+1,\kappa}(\Gamma_{34}) \times H^{\mathcal{L}^{-1}}_{-1/2,l,\kappa}(\Gamma_{23}) \subset \mathcal{H}^{\mathcal{L}^{-1}}_{\mathrm{mixed},l,\kappa}(\Gamma).$$

Since the norm on $\mathcal{H}^{\mathcal{L}^{-1}}_{\mathrm{mixed},l,\kappa}(\Gamma)$ is weaker than that on

$$\mathbf{H}^{\mathcal{L}^{-1}}_{1/2,l+2,\kappa}(\Gamma_{12}) \times H^{\mathcal{L}^{-1}}_{1/2,l+2,\kappa}(\Gamma_{41}) \times \mathbf{H}^{\mathcal{L}^{-1}}_{-1/2,l+1,\kappa}(\Gamma_{34}) \times H^{\mathcal{L}^{-1}}_{-1/2,l,\kappa}(\Gamma_{23}),$$

we deduce the continuity of $\rho^{\pm}$.

If $G = (g^{\mathrm{T}}, g_4)^{\mathrm{T}} \in \mathcal{H}^{\mathcal{L}^{-1}}_{-1/2,l+2,l+1,\kappa}(\Gamma)$, then

$$F = (f^{\mathrm{T}}, f_4)^{\mathrm{T}} = (\mathcal{T}^{\pm})^{-1} G \in \mathcal{H}^{\mathcal{L}^{-1}}_{1/2,l+1,l+1,\kappa}(\Gamma).$$

It is evident that

$$\Phi = \eta^{\pm} G \in \mathbf{H}^{\mathcal{L}^{-1}}_{1/2,l+1,\kappa}(\Gamma_{12}) \times H^{\mathcal{L}^{-1}}_{1/2,l+1,\kappa}(\Gamma_{41})$$
$$\times \mathbf{H}^{\mathcal{L}^{-1}}_{-1/2,l+2,\kappa}(\Gamma_{34}) \times H^{\mathcal{L}^{-1}}_{-1/2,l+1,\kappa}(\Gamma_{23}) \subset \mathcal{H}^{\mathcal{L}^{-1}}_{\mathrm{mixed},l,\kappa}(\Gamma).$$

The continuity of $\eta^{\pm}$ now follows from the fact that the norm on

$$\mathbf{H}^{\mathcal{L}^{-1}}_{1/2,l+1,\kappa}(\Gamma_{12})\times H^{\mathcal{L}^{-1}}_{1/2,l+1,\kappa}(\Gamma_{41})\times\mathbf{H}^{\mathcal{L}^{-1}}_{-1/2,l+2,\kappa}(\Gamma_{34})\times H^{\mathcal{L}^{-1}}_{-1/2,l+1,\kappa}(\Gamma_{23})$$

is stronger than that on $\mathcal{H}^{\mathcal{L}^{-1}}_{\mathrm{mixed},l,\kappa}(\Gamma)$. Suppose that $\rho^{\pm}F=\Phi=0$. Then the solutions of (TM$^{\pm}$) with boundary data Φ are $U(x,t)=0$, $(x,t)\in G^{\pm}$. This means that $F=0$. The injectivity of $\eta^{\pm}$ is proved similarly.

Let $\Phi\in\mathcal{H}^{\mathcal{L}^{-1}}_{\mathrm{mixed},l,\kappa}(\Gamma)$, and let $U\in\mathcal{H}^{\mathcal{L}^{-1}}_{1,l,l,\kappa}(G^{\pm})$ be the solutions of (TM$^{\pm}$) with boundary data Φ. The continuity of $(\rho^{\pm})^{-1}$ and $(\eta^{\pm})^{-1}$ follows from (14), the trace theorem, and (13).

The fact that $\mathcal{H}^{\mathcal{L}^{-1}}_{\mathrm{mixed},l+1,\kappa}(\Gamma)$ is dense in $\mathcal{H}^{\mathcal{L}^{-1}}_{\mathrm{mixed},l,\kappa}(\Gamma)$ implies that the ranges of $\rho^{\pm}$ are dense in $\mathcal{H}^{\mathcal{L}^{-1}}_{\mathrm{mixed},l,\kappa}(\Gamma)$. The corresponding statement about the ranges of $\eta^{\pm}$ is proved analogously. □

5. Properties of the Layer Potentials. The single-layer and double-layer potentials were introduced and studied in [10]. For convenience, here we list their basic properties and the properties of the boundary operators generated by them.

Let $\mathcal{D}(x,t)$ be a matrix of fundamental solutions for (1), which vanishes for $t<0$. This means that the $(4{\times}4)$-matrix $\mathcal{D}(x,t)$ satisfies

$$\begin{gathered}\mathsf{B}_0(\partial_t^2\mathcal{D})(x,t)+(\mathsf{B}_1\partial_t\mathcal{D})(x,t)+(\mathsf{A}\mathcal{D})(x,t)=\delta(x,t)I,\quad (x,t)\in\mathbb{R}^3,\\ \mathcal{D}(x,t)=0,\quad t<0,\end{gathered}$$

where $\delta(x,t)$ is the Dirac delta and I is the identity $(4{\times}4)$-matrix. Obviously, the Laplace transform $\hat{\mathcal{D}}(x,p)$ of $\mathcal{D}(x,t)$ satisfies the transformed equation

$$p^2\mathsf{B}_0\hat{\mathcal{D}}(x,p)+p\,(\mathsf{B}_1\hat{\mathcal{D}})(x,p)+(\mathsf{A}\hat{\mathcal{D}})(x,p)=\delta(x)I,\quad x\in\mathbb{R}^2.$$

The explicit form of $\hat{\mathcal{D}}(x,p)$ and its asymptotic behavior as $|x|\to 0$ and as $|x|\to\infty$ can be found in [11].

Let $\mathcal{A}=(\alpha^{\mathrm{T}},\alpha_4)^{\mathrm{T}}$, where $\alpha=(\alpha_1,\alpha_2,\alpha_3)^{\mathrm{T}}$, be a smooth function with compact support in $\partial S\times\mathbb{R}$, which is equal to zero for $t<0$. We define the single-layer thermoelastic potential $\mathcal{V}\mathcal{A}$ of density $\mathcal{A}$ by

$$\mathcal{V}\mathcal{A}(x,t) = \int\limits_{\Gamma} \mathcal{D}(x-y,t-\tau)\mathcal{A}(y,\tau)\,ds_y\,d\tau, \quad (x,t)\in\mathbb{R}^3.$$

In [10] it was proved that for any $\kappa > 0$ and $l \in \mathbb{R}$, the single-layer potential can be extended by continuity to densities $\mathcal{A} \in \mathcal{H}^{\mathcal{L}^{-1}}_{-1/2,l+1,l,\kappa}(\Gamma)$ and the vector-valued function $U = \mathcal{V}\mathcal{A} \in \mathcal{H}^{\mathcal{L}^{-1}}_{1,l,l,\kappa}(G^{\pm})$ satisfies

$$(\mathsf{L}U)(x,t) = 0, \quad (x,t) \in G^{\pm}.$$

The limiting values (traces) of $(\mathcal{V}\mathcal{A})(x,t)$, as $(x,t) \to \Gamma$ from inside $G^{\pm}$, coincide, and we write

$$(\mathcal{V}\mathcal{A})^{+}(x,t) = (\mathcal{V}\mathcal{A})^{-}(x,t) = (\mathcal{V}\mathcal{A})(x,t), \quad (x,t)\in\Gamma; \tag{15}$$

therefore, we may define a boundary operator $\mathcal{V}$ for densities $\mathcal{A} \in \mathcal{H}^{\mathcal{L}^{-1}}_{-1/2,l+1,l,\kappa}(\Gamma)$ by means of (15).

Lemma 7. (i) *For any $\kappa > 0$ and $l \in \mathbb{R}$, the operator*

$$\mathcal{V} : \mathcal{H}^{\mathcal{L}^{-1}}_{-1/2,l+1,l,\kappa}(\Gamma) \to \mathcal{H}^{\mathcal{L}^{-1}}_{-1/2,l,l,\kappa}(\Gamma)$$

is continuous and injective, its range is dense in $\mathcal{H}^{\mathcal{L}^{-1}}_{-1/2,l,l,\kappa}(\Gamma)$, and its inverse, extended by continuity from its range, defines a continuous and injective mapping

$$\mathcal{V}^{-1} : \mathcal{H}^{\mathcal{L}^{-1}}_{1/2,l,l,\kappa}(\Gamma) \to \mathcal{H}^{\mathcal{L}^{-1}}_{-1/2,l-1,l-2,\kappa}(\Gamma)$$

whose range is dense in $\mathcal{H}^{\mathcal{L}^{-1}}_{1/2,l-1,l-2,\kappa}(\Gamma)$.

(ii) *If $\mathcal{A} \in \mathcal{H}^{\mathcal{L}^{-1}}_{-1/2,l+1,l,\kappa}(\Gamma)$ and $U = \mathcal{V}\mathcal{A} \in \mathcal{H}^{\mathcal{L}^{-1}}_{1,l,l,\kappa}(G^{\pm})$, then there hold the jump formulas*

$$\gamma^{+}U = \gamma^{-}U = \mathcal{V}\mathcal{A}, \quad \mathcal{T}^{+}\mathcal{V}\mathcal{A} - \mathcal{T}^{-}\mathcal{V}\mathcal{A} = \mathcal{A}.$$

Let

$$\mathcal{T}' = \begin{pmatrix} & & & \eta n_1(y)\partial_t \\ & T_y & & \eta n_2(y)\partial_t \\ & & & 0 \\ 0 & 0 & 0 & \partial/\partial n(y) \end{pmatrix},$$

where T_y is the boundary operator defined by (5) with $n=(n_1,n_2)^{\mathrm{T}}$ and $\partial_\alpha = \partial/\partial y_\alpha$, $\alpha=1,2$. We construct the (4×4)-matrix kernel

$$\mathcal{P}(x,y,t) = \left[\mathcal{T}'\mathcal{D}^{\mathrm{T}}(x-y,t)\right]^{\mathrm{T}}$$

and define the double-layer potential of density $\mathcal{B} = (\beta^{\mathrm{T}},\beta_4)^{\mathrm{T}}, \beta = (\beta_1,\beta_2,\beta_3)^{\mathrm{T}}$, which is smooth, has compact support in $\partial S\times\mathbb{R}$, and is equal to zero for $t<0$, by

$$(\mathcal{W}\mathcal{B})(x,t) = \int\limits_{\Gamma} \mathcal{P}(x,y,t-\tau)\mathcal{B}(y,\tau)\,ds_y\,d\tau, \quad (x,t)\in\mathbb{R}^3.$$

In [10] it was proved that for any $\kappa>0$ and $l\in\mathbb{R}$, the double-layer potential can be extended by continuity to densities $\mathcal{B}\in\mathcal{H}^{\mathcal{L}^{-1}}_{1/2,l,l,\kappa}(\Gamma)$, and that the vector-valued function $U=\mathcal{W}\mathcal{B}\in\mathcal{H}^{\mathcal{L}^{-1}}_{1,l-3,l-3,\kappa}(G^\pm)$ satisfies

$$(\mathsf{L}U)(x,t) = 0, \quad (x,t)\in G^\pm.$$

We introduce the operators $\mathcal{W}^\pm$ generated by the limiting values (traces) of the double-layer potential by

$$(\mathcal{W}^\pm\mathcal{B})(x,t) = (\gamma^\pm U)(x,t), \quad (x,t)\in\Gamma.$$

Since, as shown in [10], $\mathcal{T}^+\mathcal{W}^+\mathcal{B} = \mathcal{T}^-\mathcal{W}^-\mathcal{B}$, we can define a boundary operator $\mathcal{F}$ for densities $\mathcal{B}\in\mathcal{H}^{\mathcal{L}^{-1}}_{1/2,l,l,\kappa}(\Gamma)$ by

$$\mathcal{F}\mathcal{B} = \mathcal{T}^+\mathcal{W}^+\mathcal{B} = \mathcal{T}^-\mathcal{W}^-\mathcal{B}.$$

Also [10], $\mathcal{W}^\pm = \mathcal{V}\mathcal{T}^\mp$.

Lemma 8. (i) *For any $\kappa>0$ and $l\in\mathbb{R}$, the operators*

$$\mathcal{W}^\pm : \mathcal{H}^{\mathcal{L}^{-1}}_{1/2,l+2,l+2,\kappa}(\Gamma)\to\mathcal{H}^{\mathcal{L}^{-1}}_{1/2,l,l,\kappa}(\Gamma),$$
$$\mathcal{F} : \mathcal{H}^{\mathcal{L}^{-1}}_{1/2,l+2,l+2,\kappa}(\Gamma)\to\mathcal{H}^{\mathcal{L}^{-1}}_{-1/2,l-1,l-2,\kappa}(\Gamma)$$

are continuous and injective, their ranges are dense, respectively, in $\mathcal{H}^{\mathcal{L}^{-1}}_{1/2,l,l,\kappa}(\Gamma)$ and $\mathcal{H}^{\mathcal{L}^{-1}}_{-1/2,l-1,l-2,\kappa}(\Gamma)$, and their inverses, extended by continuity from their ranges, define continuous and injective mappings

$$(\mathcal{W}^\pm)^{-1} : \mathcal{H}^{\mathcal{L}^{-1}}_{1/2,l,l,\kappa}(\Gamma)\to\mathcal{H}^{\mathcal{L}^{-1}}_{1/2,l-2,l-2,\kappa}(\Gamma),$$
$$\mathcal{F}^{-1} : \mathcal{H}^{\mathcal{L}^{-1}}_{-1/2,l-1,l-2,\kappa}(\Gamma)\to\mathcal{H}^{\mathcal{L}^{-1}}_{1/2,l-2,l-2,\kappa}(\Gamma),$$

whose ranges are dense in $\mathcal{H}^{\mathcal{L}^{-1}}_{1/2,l-2,l-2,\kappa}(\Gamma)$.

(ii) *If* $\mathcal{B} \in \mathcal{H}^{\mathcal{L}^{-1}}_{1/2,l+2,l+2,\kappa}(\Gamma)$ *and* $U = \mathcal{W}\mathcal{B} \in \mathcal{H}^{\mathcal{L}^{-1}}_{1,l-1,l-1,\kappa}(G^{\pm})$, *then there hold the jump formulas*

$$\gamma^{+}U - \gamma^{-}U = -\mathcal{B}, \qquad \mathcal{T}^{+}\mathcal{W}^{+}\mathcal{B} = \mathcal{T}^{-}\mathcal{W}^{-}\mathcal{B} = \mathcal{F}\mathcal{B}.$$

6. Boundary Integral Equations and Existence Theorems. Let $U \in \mathcal{H}^{\mathcal{L}^{-1}}_{1,l,l,\kappa}(G^{\pm})$ be the (unique) solutions of $(\mathrm{TM}^{\pm})$ with boundary data

$$\Phi = \{f_{12}, f_{4,41}, g_{34}, g_{4,23}\} \in \mathcal{H}^{\mathcal{L}^{-1}}_{\mathrm{mixed},l,\kappa}(\Gamma),$$

$F = \gamma^{\pm}U = (\rho^{\pm})^{-1}\Phi$, and $G = \mathcal{T}^{\pm}F = (\eta^{\pm})^{-1}\Phi$. We consider three representations for $U(x,t)$ in terms of layer potentials. First, we seek $U(x,t)$ in the form

$$U(x,t) = (\mathcal{V}\mathcal{A})(x,t), \quad (x,t) \in G^{\pm}. \tag{16}$$

This leads to the systems of boundary integral equations

$$\rho^{\pm}\mathcal{V}\mathcal{A} = \Phi. \tag{17}$$

Theorem 2. (i) *For any* $\kappa > 0$ *and* $l \in \mathbb{R}$, *systems* (17) *have unique solutions*

$$\mathcal{A} = \mathcal{V}^{-1}(\rho^{\pm})^{-1}\hat{F},$$

and the resolving operators define continuous mappings

$$\mathcal{V}^{-1}(\rho^{\pm})^{-1} : \mathcal{H}^{\mathcal{L}^{-1}}_{\mathrm{mixed},l,\kappa}(\Gamma) \to \mathcal{H}^{\mathcal{L}^{-1}}_{-1/2,l-1,l-2,\kappa}(\Gamma). \tag{18}$$

(ii) *If* $\mathcal{A} \in \mathcal{H}^{\mathcal{L}^{-1}}_{-1/2,l-1,l-2,\kappa}(\Gamma)$ *are the solutions of* (17), *then* U *given by* (16) *belong to* $\mathcal{H}^{\mathcal{L}^{-1}}_{1,l,l,\kappa}(G^{\pm})$ *and are the solutions of* $(\mathrm{TM}^{\pm})$ *for* $l \geq 0$.

Proof. The continuity of the mappings (18) follows from the properties of the operators $\mathcal{V}^{-1}$ and $(\rho^{\pm})^{-1}$ listed in Lemmas 7 and 6, respectively. From (10) it follows that $U = \mathcal{V}\mathcal{A} \in \mathcal{H}^{\mathcal{L}^{-1}}_{1,l,l,\kappa}(G^{\pm})$. Finally, by Theorem 1, $U(x,t)$ is the solution of (TM$^{\pm}$) for $l \geq 0$. □

Representing the solutions of (TM$^{\pm}$) in the form

$$U(x,t) = (\mathcal{W}\mathcal{B})(x,t), \quad (x,t) \in G^{\pm}, \tag{19}$$

we arrive at the systems of boundary integral equations

$$\eta^{\pm}\mathcal{F}\mathcal{B} = \Phi. \tag{20}$$

Theorem 3. (i) *For any* $\kappa > 0$ *and* $l \in \mathbb{R}$, *systems* (20) *have unique solutions*

$$\mathcal{B} = \mathcal{F}^{-1}(\eta^{\pm})^{-1}\Phi,$$

and the resolving operators define continuous mappings

$$\mathcal{F}^{-1}(\eta^{\pm})^{-1} : \mathcal{H}^{\mathcal{L}^{-1}}_{\mathrm{mixed},l,\kappa}(\Gamma) \to \mathcal{H}^{\mathcal{L}^{-1}}_{1/2,l-1,l-1,\kappa}(\Gamma).$$

(ii) *If* $\mathcal{B} \in \mathcal{H}^{\mathcal{L}^{-1}}_{1/2,l-1,l-1,\kappa}(\Gamma)$ *are the solutions of* (20), *then* U *given by* (19) *belong to* $\mathcal{H}^{\mathcal{L}^{-1}}_{1,l,l,\kappa}(G^{\pm})$ *and are the solutions of* (TM$^{\pm}$) *for* $l \geq 0$.

The proof of this assertion follows from Lemmas 6 and 8, (10), and Theorem 1.

Let $\partial\tilde{S}_1$ and $\partial\tilde{S}_2$ be two open arcs of ∂S such that $\partial\tilde{S}_1 \cap \partial\tilde{S}_2 = \varnothing$ and $\overline{\partial\tilde{S}_1} \cup \overline{\partial\tilde{S}_2} = \partial S$. Once again, we write $\tilde{\Gamma}_i = \partial S_i \times (0,\infty)$, $i = 1, 2$. The last representation for U is

$$U(x,t) = (\mathcal{V}\mathcal{A})(x,t) + (\mathcal{W}\mathcal{B})(x,t), \quad (x,t) \in G^{\pm}, \tag{21}$$

with densities $\mathcal{A}$ and $\mathcal{B}$ such that $\operatorname{supp}\mathcal{A} \subset \bar{\tilde{\Gamma}}_1$ and $\operatorname{supp}\mathcal{B} \subset \bar{\tilde{\Gamma}}_2$. This yields the systems

$$\rho^{\pm}(\mathcal{V}\mathcal{A} + \mathcal{W}^{\pm}\mathcal{B}) = \Phi, \tag{22}$$

or, equivalently,

$$\eta^{\pm}(\mathcal{T}^{\pm}\mathcal{V}\mathcal{A}+\mathcal{F}\mathcal{B})=\Phi. \tag{23}$$

Let $\hat{F}\in\mathcal{H}^{\mathcal{L}^{-1}}_{\mathrm{mixed},l,\kappa}(\Gamma)$. From Lemma 6 it follows that $F=(\rho^{\pm})^{-1}\Phi\in\mathcal{H}^{\mathcal{L}^{-1}}_{1/2,l,l,\kappa}(\Gamma)$ and $G=\mathcal{T}^{\pm}F=(\eta^{\pm})^{-1}\Phi\in\mathcal{H}^{\mathcal{L}^{-1}}_{-1/2,l,l-1,\kappa}(\Gamma)$. We make the notation $F_1=\tilde{\pi}_1F\in\mathcal{H}^{\mathcal{L}^{-1}}_{1/2,l,l,\kappa}(\tilde{\Gamma}_1)$ and $G_2=\tilde{\pi}_2G\in\mathcal{H}^{\mathcal{L}^{-1}}_{-1/2,l,l-1,\kappa}(\tilde{\Gamma}_2)$, and introduce the pair $\{F_1,G_2\}\in\mathcal{H}^{\mathcal{L}^{-1}}_{1/2,l,l,\kappa}(\tilde{\Gamma}_1)\times\mathcal{H}^{\mathcal{L}^{-1}}_{-1/2,l,l-1,\kappa}(\tilde{\Gamma}_2)$.

Theorem 4. (i) *For any $\kappa>0$ and $l\in\mathbb{R}$, systems* (22) *and* (23) *have unique solutions*

$$\mathcal{A}=[(\varepsilon^{\pm}_{12})^{-1}-(\varepsilon^{\mp}_{12})^{-1}]\{F_1,G_2\}\in\mathring{\mathcal{H}}^{\mathcal{L}^{-1}}_{-1/2,l-1,l-2,\kappa}(\Gamma), \tag{24}$$

$$\mathcal{B}=[(\pi^{\mp}_{12})^{-1}-(\pi^{\pm}_{12})^{-1}]\{F_1,G_2\}\in\mathring{\mathcal{H}}^{\mathcal{L}^{-1}}_{1/2,l-1,l-1,\kappa}(\Gamma), \tag{25}$$

which satisfy the estimate

$$|||\mathcal{A}|||_{-1/2,l-1,l-2,\kappa;\Gamma}+|||\mathcal{B}|||_{1/2,l-1,l-1,\kappa;\Gamma}\le c|||\Phi|||_{\mathrm{mixed},l,\kappa,\Gamma}. \tag{26}$$

(ii) *If $\mathcal{A}\in\mathring{\mathcal{H}}^{\mathcal{L}^{-1}}_{-1/2,l-1,l-2,\kappa}(\tilde{\Gamma}_1)$ and $\mathcal{B}\in\mathring{\mathcal{H}}^{\mathcal{L}^{-1}}_{1/2,l-1,l-1,\kappa}(\tilde{\Gamma}_2)$ are the solutions of* (22) *and* (23), *then U given by* (21) *belong to $\mathcal{H}^{\mathcal{L}^{-1}}_{1,l,l,\kappa}(G^{\pm})$ and are the solutions of* ($\mathrm{TM}^{\pm}$) *for $l\ge 0$.*

Proof. Consider the case of (TM^{+}); the exterior problem (TM^{-}) is treated analogously.

It is clear that

$$\tilde{\pi}_2(\varepsilon^{+}_{12})^{-1}\{F_1,G_2\}=\tilde{\pi}_2(\varepsilon^{-}_{12})^{-1}\{F_1,G_2\}=G_2;$$

therefore, $\operatorname{supp}\mathcal{A}\subset\bar{\tilde{\Gamma}}_1$. For a similar reason, $\operatorname{supp}\mathcal{B}\subset\bar{\tilde{\Gamma}}_2$. From (24), (5), Lemmas 4 and 5, and the inclusion $\{F_1,G_2\}\in\mathcal{H}^{\mathcal{L}^{-1}}_{1/2,l,l,\kappa}(\tilde{\Gamma}_1)\times\mathcal{H}^{\mathcal{L}^{-1}}_{-1/2,l,l-1,\kappa}(\tilde{\Gamma}_2)$ it follows that $\mathcal{A}\in\mathring{\mathcal{H}}^{\mathcal{L}^{-1}}_{-1/2,l-1,l-2,\kappa}(\tilde{\Gamma}_1)$ and $\mathcal{B}\in\mathring{\mathcal{H}}^{\mathcal{L}^{-1}}_{1/2,l-1,l-1,\kappa}(\tilde{\Gamma}_1)$, and that (26) holds. We now prove (22), which in the case of the interior problem (TM^+) may be written in the equivalent form

$$\mathcal{V}\mathcal{A} + \mathcal{W}^+\mathcal{B} = F. \tag{27}$$

Replacing (24) and (25) in (27) and recalling that $\mathcal{W}^+ = \mathcal{V}\mathcal{T}^-$, $\mathcal{V}^{-1} = \mathcal{T}^+ - \mathcal{T}^-$, and $\pi_{12}^-(\mathcal{T}^-)^{-1} = \varepsilon_{12}^-$, we obtain the sequence of equalities

$$\begin{aligned}\mathcal{V}\mathcal{A} + \mathcal{W}^+\mathcal{B} &= \mathcal{V}\left([I - (\varepsilon_{12}^-)^{-1}\varepsilon_{12}^+]G + \mathcal{T}^-[(\pi_{12}^-)^{-1}\pi_{12}^+ - I]F\right)\\ &= \mathcal{V}(\mathcal{T}^+ - \mathcal{T}^-)F + \mathcal{V}\left(\mathcal{T}^-(\pi_{12}^-)^{-1}\pi_{12}^+F - (\varepsilon_{12}^-)^{-1}\varepsilon_{12}^+G\right)\\ &= F + \mathcal{V}\left(\mathcal{T}^-(\pi_{12}^-)^{-1} - (\varepsilon_{12}^-)^{-1}\right)\{F_1, G_2\} = F.\end{aligned}$$

To prove uniqueness, let the densities $\mathcal{A} \in \mathring{\mathcal{H}}^{\mathcal{L}^{-1}}_{-1/2,l-1,l-2,\kappa}(\tilde{\Gamma}_1)$ and $\mathcal{B} \in \mathring{\mathcal{H}}^{\mathcal{L}^{-1}}_{1/2,l-1,l-1,\kappa}(\tilde{\Gamma}_2)$ be the solutions of $\mathcal{V}\mathcal{A} + \mathcal{W}^+\mathcal{B} = 0$, and consider the function $V(x,t) = (\mathcal{V}\mathcal{A})(x,t) + (\mathcal{W}\mathcal{B})(x,t)$, $(x,t) \in G^-$. Obviously, V is the solution of ($\mathrm{T'M}^-$) with boundary data $\tilde{\pi}_1\gamma^-V = \tilde{\pi}_1(\mathcal{V}\mathcal{A} + \mathcal{W}^+\mathcal{B}) = 0$ and $\tilde{\pi}_2\mathcal{T}^-\gamma^-V = \tilde{\pi}_2\mathcal{T}^+(\mathcal{V}\mathcal{A} + \mathcal{W}^+\mathcal{B}) = 0$. From Corollary 2 it follows that $V(x,t) = 0$, $(x,t) \in G^-$, and the jump formulas for the layer potentials now yield $\mathcal{A} = 0$ and $\mathcal{B} = 0$.

Statement (ii) follows from Lemmas 7 and 8 and Theorem 1. □

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