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ON THE CORRECTNESS OF THE PROBLEM OF INVERTING THE FINITE HILBERT TRANSFORM IN CERTAIN AEROELASTIC MODELS

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ABSTRACT. We indicate methods of ensuring that the problem in the title is correctly posed in the L^p sense whenever the derivative of the circulation function satisfies certain mild conditions.

1. Introduction. In the theory of aeroelastic control systems it is required to solve

$$f(x) = \int_{-1}^{1} \frac{\gamma(y)}{y-x}\,dy \tag{1}$$

for $\gamma(y), -1 < y < 1$, in terms of $f(x), -1 < x < 1$. The function f is assumed to be of the form

$$f(x) = w(x) + g(x), \tag{2}$$

where

$$g(x) = \int_{0}^{\infty} \frac{G(s)}{1-x+s}\,ds. \tag{3}$$

Here w and G are constant multiples of the so-called downwash function and the derivative of the circulation function respectively.

Formula (1) is often referred to as the finite Hilbert transform of γ, see [**3**]. For more detail concerning this model of aeroelasticity see [**1**] and the references cited there. In particular, $f = f^t$ and $G = \dot{\Gamma}_t$ in the notation of [**1**].

In order to guarantee the correctness of the problem of solving (1) via the methods in [**3**], it is necessary to assume that f is in some $L^p(-1,1)$

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class. (If J is an interval and p is a positive number then $L^p(J)$ is the class of those Lebesgue measurable functions for which $\int_J |G(x)|^p dx$ is finite. When $p = \infty$, $L^\infty(J)$ is the class of essentially bounded functions on J.) In view of this and the fact that w can usually be taken to be in any class $L^p(-1,1)$, it is important to obtain a fairly general answer to the following question: What conditions on G ensure that g is in $L^p(-1,1)$?

It is the purpose of this note to point out certain natural methods for obtaining such conditions. Propositions 2 and 3 below contain several typical results.

2. Discussion. Observe that the integral defining $g(x)$ is a smooth function of x for $x < 1$ whenever G is locally integrable and satisfies a mild condition at infinity. In particular, it is clear that $g(x)$ is infinitely differentiable for $x < 1$ if

$$\int_0^\infty \frac{|G(s)|}{1+s}\, ds < \infty. \tag{4}$$

In fact, if $0 < x < 1$, we may write

$$|g(x)| \leq (1-x)^{-1} \int_0^\infty \frac{|G(s)|}{1+s}\, ds$$

and conclude that the only questionable behavior of g occurs in arbitrarily small neighborhoods of $x = 1$ whenever G satisfies (4).

It should be noted that condition (4) is quite mild and general. For example, if $G = G_1 + G_2$ where G_1 is in $L^1(0,\infty)$ and G_2 is in $L^p(0,\infty)$ for some $p, p < \infty$, then G satisfies (4).

To understand how G influences the behavior of g in neighborhoods of $x = 1$, express the integral defining g as a sum, $\int_0^\varepsilon + \int_\varepsilon^\infty$, where ε is any positive number ≤ 1. Since

$$\left| \int_\varepsilon^\infty \frac{G(s)}{1-x+s}\, ds \right| \leq \frac{2}{\varepsilon} \int_0^\infty \frac{|G(s)|}{1+s}\, ds, \tag{5}$$

it should be clear that the behavior of g at $x = 1$ is determined by the behavior of G at the origin. Indeed, if $s^{-\alpha} G(s)$ is in $L^p(0,\varepsilon)$ for some value of $p, 1 \le p \le \infty$, then by virtue of Hölder's inequality we may write

$$\left| \int_0^\varepsilon \frac{G(s)}{1-x+s}\, ds \right| \le I(x,p,\alpha) \left[\int_0^\varepsilon \left(s^{-\alpha} G(s) \right)^p ds \right]^{1/p}, \tag{6}$$

where

$$I(x,p,\alpha) = \left[\int_0^\varepsilon \left(\frac{s^\alpha}{1-x+s} \right)^{p/(p-1)} ds \right]^{1-1/p}$$

Now, using a change of variable, I may be expressed as

$$I(x,p,\alpha) = (1-x)^{\alpha-1/p} \left[\int_0^{\varepsilon/(1-x)} \left(\frac{s^\alpha}{1+s} \right)^{p/(p-1)} ds \right]^{1-1/p},$$

from which we may easily estimate its size.

We summarize these observations as

PROPOSITION 1. *Suppose g is related to G via* (3), *G satisfies condition* (4), *and $s^{-\alpha}G(s)$ is in $L^p(0,\varepsilon)$ for some positive ε, some $\alpha, \alpha \ge 0$, and some p, $1 \le p \le \infty$. Then, for $-1 < x < 1$,*

$$|g(x)| \le \begin{cases} C(1-x)^{\alpha-1/p} & \text{if } \alpha < 1/p \\ C(1+\log(1-x)) & \text{if } \alpha = 1/p \\ C & \text{if } \alpha > 1/p \end{cases} \tag{7}$$

where C is independent of x.

A result concerning the L^p class of g follows as an immediate corollary.

PROPOSITION 2. *Suppose G and g satisfy the hypothesis of Proposition 1. If $\alpha \ge 1/p$ then g is in $L^q(-1,1)$ for all positive q. If $\alpha < 1/p$ then g is in $L^q(-1,1)$ for all positive q which satisfy $q < p/(1-\alpha p)$.*

By using a slightly more delicate argument, the inequality $q < p/(1-\alpha p)$ in the second half of the above proposition can be tightened to $q \le p/(1-\alpha p)$ in the case $1 < p < \infty$. To see this, use the fact that if $\alpha \ge 0$ and $x < 1$ then

$$\left(\frac{s}{1-x+s}\right)^{\alpha} \le 1$$

to observe that

$$\left|\int_0^{\varepsilon} \frac{G(s)}{1-x+s}\,ds\right| \le I_\alpha G_\varepsilon^\alpha(x-1), \tag{8}$$

where

$$I_0\phi(y) = \int_{-\infty}^{\infty} \frac{\phi(s)}{s-y}\,ds$$

is the classical Hilbert transform of ϕ and, when $\alpha > 0$,

$$I_\alpha\phi(y) = \int_{-\infty}^{\infty} \frac{\phi(s)}{|s-y|^{1-\alpha}}\,ds$$

is the fractional integral or Riesz potential of ϕ. Here

$$G_\varepsilon^\alpha(s) = \begin{cases} |s^{-\alpha}G(s)| & \text{if } 0 < s < \varepsilon \\ 0 & \text{if } s < 0 \text{ or } s > \varepsilon. \end{cases}$$

The mapping properties of the transformation $\phi \to I_\alpha\phi$ are well known and, in view of (8), can be used to make conclusions concerning the behavior of g. For instance, if $1 < p < \infty, 0 \le \alpha < 1/p$, and ϕ is in $L^p(-\infty,\infty)$ then $I_\alpha\phi$ is in $L^q(-\infty,\infty)$ where $q = p/(1-\alpha p)$; see [**2**]. This, together with (5), (8), and Proposition 1, allows us to conclude

PROPOSITION 3. *Suppose G and g satisfy the hypothesis of Proposition 1 with the restriction that $1 < p < \infty$ and $0 \le \alpha < 1/p$. Then g is in $L^q(-1,1)$ for all positive q which satisfy $q \le p/(1-\alpha p)$.*

Another method of estimating the left hand side of (8) involves writing $\int_0^\varepsilon$ as $\int_0^{(1-x)} + \int_{(1-x)}^\varepsilon$ when $1 - x < \varepsilon$, using the change of variables $y = 1 - x$, and applying variants of Hardy's inequality, see [**2**,

p. 245] to each of the resulting integrals. This leads to generalizations of Proposition 1 for certain values of the parameters α and p.

For the sake of completeness we mention that, using similar methods involving fractional integrals, it is possible to obtain results concerning the behavior of g in neighborhoods of 1 for other values of the parameters α and p. Since such estimates require the introduction of certain technical machinery and the conclusions do not involve L^p, we will not pursue the details here.

It may be worth noting that, in the notation of [**1**], $G(s)$ is equal to $\psi(t-s)$ for $t < s$; it is equal to another expression when $s < t$. Furthermore ψ is assumed to be in $L^1(-\infty, 0)$. In this case our observations imply that if $t > 0$ then ψ does not affect the L^p class of f^t. This should be compared with the conclusions in [**1**].

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