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ON AN INVERSE PROBLEM FOR A MODEL OF LINEAR VISCOELASTIC KIRCHHOFF PLATE

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ABSTRACT. A linear viscoelastic Kirchhoff plate model with a rotational inertia term is considered. In this model, the vertical deflection u of a viscoelastic plate is governed by a linear integrodifferential evolution equation which contains a time convolution term. The convolution kernel, D, named viscoelastic flexural rigidity, is supposed to depend on time only. Provided that u is a solution to a suitable initial and boundary value problem for the motion equation, the inverse problem of determining D from supplementary information is analyzed. Three possible additional measurements and the corresponding inverse problems are examined. The main theorems are concerned with existence of solutions on a given bounded time interval. Continuous dependence on data is also discussed. These results extend the ones contained in a previous authors' paper.

1. Introduction. Consider a homogeneous and isotropic plate of uniform thickness $h > 0$ which occupies, for any $t \in [0,T]$, $T > 0$, a domain $\Omega \times (-h/2, h/2) \subset \mathbf{R}^3$, where Ω is an open, connected, and bounded subset of $\mathbf{R}^2$ with a smooth boundary Γ. Denote by $u(x,t)$ the vertical deflection of the plate from its equilibrium position $u \equiv 0$, at point $x \in \Omega$, at time $t \in [0,T]$. Assume that the plate is made from a viscoelastic material and lies free of stresses and strains up to the initial time $t = 0$. Besides, suppose that the mass density is equal to 1, just for the sake of simplicity. Then, neglecting thermal effects, using the stress-strain relationship characterizing the three-dimensional linear viscoelasticity of Boltzmann type (see, e.g., [**5, 8**] and references therein) and imposing the *Kirchhoff hypothesis*, it can be shown that the evolution of u is ruled by (cf. [**9**, Chapter I, Section 7], [**10**] and [**11**, Chapter 2, Section 1.4 and Chapter 6], see also [**15**] and its references

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for the elastic case)

$$hu'' - \frac{h^3}{12}\Delta u'' + D(0)\Delta^2 u + D' * \Delta^2 u = f \quad \text{in } \Omega \times (0,T) \tag{1.1}$$

where prime denotes the time derivative, Δ is the Laplace operator, Δ^2 indicates the biharmonic operator, and $*$ stands for the usual time convolution product over $(0,t)$. Here $D : (0,+\infty) \to \mathbf{R}$ is the so-called *viscoelastic flexural rigidity* and $D(0) > 0$, while $f : \Omega \times (0,T) \to \mathbf{R}$ is an external force acting along the vertical direction, i.e., orthogonal to Ω.

Let us associate with (1.1) a set of initial and boundary conditions. We first suppose $\Gamma = \overline{\Gamma}_0 \cup \overline{\Gamma}_1$, where Γ_0 and Γ_1 are relatively open and disjoint subsets of Γ. Also, we assume from now on that $|\Gamma_0| > 0$, $|\Gamma_0|$ being the one-dimensional Lebesgue measure of Γ_0, while it may be $|\Gamma_1| = 0$. In addition, we indicate by $\nu = (\nu_1, \nu_2)$ and $\tau = (-\nu_2, \nu_1)$ the unit normal vector to Γ pointing outward and the unit positively oriented tangent vector to Γ, respectively. Then we introduce

$$u(\cdot,0) = u_0, \qquad u'(\cdot,0) = u_1 \quad \text{in } \Omega \tag{1.2}$$

$$u = u_\nu = 0 \quad \text{on } \Gamma_0 \times [0,T] \tag{1.3}$$

$$D(0)\mathcal{B}_1 u + D' * \mathcal{B}_1 u = 0 \quad \text{on } \Gamma_1 \times [0,T] \tag{1.4}$$

$$D(0)\mathcal{B}_2 u + D' * \mathcal{B}_2 u - \frac{h^3}{12} u''_\nu = 0 \quad \text{on } \Gamma_1 \times [0,T] \tag{1.5}$$

$u_0, u_1 : \Omega \to \mathbf{R}$ being given initial conditions. Here $(\cdot)_\nu$ represents the derivative in the ν-direction and $\mathcal{B}_1, \mathcal{B}_2$ are linear operators defined by

$$\begin{aligned} \mathcal{B}_1 z &:= \Delta z + (1-\mu) B_1 z, \\ \mathcal{B}_2 z &:= (\Delta z)_\nu + (1-\mu)(B_2 z)_\tau \end{aligned} \tag{1.6}$$

where

$$\begin{aligned} B_1 z &:= 2\nu_1\nu_2 z_{xy} - \nu_1^2 z_{yy} - \nu_2^2 z_{xx}, \\ B_2 z &:= (\nu_1^2 - \nu_2^2) z_{xy} + \nu_1\nu_2 (z_{yy} - z_{xx}). \end{aligned} \tag{1.7}$$

Here $\mu \in (0, 1/2)$ is the *viscoelastic Poisson ratio* and $(\cdot)_\tau$ stands for he derivative in the τ-direction. From a mechanical viewpoint, boundary conditions (1.3)–(1.5) say that the plate is clamped on Γ_0 and free of bending and twisting on Γ_1.

The *direct* problem of finding u satisfying (1.1)–(1.5) has already been studied both from the well-posedness and controllability viewpoints (see [**2, 9, 10, 11, 13**]). Concerning applications, a further interesting problem consists in identifying the kernel D via some additional information. This kind of inverse problem has been analyzed in [**2**] (see also [**3, 7**] for similar problems for three-dimensional bodies). There, two possible supplementary measurements are considered, i.e., the bending or the twisting moment exerted on a given subset $\widetilde{\Gamma} \subseteq \Gamma_0$ with positive measure. This corresponds to assuming that, cf., [**2**, Equation (1.9)],

$$\Phi^1[D, u] := \int_{\widetilde{\Gamma}} \{D(0)\mathcal{B}_1 u + D' * \mathcal{B}_1 u\}\, d\Gamma = g^1 \quad \text{in } [0, T] \tag{1.8}$$

for the bending moment, and, cf., [**2**, Equation (7.29)],

$$\Phi^2[D, u] := \int_{\widetilde{\Gamma}} \{D(0)\mathcal{B}_2 u + D' * \mathcal{B}_2 u - \frac{h^3}{12} u''_\nu\}\, d\Gamma = g^2 \quad \text{in } [0, T] \tag{1.9}$$

for the twisting moment. Here $d\Gamma$ denotes the Lebesgue measure on Γ and $g^i : [0, T] \to \mathbf{R}$ are known functions, $i = 1, 2$. To be more precise, we remark that g^2 may also depend on a *shear* force which acts along the vertical direction. Also, it is worth noting that, owing to (1.3), the term $-(h^3/12)u''_\nu$ in (1.9) disappears.

The inverse problem of finding (u, D) satisfying (1.1)–(1.5) and (1.8) (or (1.9)) is investigated in [**2**]. In particular, local (in time) existence, uniqueness, and continuous dependence estimates are obtained. Here we mainly prove a global existence and uniqueness result, i.e., on the whole $[0, T]$, taking advantage of the technique devised in [**1**]. Also, we consider further possible additional information which consists in measuring the vertical deflection at a fixed point $x_0 \in \Omega \cup \overline{\Gamma}_1$, for any $t \in [0, T]$, i.e.,

$$\Phi^3[D, u](t) := u(x_0, t) = g^3(t), \quad \forall\, t \in [0, T], \tag{1.10}$$

where $g^3 : [0, T] \to \mathbf{R}$ is given.

To sum up, this paper is concerned with the study of

Problem (IP_i). *Find* u *and* D *satisfying* (1.1)–(1.5) *and*

$$\Phi^i[D,u] = g^i \quad in\ [0,T], \quad i \in \{1,2,3\}. \tag{1.11}$$

Our main result ensures that, for any $i = 1,2,3$, (IP_i) has a unique solution on the whole $[0,T]$, provided that Γ, u_0, u_1, f and g^i are smooth enough.

More precisely, the plan goes as follows. In Section 2 we recall some results about the well-posedness of the so-called *direct* problem, i.e., finding u satisfying (1.1)–(1.5) provided that D is given. Section 3 contains our main results. In Sections 4 and 5, we show that rigorous formulations of (IP_i) are equivalent to a system of nonlinear integral equations in fixed-point form. This system is solved in Section 6. Section 7 is devoted to discussing continuous dependence on data, while in Section 8 a basic technical lemma used in Section 6 is proved.

2. The direct problem. Here we recall some results concerning the well-posedness of what we call the *direct* problem, i.e., the initial and boundary value problem (1.1)–(1.5) where D is given. These results turn out to be useful in Section 4 for a better understanding of the inverse problem formulations (see Remark 4.1 below).

Let us introduce first some notation. The space of all functions $z : [0,T] \to X$, X being a real Banach space, which are strongly continuous along with their first m time derivatives is denoted by $C^m([0,T];X)$. Endowing it with the norm

$$\|z\|_{m,X} := \sum_{i=0}^{m} \operatorname{ess} \sup_{t\in[0,T]} \|z^{(i)}(t)\|_X$$

we obtain a Banach space. Besides, if $X \equiv \mathbf{R}$, then we set $\|\cdot\|_m := \|\cdot\|_{m,X}$.

Consider now $H := L^2(\Omega)$ and define

$$V_1 := \{v \in H^1(\Omega) \mid v = 0 \quad \text{on } \Gamma_0\} \tag{2.1}$$

$$V_2 := \{w \in H^2(\Omega) \mid v = v_\nu = 0 \quad \text{on } \Gamma_0\} \tag{2.2}$$

$$V_3 := H^3(\Omega) \cap V_2 \tag{2.3}$$

$$V_4 := H^4(\Omega) \cap V_2. \tag{2.4}$$

We endow V_1 and V_2 with the inner products

$$c(v, \tilde{v}) := h \int_\Omega v\tilde{v} + \gamma \int_\Omega \nabla v \cdot \nabla \tilde{v}, \quad \forall\, v, \tilde{v} \in V_1 \tag{2.5}$$

$$\begin{aligned} a(w, \tilde{w}) := \int_\Omega \{ & w_{xx}\tilde{w}_{xx} + w_{yy}\tilde{w}_{yy} + \mu(w_{xx}\tilde{w}_{yy} + w_{yy}\tilde{w}_{xx}) \\ & + 2(1-\mu)w_{xy}\tilde{w}_{xy}\}\, dx\, dy, \quad \forall\, w, \tilde{w} \in V_2. \end{aligned} \tag{2.6}$$

Here $\gamma := h^3/12$ and the dot stands for the usual scalar product in $\mathbf{R}^2$. Also, we recall that

$$V_2 \hookrightarrow V_1 \hookrightarrow H \hookrightarrow V_1' \hookrightarrow V_2' \tag{2.7}$$

with dense and compact injections, V_1' and V_2' being the dual spaces of V_1 and V_2.

The bilinear form $c(\cdot,\cdot)$ is obviously coercive and the same holds for $a(\cdot,\cdot)$ by virtue of Korn's lemma, as $|\Gamma_0| > 0$ (see, e.g., [**4**, Chapter III, Section 3.3]). On account of that, we introduce the linear operators $C : V_1 \to V_1'$ and $A : V_2 \to V_2'$ by setting

$$\langle Cv, \tilde{v}\rangle := c(v, \tilde{v}), \quad \forall\, v, \tilde{v} \in V_1 \tag{2.8}$$

$$\langle Aw, \tilde{w}\rangle := a(w, \tilde{w}), \quad \forall\, w, \tilde{w} \in V_2, \tag{2.9}$$

and we remark that C and A are nothing but the canonical isomorphisms of V_1 onto V_1' and of V_2 onto V_2'.

Assume now

$$D \in C^2([0,T]) \tag{2.10}$$

$$D(0) > 0. \tag{2.11}$$

Suppose, moreover,

$$\overline{\Gamma}_0 \cap \overline{\Gamma}_1 = \varnothing \tag{2.12}$$

$$\Gamma \in \mathcal{C}^4 \tag{2.13}$$

$$F^1 \in C^1([0,T]; V_1') \tag{2.14}$$

$$C^{-1}F^2 \in C^0([0,T]; V_2) \tag{2.15}$$

$$w_0 \in V_3, \qquad \mathcal{B}_1 w_0 = 0 \quad \text{on } \Gamma_1 \tag{2.16}$$

$$w_1 \in V_2 \tag{2.17}$$

and consider

Problem (DP). *Find* $u : [0,T] \to V_2$ *satisfying*

$$u \in C^0([0,T]; V_3) \cap C^1([0,T]; V_2) \cap C^2([0,T]; V_1) \tag{2.18}$$

$$\begin{gathered} Cu''(t) + A(D(0)u + D' * u)(t) = (F^1 + F^2)(t) \\ \text{in } V_2', \quad \forall\, t \in (0,T) \end{gathered} \tag{2.19}$$

$$u(0) = w_0, \qquad u'(0) = w_1. \tag{2.20}$$

Hence we have, cf., [**2**, Theorem 3.1],

Theorem 2.1. *Let assumptions* (2.12)–(2.17) *hold. Then, for any* D *fulfilling* (2.10)–(2.11), *Problem* (DP) *admits a unique solution.*

Remark 2.1. Problem (DP) is a variational formulation of the initial and boundary value problem (1.1)–(1.5). Existence and uniqueness of weaker solutions can be proved under less stringent hypotheses on the data and, in particular, on Γ, see [**10**], [**11**, Chapter 6, Sections 1, 2]. Nevertheless, in our approach, u has to be sufficiently smooth (see next sections). Thus, in the presence of mixed boundary conditions, i.e., $|\Gamma_1| > 0$, condition (2.12) imposes some restrictions on Ω, which hold, e.g., for annular plates.

Strengthening a bit the assumptions (2.13)–(2.17), one can prove that the solution u given by Theorem 2.1 is in fact a *strong* solution, i.e., it fulfills equations (1.1)–(1.5) almost everywhere. Let

$$\Gamma \in \mathcal{C}^5 \tag{2.21}$$

$$C^{-1}F^1 \in C^1([0,T];V_2) \tag{2.22}$$

$$C^{-1}F^2 \in C^0([0,T];V_3) \tag{2.23}$$

$$\begin{gathered} w_0 \in V_4, \\ \mathcal{B}_1 w_0 = \mathcal{B}_2 w_0 = [C^{-1}(-D(0)Aw_0 + f(0))]_\nu = 0 \quad \text{on } \Gamma_1 \end{gathered} \tag{2.24}$$

$$w_1 \in V_3, \qquad \mathcal{B}_1 w_1 = 0 \quad \text{on } \Gamma_1. \tag{2.25}$$

It is worth remarking that the compatibility relation on Γ_1 deriving from (1.5) reads

$$D(0)\mathcal{B}_2 w_0 = \frac{h^3}{12}[C^{-1}(-D(0)Aw_0 + f(0))]_\nu \quad \text{on } \Gamma_1.$$

To obtain that, one has to use equation (2.19). Clearly, this relationship holds because of the boundary conditions written in (2.24). Our choice has been made just for sake of simplicity. Moreover, observe that in [**2**] the compatibility relations coming from (1.4)–(1.5) were completely forgotten.

The regularity result reads, cf. [**2**, Theorem 7.2], see also [**11**, Chapter 6, Section 2],

Theorem 2.2. *Let assumptions* (2.12) *and* (2.21)–(2.25) *hold. Then, for any D fulfilling* (2.10)–(2.11)*, the unique solution u to Problem* (DP) *satisfies*

$$u \in C^0([0,T];V_4) \cap C^1([0,T];V_3) \cap C^2([0,T];V_2). \tag{2.26}$$

Remark 2.2. Standard arguments based on Green formulas, see, e.g., [**2**] or [**10**], allow us to check that if u solves Problem (DP) and fulfills (2.26), then $u(t)$ satisfies (1.1) almost everywhere in Ω, (1.3) almost everywhere on Γ_0 and (1.4)–(1.5) almost everywhere on Γ_1, for any $t \in [0,T]$.

3. Main results. Before stating the main theorems, we give a rigorous formulation of Problem (P_i) for $i = 1, 2, 3$.

Consider first Problem (P_1). Assume

$$f = f^1 + f^2 \in C^3([0,T]; V_1') + C^2([0,T]; H) \tag{3.1}$$

$$C^{-1} f^2 \in C^2([0,T]; V_2) \tag{3.2}$$

$$u_0 \in V_4, \quad u_1 \in V_3 \tag{3.3}$$

$$C^{-1} A u_0 \in V_2 \tag{3.4}$$

$$\begin{gathered} C^{-1}(-D_0^1 A u_0 + f(0)) \in V_3, \\ \mathcal{B}_1 C^{-1}(-D_0^1 A u_0 + f(0)) = 0 \quad \text{on } \Gamma_1 \end{gathered} \tag{3.5}$$

$$C^{-1}(-D_0^1 A u_1 + f'(0)) \in V_2 \tag{3.6}$$

$$g^1 \in C^2([0,T]) \tag{3.7}$$

$$\begin{aligned} &\mathcal{B}_1 u_0 = \mathcal{B}_1 u_1 = 0, \\ &\mathcal{B}_2 u_0 = [C^{-1}(-D_0^1 A u_0 + f(0))]_\nu = 0 \quad \text{on } \Gamma_1 \end{aligned} \tag{3.8}$$

$$\delta_1 := \int_{\widetilde{\Gamma}} \mathcal{B}_1 u_0 \, d\Gamma \neq 0 \tag{3.9}$$

$$D_0^1 := g^1(0)\delta_1^{-1} > 0 \tag{3.10}$$

and set

$$D_1^1 := \delta_1^{-1}\left((g^1)'(0) - D_0^1 \int_{\widetilde{\Gamma}} \mathcal{B}_1 u_1 \, d\Gamma\right). \tag{3.11}$$

Then we introduce

Problem (P_1). *Find* $u : [0,T] \to V_2$ *and* $D : [0,T] \to \mathbf{R}$ *satisfying*

$$u \in C^2([0,T];V_3) \cap C^3([0,T];V_2) \cap C^4([0,T];V_1) \tag{3.12}$$

$$D \in C^2([0,T]) \tag{3.13}$$

$$\begin{gathered} Cu''(t) + A(D(0)u + D' * u)(t) = f(t) \\ \text{in} \quad V_2', \quad \forall\, t \in (0,T) \end{gathered} \tag{3.14}$$

$$u(0) = u_0, \qquad u'(0) = u_1 \tag{3.15}$$

$$D(0) = D_0^1, \qquad D'(0) = D_1^1 \tag{3.16}$$

$$\Phi^1[D,u] = g^1 \quad \text{in } [0,T]. \tag{3.17}$$

Remark 3.1. Initial conditions (3.16) are derived from conditions (3.9)–(3.10) and position (3.11). More precisely, they come out by setting $t = 0$ in equation (3.17) and in the one obtained by differentiating (3.17) in time, cf. (3.10)–(3.11). A similar remark holds for Problems (P_2) and (P_3), see below (3.27)–(3.28), (3.33) and (3.40)–(3.41), (3.46), respectively.

Concerning Problem (P_1), we show

Theorem 3.1. *Let (2.12)–(2.13) and (3.1)–(3.10) hold. Then Problem (P_1) has one and only one solution.*

Problems (P_2) and (P_3) require stronger regularity hypotheses on the data. Indeed, regarding Problem (P_2), we suppose

$$f = f^1 + f^2 \in C^3([0,T]; V_1') + C^2([0,T]; H) \tag{3.18}$$

$$C^{-1} f^1 \in C^3([0,T]; V_2) \tag{3.19}$$

$$C^{-1} f^2 \in C^2([0,T]; V_3) \tag{3.20}$$

$$\begin{aligned} u_0, u_1 \in V_4, \quad \mathcal{B}_1 u_0 = \mathcal{B}_1 u_1 = 0 \quad \text{on } \Gamma_1 \\ C^{-1} A u_0 \in V_3, \quad C^{-1} A u_1 \in V_2, \qquad \mathcal{B}_1 C^{-1} A u_0 = 0 \quad \text{on } \Gamma_1 \end{aligned} \tag{3.21}$$

$$\begin{aligned} \mathcal{B}_2 u_0 = [C^{-1}(-D_0^2 A u_0 + f(0))]_\nu = 0 \quad \text{on } \Gamma_1 \\ \mathcal{B}_2 u_1 = [C^{-1}(-D_0^2 A u_1 - D_1^2 A u_0 + f'(0))]_\nu = 0 \quad \text{on } \Gamma_1 \end{aligned} \tag{3.22}$$

(3.23)

$$\begin{aligned} & C^{-1}(-D_0^2 A u_0 + f(0)) \in V_4, \\ & \mathcal{B}_r C^{-1}(-D_0^2 A u_0 + f(0)) = 0 \quad \text{on } \Gamma_1 \\ & \qquad [C^{-1} A u_0]_\nu = [C^{-1}(-D_1^2 A u_1 + f''(0))]_\nu = 0 \quad \text{on } \Gamma_1 \end{aligned}$$

$$\begin{aligned} C^{-1}(-D_0^2 A u_1 + f'(0)) \in V_3, \\ \mathcal{B}_1 C^{-1}(-D_0^2 A u_1 + f'(0)) = 0 \quad \text{on } \Gamma_1 \end{aligned} \tag{3.24}$$

$$g^2 \in C^2([0,T]) \tag{3.25}$$

$$\delta_2 := \int_{\widetilde{\Gamma}} \mathcal{B}_2 u_0 \, d\Gamma \neq 0 \tag{3.26}$$

$$D_0^2 := g^2(0)\delta_2 > 0 \tag{3.27}$$

where $r = 1, 2$, and set

$$D_1^2 := \delta_2^{-1}\left((g^2)'(0) - D_0^2 \int_{\widetilde{\Gamma}} \mathcal{B}_2 u_1 \, d\Gamma\right). \tag{3.28}$$

Then Problem (P_2) can be formulated as

Problem (P_2). *Find* $u : [0,T] \to V_2$ *and* $D : [0,T] \to \mathbf{R}$ *satisfying*

$$u \in C^2([0,T];V_4) \cap C^3([0,T];V_3) \cap C^4([0,T];V_2) \tag{3.29}$$

$$D \in C^2([0,T]) \tag{3.30}$$

$$\begin{gathered} Cu''(t) + A(D(0)u + D' * u)(t) = f(t) \\ \textit{in } V_2', \quad \forall\, t \in (0,T) \end{gathered} \tag{3.31}$$

$$u(0) = u_0, \qquad u'(0) = u_1 \tag{3.32}$$

$$D'(0) = D_1^2, \qquad D(0) = D_0^2 \tag{3.33}$$

$$\Phi^2[D,u] = g^2 \quad \textit{in } [0,T]. \tag{3.34}$$

Our result reads

Theorem 3.2. *Let* (2.12), (2.21) *and* (3.18)–(3.27) *hold. Then Problem* (P_2) *has one and only one solution.*

Consider now Problem (P_3). Let

(3.35)

$$\begin{gathered} C^{-1}(-D_0^3 A u_0 + f(0)) \in V_4, \\ \mathcal{B}_r C^{-1}(-D_0^3 A u_0 + f(0)) = 0 \quad \text{on } \Gamma_1 \\ [C^{-1} A u_0]_\nu = [C^{-1}(-D_1^3 A u_1 + f''(0))]_\nu = 0 \quad \text{on } \Gamma_1 \end{gathered}$$

$$C^{-1}(-D_0^3 Au_1 + f'(0)) \in V_3, \quad \mathcal{B}_1 C^{-1}(-D_0^3 Au_1 + f'(0)) = 0 \quad \text{on } \Gamma_1 \tag{3.36}$$

$$g^3 \in C^4([0,T]) \tag{3.37}$$

$$g^3(0) = u_0(x_0), \qquad (g^3)'(0) = u_1(x_0) \tag{3.38}$$

$$\delta_3 := (C^{-1}Au_0)(x_0) \neq 0 \tag{3.39}$$

$$D_0^3 := \delta_3^{-1}((C^{-1}f)(x_0,0) - (g^3)''(0)) > 0 \tag{3.40}$$

where $r = 1, 2$, and set

$$D_1^3 := \delta_3^{-1}((C^{-1}f')(x_0,0) - D_0^3(C^{-1}Au_1)(x_0) - (g^3)^{(3)}(0)). \tag{3.41}$$

We have

Problem (P_3). *Find* $u : [0,T] \to V_2$ *and* $D : [0,T] \to \mathbf{R}$ *satisfying*

$$u \in C^2([0,T]; V_4) \cap C^3([0,T]; V_3) \cap C^4([0,T]; V_2) \tag{3.42}$$

$$D \in C^2([0,T]) \tag{3.43}$$

$$Cu''(t) + A(D(0)u + D' * u)(t) = f(t) \quad \textit{in } V_2', \quad \forall\, t \in (0,T) \tag{3.44}$$

$$u(0) = u_0, \qquad u'(0) = u_1 \tag{3.45}$$

$$D(0) = D_0^3, \qquad D'(0) = D_1^3 \tag{3.46}$$

$$\Phi^3[D,u] = g^3 \quad \textit{in } [0,T]. \tag{3.47}$$

Existence and uniqueness is given by

Theorem 3.3. *Let* (2.12)*,* (2.21)*,* (3.18)–(3.21) *and* (3.35)–(3.40) *hold. Moreover, let* (3.22) *hold with* D_i^3 *in place of* D_i^2*,* $i = 0, 1$*. Then Problem* (P_3) *has one and only one solution.*

Remark 3.2. If, for example, u_0 identically vanishes so that (3.9) fails, then Theorem 3.1 still holds provided that smoother data are considered and u_1 (or f) plays the role of u_0 in (3.9) (see [**2**, Section 7, Remark 7.1]). Similar remarks can be made for (3.26) and Theorem 3.2 or (3.39) and Theorem 3.3.

Remark 3.3. Note that, owing to the regularity of u and f, from equation (3.14) (or (3.31) or (3.44)) we can deduce that equation (1.1) is satisfied almost everywhere in $\Omega \times (0, T)$ and the same holds for the boundary conditions (1.3)–(1.5), cf. Remark 2.2.

Remark 3.4. Analogous results can be formulated for the viscoelastic beam model considered in [**2**, Section 7, Remark 7.4]. However, it is worth noting that in Problem (P_3) for the beam we can look for solutions u satisfying (3.12) instead of (3.42); consequently, the regularity assumptions (3.18)–(3.22) and (3.35)–(3.36) can be substituted with (3.1)–(3.5). In fact, referring to Section 4, as we are in dimension one, to compute the trace of equation (4.6) at x_0 we just need the injection $H^1(\Omega) \hookrightarrow C^0(\overline{\Omega})$.

Remark 3.5. Some hypotheses which ensure the validity of compatibility relationships deriving from (1.5) are quite involved, cf., (3.23) and (3.35). Anyway, they can be avoided looking for weaker solutions in time (see Remark 6.2). Moreover, it is worth noting that if the plate is clamped along Γ, i.e., $\Gamma_1 = \varnothing$, then the assumption turns out to be simpler.

4. Equivalent problems. Here we show that Problems (P_i), $i = 1, 2, 3$, are equivalent to suitable Cauchy problems which can be further reduced to systems of nonlinear Volterra integral equations (see Section 5). All the equivalence results contained in this section and in the next one bear upon the nonvanishing conditions (3.9), (3.26) and (3.39).

Consider first Problem (P_1) and suppose it admits a solution (u, D). To find an equation for D'', we differentiate (3.17) twice in time. We have, cf. (1.8) and (3.15),

$$\left(\int_{\widetilde{\Gamma}} \mathcal{B}_1 u_0 \, d\Gamma\right) D'' = (g^1)'' - \left(\int_{\widetilde{\Gamma}} \mathcal{B}_1 u_1 \, d\Gamma\right) D' - \Phi^1[D, u''] \quad \text{in } [0,T]. \tag{4.1}$$

Then, taking (3.9) into account, equation (4.1) yields

$$\begin{aligned} D'' &= \Psi^1[D, D', u''] \\ &:= \delta_1^{-1} \left\{ (g^1)'' - \left(\int_{\widetilde{\Gamma}} \mathcal{B}_1 u_1 \, d\Gamma\right) D' - \Phi^1[D, u''] \right\} \quad \text{in } [0,T]. \end{aligned} \tag{4.2}$$

Regarding Problem (P_2), starting from (3.34) and arguing as before, we obtain, cf. also (1.9) and (3.32),

$$\left(\int_{\widetilde{\Gamma}} \mathcal{B}_2 u_0 \, d\Gamma\right) D'' = (g^2)'' - \left(\int_{\widetilde{\Gamma}} \mathcal{B}_2 u_1 \, d\Gamma\right) D' - \Phi^2[D, u''] \quad \text{in } [0,T]. \tag{4.3}$$

Using now (3.26), from equation (4.3) we infer

$$\begin{aligned} D'' &= \Psi^2[D, D', u''] \\ &:= \delta_2^{-1} \left\{ (g^2)'' - \left(\int_{\widetilde{\Gamma}} \mathcal{B}_2 u_1 \, d\Gamma\right) D' - \Phi^2[D, u''] \right\} \quad \text{in } [0,T]. \end{aligned} \tag{4.4}$$

In Problem (P_3), additional information (3.47) does not contain the delay kernel D explicitly (see (1.10)). Therefore, we have to derive an equation for D'' from (3.44). More precisely, we rewrite (3.44) as

$$u''(t) + C^{-1}A(D(0)u + D' * u)(t) = C^{-1}f(t) \quad \text{in } V_2', \quad \forall\, t \in (0,T). \tag{4.5}$$

Differentiating both sides of (4.5) twice in time, we obtain, cf. also (3.45),

$$\begin{aligned} &u^{(4)}(t) + C^{-1}A(D(0)u'' + D' * u'')(t) \\ &\quad = -(C^{-1}Au_0)D''(t) - (C^{-1}Au_1)D'(t) + C^{-1}f''(t) \end{aligned} \tag{4.6}$$

in V_2', for any $t \in (0,T)$.

Recalling that, in dimension two, we have $H^2(\Omega) \hookrightarrow C^0(\overline{\Omega})$, the regularity of u and the assumptions on u_0, u_1, f allow us to compute the pointwise trace of equation (4.6) at x_0. Then, on account of (3.47), we deduce, for any $t \in [0,T]$,

$$(4.7) \quad \begin{aligned}(C^{-1}Au_0)(x_0)D''(t) = (g^3)^{(4)}(t) \\ - (C^{-1}A(D(0)u''+D' * u''))(x_0,t) \\ - (C^{-1}Au_1)(x_0)D'(t)+(C^{-1}f'')(x_0,t).\end{aligned}$$

Hence, owing to (3.39) and (3.46), equation (4.7) becomes

$$(4.8) \quad \begin{aligned} D''(t) &= \Psi^3[D, D', u''] \\ &:= \delta_3^{-1}\{(g^3)^{(4)}(t) - (C^{-1}(AD_0^3 u'' + D' * u''))(x_0,t) \\ &\qquad - (C^{-1}Au_1)(x_0)D'(t) + (C^{-1}f'')(x_0,t)\} \\ &\qquad \forall\, t \in [0,T]. \end{aligned}$$

Observe that in the obtained equations for D'', cf. (4.2), (4.4) and (4.8), the *acceleration* field $w := u''$ appears. It is not difficult to check that w satisfies a direct problem which is quite similar to (3.14)–(3.15), or (3.31)–(3.32) or (3.44)–(3.45). In fact, differentiating twice in time the evolution equation for u and taking advantage of (3.8) and of the initial conditions for u and D, cf. (3.15)–(3.16), (3.32)–(3.33) and (3.45)–(3.46), we obtain, for any $i \in \{1,2,3\}$,

$$(4.9) \quad \begin{aligned} Cw''(t) + A(D_0^i w + D' * w)(t) = F(D', D'')(t) \\ \text{in } V_2', \quad \forall\, t \in (0,T) \end{aligned}$$

$$(4.10) \quad w(0) = w_0, \qquad w'(0) = w_1$$

where, cf. also [**2**, Equations (4.5)–(4.7)],

$$(4.11) \quad F(D', D'') := -Au_0 D'' - Au_1 D' + f''$$

$$(4.12) \quad w_0 := C^{-1}(-D_0^i Au_0 + f(0))$$

$$(4.13) \quad w_1 := C^{-1}(-D_0^i Au_1 - D_1^i Au_0 + f'(0)).$$

Consider now Problem (P_1). We have just shown that the pair (w, D) solves, cf. (4.2), (4.9)–(4.10),

Problem (P_1^1). *Find* $w : [0,T] \to V_2$ *and* $D : [0,T] \to \mathbf{R}$ *satisfying*

$$w \in C^0([0,T];V_3) \cap C^1([0,T];V_2) \cap C^2([0,T];V_1) \tag{4.14}$$

$$D \in C^2([0,T]) \tag{4.15}$$

$$Cw''(t) + A(D_0^1 w + D' * w)(t) = F(D', D'')(t) \quad \text{in } V_2', \quad \forall\, t \in (0,T) \tag{4.16}$$

$$D'' = \Psi^1[D, D', w] \quad \text{in } [0,T] \tag{4.17}$$

$$w(0) = w_0, \qquad w'(0) = w_1 \tag{4.18}$$

$$D(0) = D_0^1, \qquad D'(0) = D_1^1. \tag{4.19}$$

Analogously, taking (4.4), (4.9)–(4.10) and (4.8)–(4.10) into account, respectively, we introduce

Problem (P_2^1). *Find* $w : [0,T] \to V_2$ *and* $D : [0,T] \to \mathbf{R}$ *satisfying*

$$w \in C^0([0,T];V_4) \cap C^1([0,T];V_3) \cap C^2([0,T];V_2) \tag{4.20}$$

$$D \in C^2([0,T]) \tag{4.21}$$

$$Cw''(t) + A(D_0^2 w + D' * w)(t) = F(D', D'')(t) \quad \text{in } V_2', \quad \forall\, t \in (0,T) \tag{4.22}$$

$$D'' = \Psi^2[D, D', w] \quad \text{in } [0,T] \tag{4.23}$$

$$w(0) = w_0, \qquad w'(0) = w_1 \tag{4.24}$$

$$D(0) = D_0^2, \qquad D'(0) = D_1^2. \tag{4.25}$$

Problem (P_3^1). *Find* $w : [0,T] \to V_2$ *and* $D : [0,T] \to \mathbf{R}$ *satisfying*

$$w \in C^0([0,T]; V_4) \cap C^1([0,T]; V_3) \cap C^2([0,T]; V_2) \tag{4.26}$$

$$D \in C^2([0,T]) \tag{4.27}$$

$$\begin{gathered} Cw''(t) + A(D_0^3 w + D' * w)(t) = F(D', D'')(t) \\ \text{in } V_2', \quad \forall\, t \in (0,T) \end{gathered} \tag{4.28}$$

$$D'' = \Psi^3[D, D', w] \quad \text{in } [0,T] \tag{4.29}$$

$$w(0) = w_0, \qquad w'(0) = w_1 \tag{4.30}$$

$$D(0) = D_0^3, \qquad D'(0) = D_1^3. \tag{4.31}$$

Conversely, if for any $i \in \{1,2,3\}$, Problem (P_i^1) admits a unique solution, then setting

$$u(t) := u_0 + tu_1 + \int_0^t (t-s) w(s)\, ds \quad \forall\, t \in [0,T] \tag{4.32}$$

one can realize that the pair (u, D) solves Problem (P_i), cf. (3.10)–(3.11), (3.27)–(3.28), (3.40)–(3.41).

To sum up, we have

Proposition 4.1. *For any* $i \in \{1,2,3\}$, *let the assumptions of Theorem* 3.i *hold. Then Problem* (P_i) *has a unique solution* (u, D)

if and only if Problem (P_i^1) *has a unique solution* (w, D)*. Moreover,* u *and* w *are related by* (4.32)*.*

Remark 4.1. Theorems 2.1–2.2 tell us what kind of regularity hypotheses on w_0, w_1 and $F(D', D'')$ we have to assume. In fact, consider for instance Problem (P_1). Then, on account of (4.11)–(4.13), it suffices to compare (2.14)–(2.17) with (3.1)–(3.5). Regarding (P_2) and (P_3), strong solutions to the Cauchy problem (4.9)–(4.10) are needed, so that we have to use Theorem 2.2.

5. Reduction to systems of integral equations. Our formulation of the equivalent Problems (P_i^1) as systems of integral equations requires a representation formula of the solution to the Cauchy problem (4.9)–(4.10). To obtain that, we further transform equation (4.9) and conditions (4.10), see [**2**, Section 3].

For any $i \in \{1, 2, 3\}$, let us set, cf. (3.16), (3.33) and (3.46),

$$z = D_0^i w + D' * w \quad \text{in } \Omega \times (0, T) \tag{5.1}$$

and observe that

$$D_0^i w = z + R * z \quad \text{in } \Omega \times (0, T) \tag{5.2}$$

where R is the *resolvent* kernel of $(D_0^i)^{-1} D'$, cf. (3.9)–(3.10), (3.26)–(3.27) and (3.39)–(3.40). We recall that R solves

$$D_0^i R + R * D' = -D' \quad \text{in } [0, T]. \tag{5.3}$$

Therefore, setting $t = 0$ in (5.3) we obtain, cf. (3.16), (3.33) and (3.46),

$$R(0) = R_0^i = -(D_0^i)^{-1} D_1^i. \tag{5.4}$$

Moreover, on account of (5.4), differentiating (5.3) in time we have

$$R' = -(D_0^i)^{-1}(D'' + R_0^i D' + R' * D') \quad \text{in } [0, T]. \tag{5.5}$$

By using (5.1)–(5.2) and (5.4), Cauchy problem (4.9)–(4.10) turns out to be

$$\begin{aligned} Cz''(t) + D_0^i Az(t) = {} & \mathcal{F}^1(R', z')(t) \\ & + \mathcal{F}^2(D', D'', R')(t) \quad \text{in } V_2' \quad \forall\, t \in (0, T) \end{aligned} \tag{5.6}$$

$$z(0) = z_0, \qquad z'(0) = z_1 \tag{5.7}$$

where, cf. (4.11)–(4.13),

$$\mathcal{F}^1(R', z') := -C(R_0^i z' + R' * z') \tag{5.8}$$

$$\mathcal{F}^2(D', D'', R') := -R' C z_0 + D_0^i F(D', D'') \tag{5.9}$$

$$z_0 := D_0^i w_0 \tag{5.10}$$

$$z_1 := D_0^i w_1 + D_1^i w_0 \tag{5.11}$$

for any $i \in \{1, 2, 3\}$.

Consider the linear operators $\mathbf{C}, \mathbf{A}$ defined by

$$\mathbf{C} := \begin{pmatrix} D_0^i A & 0 \\ 0 & C \end{pmatrix}, \qquad \mathbf{A} := \begin{pmatrix} 0 & -D_0^i A \\ D_0^i A & 0 \end{pmatrix}. \tag{5.12}$$

One can easily check that $\mathbf{C}$ and $\mathbf{A}$ are isomorphisms of $V_2 \times V_1$ onto $V_2' \times V_1'$ and of $V_2 \times V_2$ onto $V_2' \times V_2'$, respectively. On account of (5.12), introduce the linear operator $\mathbf{B} := \mathbf{C}^{-1}\mathbf{A} : \mathcal{D}(\mathbf{B}) \to V_2 \times V_1$ with

$$\mathcal{D}(\mathbf{B}) := \{\mathbf{z} = (z^1, z^2)^* \in V_2 \times V_1 \mid \mathbf{A}\mathbf{z} \in V_2' \times V_1'\} \hookrightarrow V_2 \times V_1$$

the superscript $*$ denoting the transposition. It is worth observing that, thanks to (2.12)–(2.13), we can characterize $\mathcal{D}(\mathbf{B})$ as, cf. [**12**, Remark 2.4],

$$\mathcal{D}(\mathbf{B}) \equiv \{\mathbf{z} \in V_2 \times V_2 \mid z^1 \in V_3,\ \mathcal{B}_1 z^1 = 0 \quad \text{on } \Gamma_1\}. \tag{5.13}$$

Then we can endow $\mathcal{D}(\mathbf{B})$ with the norm $\|\mathbf{z}\|_{\mathcal{D}(\mathbf{B})} := \|z^1\|_{V_3} + \|z^2\|_{V_2}$, which is equivalent to the graph norm.

Moreover, we remark that $\mathbf{B}$ is invertible, namely, there exists $\mathbf{B}^{-1} : V_2 \times V_1 \to \mathcal{D}(\mathbf{B})$, and one has

$$((\mathbf{B}^{-1}\mathbf{v})^1, (\mathbf{B}^{-1}\mathbf{v})^2) = ((D_0^i)^{-1} A^{-1} C v^2, -v^1) \tag{5.14}$$

for any $\mathbf{v} = (v^1, v^2)^* \in V_2 \times V_1$.

Taking advantage of (5.12), we now transform the second order equation (5.6) into a first order system by setting

$$\mathbf{z} := (z, z')^*. \tag{5.15}$$

More precisely, Cauchy problem (5.6)–(5.7) can be rewritten as

$$\mathbf{z}' + \mathbf{B}\mathbf{z} = \mathbf{F}^1(R', \mathbf{z}) + \mathbf{F}^2(D', D'', R') \quad \text{in } \Omega \times (0,T) \tag{5.16}$$

$$\mathbf{z}(0) = \mathbf{z}_0 \tag{5.17}$$

where, cf. (5.8)–(5.11),

$$\mathbf{F}^1(R', \mathbf{z}) := (0, C^{-1}\mathcal{F}^1(R', z'))^* \tag{5.18}$$

$$\mathbf{F}^2(D', D'', R') := (0, C^{-1}\mathcal{F}^2(D', D'', R'))^* \tag{5.19}$$

$$\mathbf{z}_0 := (z_0, z_1)^*. \tag{5.20}$$

On the other hand, we know that $-\mathbf{B}$ generates a strongly continuous semigroup of contractions, say $\{S(t)\}_{t\geq 0}$, on $V_2 \times V_1$, cf. [**12**, Theorem 2.1]. Therefore, we have

$$\mathbf{z}(t) = S(t)\mathbf{z}_0 + (S * [\mathbf{F}^1(R', \mathbf{z}) + \mathbf{F}^2(D', D'', R')])(t) \quad \forall\, t \in [0,T]. \tag{5.21}$$

Let us set

$$G := D'', \qquad P := R' \quad \text{in } [0,T] \tag{5.22}$$

and observe that, cf. (3.16), (3.33), (3.46),

$$D(t) = D_0^i + tD_1^i + \int_0^t (t-s)G(s)\, ds \tag{5.23}$$

$$R(t) = R_0^i + \int_0^t P(s)\, ds \tag{5.24}$$

for any $t \in [0,T]$ and $i \in \{1,2,3\}$. Then, on account of (4.2), (5.1), (5.5), (5.15) and (5.21)–(5.24), we can say that if (w, D) is a solution to Problem (P_1^1), then $(\mathbf{z}, G, P)$ is a solution to

Problem (P_1^2). *Find* $\mathbf{z} : [0,T] \to V_2 \times V_1$, $G : [0,T] \to \mathbf{R}$ *and* $P : [0,T] \to \mathbf{R}$ *satisfying*

$$\mathbf{z} \in C^0([0,T]; \mathcal{D}(\mathbf{B})) \cap C^1([0,T]; V_2 \times V_1) \tag{5.25}$$

$$G, P \in C^0([0,T]) \tag{5.26}$$

$$\mathbf{z} = S\mathbf{z}_0 + S * [\mathbf{F}^1(P, \mathbf{z}) + \mathbf{F}^2(D_1^1 + 1 * G, G, P)] \tag{5.27}$$

$$\begin{aligned} G &= \tilde{\Psi}^1[\mathbf{z}, G, P] \\ &:= \delta_1^{-1}\left\{(g^1)'' - \left(\int_{\widetilde{\Gamma}} \mathcal{B}_1 u_1 \, d\Gamma\right)(D_1^1 + 1 * G) - \int_{\widetilde{\Gamma}} \mathcal{B}_1 \mathbf{z}^1\right\} \\ &\qquad \text{in } [0,T] \end{aligned} \tag{5.28}$$

$$P = -(D_0^1)^{-1}[G + (R_0^1 + P*)(D_1^1 + 1 * G)] \quad \text{in } [0,T]. \tag{5.29}$$

Clearly, if $(\mathbf{z}, G, P)$ solves Problem (P_1^2), then (w, D) solves Problem (P_1^1), where, cf. (5.2),

$$w := (D_0^1)^{-1}(z^1 + (R_0^1 + 1 * P) * z^1) \quad \text{in } \Omega \times (0,T) \tag{5.30}$$

and D is given by (5.23) with $i = 1$.

Observe that $\tilde{\Psi}^1$ is well defined owing to the regularity of $\mathbf{z}$, cf. (1.8), (4.2), (5.13), (5.25) and (5.28). In comparison with Problem (P_1^1), both Problems (P_2^1) and (P_3^1) require a higher smoothness for w because of Ψ_2 and Ψ_3, cf. (1.9)–(1.10), (4.4), (4.8), see also [**2**, Section 7, Remark 7.2]. Hence, to formulate equivalent Problems (P_2^2) and (P_3^2), it is convenient to set

$$\mathbf{v} := \mathbf{B}\mathbf{z} \tag{5.31}$$

and to consider the Cauchy problem derived by (5.16)–(5.17), cf. [**2**, Equations (7.23)–(7.24)],

$$\mathbf{v}' + \mathbf{B}\mathbf{v} = \mathbf{B}[\mathbf{F}^1(R', \mathbf{B}^{-1}\mathbf{v}) + \mathbf{F}^2(D', D'', R')] \quad \text{in } \Omega \times (0,T) \tag{5.32}$$

$$\mathbf{v}(0) = \mathbf{v}_0 := \mathbf{B}_0 \mathbf{z}_0. \tag{5.33}$$

Therefore, we have, cf. (5.21),

$$\mathbf{v}(t) = S(t)\mathbf{v}_0 + (S * \mathbf{B}[\mathbf{F}^1(R', \mathbf{B}^{-1}\mathbf{v}) + \mathbf{F}^2(D', D'', R')])(t) \tag{5.34}$$

for any $t \in [0,T]$.

Hence, recalling (4.4), (4.8), (4.23), (4.29), (5.1), (5.5), (5.14), (5.22)–(5.24), and taking (5.31)–(5.34) into account, if (w, D) solves Problem (P_i), $i = 2, 3$, then $(\mathbf{v}, G, P)$ solves, respectively,

Problem (P_2^2). *Find* $\mathbf{v} : [0,T] \to V_2 \times V_1$, $G : [0,T] \to \mathbf{R}$, *and* $P : [0,T] \to \mathbf{R}$ *satisfying*

$$\mathbf{v} \in C^0([0,T] : \mathcal{D}(\mathbf{B})) \cap C^1([0,T]; V_2 \times V_1) \tag{5.35}$$

$$G, P \in C^0([0,T]) \tag{5.36}$$

$$\mathbf{v} = S\mathbf{v}_0 + S * \mathbf{B}[\mathbf{F}^1(P, \mathbf{B}^{-1}\mathbf{v}) + \mathbf{F}^2(D_1^1 + 1 * G, G, P)] \tag{5.37}$$

$$G = \tilde{\Psi}^2[\mathbf{v}, G, P] := \delta_2^{-1}\left\{(g^2)'' - \left(\int_{\widetilde{\Gamma}} \mathcal{B}_2 u_1 \, d\Gamma\right)(D_1^2 + 1 * G) - \int_{\widetilde{\Gamma}} \mathcal{B}_1 D_0^2 (A^{-1}C)\mathbf{v}^2\right\} \quad \textit{in } [0,T] \tag{5.38}$$

$$P = -(D_0^2)^{-1}[G + (R_0^2 + P*)(D_1^2 + 1 * G)] \quad \textit{in } [0,T]. \tag{5.39}$$

Problem (P_3^2). *Find* $\mathbf{v} : [0,T] \to V_2 \times V_1$, $G : [0,T] \to \mathbf{R}$ *and* $P : [0,T] \to \mathbf{R}$ *satisfying*

$$\mathbf{v} \in C^0([0,T]; \mathcal{D}(\mathbf{B})) \cap C^1([0,T]; V_2 \times V_1) \tag{5.40}$$

$$G, P \in C^0([0,T]) \tag{5.41}$$

$$\mathbf{v} = S\mathbf{v}_0 + S * \mathbf{B}[\mathbf{F}^1(P, \mathbf{B}^{-1}\mathbf{v}) + \mathbf{F}^2(D_1^1 + 1 * G, G, P)] \tag{5.42}$$

(5.43)
$$\begin{aligned} G &= \tilde{\Psi}^3[\mathbf{v}, G, P] \\ &:= \delta_3^{-1}\{(g^3)^{(4)} - D_0^3\mathbf{v}^2(x_0,\cdot) \\ &\qquad - (C^{-1}Au_1)(x_0)(D_1^3 + 1*G) + (C^{-1}f'')(x_0,\cdot)\} \quad \textit{in } [0,T] \end{aligned}$$

$$P = -(D_0^3)^{-1}[G + (R_0^3 + P*)(D_1^3 + 1 * G)] \quad \textit{in } [0,T]. \tag{5.44}$$

Conversely, if $(\mathbf{v}, G, P)$ is a solution to Problem (P_i^2), $i = 2, 3$, recalling [**2**, Section 7, Remark 7.2], one can also prove that (w, D) is a solution to Problem (P_i^1), where, cf. (5.14) and (5.30)–(5.31),

$$w := A^{-1}C(v^2 + (R_0^i + 1 * P) * v^2) \quad \text{in } \Omega \times (0,T) \tag{5.45}$$

and D is given by (5.23) with $i \in \{2, 3\}$.

On account of Proposition 4.1, we can summarize our equivalence results as

Proposition 5.1. *Let the assumptions of Theorem* 3.1 *hold. Then Problem* (P_1) *has a unique solution* (u, D) *if and only if Problem* (P_1^2) *has a unique solution* $(\mathbf{z}, G, P)$*. Let the assumptions of Theorem* 3.i*,* $i = 2, 3$*, hold. Then Problem* (P_i) *has a unique solution* (u, D) *if and only if Problem* (P_i^2) *has a unique solution* $(\mathbf{v}, G, P)$*.*

We are now in a position to formulate problems (P_i^2), $i = 1, 2, 3$, in a fixed-point form. Indicate, for the sake of simplicity, by $\mathbf{U}$ the possible solution to Problem (P_i^2), that is,

$$\mathbf{U} := (U^1, U^2, U^3, U^4)^* \tag{5.46}$$

where, for Problem (P_1^2),

$$U^1 := z^1, \qquad U^2 := z^2, \qquad U^3 := G, \qquad U^4 := P, \tag{5.47}$$

while, for Problem (P_2^2) or (P_3^2),

$$U^1 := v^1, \qquad U^2 := v^2, \qquad U^3 := G, \qquad U^4 := P, \tag{5.48}$$

and set, cf. (5.14), (5.27), (5.37),

$$(J_1^1(\mathbf{U}), J_1^2(\mathbf{U}))^* := S * (0, -(R_0^1 + U^4 *)U^2 - U^4 z_0 \\ - D_0^1(C^{-1}Au_0U^3 + C^{-1}Au_1(1 * U^3))^* \tag{5.49}$$

$$(K_1^1, K_1^2)^* := S\mathbf{z}_0 + S * (0, C^{-1}(f'' - D_1^1 Au_1))^* \tag{5.50}$$

$$(J_i^1(\mathbf{U}), J_i^2(\mathbf{U}))^* := S * ((R_0^i + U^4 *)U^1 + U^4 z_0 \\ + D_0^i(C^{-1}Au_0U^3 + C^{-1}Au_1(1 * U^3), 0)^* \tag{5.51}$$

$$(K_i^1, K_i^2)^* := S\mathbf{B}\mathbf{z}_0 + S * (-C^{-1}(f'' - D_1^i Au_1), 0)^* \tag{5.52}$$

where $i = 2, 3$. Using the assumptions of Theorem 3.i and standard properties of S, cf., e.g., [**6**, Chapter 1], one can check that (see also (8.1)–(8.5) below)

$$(J_i^1(\mathbf{U}), J_i^2(\mathbf{U}))^* \in C^0([0,T]; \mathcal{D}(\mathbf{B})) \tag{5.53}$$

$$(K_i^1, K_i^2)^* \in C^0([0,T]; \mathcal{D}(\mathbf{B})) \tag{5.54}$$

for any $\mathbf{U} \in C^0([0,T]; \mathcal{D}(\mathbf{B}))$, $i = 1, 2, 3$.

On account of (5.49), (5.51), (5.53)–(5.54), relationships (5.28), (5.38) and (5.43) lead us to introduce

$$(J_1^3(\mathbf{U}) := -\delta_1^{-1}\left\{ \left(\int_{\widetilde{\Gamma}} \mathcal{B}_1 u_1 \, d\Gamma \right)(1 * U^3) + \int_{\widetilde{\Gamma}} \mathcal{B}_1 J_1^1(\mathbf{U}) \, d\Gamma \right\} \tag{5.55}$$

$$J_2^3(\mathbf{U}) := -\delta_2^{-1}\bigg\{\bigg(\int_{\widetilde{\Gamma}} \mathcal{B}_2 u_1\, d\Gamma\bigg)(1 * U^3) + \int_{\widetilde{\Gamma}} \mathcal{B}_2 D_0^2 (A^{-1}C) J_2^2(\mathbf{U})\, d\Gamma\bigg\} \tag{5.56}$$

$$J_3^3(\mathbf{U}) := -\delta_3^{-1}\{D_0^3 J_3^2(\mathbf{U})(x_0,\cdot) + (C^{-1}Au_1)(x_0)(1 * U^3)\} \tag{5.57}$$

and

$$K_1^3 := \delta_1^{-1}\bigg\{(g^1)'' - D_1^1 \int_{\widetilde{\Gamma}} \mathcal{B}_1 u_1\, d\Gamma - \int_{\widetilde{\Gamma}} \mathcal{B}_1 K_1^1\, d\Gamma\bigg\} \tag{5.58}$$

$$K_2^3 := \delta_2^{-1}\bigg\{(g^2)'' - D_1^2 \int_{\widetilde{\Gamma}} \mathcal{B}_2 u_1\, d\Gamma - D_0^2 \int_{\widetilde{\Gamma}} \mathcal{B}_1 (A^{-1}C) K_2^2\, d\Gamma\bigg\} \tag{5.59}$$

$$K_3^3 := \delta_3^{-1}\{(g^3)^{(4)} - D_0^3 K_3^2(x_0,\cdot) - D_1^3 (C^{-1}Au_1)(x_0) + (C^{-1}f'')(x_0,\cdot)\}. \tag{5.60}$$

Also, taking advantage of (5.29), (5.39) and (5.44), we define, for $i = 1, 2, 3$,

$$J_i^4(\mathbf{U}) := -(D_0^i)^{-1}[J_i^3(\mathbf{U}) + (R_0^i + U^4 *)(1 * U^3) + D_1^i * U^4] \tag{5.61}$$

$$K_i^4 := -(D_0^i)^{-1}[R_0^i D_1^i + K_i^3]. \tag{5.62}$$

Summing up, recalling Problem (P_i^2) and Proposition 5.1, positions (5.46)–(5.52) and (5.55)–(5.62) allow us to state

Proposition 5.2. *For any $i \in \{1, 2, 3\}$, let the assumptions of Theorem* 3.i *hold. Then Problem (P_i) has a unique solution (u, D) if and only if there exists a unique $\mathbf{U}_i \in C^0([0,T]; \mathcal{D}(\mathbf{B}) \times \mathbf{R}^2)$ solution to the fixed-point equation*

$$\mathbf{U} = \mathbf{J}_i(\mathbf{U}) + \mathbf{K}_i \quad \textit{in } [0,T] \tag{5.63}$$

where $\mathbf{J}_i := (J_i^1, J_i^2, J_i^3, J_i^4)^*$ and $\mathbf{K}_i := (K_i^1, K_i^2, K_i^3, K_i^4)^*$ are defined by (5.49)–(5.52) *and* (5.55)–(5.62).

6. Proof of Theorems 3.1–3.3. Our proof is based on the fixed-point formulation given in Proposition 5.2. We begin by stating a basic technical lemma about the local Lipschitz continuity of the mapping $\mathbf{J}_i$, $i \in \{1,2,3\}$. Set

$$Y := \mathcal{D}(\mathbf{B}) \times \mathbf{R}^2$$

and endow Y with the norm

$$\|\mathbf{Z}\|_Y := \|(Z^1, Z^2)^*\|_{\mathcal{D}(\mathbf{B})} + |Z^3| + |Z^4|$$

for any $\mathbf{Z} := (Z^1, Z^2, Z^3, Z^4)^* \in Y$. The result we need is (see Section 8 for its proof)

Lemma 6.1. *For any $i \in \{1,2,3\}$, let the assumptions of Theorem* 3.i *hold. Then, for any $\mathbf{U}, \tilde{\mathbf{U}} \in C^0([0,T];Y)$, we have, for any $t \in [0,T]$,*

$$\|(\mathbf{J}_i(\mathbf{U}) - \mathbf{J}_i(\tilde{\mathbf{U}}))(t)\|_Y \le \int_0^t \Lambda_i(\mathbf{U}, \tilde{\mathbf{U}})(t-s)\|(\mathbf{U}-\tilde{\mathbf{U}})(s)\|_Y \, ds \tag{6.1}$$

where

$$\Lambda_i(\mathbf{U}, \tilde{\mathbf{U}})(t) := c_1^i \left[1 + \int_0^t (\|\mathbf{U}(s)\|_Y + \|\tilde{\mathbf{U}}(s)\|_Y) \, ds\right] \tag{6.2}$$

for any $t \in [0,T]$, c_1^i being a positive constant depending on suitable norms of u_0, u_1, f, g^i and on $\widetilde{\Gamma}$ (if $i = 1,2$), $T, \Omega, h, \mu, \delta_i$.

To apply our fixed-point argument, let us rewrite equation (5.63) in a more appropriate form. First, observe that, recalling (5.54), (5.58)–(5.60), (5.62) and the assumptions of Theorem 3.i, one has

$$\mathbf{K}_i \in C^0([0,T];Y) \quad i = 1,2,3. \tag{6.3}$$

Then, set

$$\mathbf{V} := \mathbf{U} - \mathbf{K}_i \tag{6.4}$$

$$\tilde{\mathbf{J}}_i(\mathbf{V}) := \mathbf{J}_i(\mathbf{V} + \mathbf{K}_i) - \mathbf{J}_i(\mathbf{K}_i) \tag{6.5}$$

$$\tilde{\mathbf{K}}_i := \mathbf{J}(\mathbf{K}_i) \tag{6.6}$$

for any $i \in \{1, 2, 3\}$. Consequently, equation (5.63) becomes

$$\mathbf{V} = \tilde{\mathbf{J}}_i(\mathbf{V}) + \tilde{\mathbf{K}}_i \quad \text{in } [0, T]. \tag{6.7}$$

Therefore, taking advantage of Proposition 5.2 and (6.7), our proof consists in showing that the mappings

$$\mathbf{L}_i(\mathbf{V}) := \tilde{\mathbf{J}}_i(\mathbf{V}) + \tilde{\mathbf{K}}_i, \quad i = 1, 2, 3, \tag{6.8}$$

have a fixed-point in Y_T, where $Y_\tau := C^0([0, \tau]; Y)$, $\tau \in (0, T]$. Note that, on account of (6.1), (6.3) and (6.5)–(6.6), one can check that $\mathbf{L}_i : Y_T \to Y_T$ is well defined.

Following [**1**, Theorem 2.2 (ii)], we introduce in Y_τ the weighted norm

$$\|\mathbf{V}\|_\tau^\sigma := \operatorname{ess} \sup_{\tau \in [0,\tau]} e^{-\sigma t} \|\mathbf{V}(t)\|_Y, \quad \sigma \in [0, +\infty) \tag{6.9}$$

for any $\mathbf{V} \in Y_\tau$, $\tau \in (0, T]$. Of course, Y_τ turns out to be a Banach space for any fixed $\sigma \in [0, +\infty)$. In particular, if $\tau = T$, and $\sigma = 0$, we have, cf. Section 2, $\|\mathbf{U}\|_T^0 \equiv \|\mathbf{U}\|_{0,Y}$. Besides, one can easily realize that the norms defined by (6.9) are all equivalent in Y_τ, for any fixed $\tau \in (0, T]$.

Consider now the closed and bounded subset of Y_T

$$E_{r,\sigma} := \{\mathbf{V} \in Y_T : \|\mathbf{V}\|_T^\sigma \le r\}$$

for some $(r, \sigma) \in (0, +\infty)^2$ and let $\mathbf{V} \in E_{r,\sigma}$.

From (6.1) and (6.5), we derive, for any $t \in [0, T]$,

$$\begin{aligned} &e^{-\sigma t} \|\tilde{\mathbf{J}}_i(\mathbf{V})(t)\|_Y \\ &\qquad \le \int_0^t e^{-\sigma(t-s)} \Lambda_i(\mathbf{V} + \mathbf{K}_i, \mathbf{K}_i)(t - s) e^{-\sigma s} \|\mathbf{V}(s)\|_Y \, ds. \end{aligned} \tag{6.10}$$

An easy computation shows that, cf. (6.2),

$$\int_0^T e^{-\sigma t} \Lambda_i(\mathbf{Z}, \tilde{\mathbf{Z}})(t) \, dt \le \lambda(\|\mathbf{Z}\|_T^\sigma, \|\tilde{\mathbf{Z}}\|_T^\sigma, \sigma) \tag{6.11}$$

where

$$\lambda(\|\mathbf{Z}\|_T^\sigma, \|\tilde{\mathbf{Z}}\|_T^\sigma, \sigma) := c_1^i\{(1-e^{-\sigma T})\sigma^{-1} + [T\sigma^{-1}+(e^{-\sigma T}-1)\sigma^{-2}](\|\mathbf{Z}\|_T^\sigma+\|\tilde{\mathbf{Z}}\|_T^\sigma)\}. \tag{6.12}$$

Hence, a combination of (6.10) and (6.11) yields

$$\|\tilde{\mathbf{J}}_i(\mathbf{V})(t)\|_T^\sigma \leq \lambda(r+\|\mathbf{K}_i\|_T^\sigma, \|\mathbf{K}_i\|_T^\sigma, \sigma)r, \quad \forall\, \mathbf{V} \in E_{r,\sigma}. \tag{6.13}$$

Taking (6.13) into account, from (6.8) one infers

$$\|\mathbf{L}_i(\mathbf{V})\|_T^\sigma \leq \lambda(r+\|\mathbf{K}_i\|_T^\sigma, \|\mathbf{K}_i\|_T^\sigma, \sigma)r + \|\tilde{\mathbf{K}}_i\|_T^\sigma \tag{6.14}$$

for any $\mathbf{V} \in E_{r,\sigma}$. Therefore, recalling (6.12) and picking $(\tilde{r}, \tilde{\sigma}) \in (0,+\infty)^2$ such that (see Remark 7.1 below)

$$c_1^i\{(1-e^{-\tilde{\sigma} T})\tilde{\sigma}^{-1} + [T\tilde{\sigma}^{-1} + (e^{-\tilde{\sigma} T}-1)\tilde{\sigma}^{-2}](\tilde{r}+2\|\mathbf{K}_i\|_T^{\tilde{\sigma}})\} + \|\tilde{\mathbf{K}}_i\|_T^{\tilde{\sigma}} \leq \tilde{r} \tag{6.15}$$

we deduce from (6.14)

$$\mathbf{L}_i(E_{\tilde{r},\tilde{\sigma}}) \subseteq E_{\tilde{r},\tilde{\sigma}}. \tag{6.16}$$

On the other hand, thanks to (6.1) and (6.5), we obtain

$$e^{-\tilde{\sigma} t}\|\tilde{\mathbf{J}}_i(\mathbf{V}) - \tilde{\mathbf{J}}_i(\tilde{\mathbf{V}}))(t)\|_Y \leq \int_0^t e^{-\tilde{\sigma}(t-s)}\Lambda_i(\mathbf{V}+\mathbf{K}_i, \tilde{\mathbf{V}}+\mathbf{K}_i)(t-s)e^{-\tilde{\sigma} s}\|(\mathbf{V}-\tilde{\mathbf{V}})(s)\|_Y\, ds \tag{6.17}$$

which implies

$$e^{-\tilde{\sigma} t}\|\tilde{\mathbf{J}}_i(\mathbf{V})(t) - \tilde{\mathbf{J}}_i(\tilde{\mathbf{V}})(t)\|_Y \leq \|\Lambda_i(\mathbf{V}+\mathbf{K}_i, \tilde{\mathbf{V}}+\mathbf{K}_i)\|_T^{\tilde{\sigma}} \int_0^t \|\mathbf{V}-\tilde{\mathbf{V}}\|_s^{\tilde{\sigma}}\, ds \tag{6.18}$$

for any $\mathbf{V}, \tilde{\mathbf{V}} \in E_{\tilde{r},\tilde{\sigma}}$ and any $t \in [0,T]$.

Recalling (6.2), one can find a positive constant c_2^i, which depends on c_1^i, $\|\mathbf{K}_i\|_T^{\tilde\sigma}$, $\tilde r$, $\tilde\sigma$ and T, such that

$$\|\Lambda_i(\mathbf{V}+\mathbf{K}_i,\tilde{\mathbf{V}}+\mathbf{K}_i)\|_T^{\tilde\sigma} \le c_2^i \tag{6.19}$$

for any $i \in \{1,2,3\}$, whenever $\mathbf{V},\tilde{\mathbf{V}} \in E_{\tilde r,\tilde\sigma}$. Hence, owing to (6.8), inequalities (6.18) and (6.19) give

$$\|\mathbf{L}_i(\mathbf{V}) - \mathbf{L}_i(\tilde{\mathbf{V}})\|_t^{\tilde\sigma} \le c_2^i \int_0^t \|\mathbf{V}-\tilde{\mathbf{V}}\|_s^{\tilde\sigma}\,ds \qquad \forall\, t \in [0,T] \tag{6.20}$$

for any $\mathbf{V},\tilde{\mathbf{V}} \in E_{\tilde r,\tilde\sigma}$.

On account of (6.8) and (6.16), inequality (6.20) entails that some power $(\mathbf{L}_i)^l$, $l \in \mathbf{N}$, is a contraction from $E_{\tilde r,\tilde\sigma}$ into itself. Then, using the Picard-Banach fixed-point theorem, see, e.g., [**6**, Chapter 2, Theorem 2.2], we deduce that $\mathbf{L}_i$ has a unique fixed-point in $E_{\tilde r,\tilde\sigma}$.

Remark 6.1. Note that there is no need for $\mathbf{L}_i$ to be a contraction itself (compare with [**1**, Theorem 2.1 (i), Equation (2.24)]).

Remark 6.2. In Theorem 3.i, $i = 1,2,3$, the spaces of type $C^m([0,T];X)$, $m \in \mathbf{N}\cup\{0\}$, (see the assumptions regarding f and g^i) can be replaced by spaces of type $W^{m,p}(0,T;X)$, $p \in [1,+\infty]$, where $W^{0,p} \equiv L^p$. In this case the functional spaces in which we are looking for u and D must be modified accordingly, cf. [**2**, Section 2, Remark 2.4]. In particular, we have to find $D \in W^{2,p}(0,T)$ and equation (5.63) has to be solved in $L^p(0,T;Y)$, cf. also [**1**, Theorem 2.1(i)].

7. Continuous dependence on data. A Lipschitz continuous dependence result is proved for problem (P_1) in [**2**], see [**2**, Theorem 2.2]. There, the Lipschitz constant depends both on known quantities (data, T, Ω, and the like) and on some norm of D. We can get rid of the latter dependence, taking advantage of our global existence results. Indeed, the solution $\mathbf{U}$ to the fixed-point equation (5.63) satisfies the bound, cf. (6.4), (6.9) and (6.16),

$$\|\mathbf{U}\|_{0,Y} \le \tilde r e^{\tilde\sigma T} + \|\mathbf{K}_i\|_T^{\tilde\sigma} \tag{7.1}$$

where $(\tilde{\sigma}, \tilde{r}) \in (0, +\infty)^2$ only depends on known quantities, cf. Remark 7.1 below. Therefore, thanks to Proposition 5.2, one can deduce that the corresponding solution (u_i, D_i) to Problem (P_i) is bounded in the appropriate norm.

To illustrate this improvement, we state and prove an analog of [**2**, Theorem 2.2] for Problem (P_3). Of course, similar results hold for Problems (P_1) and (P_2).

Theorem 7.1. *Let* (2.12) *and* (2.21) *hold, and let* $(f_j, u_{0j}, u_{1j}, g_j^3)$, $j = 1, 2$, *be two sets of data satisfying hypotheses* (3.18)–(3.22) *and* (3.35)–(3.40). *Assume that* $D_{01}^3 = D_{02}^3$ *where* D_{0j}^3 *is defined by* (3.40) *with* $u_{0j}, \delta_{3j}, f_j, g_j^3$ *in place of* u_0, δ_3, f, g^3, *respectively.*

Denote by (u_j, D_j), $j = 1, 2$, *the corresponding solutions to Problem* (P_3), *and let* K *be a positive constant such that*

$$\|f_j^1\|_{3,H} + \|f_j^2\|_{2,H^1(\Omega)} + \|f_j(0)\|_{H^2(\Omega)} + \|f_j'(0)\|_{H^1(\Omega)} + \|u_{0j}\|_{H^6(\Omega)} + \|u_{1j}\|_{H^5(\Omega)} + \|g_j^3\|_4 + (\delta_{0j}^3)^{-1} \le K \tag{7.2}$$

for $j = 1, 2$. *Then there exists a function* $M_1 \in C^0((0, +\infty)^2; (0, +\infty))$ *such that*

$$\begin{aligned} &\|u_1 - u_2\|_{2,V_4} + \|u_1 - u_2\|_{3,V_3} + \|u_1 - u_2\|_{4,V_2} + \|D_1 - D_2\|_2 \\ &\quad \le M_1(K, T)\{\|f_1^1 - f_2^1\|_{3,H} + \|f_1^2 - f_2^2\|_{2,H^1(\Omega)} \\ &\qquad + \|u_{01} - u_{02}\|_{H^6(\Omega)} + \|u_{11} - u_{12}\|_{H^5(\Omega)} \\ &\qquad + \|(f_1 - f_2)(0)\|_{H^2(\Omega)} + \|(f_1' - f_2')(0)\|_{H^1(\Omega)} \\ &\qquad + \|g_1^3 - g_2^3\|_4\}. \end{aligned} \tag{7.3}$$

Moreover, the function M_1 *is nondecreasing in each of its arguments and also depends on* Ω, h, μ.

Proof. Observe first that, thanks to Proposition 5.2, for any $j \in \{1, 2\}$, there exists a unique solution $\mathbf{U}_j$ to

$$\mathbf{U}_j = \mathbf{J}_{3j}(\mathbf{U}_j) + \mathbf{K}_{3j} \quad \text{in } [0, T] \tag{7.4}$$

which corresponds to (u_j, D_j). Here $\mathbf{J}_{3j}$ and $\mathbf{K}_{3j}$ are associated with the set of data $(f_j, u_{0j}, u_{1j}, g_j^3)$. Therefore, from (7.4) we deduce

$$\begin{aligned} \mathbf{U}_1 - \mathbf{U}_2 &= \mathbf{J}_{31}(\mathbf{U}_1) - \mathbf{J}_{31}(\mathbf{U}_2) \\ &\quad + \mathbf{J}_{31}(\mathbf{U}_2) - \mathbf{J}_{32}(\mathbf{U}_2) + \mathbf{K}_{31} - \mathbf{K}_{32} \quad \text{in } [0, T]. \end{aligned} \tag{7.5}$$

From now on, $M_m(\cdot,\cdot)$, $m \in \mathbf{N}$, stands for a positive and continuous function which depends on Ω, h, μ, at most.

Indicating by c_{1j}^3 the constant appearing in (6.2) and associated with $\mathbf{J}_{3j}$ and using (7.2), one can obtain

$$c_{1j}^3 \le M_2(K,T), \quad j = 1,2. \tag{7.6}$$

On the other hand, recalling (5.52), (5.60), (5.62) and using again (7.2), we get

$$\|\mathbf{K}_{3j}\|_T^\sigma + \|\tilde{\mathbf{K}}_{3j}\|_T^\sigma \le M_3(K,T), \quad j = 1,2, \tag{7.7}$$

for any $\sigma \in [0,+\infty)$.

On account of (7.6)–(7.7), looking at (6.15) one realizes that the pair $(\tilde{\sigma},\tilde{r}) \in (0,+\infty)^2$ fulfilling (6.15) can be chosen in a way that it only depends on K,T and Ω, h, μ at most, see Remark 7.1 below. Consequently, we have, cf. (6.16) and (7.1),

$$\|\mathbf{U}_j\|_{0,Y} \le M_4(K,T). \tag{7.8}$$

Then, thanks to (6.1)–(6.2), (7.6) and (7.8), we can obtain

$$\|\mathbf{J}_{31}(\mathbf{U}_1) - \mathbf{J}_{31}(\mathbf{U}_2)\|_t^0 \le M_5(K,T)\int_0^t \|\mathbf{U}_1 - \mathbf{U}_2\|_s^0\,ds \tag{7.9}$$

for any $t \in [0,T]$.

Recalling now (5.51)–(5.52), (5.57), (5.60)–(5.62) for $i = 3$ and taking advantage of (7.2) and (7.8), computations similar to the ones done in [**2**, Section 6] (see also [**2**, Remark 2.2]) lead to

$$\begin{aligned} &\|\mathbf{J}_{31}(\mathbf{U}_2) - \mathbf{J}_{32}(\mathbf{U}_2)\|_{0,Y} + \|\mathbf{K}_{31} - \mathbf{K}_{32}\|_{0,Y} \\ &\quad \le M_6(K,T)\{\|f_1^1 - f_2^1\|_{3,H} + \|f_1^2 - f_2^2\|_{2,H^1(\Omega)} \\ &\qquad + \|u_{01} - u_{02}\|_{H^6(\Omega)} + \|u_{11} - u_{12}\|_{H^5(\Omega)} \\ &\qquad + \|(f_1 - f_2)(0)\|_{H^2(\Omega)} + \|(f_1' - f_2')(0)\|_{H^1(\Omega)} \\ &\qquad + \|g_1^3 - g_2^3\|_4\}. \end{aligned} \tag{7.10}$$

Hence, considering (7.5), owing to (7.9) and (7.10), the Gronwall lemma yields

$$
\begin{aligned}
(7.11)\quad & \|\mathbf{U}_1-\mathbf{U}_2\|_{0,Y} \\
&\leq M_7(K,T)\{\|f_1^1-f_2^1\|_{3,H}+\|f_1^2-f_2^2\|_{2,H^1(\Omega)} \\
&\qquad +\|u_{01}-u_{02}\|_{H^6(\Omega)}+\|u_{11}-u_{12}\|_{H^5(\Omega)} \\
&\qquad +\|(f_1-f_2)(0)\|_{H^2(\Omega)} \\
&\qquad +\|(f_1'-f_2')(0)\|_{H^1(\Omega)}+\|g_1^3-g_2^3\|_4\}.
\end{aligned}
$$

Finally, on account of (4.32), (5.23)–(5.24), (5.45) and (5.48), it is straightforward to infer (7.3) from (7.11).

Remark 7.1. A possible explicit choice of $(\tilde{\sigma},\tilde{r})\in(0,+\infty)^2$ satisfying (6.15), can be made as follows. Observe that, thanks to (7.6)–(7.7), we have

$$
\begin{aligned}
(7.12)\quad & c_{1j}^i\{(1-e^{-\tilde{\sigma}T})\tilde{\sigma}^{-1} \\
&\quad +[T\tilde{\sigma}^{-1}+(e^{-\tilde{\sigma}T}-1)\tilde{\sigma}^{-2}](\tilde{r}+2\|\mathbf{K}_3^j\|_T^{\tilde{\sigma}})\}+\|\tilde{\mathbf{K}}_3^j\|_T^{\tilde{\sigma}} \\
&\leq M_2(K,T)\{(1-e^{-\tilde{\sigma}T})\tilde{\sigma}^{-1} \\
&\quad +[T\tilde{\sigma}^{-1}+(e^{-\tilde{\sigma}T}-1)\tilde{\sigma}^{-2}](\tilde{r}+2M_3(K,T))\}+M_3(K,T) \\
&\leq M_2(K,T)\tilde{\sigma}^{-1}(1+T(\tilde{r}+2M_3(K,T))+M_3(K,T)
\end{aligned}
$$

for $j=1,2$. Then, choosing, e.g.,

$$
\begin{aligned}
\tilde{r} &:= 2M_3(K,T) \\
\tilde{\sigma} &:= M_2(K,T)(M_3(K,T))^{-1}(1+4TM_3(K,T)),
\end{aligned}
$$

from (7.12) we deduce (6.15).

8. Proof of Lemma 6.1. We begin to state a preliminary estimate which turns out to be useful in the sequel.

Let us consider a function $\mathbf{u}:[0,T]\to V_2\times V_1$ defined by

$$
(8.1)\quad \mathbf{u}(t):=S(t)\mathbf{w}_0+S*[(\delta I+Q*)\mathbf{w}+\mathbf{N}_1+\mathbf{N}_2](t),\quad t\in(0,T)
$$

where S is the strongly continuous semigroup generated by $-B$, see (5.21), $\delta \in \mathbf{R}$ and

$$\mathbf{w}_0 \in \mathcal{D}(B), \quad \mathbf{w} \in C^0([0,T];\mathcal{D}(B)) \tag{8.2}$$

$$Q \in C^0([0,T]) \tag{8.3}$$

$$\mathbf{N}_1 \in C^1([0,T];V_2 \times V_1), \qquad \mathbf{N}_2 \in C^0([0,T];\mathcal{D}(B)). \tag{8.4}$$

Then, reasoning as in [**2**, Section 3], it is possible to prove the estimate

$$\begin{aligned}\|\mathbf{u}(t)\|_{\mathcal{D}(B)} \le \Lambda\bigg\{ &\|\mathbf{w}_0\|_{\mathcal{D}(B)} + \|\mathbf{N}_1(0)\|_{V_2\times V_1} \\ &+ \int_0^t \bigg[\int_0^s |Q(s-\tau)|\, \|\mathbf{w}(\tau)\|_{\mathcal{D}(B)}\, d\tau + \|\mathbf{w}(s)\|_{\mathcal{D}(B)} \\ &\qquad + \|\mathbf{N}_1'(s)\|_{V_2\times V_1} + \|\mathbf{N}_2(s)\|_{\mathcal{D}(B)}\bigg]\, ds\bigg\}\end{aligned} \tag{8.5}$$

for any $t \in [0,T]$, where Λ is a positive constant depending on T, Ω, h, μ and δ.

Consider first the case $i = 1$. Recalling (5.49), (5.55) and (5.61), observe that

$$\begin{aligned}&(J_1^1(\mathbf{U}) - J_1^1(\tilde{\mathbf{U}}), J_1^2(\mathbf{U}) - J_1^2(\tilde{\mathbf{U}}))^* \\ &\quad = S * (0, -(R_0^1 + U^4 *)(U^2 - \tilde{U}^2) \\ &\qquad - (U^4 - \tilde{U}^4) * \tilde{U}^2 - (U^4 - \tilde{U}^4) z_0 \\ &\qquad - D_0^1[C^{-1}Au_0(U^3 - \tilde{U}^3) + C^{-1}Au_1(1 * (U^3 - \tilde{U}^3))])^*\end{aligned} \tag{8.6}$$

$$\begin{aligned}&J_1^3(\mathbf{U}) - J_1^3(\tilde{\mathbf{U}}) \\ &\quad = -\delta_1^{-1}\bigg\{\bigg(\int_{\widetilde{\Gamma}} \mathcal{B}_1 u_1\, d\Gamma\bigg)(1 * (U^3 - \tilde{U}^3)) \\ &\qquad + \int_{\widetilde{\Gamma}} \mathcal{B}_1[J_1^1(\mathbf{U}) - J_1^1(\tilde{\mathbf{U}})]\, d\Gamma\bigg\}\end{aligned} \tag{8.7}$$

$$\begin{aligned}&J_1^4(\mathbf{U}) - J_1^4(\tilde{\mathbf{U}}) \\ &\quad = -(D_0^1)^{-1}[(J_1^3(\mathbf{U}) - J_1^3(\tilde{\mathbf{U}})) \\ &\qquad + (R_0^1 + U^4 *)(1 * (U^3 - \tilde{U}^3)) \\ &\qquad + (U^4 - \tilde{U}^4) * 1 * \tilde{U}^3 + D_1^1 * (U^4 - \tilde{U}^4)].\end{aligned} \tag{8.8}$$

On account of assumptions (3.1), (3.4) and positions (4.12) and (5.10), since $\mathbf{U}, \tilde{\mathbf{U}} \in C^0([0,T];Y)$, one easily checks that

$$C^{-1}Au_1(1 * (U^3 - \tilde{U}^3)) \in C^1([0,T];V_1) \tag{8.9}$$

$$(U^4 - \tilde{U}^4) * \tilde{U}^2 + (U^4 - \tilde{U}^4)z_0 + D_0^1 C^{-1}Au_0(U^3 - \tilde{U}^3) \in C^0([0,T];V_2) \tag{8.10}$$

then assumptions (8.2)–(8.4) are satisfied. Thus, applying estimate (8.5) to (8.6), we deduce, for any $t \in [0,T]$,

$$\begin{aligned} &\|(J_1^1(\mathbf{U}) - J_1^1(\tilde{\mathbf{U}}), J_1^2(\mathbf{U}) - J_1^2(\tilde{\mathbf{U}}))(t)\|_{\mathcal{D}(B)} \\ &\le c_2^1 \bigg\{ \int_0^t \bigg[\int_0^s |U^4(s-\tau)| \, \|(U^1 - \tilde{U}^1, U^2 - \tilde{U}^2)(\tau)\|_{\mathcal{D}(B)} \, d\tau \\ &\qquad + \int_0^s |(U^4 - \tilde{U}^4)(s-\tau)| \, \|\tilde{U}^2(\tau)\|_{V_2} \, d\tau \\ &\qquad + \|(U^1 - \tilde{U}^1, U^2 - \tilde{U}^2)(s)\|_{\mathcal{D}(B)} \\ &\qquad + \|C^{-1}Au_1\|_{V_1} |(U^3 - \tilde{U}^3)(s)| \\ &\qquad + \|z_0\|_{V_2} |(U^4 - \tilde{U}^4)(s)| \\ &\qquad + |D_0^1| \, \|C^{-1}Au_0\|_{V_2} |(U^3 - \tilde{U}^3)(s)| \bigg] \, ds \bigg\}, \end{aligned} \tag{8.11}$$

where c_2^1 is a positive constant depending on T, Ω, h, μ and R_0^1. In the sequel of the proof, c_m^i, $m \in \mathbf{N}$, $i \in \{1,2,3\}$, will stand for a positive constant depending on suitable norms of u_0, u_1, f, g^i and on $\widetilde{\Gamma}$ (if $i = 1,2$), $T, \Omega, h, \mu, \delta_i$, at most.

From (8.11), one infers

$$
\begin{aligned}
(8.12)\quad & \|(J_1^1(\mathbf{U}) - J_1^1(\tilde{\mathbf{U}}), J_1^2(\mathbf{U}) - J_1^2(\tilde{\mathbf{U}}))(t)\|_{\mathcal{D}(B)} \\
& \le c_3^1 \bigg\{ \int_0^t \bigg[\int_0^s |U^4(s-\tau)| \, \|(U^1 - \tilde{U}^1, U^2 - \tilde{U}^2)(\tau)\|_{\mathcal{D}(B)} \, d\tau \\
& \qquad + \int_0^s |(U^4 - \tilde{U}^4)(s-\tau)| \, \|\tilde{U}^2(\tau)\|_{V_2} \, d\tau \\
& \qquad + \|(U^1 - \tilde{U}^1, U^2 - \tilde{U}^2)(s)\|_{\mathcal{D}(B)} \\
& \qquad + |(U^3 - \tilde{U}^3)(s)| + |(U^4 - \tilde{U}^4)(s)| \bigg] \, ds \bigg\} \\
& \le c_4^1 \bigg\{ \int_0^t \bigg[\int_0^s (\|\mathbf{U}(s-\tau)\|_Y \\
& \qquad\qquad + \|\tilde{\mathbf{U}}(s-\tau)\|_Y) \|(\mathbf{U} - \tilde{\mathbf{U}})(\tau)\|_Y \, d\tau \\
& \qquad + \|(\mathbf{U} - \tilde{\mathbf{U}})(s)\|_Y \big] \, ds \bigg\}, \quad \forall\, t \in [0, T].
\end{aligned}
$$

Recall now that, cf. (1.6)–(1.7), and see, e.g., [**14**, Chapter 1, Section 8, Theorem 8.3],

$$
(8.13)\qquad \left| \int_{\widetilde{\Gamma}} \mathcal{B}_1 z \, d\Gamma \right| \le c_5^1 |\widetilde{\Gamma}|^{1/2} \|z\|_{V_3}, \quad \forall\, z \in V_3.
$$

Then, on account of (8.13), from (8.7) we get

$$
\begin{aligned}
(8.14)\quad |(J_1^3(\mathbf{U}) - J_1^3(\tilde{\mathbf{U}})(t)| \le c_6^1 \{ & (1 * |U^3 - \tilde{U}^3|)(t) \\
& + \|(J_1^1(\mathbf{U}) - J_1^1(\tilde{\mathbf{U}}))(t)\|_{V_3} \}, \quad \forall\, t \in [0, T],
\end{aligned}
$$

and, combining (8.14) and (8.12), one obtains

$$
\begin{aligned}
(8.15)\quad & |(J_1^3(\mathbf{U}) - J_1^3(\tilde{\mathbf{U}}))(t)| \\
& \le c_7^1 \bigg\{ \int_0^t \bigg[\int_0^s (\|\mathbf{U}(s-\tau)\|_Y + \|\tilde{\mathbf{U}}(s-\tau)\|_Y) \|(\mathbf{U} - \tilde{\mathbf{U}})(\tau)\|_Y \, d\tau \\
& \qquad\qquad + \|(\mathbf{U} - \tilde{\mathbf{U}})(s)\|_Y \bigg] \, ds \bigg\}, \quad \forall\, t \in [0, T].
\end{aligned}
$$

On the other hand, from (8.8), it is not difficult to deduce

$$(8.16)\quad \begin{aligned} |(J_1^4(\mathbf{U}) - J_i^4(\tilde{\mathbf{U}}))(t)| \le c_8^1\{&|(J_1^3(\mathbf{U}) - J_1^3(\tilde{\mathbf{U}}))(t)| \\ &+ ((1+|U^4|*)(1*|U^3 - \tilde{U}^3|))(t) \\ &+ (1*|U^4 - \tilde{U}^4|*|\tilde{U}^3|)(t) \\ &+ (1*|U^4 - \tilde{U}^4|)(t)\}, \\ \forall\, t \in [0,T].& \end{aligned}$$

Hence, (8.15) and (8.16) yield

$$(8.17)\quad \begin{aligned} |(J_1^4(\mathbf{U}) - J_1^4(\tilde{\mathbf{U}}))(t)| \le c_9^1\bigg\{ \int_0^t \bigg[\int_0^s (&\|\mathbf{U}(s-\tau)\|_Y \\ &+ \|\tilde{\mathbf{U}}(s-\tau)\|_Y)\|(\mathbf{U} - \tilde{\mathbf{U}})(\tau)\|_Y \, d\tau \\ &+ \|(\mathbf{U} - \tilde{\mathbf{U}})(s)\|_Y \bigg] ds \bigg\}, \quad \forall\, t \in [0,T]. \end{aligned}$$

Therefore, owing to (8.12), (8.15) and (8.17), inequality (6.1) is proved for $i = 1$.

Consider now the cases $i = 2, 3$. From (5.51), (5.56)–(5.57) and (5.61), one easily derives

$$(8.18)\quad \begin{aligned} &(J_i^1(\mathbf{U}) - J_i^1(\tilde{\mathbf{U}}), J_i^2(\mathbf{U}) - J_i^2(\tilde{\mathbf{U}}))^* \\ &\quad = S * ((R_0^i + U^4 *)(U^1 - \tilde{U}^1) \\ &\qquad + (U^4 - \tilde{U}^4) * \tilde{U}^1 + (U^4 - \tilde{U}^4) z_0 \\ &\qquad + D_0^i[C^{-1}Au_0(U^3 - \tilde{U}^3) \\ &\qquad + C^{-1}Au_1(1 * (U^3 - \tilde{U}^3))], 0)^*, \quad i = 2,3 \end{aligned}$$

$$(8.19)\quad \begin{aligned} J_2^3(\mathbf{U}) - J_2^3(\tilde{\mathbf{U}}) = -\delta_2^{-1}\bigg\{ &\bigg(\int_{\widetilde{\Gamma}} \mathcal{B}_2 u_1 \, d\Gamma \bigg)(1 * (U^3 - \tilde{U}^3)) \\ &+ \int_{\widetilde{\Gamma}} \mathcal{B}_2 A^{-1} C[J_2^2(\mathbf{U}) - J_2^2(\tilde{\mathbf{U}})] \, d\Gamma \bigg\} \end{aligned}$$

$$(8.20)\quad \begin{aligned} J_3^3(\mathbf{U}) - J_3^3(\tilde{\mathbf{U}}) = -\delta_3^{-1}\{&D_0^3(J_3^2(\mathbf{U}) - J_3^2(\tilde{\mathbf{U}}))(x_0, \cdot) \\ &+ (C^{-1}Au_1)(x_0)(1 * (U^3 - \tilde{U}^3))\} \end{aligned}$$

$$
\begin{aligned}
(8.21)\quad & J_i^4(\mathbf{U}) - J_i^4(\tilde{\mathbf{U}}) \\
&= -(D_0^i)^{-1}[(J_i^3(\mathbf{U})-J_i^3(\tilde{\mathbf{U}}))+(R_0^i+U^4*)(1*(U^3-\tilde{U}^3)) \\
&\quad + (U^4-\tilde{U}^4)*(1*\tilde{U}^3)+D_1^i*(U^4-\tilde{U}^4)], \\
&\quad i=2,3.
\end{aligned}
$$

Reasoning as in the case $i = 1$, one gets, cf. (3.22), (4.12) and (5.10),

$$
(8.22)\qquad C^{-1}Au_1(1*(U^3-\tilde{U}^3)) \in C^1([0,T];V_2)
$$

$$
(8.23)\quad (U^4-\tilde{U}^4)*\tilde{U}^1+(U^4-\tilde{U}^4)z_0+D_0^iC^{-1}Au_0(U^3-\tilde{U}^3) \in C([0,T];V_3)
$$

where $i = 2, 3$.

Thanks to (8.22)–(8.23), assumptions (8.2)–(8.4) are satisfied and we can apply (8.5) to (8.18). Thus we have, for any $t \in [0,T]$,

$$
\begin{aligned}
(8.24)\quad & \|(J_i^1(\mathbf{U}) - J_i^1(\tilde{\mathbf{U}}), J_i^2(\mathbf{U}) - J_i^2(\tilde{\mathbf{U}}))(t)\|_{\mathcal{D}(B)} \\
&\le c_{10}^i\bigg\{\int_0^t \bigg[\int_0^s |U^4(s-\tau)|\,\|(U^1-\tilde{U}^1, U^2-\tilde{U}^2)(\tau)\|_{\mathcal{D}(B)}\,d\tau \\
&\quad + \int_0^s |(U^4-\tilde{U}^4)(s-\tau)|\,\|\tilde{U}^1(\tau)\|_{V_3}\,d\tau \\
&\quad + \|(U^1-\tilde{U}^1, U^2-\tilde{U}^2)(s)\|_{\mathcal{D}(B)} \\
&\quad + \|C^{-1}Au_1\|_{V_2}|(U^3-\tilde{U}^3)(s)| \\
&\quad + \|z_0\|_{V_3}|(U^4-\tilde{U}^4)(s)| \\
&\quad + |D_0^i|\,\|C^{-1}Au_0\|_{V_3}|(U^3-\tilde{U}^3)(s)|\bigg]\,ds\bigg\}, \quad i=2,3.
\end{aligned}
$$

From (8.24) we infer, for any $t \in [0,T]$,

$$\begin{aligned}(8.25)\quad &\|(J_i^1(\mathbf{U}) - J_i^1(\tilde{\mathbf{U}}), J_i^2(\mathbf{U}) - J_i^2(\tilde{\mathbf{U}}))(t)\|_{\mathcal{D}(B)} \\ &\leq c_{11}^i \bigg\{ \int_0^t \bigg[\int_0^s |U^4(s-\tau)| \, \|(U^1 - \tilde{U}^1, U^2 - \tilde{U}^2)(\tau)\|_{\mathcal{D}(B)} \, d\tau \\ &\qquad + \int_0^s |(U^4 - \tilde{U}^4)(s-\tau)| \, \|\tilde{U}^1(\tau)\|_{V_3} \, d\tau \\ &\qquad + \|(U^1 - \tilde{U}^1, U^2 - \tilde{U}^2)(s)\|_{\mathcal{D}(B)} \\ &\qquad + |(U^3 - \tilde{U}^3)(s)| + |(U^4 - \tilde{U}^4)(s)| \bigg] ds \bigg\} \\ &\leq c_{12}^i \bigg\{ \int_0^t \bigg[\int_0^s (\|\mathbf{U}(s-\tau)\|_Y + \|\tilde{\mathbf{U}}(s-\tau)\|_Y) \|(\mathbf{U} - \tilde{\mathbf{U}})(\tau)\|_Y \, d\tau \\ &\qquad + \|(\mathbf{U} - \tilde{\mathbf{U}})(s)\|_Y \bigg] ds \bigg\}, \quad i = 2, 3.\end{aligned}$$

Let, for instance, $i = 2$ and recall that, cf. (1.6)–(1.7) and see, e.g., [**14**, Chapter 1, Section 8, Theorem 8.3],

$$(8.26)\qquad \left| \int_{\widetilde{\Gamma}} \mathcal{B}_2 A^{-1} C z \, d\Gamma \right| \leq c_{13}^2 |\widetilde{\Gamma}|^{1/2} \|z\|_{V_2}, \quad \forall\, z \in V_2.$$

Then, using (8.26), from (8.19) we deduce

$$\begin{aligned}(8.27)\quad |(J_2^3(\mathbf{U}) - J_2^3(\tilde{\mathbf{U}}))(t)| &\leq c_{14}^2 \{(1 * |U^3 - \tilde{U}^3|)(t) \\ &\quad + \|(J_2^2(\mathbf{U}) - J_2^2(\tilde{\mathbf{U}}))(t)\|_{V_2}\}, \quad \forall\, t \in [0,T].\end{aligned}$$

Similarly, if $i = 3$, taking advantage of the injection $H^2(\Omega) \hookrightarrow C^0(\overline{\Omega})$, one has

$$(8.28)\qquad |z(x_0)| \leq c_{15}^3 \|z\|_{V_2}, \quad \forall\, z \in V_2.$$

Hence, from (8.20) we obtain, for any $t \in [0,T]$,

$$\begin{aligned}(8.29)\quad |(J_3^3(\mathbf{U}) - J_3^3(\tilde{\mathbf{U}}))(t)| &\leq c_{16}^3 \{\|(J_3^2(\mathbf{U}) - J_3^2(\tilde{\mathbf{U}}))(t)\|_{V_2} \\ &\quad + \|C^{-1} A u_1\|_{V_2} (1 * |U^3 - \tilde{U}^3|)(t)\}.\end{aligned}$$

Combining (8.25) and (8.27), if $i = 2$ (or (8.29), if $i = 3$), one infers

$$
\begin{aligned}
(8.30)\quad & |(J_i^3(\mathbf{U})-J_i^3(\tilde{\mathbf{U}}))(t)| \\
& \leq c_{17}^i\bigg\{\int_0^t\bigg[\int_0^s(\|\mathbf{U}(s-\tau)\|_Y+\|\tilde{\mathbf{U}}(s-\tau)\|_Y)\|(\mathbf{U}-\tilde{\mathbf{U}})(\tau)\|_Y\,d\tau \\
& \qquad +\|(\mathbf{U}-\tilde{\mathbf{U}})(s)\|_Y\bigg]\,ds\bigg\},\quad \forall\, t\in[0,T],\quad i=2,3.
\end{aligned}
$$

Taking now (8.16) into account, from (8.21) and (8.29) we get

$$
\begin{aligned}
(8.31)\quad & |(J_i^4(\mathbf{U})-J_i^4(\tilde{\mathbf{U}}))(t)| \\
& \leq c_{18}^i\bigg\{\int_0^t\bigg[\int_0^s(\|\mathbf{U}(s-\tau)\|_Y+\|\tilde{\mathbf{U}}(s-\tau)\|_Y)\|(\mathbf{U}-\tilde{\mathbf{U}})(\tau)\|_Y\,d\tau \\
& \qquad +\|(\mathbf{U}-\tilde{\mathbf{U}})(s)\|_Y\bigg]\,ds\bigg\},\quad \forall\, t\in[0,T],\quad i=2,3.
\end{aligned}
$$

Finally, inequality (6.1) for $i=2,3$ follows from (8.25) and (8.30)–(8.31). The proof is thus complete. □

REFERENCES

1. A.L. Bukhgeim, *Inverse problems of memory reconstruction*, J. Inv. Ill-Posed Problems **1** (1993), 193–205.

2. C. Cavaterra and M. Grasselli, *An inverse problem for the linear viscoelastic Kirchhoff plate*, Quart. J. Appl. Math. **53** (1995), 9–33.

3. ———, *Identifying the memory kernels in linear thermoviscoelasticity of Boltzmann type*, M^3AS **4** (1994), 807–842.

4. G. Duvaut and J.-L. Lions, *Inequalities in mechanics and physics*, Springer-Verlag, Berlin, 1976.

5. M. Fabrizio and A. Morro, *Mathematical problems in linear viscoelasticity*, SIAM Stud. Appl. Math. **12** (1992).

6. J.E. Goldstein, *Semigroups of linear operators and applications*, Oxford University Press, New York, 1985.

7. M. Grasselli, *Determining the relaxation tensor in linear viscoelasticity of integral type*, Japan J. Appl. Math. **11** (1994), 131–153. Correction in Japan J. Appl. Math. **13** (1996), 185–186.

8. W.J. Hrusa, J.A. Nohel and M. Renardy, *Mathematical problems in viscoelasticity*, Pitman Monographs Surveys Pure Appl. Math. **35** (1987).

9. J.E. Lagnese and J.-L. Lions, *Modelling, analysis and control of thin plates*, Coll. RMA **6**, Masson, Paris, 1988.

10. J.E. Lagnese, *Asymptotic energy estimates for Kirchhoff plates subject to weak viscoelastic damping*, in *Control and estimation of distributed parameter systems* (F. Kappel, K. Kunisch and W. Schappacher, eds.), Internat. Ser. Num. Math. **91** (1989), 211–236.

11. ———, *Boundary stabilization of thin plates*, SIAM, Studies Appl. Math **10** (1989).

12. ———, *The reachability problem for thermoelastic plates*, Arch. Rat. Mech. Anal. **112** (1990), 223–267.

13. I. Lasiecka, *Controllability of a viscoelastic Kirchhoff plate*, in *Control and estimation of distributed parameter systems* (F. Kappel, K. Kunisch and W. Schappacher, eds.), Internat. Ser. Num. Math. **91** (1989), 236–247.

14. J.-L. Lions and E. Magenes, *Non-homogeneous boundary value problems and applications*, Vol. I, Springer-Verlag, Berlin, 1972.

15. A. Raoult, *Construction d'un modèle d'évolution de plaques avec terme d'inerte de rotation*, Annali Mat. Pura Appl. **139** (1985), 361–400.

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