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ASYMPTOTIC BEHAVIOR OF THE SOLUTIONS OF THE INTEGRO-DIFFERENTIAL EQUATIONS ON POSITIVE HALF-AXIS WITH NON-DIFFERENCE KERNEL OF A CERTAIN TYPE

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ABSTRACT. The asymptotic behavior of the solutions of a class of integro-differential equations on positive half-axis with non-difference kernel is investigated. The solutions are equal asymptotically to the sum of terms having the form of product of an exponent $e^{p_k x}$ and a polynomial. Numbers p_k are zeros of a function which is found in explicit form. Location of these zeros in the complex plane is also investigated. To obtain these results the technique of analytical continuation is used.

1. Introduction. Formulation of the problem, and the main result. There are many applied problems which lead to the equations of the form:

$$(1)\quad -\frac{d^2y}{dx^2}+y=\int_0^\infty R(x-t)y(t)\,dt+\int_0^\infty R_1(x+t)y(t)\,dt, \qquad x>0.$$

Equations of this kind arise in various fields of physics. As such, we may mention radiative equilibrium of stars [**3**], anomalous skin-effect in metals [**8,4,2**] stationary neutron density in multiplying media [**1, 5**], wave propagation in acoustic and electrodynamic waveguides [**10, 11, 7, 9**]. In all these fields of research there are many particular problems and cases which lead to the equation (1) with $R_1 \equiv 0$. These cases have been exhaustively treated with the standard Wiener-Hopf technique. However, there are many problems which cannot be simplified in this way. That is why the equation (1) in its general form deserves an independent investigation. It turned out rather unexpectedly that in many cases the results obtained in this paper as far as asymptotic behavior is concerned justify replacing of equation (1) with the equation with kernel $R_1 \equiv 0$. However, as is often the case, the more subtle features of the solution essentially depend on the kernel R_1. As an

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example, we may mention a problem in theory of neutron physics considered by the author [**6**] several years ago. The solution of the problem considered in [**6**] had a linear asymptotic behavior $y = ax + b$ and the problem of physical interest was to find the ratio a/b. It was shown [**6**] that this ratio was essentially affected by the second integral in (1), so the results obtained in [**6**] were new ones. It is the author's intention to apply the technique introduced in this paper to the investigation of some physical problems in the above mentioned fields.

We restrict the class of considered equations by the following three conditions:

(i) $R(x)$ is an even function with positive range;

(ii) the set of all numbers s such that:

$$\int_{-\infty}^{\infty} R(x)e^{sx}\,dx < \infty \tag{2}$$

has a positive or infinite superior boundary s^*, we also assume that $s^* > 1$;

(iii) the integrals $\int_{-\infty}^{\infty} |R_1(x)|e^{sx}\,dx$ converge for all $0 \le s < s^*$:

$$\int_{-\infty}^{\infty} |R_1(x)|e^{sx}\,dx < \infty, \qquad 0 \le s < s^*. \tag{3}$$

In many applied problems there is a natural restriction on desired solution, i.e.,

$$|y(x)| < \text{const} \cdot e^{\lambda x}, \qquad x > 0,$$

where λ is an a priori determined positive or negative number.

Let us consider the class of all twice differentiable functions satisfying the last condition with the same λ but not necessarily the same constant. We denote this class by Q_λ. In this paper we are concerned with solutions of equation (1) only in the class Q_λ provided $\lambda < s^*$.

The problem of solving the equation (1) is rather a formidable one. Fortunately, in many cases it is enough to investigate the qualitative side of the question. In particular, it is often of principal interest to determine the asymptotic behavior of solutions of equation (1) as $x \to \infty$. This is precisely the problem we are going to solve. Under three additional restrictions this problem was solved in [**1**]. These restrictions were:

(i) the function $R_1(x)$ was assumed to be proportional to the function $R(x)$;

(ii) $R(x)$ was normalized by the condition $\int_{-\infty}^{\infty} R(x)\,dx = 1$;

(iii) sought solutions were assumed to grow not faster than a polynomial. In this article we dispense with the first two of these restrictions and essentially weaken the last one.

The main result of the present paper is comprised in the following theorem.

Theorem 1. *If the conditions* (2) *and* (3) *are satisfied and* $\lambda > -s^*$, *then all solutions for* (1) *in the class* Q_λ (*if any*) *have the asymptotic behavior of the form*:

$$y(x) = \sum_k P_k(x) e^{p_k x} + O(e^{\mu x}).$$

The summation is over all zeros p_k *of the function*

$$G(p) = 1 - p^2 - \phi(ip) \tag{4}$$

lying in the open strip

$$\mu < \operatorname{Re} p < \lambda, \qquad \mu > -s^*,$$

ϕ *being the Fourier transform of the function* $R(x)$

$$\phi(t) = \int_{-\infty}^{\infty} e^{itx} R(x)\,dx, \qquad |\operatorname{Im} t| < s^*.$$

Function $P_k(x)$ *is a polynomial of degree* $\nu_k = m_k - 1$, m_k *being the multiplicity of the zero* p_k.

Notice that the main characteristics of the asymptotic behavior of the desired solution, i.e., numbers p_k, do not depend on the function R_1 and may be fully determined as soon as we know the function R. The same is true for the degrees of the polynomials $P_k(x)$ but not for their coefficients. They essentially depend on both kernels R and R_1 and cannot be evaluated without knowing the solution $y(x)$ on positive x-axis. Nevertheless, the formula for the coefficients of polynomials P_k which we obtain in the process of proving the theorem permits us to get their approximate values as soon as we have obtained some approximation of the solution $y(x)$.

2. Proof of Theorem 1. To prove Theorem 1 we use the technique of Laplace and Fourier transforms. With the help of equation (1) we deduce the important analytical properties of Laplace transform

$$Y(p) = \int_0^\infty y(t)e^{-pt}\,dt, \qquad \operatorname{Re} p > \lambda$$

assuming the existence of the solution $y(x) \in Q_\lambda$. One can easily see that under the conditions (2) and (3) the following inequalities hold:

$$\int_0^\infty R(x-t)y(t)\,dt \le \text{const} \cdot e^{\lambda x}$$
$$\int_0^\infty R_1(x+t)y(t)\,dt \le \text{const} \cdot e^{\lambda x}.$$

Thus, both integrals in the right-hand side of the equation (1) have Laplace transforms analytic in the half-plane $\operatorname{Re} p > \lambda$. Therefore we may apply Laplace transformation to both sides of equation (1) and obtain the following relation:

(5)
$$\begin{aligned} &Y(p)(1-p^2) + y'(0) + py(0) \\ &\quad = \int_0^\infty e^{-px}\left[\int_0^\infty R(x-t)y(t)\,dt + R_1(x+t)y(t)\,dt\right]dx. \end{aligned}$$

Let us consider Fourier transforms of functions $R(t)$ and $R_1(t)$, $R_1(t) \equiv 0$ if $t < 0$,

$$\phi(\alpha) = \int_{-\infty}^\infty e^{i\alpha t}R(t)\,dt, \qquad |\operatorname{Im}\alpha| < s^*$$
$$\phi_1(\alpha) = \int_{-\infty}^\infty e^{i\alpha t}R_1(t)\,dt, \qquad \operatorname{Im}\alpha < s^*.$$

Inverting these formulas we may write:

$$R(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\alpha t} \phi(\alpha)\, d\alpha$$
$$R_1(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\alpha t} \phi_1(\alpha)\, d\alpha.$$

The path of integration in expression for $R(t)$ may be shifted upward within the strip $|\operatorname{Im}\alpha| < s^*$:

$$R(t) = \frac{1}{2\pi} \int_{-\infty+is_0}^{\infty+is_0} e^{-i\alpha t} \phi(\alpha)\, d\alpha$$

where s_0 is an arbitrary number in the interval $\lambda < s_0 < s^*$. Similarly:

$$R_1(t) = \frac{1}{2\pi} \int_{-\infty-is_1}^{\infty-is_1} e^{-i\alpha t} \phi_1(\alpha)\, d\alpha, \qquad \lambda < s_1 < s^*.$$

Substituting $R(t)$ and $R_1(t)$ into (5) and reversing the order of integration which is permissible due to the conditions $\lambda < s_0 < s^*$ and $\lambda < s_1 < s^*$ we obtain for $\operatorname{Re} p > s_0$:

$$\int_0^\infty e^{-px}\, dx \int_0^\infty R(x-t) y(t)\, dt = \frac{1}{2\pi i} \int_{-\infty+is_0}^{\infty+is_0} \frac{\phi(\alpha) Y(-i\alpha)}{p+i\alpha}\, d\alpha, \qquad \operatorname{Re} p > s_0$$

$$\int_0^\infty e^{-px}\, dx \int_0^\infty R_1(x+t) y(t)\, dt = \frac{1}{2\pi i} \int_{-\infty-is_1}^{\infty-is_1} \frac{\phi_1(\alpha) Y(i\alpha)}{p+i\alpha}\, d\alpha, \qquad \operatorname{Re} p > s_1.$$

Therefore the relation (5) may be rewritten in the form

$$Y(p)(1-p^2) + y'(0) + py(0) = \frac{1}{2\pi i} \int_{-\infty+is_0}^{\infty+is_0} \frac{\phi(iz) Y(z)}{p-z}\, dz + \frac{1}{2\pi i} \int_{-\infty-is_1}^{\infty-is_1} \frac{\phi_1(iz) Y(-z)}{p-z}\, dz,$$

for all p in the half-plane $\operatorname{Re} p > \max\{s_0, s_1\}$. The remarkable fact is that the second term in the right-hand side of the equation (5) is an analytic function in the half-plane $\operatorname{Re} p > -s_1$ while the first one may be analytically continued onto the strip $\lambda < \operatorname{Re} p < s_0$, its analytical continuation being:

$$\Phi_1(p) = \frac{1}{2\pi i} \int_{-\infty+s_0}^{i\infty+s_0} \frac{\phi(iz)Y(z)}{p-z}\, dz + Y(p)\phi(ip),$$
$$\lambda < \operatorname{Re} p < s_0.$$

So in the strip $\lambda < \operatorname{Re} p < s_0$ we obtain:

$$\begin{aligned} Y(p)(1-p^2) + y'(0) + py(0) = {}& \frac{1}{2\pi i} \int_{-i\infty+s_0}^{i\infty+s_0} \frac{\phi(iz)Y(z)}{p-z}\, dz \\ & + Y(p)\phi(ip) \\ & + \frac{1}{2\pi i} \int_{-i\infty-s_1}^{i\infty-s_1} \frac{\phi_1(iz)Y(-z)}{p-z}\, dz \end{aligned}$$

or

$$\begin{aligned} (6) \quad Y(p) = \frac{1}{2\pi i G(p)} \Bigg[& \int_{-i\infty+s_0}^{i\infty+s_0} \frac{Y(z)\phi(iz)}{p-z}\, dz \\ & + \int_{-i\infty-s_1}^{i\infty-s_1} \frac{\phi_1(iz)Y(-z)}{p-z}\, dz \Bigg] - \frac{y'(0) + py(0)}{G(p)} \end{aligned}$$

where $\lambda < \operatorname{Re} p < s_0$ and

$$G(p) = 1 - p^2 - \phi(ip).$$

The expression in brackets is an analytical function in the strip $-s_1 < \operatorname{Re} p < s_0$. Thus the only singularities of the right-hand side of the last equation in the strip $-s_1 < \operatorname{Re} p < s_0$ are the zeros of the function $G(p)$. It is obvious that $G(p)$ has no singular points in this strip. Thus, we have proved the following

Lemma 1. *Laplace transform of any solution $y(x)$ of equation* (1) *such that $y(x) \in Q_\lambda$ has an analytical continuation onto the strip $-s^* < \operatorname{Re} p < \lambda$ with no other singularities except possibly zeros of* *the function* $G(p) = 1 - p^2 - \phi(ip)$ *where* $\phi(z)$ *is a Fourier transform of the kernel* $R(x)$.

Notice that $G(p)$ has no more than finite number of zeros in any closed strip within the strip $-s^* < \operatorname{Re} p < \lambda$. Moreover, according to the representation (6) function $Y(p)$ satisfies in the said closed strip the following inequality:

$$|Y(p)| < \text{const}/|p| \tag{7}$$

(if $|p|$ is sufficiently large).

Lemma 1 allows us to investigate the asymptotic behavior of the solution $y(x)$, $x \to \infty$. For this purpose we use the inversion formula

$$y(x) = \frac{1}{2\pi i}\int_{-i\infty+s}^{i\infty+s} Y(p)e^{px}\,dp, \qquad s > s^*.$$

Due to (7) we may shift to the left the contour of integration and get the formula:

$$y(x) = \frac{1}{2\pi i}\int_{-i\infty+\mu}^{i\infty+\mu} Y(P)e^{px}\,dp + \sum_k P_k(x)e^{p_k x}, \qquad -s^* < \mu < \lambda \tag{8}$$

where

$$P_k(x) = e^{-p_k x}\operatorname{Re} s_{p=p_k} Y(p)e^{px}.$$

The sum at the right-hand side of (8) accounts for all poles of the integrand lying in the strip $\mu < \operatorname{Re} p < \lambda$. It is not difficult to see that for the first term of (8) the following estimation is valid:

$$\frac{1}{2\pi i}\int_{-i\infty+\mu}^{i\infty+\mu} Y(p)e^{px}\,dp = O(e^{\mu x})$$

which concludes the proof of Theorem 1. □

Remark. Theorem 1 remains valid if the second term in the right-hand side of equation (1) is replaced with the integral

$$\int_0^\infty R_1(x+t, y(t))\,dt$$

provided it has Laplace transform analytic in the half-plane $\operatorname{Re} p > -s^*$.

3. Zeros of the function $G(p)$. According to Theorem 1 it is sufficient to locate zeros of the function $G(p)$ in the complex plane to obtain the most important characteristic of the asymptotic behavior of the solutions $y(x) \in Q_\lambda$. Therefore, it seems worthwhile to investigate this problem. We will begin with

Lemma 2. *The function $G(p)$ takes on only real values on the imaginary axis and on the interval $-s^* < p < s^*$. The function $G(p)$ decreases on the interval $0 \le p < s^*$ and*

$$\lim_{p \to \pm i\infty} G(p) = +\infty.$$

Also,

$$\operatorname{Re} G(\alpha + i\beta) \ge G(\alpha), \qquad -s^* < \alpha < s^*,\ -\infty < \beta < \infty.$$

To prove Lemma 2 it is sufficient to rewrite the definition (4) of the function $G(p)$ in the following way

$$G(\alpha + i\beta) = 1 - \alpha^2 + \beta^2 - 2i\alpha\beta - \int_{-\infty}^{\infty} R(t) e^{-\alpha t} (\cos \beta t - i \sin \beta t)\, dt.$$

Thus, since $R(t)$ is an even function, we have

$$\text{(9)}\quad \operatorname{Re} G(\alpha + i\beta) = 1 - \alpha^2 + \beta^2 - 2 \int_0^{\infty} R(t) \cosh \alpha t \cos \beta t\, dt, \qquad |\alpha| < s^*.$$

The last representation of the function $\operatorname{Re} G(\alpha + i\beta)$ makes evident all three assertions of Lemma 2. The following properties of the zeros of the function $G(p)$ now become obvious.

i) If the integral

$$\phi(0) = \int_{-\infty}^{\infty} R(t)\, dt$$

is greater than the unity, then the interval $(-s^*, s^*)$ has no zeros of the function $G(p)$. The imaginary axis contains at least two zeros.

ii) If $\phi(0) = 1$, then the single zero lying in the interval $(-s^*, s^*)$ is the point $p = 0$. Its multiplicity equals two.

iii) If $\phi(0) < 1$, the interval $(-s^*, s^*)$ contains exactly two zeros $p = -p^*$ and $p = p^*$. $p^* < 1$ because $G(1) = -\phi(i) < 0$ and $G(0) = 1 - \phi(0) > 0$. The strip $|\operatorname{Re} p| < p^*$ is free from zeros of $G(p)$.

The relation (9) allows us to reveal more properties of real and pure imaginary zeros of the function $G(p)$.

Let $\phi(0) < 1$ and p^* be the sole positive zero of $G(p)$. Setting $\alpha = p^*$, $\beta = 0$ in relation (9) we get:

$$0 = 1 - (p^*)^2 - 2\int_0^\infty R(t)\cosh p^* t\, dt.$$

Hence,

$$p^* < \frac{1 - \phi(0)}{1 + \int_0^\infty R(t)t^2\, dt} \tag{10}$$

since

$$\cosh x \geq 1 + x^2/2.$$

Inequality (10) simplifies the numerical evaluation of the quantity p^*.

If $\phi(0) > 1$ and $p^* = i\beta$, $\beta > 0$, is a pure imaginary zero of $G(p)$, the relation (9) yields in a similar way two inequalities:

$$\frac{\phi(0) - 1}{1 + \int_0^\infty R(t)t^2\, dt} < \beta^2 < \phi(0) - 1.$$

Relation (9) yields a sufficient condition for imaginary axis having precisely two zeros of $G(p)$. Indeed, setting in relation (9) $\alpha = 0$ and differentiating with respect to β, we get

$$\frac{d}{d\beta}G(i\beta) = 2\beta + 2\int_0^\infty R(t)t\sin\beta t\, dt < 2\beta\left(1 - \int_0^\infty R(t)t^2\, dt\right).$$

So if the inequality

$$\int_0^\infty R(t)t^2\,dt < 1$$

holds, the function $G(p)$ increases along the positive imaginary axis and therefore cannot have there more than one zero, $\phi(0) > 1$.

4. The non-increasing and slowly increasing solutions of the equation (1). The last two properties of zeros of $G(p)$ allow us to strengthen Theorem 1 in case when $\phi(0) \le 1$:

Theorem 2. *If $\phi(0) < 1$ and $|\lambda| < p^*$ where p^* is the sole zero of $G(p)$ lying on the positive axis then all solutions $y(x)$ of the equation (1) belonging to the class Q_λ asymptotically decrease not slower than e^{-p^*x}. If $\phi(0) = 1$ and λ is any positive number such that the strip $0 < \operatorname{Re} p \le \lambda$ is free from zeros of function $G(p)$, then all solutions of (1) from Q_λ have the following asymptotic representation*

$$y(x) = a + bx + O(e^{-\mu x})$$

where μ is an arbitrary positive number such that the strip $-\mu < \operatorname{Re} p < 0$ is free from zeros of function $G(p)$, i.e., they cannot increase faster than some polynomial of the first degree.

Proof. If $\phi(0) < 1$, the function $G(p)$ has no zeros in the strip $-p^* < \operatorname{Re} p < p^*$. Thus, all zeros lying to the left from the line $\operatorname{Re} p = \lambda$ ($\lambda < p^*$) are situated in the half-plane $\operatorname{Re} p \le -p^*$. Therefore the first statement of the theorem is an immediate consequence of Theorem 1. If $\phi(0) > 1$, then at least one zero of the function $G(p)$ belongs to the imaginary axis. Since the function $G(p)$ has only finite number of zeros there exist $\lambda > 0$ such that all zeros with positive real part are lying to the right from the line $\operatorname{Re} p = \lambda$. Applying once more Theorem 1 we obtain the second statement of Theorem 2. □

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