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FAST SOLUTION OF A CLASS OF PERIODIC PSEUDODIFFERENTIAL EQUATIONS

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ABSTRACT. This paper presents a quadrature method for discretizing periodic pseudodifferential equations. The principal part is discretized by a product rule and the smooth remaining part is discretized by the rectangular rule. This discretization yields as rapid convergence of the truncation error as discretization by global spectral methods, and gives rise to a linear system of algebraic equations with a structure that enables rapid solution by iterative methods. We present error bounds for the discretization and discuss the convergence of an iterative method.

1. Introduction. The solution of boundary value problems for homogeneous elliptic partial differential equations with constant coefficients on a simply connected region Ω in the plane with a smooth boundary curve Γ can often be conveniently computed by solving a boundary integral equation. This approach is particularly attractive if Ω contains the point at infinity. Properties of integral equations so obtained, as well as of more general ones, can be studied by using the theory for periodic pseudodifferential operators. This theory can also be applied to study properties of numerical schemes for the solution of the integral equations. Pseudodifferential operators are defined as follows. Let the function $u \in L^2(-\pi,\pi)$ have the Fourier coefficients

$$\hat{u}(m) = \frac{1}{2\pi}\int_{-\pi}^{\pi} u(\sigma)e^{-im\sigma}\,d\sigma, \qquad m \in \mathbf{Z},$$

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and introduce the projections

$$(\mathcal{P}_+u)(s) = \sum_{m\geq 1} \hat{u}(m)e^{ims},$$
$$(\mathcal{P}_-u)(s) = \sum_{m\leq -1} \hat{u}(m)e^{ims},$$
$$(\mathcal{P}_0u)(s) = \hat{u}(0).$$

They satisfy

$$\mathcal{P}_+ + \mathcal{P}_- + \mathcal{P}_0 = I, \qquad \mathcal{P}_+ - \mathcal{P}_- = \mathcal{H},$$

where $\mathcal{H}$ is the periodic Hilbert transform

$$(\mathcal{H}u)(s) = \frac{1}{2\pi i} PV \int_{-\pi}^{\pi} \cot\frac{\sigma - s}{2} u(\sigma)\, d\sigma.$$

Define the Bessel potential operator $\mathcal{D}^\beta$ of order $\beta \in \mathbf{R}$ by

$$\mathcal{D}^\beta e^{ims} = |m + \delta_{0m}|^\beta e^{ims}, \qquad m \in \mathbf{Z},$$

with δ_{0m} denoting the Kronecker symbol. We are now in a position to introduce the 2π-periodic pseudodifferential operator of order β,

$$\mathcal{A} = (a_+\mathcal{P}_+ + a_-\mathcal{P}_- + a_0\mathcal{P}_0)\mathcal{D}^\beta. \tag{1.1}$$

Here a_+, a_- and a_0 are complex numbers. Properties of pseudodifferential operators are discussed in [**25, 27, 40, 47**]. The present paper considers the solution of pseudodifferential equations of the form

$$(\mathcal{A} + \mathcal{B})w = f, \tag{1.2}$$

where the right hand side function f and the solution w belong to appropriate function spaces, and $\mathcal{B}$ is an integral operator defined by

$$(\mathcal{B}u)(s) = \int_{-\pi}^{\pi} b(s,\sigma)u(\sigma)\, d\sigma.$$

The kernel $b(s,\sigma)$ is assumed to be continuous and 2π-periodic in both variables.

Example 1.1. Let $\beta = 0$ and $a_+ = a_0 = a_- = 1$. Then $\mathcal{A} = I$ and the equation (1.2) is a Fredholm integral equation of the second kind.

Example 1.2. Let $\beta = 0$, $a_+ = a_0 = 1$ and $a_- = -1$. Then $\mathcal{A}$ is a Cauchy integral operator of the first kind. If, instead, $a_+ = 2$, $a_0 = 1$ and $a_- = 0$, then $\mathcal{A}$ is a Cauchy integral operator of the second kind.

Example 1.3. Define the integral operator associated with a single-layer potential on the unit circle $\{\rho e^{it} : -\pi \le t < \pi\}$ of radius $\rho > 0$,

$$(\mathcal{A}\,u)(t) = \frac{1}{\pi}\int_{-\pi}^{\pi} \ln \frac{1}{|\rho e^{it} - \rho e^{is}|} u(s)\,ds, \qquad -\pi \le t < \pi.$$

Let $u_k(t) = e^{ikt}$. Then

$$(\mathcal{A}\,u_k)(t) = \begin{cases} (1/|k|)u_k(t) & \text{if } k \neq 0, \\ -2\ln(\rho)u_0(t) & \text{if } k = 0. \end{cases}$$

Thus, $\mathcal{A}$ can be written in the form (1.1) with $\beta = -1$, $a_+ = a_- = 1$ and $a_0 = -2\ln\rho$.

Example 1.4. Introduce the hypersingular integral operator

$$(\mathcal{A}\,u)(t) = -\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1}{1-\cos(s-t)} u(s)\,ds, \qquad -\pi \le t < \pi.$$

Let $u_k(t) = e^{ikt}$. Then $(\mathcal{A}\,u_k)(t) = |k|u_k(t)$. Thus, $\mathcal{A}$ can be written in the form (1.1) with $\beta = 1$, $a_+ = a_- = 1$ and $a_0 = 0$. For other examples of hypersingular integral operators, see [**16, 32**].

We will henceforth assume that $a_+a_-a_0 \neq 0$ in order to secure that $\mathcal{A}$ is invertible. Properties of integral equations of the form (1.2) are discussed in [**18, 27, 40, 47**]. These equations have many applications, such as to scattering, flows, elasticity and conformal mapping; see [**7, 12, 15, 17, 19, 20, 28, 31, 33, 45, 46, 47**] and references therein.

If we allow the righthand side f and the solution w of (1.2) to be vector-valued, then a larger class of problems can be solved. This extension is fairly straightforward but will not be discussed in the present paper. A still larger class of problems can be solved if we allow the solution w to satisfy certain constraints. Vector-valued solutions that satisfy constraints arise when solving partial differential equations on multiply connected regions with smooth periodic mutually exterior boundary curves, as well as in conformal mapping of multiply connected regions, see [**14, 33**].

The periodicity of the operators $\mathcal{A}$ and $\mathcal{B}$ makes it natural to use trigonometric polynomials to discretize equation (1.2), and many discretization methods based on trigonometric polynomials are available, such as Galerkin, Petrov-Galerkin, discrete Galerkin and collocation methods; see [**2, 4, 22, 24, 25, 30, 33, 39**] and references therein. We remark, however, that other discretization methods for (1.2) are also available, among them methods based on discretization by splines; see, e.g., [**16, 18, 44**]. A recent survey of discretization methods is presented in [**43**]. The linear system of algebraic equations obtained after discretization of (1.2) has a structure that makes iterative solution attractive. Available iterative schemes include two-grid and multigrid methods [**3, 6, 16, 34, 36, 41**].

This paper describes a solution method based on trigonometric polynomials. Our scheme is a quadrature method in which we discretize the operator $\mathcal{A}$ by a product integration rule and the operator $\mathcal{B}$ by the rectangular rule. The discretization method is described in Section 2. Error estimates for the computed solution of the discretized system of equations are given in Section 3. The linear system of algebraic equations has a special structure which can be exploited to introduce a preconditioned stationary Richardson iterative method. This structure and the preconditioned iterative method are discussed in Section 4. In Section 5 we determine the rate of convergence of the iterative scheme. Section 5 generalizes results for an iterative method for integral equations presented and discussed in [**34, 36, 48, 49**]. Properties of our iterative scheme are summarized in Section 6. The iterative scheme of this paper would appear to be particularly attractive for the solution of time dependent or nonlinear problems involving the operator $\mathcal{A}+\mathcal{B}$. For discussions on nonlinear problems, see [**5, 38**]. A time dependent problem is treated in [**14**, Example 2].

2. A quadrature method. This section introduces a quadrature method for the discretization of the pseudodifferential equation (1.2).

We regard $\mathcal{A}$ as a singular integral operator and discretize $\mathcal{A}\,w$ by a product integration rule as follows. Assume for notational convenience that N is an even positive integer, and introduce the set $\Lambda_h = \{-N/2+1, -N/2+2, \dots, N/2\}$. Let $h = 2\pi/N$ and $t_k = kh$ for $k \in \Lambda_h$. Define the interpolation operator $\mathcal{Q} : C[-\pi,\pi] \to \operatorname{span}\{e^{ims}, m \in \Lambda_h\}$ by

$$(\mathcal{Q}\,v)(t_k) = v(t_k), \qquad k \in \Lambda_h. \tag{2.1}$$

Let $v_h(m)$ denote the discrete Fourier coefficients associated with $v \in C[-\pi,\pi]$, i.e.,

$$v_h(m) = \frac{1}{N} \sum_{k \in \Lambda_h} v(t_k) e^{-imt_k}, \qquad m \in \Lambda_h. \tag{2.2}$$

Then $\mathcal{Q}$ can be written as

$$(\mathcal{Q}\,v)(s) = \sum_{m \in \Lambda_h} v_h(m) e^{ims}.$$

Our product integration rule approximates $(\mathcal{A}\,v)(s)$ by $(\mathcal{A}\,\mathcal{Q}\,v)(s)$ and evaluates the latter operator exactly. Thus, using the relation (2.2), we obtain

$$(\mathcal{A}\,v)(s) \approx (\mathcal{A}\,\mathcal{Q}\,v)(s) = \frac{1}{N} \sum_{k \in \Lambda_h} v(t_k) \rho(s - t_k), \tag{2.3}$$

where

$$\rho(s) = \sum_{m=-N/2+1}^{-1} a_-(-m)^\beta e^{ims} + a_0 + \sum_{m=1}^{N/2} a_+ m^\beta e^{ims}. \tag{2.4}$$

We discretize $\mathcal{B}\,w$ by the rectangular rule

$$\int_{-\pi}^{\pi} v(\sigma)\, d\sigma \approx h \sum_{k \in \Lambda_h} v(t_k). \tag{2.5}$$

Since the kernel $b(s,\sigma)$ is smooth and 2π-periodic in each variable, this quadrature rule yields high accuracy. Thus, in equation (1.2) we apply the product integration rule (2.3) to integrate $\mathcal{A}w$ and we use the rectangular rule (2.5) to integrate $\mathcal{B}w$. Collocating (1.2) at the nodes t_k for $k \in \Lambda_h$ yields the linear system of algebraic equations

$$\sum_{k\in\Lambda_h} \left(\frac{1}{N}\rho(t_{j-k}) + hb(t_j, t_k)\right) w_k = f_j, \qquad j \in \Lambda_h, \tag{2.6}$$

for the approximations w_k of $w(t_k)$, where $f_j = f(t_j)$. We consider this approach of approximating equation (1.2) a quadrature method. This quadrature method has several advantages over other numerical discretization schemes:

i) The computation of the matrix elements does not require the evaluation of integrals by numerical quadrature as in Galerkin and collocation methods. The only necessary computation is for the evaluation of the weights $\rho(t_k)$ for $k \in \Lambda_h$, and this can be carried out in only $O(N \log N)$ arithmetic operations by the fast Fourier transform (FFT) algorithm.

ii) The discretization error converges as quickly (polynomially or exponentially) as for global spectral methods. For properties of the latter, see [**4, 24, 25, 22, 43**]. In contrast, spline Galerkin, collocation and qualocation methods have usually low order of convergence; see [**18, 42, 43, 44**].

iii) The linear system (2.6) has a structure which can be exploited in the development of iterative methods for the numerical solution of the system. In particular, the matrix obtained by discretizing the operator $\mathcal{A}$ is a circulant matrix.

The quadrature method has been presented for N even, and results of the present paper are stated for the case when N is even only. However, the quadrature method can be modified to be applicable for N odd, and our results remain valid for this modification.

We remark that the quadrature method of the present paper is mathematically equivalent to the discrete trigonometric collocation and Galerkin methods discussed in [**1, 4, 26**]. The methods differ in the formulation of the linear system of algebraic equations, in that for the quadrature method the function values at the nodes are the unknowns to be determined, while for the collocation and Galerkin methods the unknowns to be determined are the Fourier coefficients of the computed solution. Consequently, implementations of the collocation and Galerkin methods require numerical integration for every element of the matrix, while implementation of the quadrature method does not. This makes it easier to implement the quadrature method than to implement the Galerkin and collocation methods discussed in [**1, 4, 26**].

The circulant matrix mentioned in iii) can be diagonalized explicitly without any computational work, assuming that the weights $\rho(t_k)$ are known. For future reference, we formulate this result as a lemma. Introduce the unitary matrix

$$\mathbf{F} = N^{-1/2}[\omega^{jk}]_{j,k=-N/2+1}^{N/2}, \qquad \omega = e^{ih}. \tag{2.7}$$

Thus, $\mathbf{FF}^* = \mathbf{F}^*\mathbf{F} = \mathbf{I}$, where * denotes transposition and complex conjugation. Throughout this paper, matrices and vectors are written in boldface. Define the matrix

$$\mathbf{A} = \left[\frac{1}{N}\rho(t_{j-k})\right]_{j,k=-N/2+1}^{N/2}, \qquad t_k = kh. \tag{2.8}$$

Lemma 1. *Let* $\mathbf{D}_- = \operatorname{diag}[(N/2-1)^\beta, (N/2-2)^\beta, \dots, 2^\beta, 1]$ *and* $\mathbf{D}_+ = \operatorname{diag}[1, 2^\beta, \dots, (N/2)^\beta]$. *Define*

$$\mathbf{D} = \begin{bmatrix} a_-\mathbf{D}_- & & \\ & a_0 & \\ & & a_+\mathbf{D}_+ \end{bmatrix}.$$

Then

$$\mathbf{D} = \mathbf{F}^*\mathbf{AF}.$$

Proof. The lemma follows from the observation that the matrix $\mathbf{A}$ is a circulant matrix. It therefore has the discrete Fourier coefficients of $\rho(s)$ as eigenvalues and the columns of $\mathbf{F}$ as eigenvectors. $\square$

3. Convergence of the quadrature method. Typically, quadrature methods for integral equations are analyzed using the L^∞-norm.

This approach yields pointwise convergence at the nodes t_k. Another approach is to reformulate the quadrature method as an equivalent discrete trigonometric collocation method and then apply results in [**26**] for appropriate Hölder-Zygmund function spaces. However, due to the periodicity of the operators $\mathcal{A}$ and $\mathcal{B}$, we find it more convenient to study the quadrature method, and an iteration scheme introduced in Section 4, by using Fourier analysis and periodic Sobolev space norms. Our analysis shows that the stability of the linear system of algebraic equations (2.6) is equivalent to the stability of an approximate integral operator. In Section 5 we apply the analysis of the present section to determine the rate of convergence of the iterative method for the solution of (2.6).

Our analysis uses the function space $L^2(-\pi,\pi)$ and its norm

$$||v|| = \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} |v(\sigma)|^2\,d\sigma\right)^{1/2},$$

as well as the Sobolev space of 2π-periodic functions $H^r(2\pi)$ and its norm

$$||v||_r = \left(\sum_{m\in\mathbf{Z}} |m+\delta_{0m}|^{2r}|\hat{v}(m)|^2\right)^{1/2},$$

where r is an arbitrary real number. In particular, $||v||_0 = ||v||$. We also need the space of continuous 2π-biperiodic functions

$$\begin{aligned} C(2\pi\times 2\pi) = \{v\in C(\mathbf{R}^2) : v(s+2\pi,\sigma) \\ = v(s,\sigma+2\pi) = v(s,\sigma), (s,\sigma)\in\mathbf{R}^2\}. \end{aligned}$$

We will use the notation

$$v^{(m)}(s) = \frac{d^m v}{ds}, \qquad v^{(m_1,m_2)}(s,\sigma) = \frac{\partial^{m_1+m_2} v}{\partial s^{m_1}\partial\sigma^{m_2}}.$$

Define the projection operator $\mathcal{P} : L^2(-\pi,\pi) \to \text{span}\,\{e^{ims}, m\in\Lambda_h\}$ by

$$(\mathcal{P}\,v)(s) = \sum_{m\in\Lambda_h} \hat{v}(m)e^{ims}.$$

Note that $\mathcal{P}$ is closely related to interpolation operator $\mathcal{Q}$ defined by (2.1). Throughout this section C, C', C'' and C''' denote generic positive constants which are independent of N. The following lemma reviews well-known convergence properties of the interpolation operator $\mathcal{Q}$ and the projection operator $\mathcal{P}$.

Lemma 2. *Let r be a real number. Then*

$$\begin{aligned} ||v - \mathcal{P}\, v|| &\le Ch^r ||v||_r, \qquad && v \in H^r(2\pi), r \ge 0,\\ ||v - \mathcal{Q}\, v|| &\le Ch^r ||v||_r, && v \in H^r(2\pi), r > 1/2,\\ ||(\mathcal{P} - \mathcal{Q})v|| &\le Ch^r ||v||_r, && v \in H^r(2\pi), r > 1/2. \end{aligned}$$

Proof. See, for example, [**11**, Chapter 9] for proofs. Related results can be found in [**1, 9, 25, 29**].

We apply Lemma 2 to obtain error bounds for the quadrature rules (2.3) and (2.5). These bounds are given by Lemmas 3 and 4.

Lemma 3. *Let $v \in H^r(2\pi)$ for some real number $r > 1/2$, and define*

$$E_h(v) = \int_{-\pi}^{\pi} v(\sigma)\, d\sigma - h \sum_{k \in \Lambda_h} v(t_k). \tag{3.1}$$

Then

$$|E_h(v)| \le Ch^r ||v||_r. \tag{3.2}$$

Proof. The rectangular rule (2.5) is exact for all $v \in \operatorname{span}\{e^{ins}, n \in \Lambda_h\}$, and therefore $E_h(v) = \int_{-\pi}^{\pi} (v - \mathcal{Q}\, v)\, d\sigma$. In view of Lemma 2, this yields inequality (3.2). $\square$

Lemma 4. *Let the operator $\mathcal{A}$ be of order $\beta \in \mathbf{R}$, and let $\rho(s)$ be defined by* (2.4). *Assume that $v \in H^r(2\pi)$ for some $r \in \mathbf{R}$, such that $r > \max\{1/2, 1/2 + \beta\}$. Introduce*

$$(E_h(v))(s) = (\mathcal{A}\, v)(s) - \frac{1}{N} \sum_{k \in \Lambda_h} v(t_k) \rho(s - t_k).$$

Then

$$||\mathcal{Q}\, E_h(v)|| \leq Ch^{\min\{r,r-\beta\}}||v||_r.$$

Proof. The equality in (2.3) yields

$$E_h(v) = \mathcal{A}\,(I - \mathcal{Q}\,)v,$$

and applying the relations $A\mathcal{P} = \mathcal{P}A$, $\mathcal{Q}\,\mathcal{P} = \mathcal{P}$ and $\mathcal{Q}^2 = \mathcal{Q}$ shows that

$$\begin{aligned}\mathcal{Q}\, E_h(v) &= \mathcal{Q}\,\mathcal{A}\, v - \mathcal{Q}\,\mathcal{A}\,\mathcal{P}\, v + \mathcal{Q}\, A(\mathcal{P} - \mathcal{Q}\,)v \\ &= (\mathcal{Q} - \mathcal{P}\,)\mathcal{A}\, v + \mathcal{A}\,(\mathcal{P} - \mathcal{Q}\,)v.\end{aligned}$$

Hence, Lemmas 1 and 2 yield

$$\begin{aligned}||\mathcal{A}\,(\mathcal{P} - \mathcal{Q}\,)v|| &\leq C||(\mathcal{P} - \mathcal{Q}\,)v||_\beta \\ &\leq C' N^{\max\{\beta,0\}}||(\mathcal{P} - \mathcal{Q}\,)v|| \\ &\leq C'' h^{\min\{r,r-\beta\}}||v||_r.\end{aligned} \tag{3.3}$$

The operator $\mathcal{A} : H^r(2\pi) \to H^{r-\beta}(2\pi)$ is bounded, i.e., there is a constant C' such that for all $v \in H^r(2\pi)$ the inequality

$$||\mathcal{A}\, v||_{r-\beta} \leq C'||v||_r$$

is valid. Thus, we apply Lemma 2 in order to obtain the bound

$$||(\mathcal{P} - \mathcal{Q}\,)\mathcal{A}\, v|| \leq Ch^{r-\beta}||\mathcal{A}\, v||_{r-\beta} \leq C'' h^{r-\beta}||v||_r. \tag{3.4}$$

Combining (3.3) and (3.4) completes the proof. □

Define the matrix

$$\mathbf{B} = [hb(t_j, t_k)]_{j,k=-N/2+1}^{N/2}, \tag{3.5}$$

and write equation (2.6) as

$$(\mathbf{A} + \mathbf{B})\mathbf{w} = \mathbf{f}, \tag{3.6}$$

where $\mathbf{A}$ is given by (2.8), $\mathbf{w} = [w_{-N/2+1}, \ldots, w_{N/2}]^T$ and $\mathbf{f} = [f(t_{-N/2+1}), \ldots, f(t_{N/2})]^T$. Using the error bounds for the quadrature formulas, we can determine how well the matrix $\mathbf{A}+\mathbf{B}$ approximates the integral operator $\mathcal{A}+\mathcal{B}$. This requires additional notation. For $\mathbf{v} = [v_{-N/2+1}, \dots, v_{N/2}]^T$ and $\mathbf{u} = [u_{-N/2+1}, \dots, u_{N/2}]^T$ in $\mathbf{C}^N$, introduce the inner product and the associated vector norm

$$\langle \mathbf{u}, \mathbf{v} \rangle = \frac{1}{N} \sum_{k \in \Lambda_h} v_k \bar{u}_k, \qquad ||\mathbf{v}|| = (\langle \mathbf{v}, \mathbf{v} \rangle)^{1/2}, \tag{3.7}$$

where the bar denotes complex conjugation. We also use $|| \cdot ||$ to denote the matrix norm induced by the vector norm (3.7). Define the restriction operator $\mathbf{r}_h : C[-\pi, \pi] \to \mathbf{C}^N$ by

$$\mathbf{r}_h v = [v(t_{-N/2+1}), \dots, v(t_{N/2})]^T,$$

and for future reference we also introduce the prolongation operator $p_h : \mathbf{C}^N \to C[-\pi, \pi]$,

$$\begin{aligned} (p_h \mathbf{v})(s) &= \sum_{m \in \Lambda_h} v_h(m) e^{ims}, \\ v_h(j) &= \frac{1}{N} \sum_{m \in \Lambda_h} v_m e^{-ijt_m}. \end{aligned} \tag{3.8}$$

When the integral operator $\mathcal{A}+\mathcal{B}$ is approximated by the matrix $\mathbf{A}+\mathbf{B}$, we obtain the truncation error

$$\boldsymbol{\varepsilon}_v = \mathbf{r}_h(\mathcal{A}+\mathcal{B})v - (\mathbf{A}+\mathbf{B})\mathbf{r}_h v. \tag{3.9}$$

In particular, letting v in (3.9) be the solution w of (1.2) yields in view of (3.6) that

$$\boldsymbol{\varepsilon}_w = (\mathbf{A}+\mathbf{B})(\mathbf{w} - \mathbf{r}_h w). \tag{3.10}$$

Our convergence analysis yields a bound for the truncation error as well as stability properties of the matrix $\mathbf{A}+\mathbf{B}$.

Lemma 5. *Let the operator $\mathcal{A}$ be of order $\beta \in \mathbf{R}$, and let $v \in H^r(2\pi)$ for some $r \in \mathbf{R}$, such that $r > \max\{1/2, 1/2+\beta\}$. Let $b \in C(2\pi \times 2\pi)$ be such that $b^{(0,r)} \in C(2\pi \times 2\pi)$. Then*

$$||\boldsymbol{\varepsilon}_v|| \le C h^{\min\{r, r-\beta\}} ||v||_r.$$

Proof. Split the truncation error given by (3.9) into two parts

$$\boldsymbol{\varepsilon}_v = \boldsymbol{\varepsilon}'_v + \boldsymbol{\varepsilon}''_v,$$

where

$$\begin{aligned}\boldsymbol{\varepsilon}'_v &= \mathbf{r}_h \mathcal{A}\, v - \mathbf{A}\mathbf{r}_h v = \mathbf{r}_h \mathcal{A}\, v - \mathbf{r}_h \mathcal{A}\, \mathcal{Q}\, v,\\ \boldsymbol{\varepsilon}''_v &= \mathbf{r}_h \mathcal{B}\, v - \mathbf{B}\mathbf{r}_h v.\end{aligned}$$

For $u \in C[-\pi, \pi]$, we have

$$||\mathbf{r}_h u|| = ||\mathcal{Q}\, u||,$$

and this together with an application of Lemma 4 yields that

$$\begin{aligned}||\boldsymbol{\varepsilon}'_v|| &= ||\mathbf{r}_h \mathcal{A}\,(v - \mathcal{Q}\, v)|| = ||\mathcal{Q}\,\mathcal{A}\,(v - \mathcal{Q}\, v)||\\ &= ||\mathcal{Q}\, E_h(v)|| \le Ch^{\min\{r, r-\beta\}}||v||_r.\end{aligned}$$

The entries of $\boldsymbol{\varepsilon}''_v$ are given by

$$\begin{aligned}\varepsilon''_v(j) = \int_{-\pi}^{\pi} & v(\sigma) b(t_j, \sigma)\, d\sigma\\ & - h \sum_{k \in \Lambda_h} v(t_k) b(t_j, t_k), \qquad j \in \Lambda_h.\end{aligned}$$

Since for all $v, w \in \operatorname{span}\{e^{ins}, n \in \Lambda_h\}$,

$$\int_{-\pi}^{\pi} v(\sigma) w(\sigma)\, d\sigma = h \sum_{k \in \Lambda_h} v(t_k) w(t_k),$$

$\boldsymbol{\varepsilon}''_v$ can be written as

$$\begin{aligned}\varepsilon''_v(j) &= \int_{-\pi}^{\pi} v(\sigma) b(t_j, \sigma)\, d\sigma - \int_{-\pi}^{\pi} \mathcal{Q}\, v(\sigma) \mathcal{Q}\, b(t_j, \sigma)\, d\sigma\\ &= \int_{-\pi}^{\pi} v(\sigma)(I - \mathcal{Q}\,) b(t_j, \sigma)\, d\sigma + \int_{-\pi}^{\pi} (I - \mathcal{Q}\,) v(\sigma) \mathcal{Q}\, b(t_j, \sigma)\, d\sigma.\end{aligned}$$

An application of Lemma 3 therefore yields that

$$|\varepsilon''_v(j)| \le Ch^r ||v||_r, \qquad j \in \Lambda_h.$$

This shows that

$$||\varepsilon_v''|| \le Ch^r ||v||_r.$$

The lemma now follows from $||\varepsilon_v|| \le ||\varepsilon_v'|| + ||\varepsilon_v''||$. $\square$

From the assumption $a_- a_+ a_0 \neq 0$ and Lemma 1, it follows that the matrix $\mathbf{A}$ is invertible. Since $\mathbf{A}+\mathbf{B} = \mathbf{A}(\mathbf{I}+\mathbf{A}^{-1}\mathbf{B})$, the essential part of our stability analysis is to bound the norm of the matrix $\mathbf{I}+\mathbf{A}^{-1}\mathbf{B}$. We apply the following lemma for this purpose. Related results can be found in several of the references, see, e.g., [**3**].

Lemma 6 ([**8**, Lemma 2.5]). *Let $\mathcal{G}$ be an integral operator on $L^2(-\pi,\pi)$ defined by*

$$(\mathcal{G}\, v)(s) = \int_{-\pi}^{\pi} v(\sigma) g(s,\sigma)\, d\sigma, \tag{3.11}$$

where the kernel g satisfies the Lipschitz conditions

$$|g(s,\sigma) - g(s',\sigma)| \le C|s-s'|,$$
$$|g(s,\sigma) - g(s,\sigma')| \le C|\sigma-\sigma'|,$$

for $s, s', \sigma, \sigma' \in [-\pi,\pi]$. Assume that

$$||(I+\mathcal{G}\,)v|| \ge C'||v||, \qquad v \in L^2(-\pi,\pi), \tag{3.12}$$

for some constant $C' > 0$ independent of v, and define the matrix

$$\mathbf{G} = [hg(t_j,t_k)]_{j,k=-N/2+1}^{N/2}. \tag{3.13}$$

Then, for sufficiently small $h > 0$, there is a constant $C'' > 0$ independent of h and $\mathbf{v}$, such that

$$||(\mathbf{I}+\mathbf{G})\mathbf{v}|| \ge C''||\mathbf{v}||, \qquad \mathbf{v} \in \mathbf{C}^N.$$

Define $\mathcal{G} = \mathcal{A}^{-1}\mathcal{B}$. Then $\mathcal{G}$ can be written as an integral operator of the form (3.11) with

$$g(s,\sigma) = g_\sigma(s) = (\mathcal{A}^{-1} b_\sigma)(s), \qquad b_\sigma(s) = b(s,\sigma).$$

The operator $\mathcal{A}^{-1} : H^{r-\beta} \to H^r$ is bounded, i.e., there is a constant C such that $||\mathcal{A}^{-1}u||_r \leq C||u||_{r-\beta}$ for all $u \in H^{r-\beta}(2\pi)$. Therefore,

$$||g_\sigma(t)||_r \leq C||b_\sigma(t)||_{r-\beta}. \tag{3.14}$$

Lemma 7. *Assume that the operator $\mathcal{A} + \mathcal{B} : H^0(2\pi) \to H^{-\beta}(2\pi)$ is invertible and that $b \in C(2\pi \times 2\pi)$ satisfies*

$$\begin{gathered} ||b_\sigma(s)||_{r-\beta} \leq C, \\ ||b_\sigma^{(0,1)}(s)||_{1-\beta} \leq C, \\ \sigma \in [-\pi, \pi], \end{gathered} \tag{3.15}$$

for some $r \in \mathbf{R}$, such that $r > \max\{|\beta| + 1/2, 3/2\}$. Then, for sufficiently large N,

$$||(\mathbf{A} + \mathbf{B})^{-1}|| \leq Ch^{\min\{0,\beta\}}.$$

Proof. Partition the matrices $\mathbf{G}$ and $\mathbf{B}$, defined by (3.13) and (3.5), respectively, into columns

$$\mathbf{G} = h[\mathbf{g}_{-N/2+1}, \dots, \mathbf{g}_{N/2}], \qquad \mathbf{B} = h[\mathbf{b}_{-N/2+1}, \dots, \mathbf{b}_{N/2}],$$

where $\mathbf{g}_m = \mathbf{r}_h g_{t_m}$, and $\mathbf{b}_m = \mathbf{r}_h b_{t_m}$. For each $m \in \Lambda_h$, it follows from the bound for $\boldsymbol{\varepsilon}'_v$ in Lemma 5 and from (3.14) that

$$\begin{aligned} ||\mathbf{A}\mathbf{g}_m - \mathbf{b}_m|| &= ||\mathbf{A}\mathbf{r}_h g_{t_m} - \mathbf{r}_h \mathcal{A}\, g_{t_m}|| \\ &= ||\boldsymbol{\varepsilon}'_{g_{t_m}}|| \\ &\leq Ch^{\min\{r, r-\beta\}}||g_{t_m}||_r \\ &\leq C'h^{\min\{r, r-\beta\}}||b_{t_m}||_{r-\beta} \\ &\leq C''h^{\min\{r, r-\beta\}}. \end{aligned}$$

Thus,

$$\begin{aligned} ||(\mathbf{AG} - \mathbf{B})\mathbf{v}|| &\leq hN \max_{m \in \Lambda_h} \{||\mathbf{A}\mathbf{g}_m - \mathbf{b}_m||\}||\mathbf{v}|| \\ &\leq C'''h^{\min\{r, r-\beta\}}||\mathbf{v}||. \end{aligned} \tag{3.16}$$

From the invertibility of the operators $\mathcal{A}$ and $\mathcal{A}+\mathcal{B}$, it follows that the operator $I+\mathcal{G}$ is invertible on $L^2(-\pi,\pi)$. Let $C'=||(I+\mathcal{G})^{-1}||$. It follows from

$$||v|| = ||(I+\mathcal{G})^{-1}(I+\mathcal{G})v|| \le C'||(I+\mathcal{G})v||$$

that condition (3.12) of Lemma 6 holds. The requirements (3.15) on b and $r>3/2$ yield, in view of (3.14), that

$$\begin{aligned}
|g^{(1,0)}(s,\sigma)| &\le C||g_\sigma(s)||_r \le C'||b_\sigma(s)||_{r-\beta} \le C'',\\
|g^{(0,1)}(s,\sigma)| &\le C||g_\sigma^{(0,1)}(s)||_1 \le C'||b_\sigma^{(0,1)}(s)||_{1-\beta} \le C'',
\end{aligned}$$

for all $s,\sigma\in[-\pi,\pi]$. Hence, the conditions of Lemma 6 are satisfied. We have

$$||(\mathbf{A}+\mathbf{B})\mathbf{v}|| \ge ||\mathbf{A}(\mathbf{I}+\mathbf{G})\mathbf{v}|| - ||(\mathbf{AG}-\mathbf{B})\mathbf{v}||, \tag{3.17}$$

and application of Lemmas 1 and 6 yields

$$\begin{aligned}
||\mathbf{A}(\mathbf{I}+\mathbf{G})\mathbf{v}|| &\ge \min\{|a_-|,|a_0|,|a_+|\}(N/2)^{\min\{0,\beta\}}||(\mathbf{I}+\mathbf{G})\mathbf{v}||\\
&\ge Ch^{-\min\{0,\beta\}}||\mathbf{v}||
\end{aligned} \tag{3.18}$$

for some constant $C>0$. We obtain from (3.16)–(3.18) and the conditions on r that

$$\begin{aligned}
||(\mathbf{A}+\mathbf{B})\mathbf{v}|| &\ge Ch^{-\min\{0,\beta\}}||\mathbf{v}|| - C'''h^{\min\{r,r-\beta\}}||\mathbf{v}||\\
&\ge C''h^{-\min\{0,\beta\}}||\mathbf{v}||
\end{aligned}$$

for all h sufficiently small, where C'' is a positive constant. This last inequality shows that, for sufficiently small values of h,

$$||(\mathbf{A}+\mathbf{B})^{-1}\mathbf{v}|| \le Ch^{\min\{0,\beta\}}||\mathbf{v}||,$$

which completes the proof. □

Theorem 1. *Assume that the operator $\mathcal{A}+\mathcal{B}: H^0(2\pi)\to H^{-\beta}(2\pi)$ is invertible. If $w\in H^r(2\pi)$ and $b\in C(2\pi\times 2\pi)$ satisfies* (3.15) *and $b^{(0,r)}\in C(2\pi\times 2\pi)$ for some $r\in\mathbf{R}$, such that $r>\max\{|\beta|+1/2,3/2\}$. Then for $h>0$ sufficiently small,*

$$||\mathbf{w}-\mathbf{r}_h w|| \le Ch^{r-|\beta|}||\mathbf{w}||_r. \tag{3.19}$$

Proof. The bound follows from (3.10) and Lemmas 5 and 7. □

The polynomial rate of convergence in Theorem 1 as h converges to zero may be replaced by an exponential rate of convergence when b is an analytic function in a neighborhood of $[-\pi, \pi]^2$ and w is analytic in a neighborhood of $[-\pi, \pi]$. We remark that for $\beta < 0$, the bound (3.19) is not sharp; the results in [**26**] for Hölder-Zygmund function spaces imply that the bound (3.19) can be improved by a factor $h^{-\beta}$. For $\beta > 0$, Theorem 1 provides a result whose analog is not considered in [**26**].

4. A preconditioned iterative method. We describe an iterative method tailored for the solution of the linear system of algebraic equations (3.6). The matrix $\mathbf{A} + \mathbf{B}$ of this system is dense and non-Hermitian. Our iterative scheme is based on the observation that the discrete Fourier transform of this matrix has a structure that makes it possible to determine a good preconditioner. Our choice of preconditioner generalizes an approach in [**34, 36, 48, 49**] and is applicable to a larger class of periodic pseudodifferential operators. Related iterative schemes are also discussed in [**33, 35, 37**].

Let $\mathcal{F}$ denote the one-dimensional or two-dimensional discrete Fourier transform depending on the context, i.e.,

$$\mathcal{F}\,\mathbf{v} = N^{-1/2}\mathbf{F}^*\mathbf{v}, \qquad \mathbf{v} \in \mathbf{C}^N,$$
$$\mathcal{F}\,\mathbf{V} = \mathbf{F}^*\mathbf{V}\mathbf{F}, \qquad \mathbf{V} \in \mathbf{C}^{N\times N},$$

where the unitary matrix $\mathbf{F}$ is given by (2.7). Let $\mathbf{B}_h = \mathcal{F}\,\mathbf{B}$, $\psi = \mathcal{F}\,\mathbf{w}$ and $\mathbf{f}_h = \mathcal{F}\,\mathbf{f}$. Then equation (3.6) is equivalent to

$$\mathbf{D}\psi + \mathbf{B}_h\psi = \mathbf{f}_h, \tag{4.1}$$

where $\mathbf{D} = \operatorname{diag}[\delta_{-N/2+1}, \dots, \delta_{N/2}]$ is defined in Lemma 1.

The elements of the matrix $\mathbf{B}_h = [b_h(j,k)]_{j,k=-N/2+1}^{N/2}$ are discrete Fourier coefficients of the kernel $b(s,\sigma)$ up to a factor $1/(2\pi)$. Therefore, when the function $b(s,\sigma)$ is smooth and 2π-periodic in each variable, the elements $b_h(j,k)$ are of small magnitude when j or k are of large magnitude. This suggests that we may be able to approximate $\mathbf{B}_h$ by a low-rank matrix of the form

$$\widetilde{\mathbf{B}_h} = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{B}_{h_d} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} \in \mathbf{C}^{N \times N}, \tag{4.2}$$

where $\mathbf{B}_{h_d}$ is a $2d \times 2d$ matrix with $2d \ll N$.

We determine the matrix $\widetilde{\mathbf{B}}_h$ as follows. Let $h_d = \pi/d$, where d is chosen so that $N/(2d)$ is an integer. Define $\sigma_n = nh_d$ for $-d+1 \leq n \leq d$, and introduce the $2d \times 2d$ matrix

$$\mathbf{B}_d = [h_d b(\sigma_n, \sigma_m)]_{n,m=-d+1}^{d}.$$

Let $\omega_d = e^{ih_d}$, and define the $2d \times 2d$ unitary matrix

$$\mathbf{F}_d = (2d)^{-1/2} [\omega_d^{nm}]_{n,m=-d+1}^{d}.$$

Let

$$\mathbf{B}_{h_d} = [b_{h_d}(n,m)]_{n,m=-d+1}^{d} = \mathbf{F}_d^* \mathbf{B}_d \mathbf{F}_d.$$

We define $\widetilde{\mathbf{B}_h} = [\widetilde{b_h}(j,k)]_{j,k=-N/2+1}^{N/2}$ by

$$\widetilde{b_h}(j,k) = \begin{cases} b_{h_d}(j,k), & \text{if } -d+1 \leq j,k \leq d, \\ 0, & \text{otherwise.} \end{cases}$$

In our iterative scheme we use $\mathbf{D} + \widetilde{\mathbf{B}_h}$ as a preconditioner and define the iterates as follows. Determine an initial approximate solution $\boldsymbol{\psi}^{(0)}$ by solving

$$(\mathbf{D} + \widetilde{\mathbf{B}_h})\boldsymbol{\psi}^{(0)} = \mathbf{f}_h, \tag{4.3}$$

and compute subsequent iterates $\boldsymbol{\psi}^{(m)}$ by

$$\begin{aligned} \mathbf{r}^{(m)} &= \mathbf{f}_h - (\mathbf{D} + \mathbf{B}_h)\boldsymbol{\psi}^{(m)}, \\ \boldsymbol{\psi}^{(m+1)} &= \boldsymbol{\psi}^{(m)} + (\mathbf{D} + \widetilde{\mathbf{B}_h})^{-1}\mathbf{r}^{(m)}, \end{aligned} \qquad m = 0, 1, 2, \ldots. \tag{4.4}$$

In each iteration a block diagonal linear system of the form

$$(\mathbf{D} + \widetilde{\mathbf{B}_h})\mathbf{v} = \boldsymbol{\rho} \tag{4.5}$$

has to be solved. All diagonal blocks are of order one, except for one block of order $2d$. The LU factorization of the $2d \times 2d$ diagonal block requires roughly $(2d)^3/3$ multiplications. Given this LU factorizations, the solution of (4.5) requires about $(2d)^2$ multiplications.

The computation of the matrix $\mathbf{B}_h$ by the FFT method would require $O(N^2 \log N)$ multiplications. Therefore, Algorithm 1 carries out the iterations (4.4) without explicitly forming $\mathbf{B}_h$.

Algorithm 1 (Preconditioned stationary Richardson iteration).

(I) *Matrix generation. Compute* $\mathbf{B}_{h_d} := \mathbf{F}_d^* \mathbf{B}_d \mathbf{F}_d$ *using* 2-*d* FFT *and* $\mathbf{D}$ *using* 1-*d* FFT. *Determine the LU factorization of the* $2d \times 2d$ *diagonal block of* $\mathbf{D} + \widetilde{\mathbf{B}_h}$.

(II) *Preliminary calculation. Compute* $\mathbf{f}_h := N^{-1/2}\mathbf{F}^*\mathbf{f}$ *using* 1-*d* FFT, *and determine* $\boldsymbol{\psi}^{(0)} := (\mathbf{D} + \widetilde{\mathbf{B}_h})^{-1}\mathbf{f}_h$.

(III) *Iteration.*

$$
\begin{aligned}
&\textit{for } m := 0, 1, 2, \ldots \textit{ until convergence do}\\
\mathbf{r}^{(m)} &:= \mathbf{f}_h - \mathbf{D}\boldsymbol{\psi}^{(m)} - \mathbf{F}^*(\mathbf{B}(\mathbf{F}\boldsymbol{\psi}^{(m)}));\\
\boldsymbol{\delta}^{(m)} &:= (\mathbf{D} + \mathbf{B}_h)^{-1}\mathbf{r}^{(m)};\\
\boldsymbol{\psi}^{(m+1)} &:= \boldsymbol{\psi}^{(m)} + \boldsymbol{\delta}^{(m)};\\
&\textit{end } m;
\end{aligned}
$$

(IV) *Postprocessing. Compute* $\mathbf{w}^{(m)} := N^{1/2}\mathbf{F}\boldsymbol{\psi}^{(m)}$ *using* 1-*d FFT.*

This iteration method may also be thought of as preconditioned Picard iteration. Step I of the algorithm requires $(2d)^3/3 + O(d^2 \log d)$ multiplications, and Step II can be carried out using $O(N \log N) + O(d^2)$ multiplications. Each iteration in Step III can be carried out with $N^2 + O(N \log N) + O(d^2)$ multiplications. Only $O(N \log N)$ multiplications are necessary for Step IV. Thus, m iterations by Algorithm 1, including Steps I, II and IV can be carried out in roughly

$$(2d)^3/3 + m(N^2 + (2d)^2)$$

multiplications. The number of additions required is about the same. We choose d so that $(2d)^3 \approx N^2$. Then the total number of multiplications required is roughly $(m+1)N^2$. Our analysis of Section 5 shows that if the kernel $b(s, \sigma)$ is 2π-periodic and sufficiently smooth, then the number of iterations required with this choice of d is independent of N. This yields a multiplication count of only $O(N^2)$ for Algorithm 1. Computed examples using Algorithm 1 are presented in [**49, 50**]. Also some computed examples with an iterative scheme closely related to Algorithm 1 are presented in [**36**]. We therefore omit numerical examples in the present paper.

We remark that the preconditioner of the present paper can also be used together with other iterative methods than the stationary Richardson iteration method, such as the Chebyshev iteration method [**10, 23**] or the QMR method [**13**]. However, the rapid convergence obtained by Algorithm 1, as well as the simplicity of the iterative scheme, suggests that Algorithm 1 often is appropriate for the solution of (3.6).

5. Convergence of the iterative method. We present a convergence analysis for Algorithm 1. Central for our analysis is an error estimate that shows how well the matrix $\widetilde{\mathbf{B}_h}$ approximates $\mathbf{B}_h$. The convergence properties of Algorithm 1 are stated in Theorem 2 and show that, under suitable conditions, the number of iterations necessary is independent of N.

Let $\boldsymbol{\psi}$ solve (4.1). Then (4.4) yields

$$(\mathbf{D} + \widetilde{\mathbf{B}_h})(\boldsymbol{\psi}^{(m)} - \boldsymbol{\psi}) = (\widetilde{\mathbf{B}_h} - \mathbf{B}_h)(\boldsymbol{\psi}^{(m-1)} - \boldsymbol{\psi}), \quad m = 1, 2, 3, \ldots . \tag{5.1}$$

Define

$$\mathbf{M} = (\mathbf{D} + \widetilde{\mathbf{B}_h})^{-1}(\widetilde{\mathbf{B}_h} - \mathbf{B}_h).$$

Then (5.1) yields that

$$\boldsymbol{\psi}^{(m)} - \boldsymbol{\psi} = \mathbf{M}(\boldsymbol{\psi}^{(m-1)} - \boldsymbol{\psi}) = \mathbf{M}^m(\boldsymbol{\psi}^{(0)} - \boldsymbol{\psi}). \tag{5.2}$$

It follows from equations (4.1) and (4.3) that

$$\boldsymbol{\psi}^{(0)} - \boldsymbol{\psi} = -\mathbf{M}\boldsymbol{\psi}, \tag{5.3}$$

and substituting (5.3) into (5.2) yields

$$||\boldsymbol{\psi}^{(m)} - \boldsymbol{\psi}|| \leq ||\mathbf{M}||^{m+1}||\boldsymbol{\psi}||. \tag{5.4}$$

We now bound the norm of the matrix $\mathbf{M}$. Introduce the error matrix

$$\mathbf{E} = \widetilde{\mathbf{B}_h} - \mathbf{B}_h.$$

Then $\mathbf{M}$ can be written as

$$\begin{aligned}\mathbf{M} &= (\mathbf{D} + \mathbf{B}_h + \mathbf{E})^{-1}\mathbf{E} \\ &= \mathbf{F}^*(\mathbf{A} + \mathbf{B} + \mathbf{F}\mathbf{E}\mathbf{F}^*)^{-1}\mathbf{F}\mathbf{E},\end{aligned}$$

which in view of Lemma 7 leads to

$$\begin{aligned}||\mathbf{M}|| &\leq \frac{||(\mathbf{A} + \mathbf{B})^{-1}||\,||\mathbf{E}||}{1 - ||(\mathbf{A} + \mathbf{B})^{-1}||\,||\mathbf{E}||} \\ &\leq C\frac{h^{\min\{0,\beta\}}||\mathbf{E}||}{1 - Ch^{\min\{0,\beta\}}||\mathbf{E}||} \\ &\leq C' h^{\min\{0,\beta\}}||\mathbf{E}||,\end{aligned} \tag{5.5}$$

provided that $h^{\min\{0,\beta\}}||\mathbf{E}||$ is sufficiently small. Our convergence analysis will show that $h^{\min\{0,\beta\}}||\mathbf{E}||$ is sufficiently small for N large enough.

Introduce the function space $L^2((-\pi, \pi)^2)$ with norm

$$|||v||| = \left(\frac{1}{(2\pi)^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} |v(s,\sigma)|^2\, ds\, d\sigma\right)^{1/2},$$

and for r a nonnegative integer, define the Sobolev space

$$\begin{aligned}H^r(2\pi \times 2\pi) = \{&v \in L^2((-\pi,\pi)^2) : \text{ for } 0 \leq m_1 + m_2 \leq r, \\ &v^{(m_1,m_2)} \in L^2((-\pi,\pi)^2) \text{ in the sense of periodic distributions}\}\end{aligned}$$

with norm

$$|||v|||_r = \left(\sum_{m_1+m_2=0}^{r} |||v^{(m_1,m_2)}|||^2\right)^{1/2}.$$

Let $\mathcal{Q} : C([-\pi,\pi]^2) \to \operatorname{span}\{e^{ins}e^{-im\sigma}, n, m \in \Lambda_h\}$ be the two-dimensional interpolation operator

$$(\mathcal{Q}\, v)(t_j, t_k) = v(t_j, t_k), \qquad j, k \in \Lambda_h,$$

and let $v_h(n,m)$ denote the discrete Fourier coefficients associated with $v \in C([-\pi,\pi]^2)$, i.e.,

$$v_h(n,m) = \frac{1}{N^2} \sum_{j\in\Lambda_h} \sum_{k\in\Lambda_h} v(t_j,t_k) e^{-int_j} e^{imt_k}, \qquad n,m \in \Lambda_h.$$

Then $\mathcal{Q}$ can be written as

$$(\mathcal{Q}\, v)(s,\sigma) = \sum_{n\in\Lambda_h} \sum_{m\in\Lambda_h} v_h(n,m) e^{ins} e^{-im\sigma}. \tag{5.6}$$

Lemma 8. *Let $r \geq 2$ be an integer. Then*

$$|||v - \mathcal{Q}\, v||| \leq Ch^r |||v|||_r, \qquad v \in H^r(2\pi \times 2\pi).$$

Proof. See [**11**, Formula (9.7.7), p. 308]. $\square$

Clearly, $\mathcal{Q}$ depends on N. We denote the operator obtained for $N = 2d$ by $\mathcal{Q}_d$. From (5.6) it follows that

$$(\mathcal{Q}\, b)(s,\sigma) = \frac{1}{2\pi} \sum_{n\in\Lambda_h} \sum_{m\in\Lambda_h} b_h(n,m) e^{ins} e^{-im\sigma},$$

where the $b_h(n,m)$ are elements of the matrix $\mathbf{B}_h = \mathbf{F}^*\mathbf{B}\mathbf{F}$. Similarly,

$$(\mathcal{Q}_d b)(s,\sigma) = \frac{1}{2\pi} \sum_{n\in\Lambda_{h_d}} \sum_{m\in\Lambda_{h_d}} b_{h_d}(n,m) e^{ins} e^{-im\sigma},$$

where $b_{h_d}(n,m)$ are elements of the matrix $\mathbf{B}_{h_d} = \mathbf{F}_d^*\mathbf{B}_d\mathbf{F}_d$. Note that

$$\sum_{n\in\Lambda_{h_d}} \sum_{m\in\Lambda_{h_d}} b_{h_d}(n,m) e^{ins} e^{-im\sigma} = \sum_{n\in\Lambda_h} \sum_{m\in\Lambda_h} \widetilde{b_{h_d}}(n,m) e^{ins} e^{-im\sigma},$$

where $\widetilde{b_{h_d}}$ are elements of the matrix $\widetilde{\mathbf{B}_h}$. Thus,

$$\begin{aligned} &((\mathcal{Q} - \mathcal{Q}_d)b)(s,\sigma) \\ &\qquad = \frac{1}{2\pi} \sum_{n\in\Lambda_h} \sum_{m\in\Lambda_h} (b_h(n,m) - \widetilde{b_{h_d}}(n,m)) e^{ins} e^{-im\sigma}. \end{aligned} \tag{5.7}$$

Let p_h be the prolongation operator (3.8). Using (5.7), a simple calculation yields that

$$\frac{1}{(2\pi)^2}\int_{-\pi}^{\pi}\left|\int_{-\pi}^{\pi}(p_h\mathbf{v})(\sigma)((\mathcal{Q}-\mathcal{Q}_d)b)(s,\sigma)d\sigma\right|^2 ds = \frac{1}{h}||(\mathbf{B}_h-\widetilde{\mathbf{B}_h})\mathbf{v}_h||^2,$$

which shows that

$$\begin{aligned}||(\mathbf{B}_h-\widetilde{\mathbf{B}_h})\mathbf{v}_h|| &\leq 2\pi N^{-1/2}|||(\mathcal{Q}-\mathcal{Q}_d)b|||\,||p_h\mathbf{v}||\\ &= 2\pi N^{-1/2}|||(\mathcal{Q}-\mathcal{Q}_d)b|||\,||v||\\ &= 2\pi|||(\mathcal{Q}-\mathcal{Q}_d)b|||\,||\mathbf{v}_h||,\end{aligned}$$

where we have used that $||p_h\mathbf{v}||=||\mathbf{v}||$ and $||\mathbf{v}||=N^{1/2}||\mathbf{v}_h||$. Therefore,

$$||\mathbf{E}||=||\mathbf{B}_h-\tilde{\mathbf{B}}_h||\leq 2\pi|||(\mathcal{Q}-\mathcal{Q}_d)b|||,$$

and Lemma 8 yields

$$||\mathbf{E}||\leq C'(h^r+h_d^r)|||b|||_r\leq Ch_d^r|||b|||_r. \tag{5.8}$$

We are now in a position to state our convergence result for the preconditioned iterative method.

Theorem 2. *Assume that the operator $\mathcal{A}+\mathcal{B}: H^0(2\pi)\to H^{-\beta}(2\pi)$ is invertible, and let r be an integer such that $r>\max\{|\beta|+1/2,3/2,-3\beta/2\}$. Assume that $b\in H^r(2\pi\times 2\pi)$ satisfies the conditions of Lemma 7, and let $\mathbf{w}$ solve (3.6). Let $\mathbf{w}^{(m)}$ be an approximate solution determined by Algorithm 1. Assume that d satisfies $C'N^{2/3}\leq d$ and $N/(2d)$ is an integer. Then for $h=2\pi/N$ sufficiently small*

$$||\mathbf{w}^{(m)}-\mathbf{w}||\leq (Ch^{2r/3+\min\{0,\beta\}})^{m+1}||\mathbf{w}||. \tag{5.9}$$

Proof. From $\mathbf{w}=N^{1/2}\mathbf{F}\boldsymbol{\psi}$ and (5.4), it follows that

$$\begin{aligned}||\mathbf{w}^{(m)}-\mathbf{w}|| &= N^{1/2}||\mathbf{F}(\boldsymbol{\psi}^{(m)}-\boldsymbol{\psi})||\\ &= N^{1/2}||\boldsymbol{\psi}^{(m)}-\boldsymbol{\psi}||\\ &\leq N^{1/2}||\mathbf{M}||^{m+1}||\boldsymbol{\psi}||\\ &= ||\mathbf{M}||^{m+1}||\mathbf{w}||.\end{aligned} \tag{5.10}$$

Substituting (5.8) into (5.5) yields that

$$||\mathbf{M}|| \le Ch^{\min\{0,\beta\}}||\mathbf{E}|| \le C'h^{2r/3+\min\{0,\beta\}}. \tag{5.11}$$

In particular, (5.11) shows that the bound (5.5) is valid for $r > -3\beta/2$ and h sufficiently small. The theorem now follows by combining (5.10) and (5.11). □

Under the assumptions of Theorem 1, we have

$$\begin{aligned} ||\mathbf{w}|| &\le ||\mathbf{w} - \mathbf{r}_h w|| + ||\mathbf{r}_h w|| \\ &\le Ch^{r-|\beta|}||w||_r + \sup_{s\in[-\pi,\pi]} |w(s)| \le C'||w||_r. \end{aligned} \tag{5.12}$$

Substitution of (5.12) into (5.9) and using Theorem 1 yields

$$||\mathbf{w}^{(m)} - \mathbf{r}_h w|| \le \begin{cases} C''(h^{r+\beta} + (Ch^{2r/3+\beta})^{m+1})||w||_r, & \text{if } \beta \le 0, \\ C''(h^{r-\beta} + (Ch^{2r/3})^{m+1})||w||_r, & \text{if } \beta > 0. \end{cases}$$

The above inequality shows that the polynomial rate of convergence for the quadrature method as $h \to 0$ can be retained in the iterative quadrature method by performing a finite number of iterations. It should be remarked again that the polynomial rate of convergence may be replaced by an exponential rate of convergence when b and w are analytic.

6. Conclusion. We have described a quadrature method for the discretization of periodic pseudodifferential equations. The matrix of the system of linear algebraic equations obtained has a structure that makes it possible to determine a simple and efficient preconditioner. The convergence of a preconditioned Richardson iteration scheme is analyzed. Let $\mathbf{w}^{(m)}$ denote the iterate obtained after m iterations with our preconditioned scheme. We show that already, after a finite number of iterations, m, independent of h, the convergence of $\mathbf{w}^{(m)}$ to the solution $\mathbf{w}$ of (3.6) as $h \to 0$ is at least as rapid as the convergence of $\mathbf{w}$ to $\mathbf{r}_h w$ as $h \to 0$, where w denotes the solution of (1.2).

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