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ON PANTOGRAPH INTEGRO-DIFFERENTIAL EQUATIONS

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ABSTRACT. The paper discusses the initial-value problem for the pantograph integro-differential equation, including as special cases the retarded functional-differential equation studied by Ockendon and Tayler [**17**], Kato and MacLeod [**11**] and the neutral differential equation studied by Kuang and Feldstein [**12**]. The main subjects of this paper are well-posedness of the initial-value problem, monotonicity and oscillation of the solution, unboundedness of the solution, and asymptotic stability of the solution, subject to different conditions.

1. Introduction. Let a be a complex constant and $\mu(q)$ and $\nu(q)$ complex-valued functions of bounded variation on $[0,1]$. The initial-value problem for pantograph integro-differential equations to be studied in this paper is of the form

$$\begin{aligned} y'(t) &= ay(t) + \int_0^1 y(qt)\, d\mu(q) \\ &\quad + \int_0^1 y'(qt)\, d\nu(q), \quad t > 0, \quad y(0) = y_0 \end{aligned} \tag{1.1}$$

where the integrals being considered are of Riemann-Stieltjes type, although most results of this paper still hold if $\mu(q)$ and $\nu(q)$ are replaced by complex-valued measures on $[0,1]$. The term *pantograph* comes from Ockendon and Tayler [**17**] and Iserles [**7**].

The pantograph integro-differential equation includes many interesting equations studied before. In the case $d\mu(q) = b\delta(q-p)\,dq$, $d\nu(q) \equiv 0$, where $p \in (0,1)$ and $\delta(\cdot)$ is a Dirac function, problem (1.1) can be written as

$$y'(t) = ay(t) + by(pt), \quad t > 0, \quad y(0) = y_0, \tag{1.2}$$

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which has been studied by Ockendon and Tayler [**17**], Kato and McLeod [**11**], etc. In the case $d\mu(q) = \sum_{i=1}^{M} b_i\delta(q-q_i)\,dq$, $d\nu(q) = \sum_{i=1}^{K} c_i\delta(q-p_i)\,dq$, where $q_i, p_i \in (0,1)$, (1.1) can be written as

$$y'(t) = ay(t) + \sum_{i=1}^{M} b_i y(q_i t) + \sum_{i=1}^{K} c_i y'(p_i t), \qquad t > 0, \quad y(0) = y_0 \tag{1.3}$$

formerly studied by Feldstein and Jackiewicz [**3**], Kuang and Feldstein [**12**], and in more general form, by Derfel [**2**] and Iserles and Liu [**8**].

The aim of this paper is not only to generalize some results of Kato and MacLeod [**11**] and of Kuang and Feldstein [**12**] to problem (1.1), but also in some cases to improve their results by using different approaches. In Section 2 we study the well-posedness of the initial-value problem (1.1). In Section 3 we study the monotonicity and oscillation of the solution in the real case. In Section 4 we study as a preliminary to the next two sections an integral equation and an integro-differential equation. In Section 5 we study the unboundedness of the solution in the case $\operatorname{Re} a \geq 0$ by using the Ahlfors theorem and some other methods. In Section 6 we study the asymptotic stability of the zero solution in the case $\operatorname{Re} a < 0$.

2. Uniqueness and existence of the solution. Let

$$\mu_k = \int_0^1 q^k\, d\mu(q), \qquad \nu_k = \int_0^1 q^k\, d\nu(q),$$
$$\mu_k^* = \int_0^1 q^k |d\mu(q)|, \qquad \nu_k^* = \int_0^1 q^k |d\nu(q)|.$$

In particular, $\mu^* := \mu_0^*$ and $\nu^* := \nu_0^*$ are the variations of $\mu(q)$ and $\nu(q)$ on $[0,1]$, respectively, whereas μ_k^* and ν_k^* are the variation of

$$\int_0^q \tau^k\, d\mu(\tau) \quad \text{and} \quad \int_0^q \tau^k\, d\nu(\tau),$$

respectively. Throughout the paper, we use the uniform-norm, i.e.,

$$||f||_{[a,b]} = \max_{t\in[a,b]} |f(t)|$$

for a function $f(t) \in C[a,b]$.

Theorem 1. *If $\nu_N^* < 1$ for some $N \in \mathbf{Z}^+$, then, in function space $C^{N+1}[0,\infty)$,*

(1) *the solution of problem* (1.1) *exists if and only if the algebraic linear system*

$$(2.1)\qquad (1-\nu_n)y_{n+1} - (a+\mu_n)y_n = 0, \quad n = 0,\dots,N-1$$

is solvable;

(2) *all solutions of problem* (1.1) *are analytic and can be expressed in the form*

$$(2.2)\qquad y(t) = \sum_{n=0}^{\infty} \frac{y_n}{n!} t^n,$$

where $\{y_n\}_{n=1}^N$ are solutions of the algebraic linear system (2.1) *and*

$$y_n = y_N \prod_{k=N}^{n-1} \frac{a+\mu_k}{1-\nu_k}, \quad n > N;$$

and

(3) *the solution of problem* (1.1) *is unique if and only if $\nu_n \neq 1$ for all $0 \le n \le N-1$. Furthermore, the uniqueness of solution implies that*

$$(2.3)\qquad y(t) = y_0 \left\{ 1 + \sum_{n=1}^{\infty} \left(\prod_{k=0}^{n-1} \frac{a+\mu_k}{1-\nu_k} \right) \frac{t^n}{n!} \right\}.$$

Proof. We commence by proving the theorem in the case $N = 0$. Since $\nu^* < 1$ implies $|\nu_k| < 1$ for all $k \ge 0$, it is easy to verify that the function on the right-hand side of (2.3) is a $C^1[0,\infty)$ solution of problem (1.1). To prove the uniqueness of this solution, it is enough to prove that the homogeneous problem of (1.1), i.e., $y_0 = 0$, has only the trivial solution. Suppose that $y(t)$ is a solution of this homogeneous problem. It follows from equation (1.1) that, for any fixed $T > 0$

$$||y'||_{[0,T]} \le (|a| + \mu^*)||y||_{[0,T]} + \nu^*||y'||_{[0,T]}.$$

Hence

$$||y'||_{[0,T]} \leq \frac{|a| + \mu^*}{1 - \nu^*} ||y||_{[0,T]}.$$

Noting that $y(0) = 0$, we derive from the preceding inequality and the equation

$$y(t) = y(0) + \int_0^t y'(\tau)\, d\tau$$

that

$$||y||_{[0,T]} \leq \frac{|a| + \mu^*}{1 - \nu^*} T ||y||_{[0,T]}.$$

Hence, $||y||_{[0,T_0]} = 0$ for $0 < T_0 < (1 - \nu^*)/(|a| + \mu^*)$, which implies that $y \equiv 0$ for $t \in [0, T_0]$. By simple inductive argument, we see that $y(t) \equiv 0$ for $t \in [nT_0, (n+1)T_0]$ and all $n \geq 0$. Hence, $y(t) \equiv 0$ for all $t \geq 0$. In the case $N \geq 1$, it is easy to see that a function $y(t)$ is a solution of problem (1.1) if and only if $y_n = y^{(n)}(0)$, $n = 1, 2, \ldots, N$, satisfy the algebraic linear system (2.1) and $y_N(t) = y^N(t) \in C^1[0, \infty)$ obeys the equation

$$y'_N(t) = a y_N(t) + \int_0^1 y_N(qt) q^N \, d\mu(q) + \int_0^1 y'_N(qt) q^N \, d\nu(q),$$
$$y_N(0) = y_N.$$

Invoking the result of $N = 0$, we see that the theorem holds for $N \geq 1$. □

Remark 1. The condition $\sum_{i=1}^K |c_i p_i^{-1}| < 1$ for the existence and uniqueness of a $C^1[0, \infty)$ solution of problem (1.3), given originally by Kuang and Feldstein [**12**], can be modified into $\sum_{i=1}^K |c_i| < 1$.

Now we give examples to show the necessity of the condition $\nu_N^* < 1$.

Example 1. Suppose that $a = 0$, $d\mu(q) \equiv 0$, $d\nu(q) = c\delta(q - p)\, dq$, where c is constant satisfying $|c|p^N > 1$, $p \in (0, 1)$. Then there is a one-to-one correspondence between the solutions $y(t) \in C^{N+1}[0, \infty)$ of problem (1.1) and the functions in the space $\{f(t) \in C[p, 1] : f(1) = cp^N f(p)\}$. For details, see Nussbaum [**16**] or Iserles and Liu [**8**].

Example 2. Assume that $a = 0$, $d\mu(q) \equiv 0$ and $\nu(q)$ satisfies

$$\int_0^1 q^N \, d\nu(q) > 1$$

and

$$\limsup_{n\to\infty} \int_0^1 q^n \, d\nu(q) < 1. \tag{2.4}$$

According to the intermediate value theorem, there exists a $\lambda > N$ such that

$$\int_0^1 q^\lambda \, d\nu(q) = 1.$$

Hence, the homogeneous problem

$$y'(t) = \int_0^1 y'(qt)\, d\nu(q), \quad y(0) = 0$$

has a nontrivial solution $y(t) = t^{\lambda+1} \in C^{N+1}[0, \infty)$. Furthermore, if λ is not an integer, then this solution is not analytic. Consider the case $\nu(q) = bq^\beta$, where $\beta > 0$, $(b-1)\beta > N$. If $(b-q)\beta$ is not an integer, then there exists a nonanalytic solution $y(t) = t^{(b-1)\beta+1} \in C^{N+1}[0, \infty)$.

It is also easy to see that, subject to the existence condition in Theorem 1, the solution $y(t)$ is an entire function which satisfies

$$|y(t)| \le \sum_{n=0}^{N-1} \frac{1}{n!}|y_n|t^n + \frac{1}{N!}|y_N|t^N \exp\left\{\frac{|a| + \mu_N^*}{1 - \nu_N^*} t\right\}, \quad t \in \mathcal{C}.$$

In the remainder of this paper, we only discuss analytic solutions of the form (2.2) and assume that $\mu(q)$ and $\nu(q)$ are continuous at $q = 1$ and that the system (2.1) is solvable. We denote by N a nonnegative integer such that $\nu_N^* < 1$. The continuity of $\mu(q)$ and $\nu(q)$ at $q = 1$ implies that

$$\lim_{n\to\infty} \mu_n^* = \lim_{n\to\infty} \nu_n^* = 0, \tag{2.5}$$

which plays an important role in the subsequent discussion. It also guarantees the existence of the integer N and the inequality (2.4). For simplicity, we exclude the trivial case of the solution being a polynomial of t, which happens if there exists an integer n such that $a + \mu_n = 0$, $1 - \nu_n \neq 0$, or $a + \mu_n = 0$, $1 - \nu_n = 0$, $y_{n+1} = 0$.

Remark 2. There is but a slight loss of generality in assuming that $\nu(q)$ is continuous at $q = 1$. This is because (1.1) can be rewritten as

$$y'(t) = \frac{1}{1-\nu(1)+\nu(1-)}\Big\{(a+\mu(1)-\mu(1-))y(t) \\ + \int_0^1 y(qt)\,d\mu_1(q) + \int_0^1 y'(qt)\,d\nu_1(q)\Big\}, \\ t > 0, \quad y(0) = y_0$$

provided that $\nu(1) - \nu(1-) \neq 1$, where

$$\mu_1(q) = \begin{cases} \mu(q), & 0 \le q < 1 \\ \mu(1-), & q = 1 \end{cases} \quad \text{and} \quad \nu_1(q) = \begin{cases} \nu(q), & 0 \le q < 1 \\ \nu(1-), & q = 1 \end{cases}$$

are continuous at $q = 1$.

3. Monotonicity and oscillation of the solution. In this section we restrict our attention to the case of a being a real constant and $\mu(q)$ and $\nu(q)$ being real functions of bounded variation. We assume that $\nu_k \neq 1$ for $k \ge 0$. This implies that the solution of the initial-value problem (1.1) is unique and is of the form (2.3).

Since $\mu(q)$ and $\nu(q)$ are real functions of bounded variation, they can be decomposed into the sums

$$\mu(q) = \mu_+(q) + \mu_-(q), \qquad \nu(q) = \nu_+(q) + \nu_-(q),$$

where $\mu_+(q)$ and $\nu_+(q)$ are monotonic increasing functions and $\mu_-(q)$ and $\nu_-(q)$ are monotonic decreasing functions. Recall from Widder [**19**] that the function $f(t)$ is said to be absolutely monotonically increasing (decreasing) if $d^k f(t)/dt^k \ge 0$ (≤ 0) for all integers $k \ge 0$ and all $t > 0$, and eventually absolutely monotonically increasing (decreasing) if there exists $T > 0$ such that $d^k f(t)/dt^k \ge 0$ (≤ 0) for all integers $k \ge 0$ and all $t > T$. The following theorem is easy to derive directly from (2.3).

Theorem 2. *Suppose that* $y_0 \neq 0$.

(1) *If* $a > 0$, *then the solution of problem* (1.1) *is eventually absolutely monotone*; *and*

(2) *If* $\int_0^1 d\mu_-(q) \geq -a$, $\int_0^1 d\nu_+(q) < 1$*; then the solution of problem* (1.1) *is absolutely monotone.*

In part 2 of the above theorem, if the condition $\int_0^1 d\mu_-(q) \geq -a$ is replaced by $\int_0^1 d\mu_-(q) > -a$, then the solution satisfies $d^k f(t)/dt^k > 0$ (< 0) for all integers $k \geq 0$ and all $t > 0$, a property stronger than absolute monotonicity. Moreover, it is easy to derive directly from (2.3) the estimate

$$|y_0| \exp\left(\frac{a + \int_0^1 d\mu_-(q)}{1 - \int_0^1 d\nu_-(q)} t\right) \leq |y(t)| \leq |y_0| \exp\left(\frac{a + \int_0^1 d\mu_+(q)}{1 - \int_0^1 d\nu_+(q)} t\right), \quad t \geq 0.$$

Theorem 3. *Suppose that* $\mu(q)$ *is monotonic decreasing and that there exists* $q_0, q_1 \in (0,1)$ *such that* $d\mu(q) = 0$ *for* $q \notin [q_0, q_1]$, $\nu_+ := \int_0^1 q^{-1}\, d\nu_+(q) < 1$ *and* $\nu_{-1}^* := \int_0^1 q^{-1} |d\nu(q)| < \infty$. *If* $a = 0$, *then every nontrivial solution of the problem* (1.1) *oscillates unboundedly.*

Proof. Suppose that a nontrivial solution $y(t)$ is nonoscillatory. Thus, $y(t)$ has at most finite number of zeros for $t \geq 0$. By the linearity of the problem, we can assume without loss of generality that the solution is eventually positive, i.e., there exists $t_0 > 0$ such that $y(t) > 0$ for $t > t_0$. Let $t^* = t_0/q_0$. We deduce from

$$\frac{d}{dt}\left\{y(t) - \int_0^1 y(qt) q^{-1}\, d\nu(q)\right\} = \int_{q_0}^{q_1} y(qt)\, d\mu(q) \leq 0, \quad t \geq t^*,$$

that

$$y(t) - \int_0^1 y(qt) q^{-1}\, d\nu(q) \leq y(t^*) - \int_0^1 y(qt^*) q^{-1}\, d\nu(q), \quad t \geq t^*.$$

Hence

$$y(t) - \int_0^1 y(qt) q^{-1}\, d\nu_+(q) \leq y(t^*) - \int_0^1 y(qt^*) q^{-1}\, d\nu(q), \quad t \geq t^*;$$

consequently,

$$y(t) - \int_{t^*/t}^{1} y(qt)q^{-1}\,d\nu_+(q) \le y(t^*) - \int_0^1 y(qt^*)q^{-1}\,d\nu(q) + \int_0^{t^*/t} y(qt)q^{-1}\,d\nu_+(q), \quad t \ge t^*.$$

We derive from the preceding inequality that

$$y(t) \le (1+\nu_{-1}^* + \nu_+) \max_{0\le\tau\le t^*} |y(\tau)| + \nu_+ \max_{t^*\le\tau\le t} y(\tau), \quad t \ge t^*.$$

Therefore

$$y(t) \le \frac{1+\nu_{-1}^*+\nu_+}{1-\nu_+} \max_{0\le\tau\le t^*} |y(\tau)|, \quad t \ge t^*,$$

which leads to a contradiction, since, according to Theorem 4 of Section 5, the solution is unbounded. Hence, every nontrivial solution oscillates unboundedly. □

Kuang and Feldstein [**12**] proved this theorem in the special case (1.3). Here we used a slightly different but more intuitive approach.

4. Related integral and integro-differential equations. Let T, T_0 and T_1 be linear operators of the form

$$Ty(t) = \int_0^1 y(qt)K(q)\,d\nu(q),$$
$$T_i y(t) = \int_0^1 y(qt)K_i(q)\,d\nu_i(q), \quad i = 0,1,$$

where $K(q)$, $K_0(q)$ and $K_1(q)$ are continuous fucntions on $[0,1]$ and $\nu(q)$, $\nu_0(q)$ and $\nu_1(q)$ are functions of bounded variation on $[0,1]$. Let

$$\chi = \int_0^1 |K(q)\,d\nu(q)|,$$
$$\chi_i = \int_0^1 |K_i(q)\,d\nu(q)|, \quad i = 0,1.$$

Consider the integral equation

$$y(t) - Ty(t) = f(t), \quad t \geq 0, \tag{4.1}$$

where $f(t) \in C[0, \infty)$.

Lemma 1. *If $\chi < 1$, then equation* (4.1) *has a unique solution in $C[0, \infty)$, which is of the form*

$$y(t) = \sum_{n=0}^{\infty} T^n f(t). \tag{4.2}$$

The proof of this lemma is straightforward.

Lemma 2. *If $\chi < 1$, $\lim_{h \to 0} \int_0^h |K(q)\, d\nu(q)| = 0$ and $\lim_{t\to\infty} f(t) = 0$, then the solution $y(t)$ of equation* (4.1) *satisfies $\lim_{t\to\infty} y(t) = 0$.*

Proof. Note that $||T||_{[0,t]} \leq \chi$; therefore, it follows from (4.2) that

$$|y(t)| \leq \frac{1}{1-\chi} ||f||_{[0,t]} \quad \text{for all } t \geq 0.$$

Hence the solution $y(t)$ is uniformly bounded. Let

$$M = \sup_{0 \leq x < \infty} |y(t)|, \qquad \phi = \limsup_{t\to\infty} |y(t)|.$$

We shall prove that $\phi > 0$ leads to a contradiction. Let $\delta = (1-\chi)/(2+\chi+M) > 0$. By our assumption, there exist $t^* > 0$ and $h \in (0, 1)$ such that $|f(t)| < \delta$, $|y(t)| < \phi + \delta$ for all $t > t^*$, and $\int_0^h |K(q)\, d\nu(q)| < \delta$. For all $t > t^*/h$, we deduce from equation (4.1) that

$$\begin{aligned} |y(t)| &< |f(t)| + M \int_0^h |K(q)\, d\nu(q)| + (\phi + \delta) \int_h^1 |K(q)\, d\nu(q)| \\ &\leq \delta + M\delta + (\phi + \delta)\chi = \phi - \delta \end{aligned}$$

which contradicts the definition of ϕ. Hence, $\phi = 0$. $\square$

Consider the initial-value problem for an integro-differential equation of the form

$$y'(t) = ay(t) + \sum_{n=0}^{\infty} T^n (T_0 + T_1) y(t), \quad t > 0, \quad y(0) = y_0. \tag{4.3}$$

Lemma 3. *If $\chi < 1$, then the problem* (4.3) *has a unique solution $y(t) \in C[0, \infty)$.*

Proof. It is trivial to verify that the function

$$y(t) = y_0 \left\{ 1 + \sum_{n=1}^{\infty} \left[\prod_{k=0}^{n-1} \left(a + \frac{\nu_{0,k} + \nu_{1,k}}{1 - \nu_k} \right) \right] \frac{t^n}{n!} \right\}$$

is a solution of (4.3), where

$$\nu_k = \int_0^1 q^k K(q)\, d\nu(q),$$
$$\nu_{i,k} = \int_0^1 q^k K_i(q)\, d\nu_i(q), \quad i = 0, 1.$$

The uniqueness of solution can be proved in a similar way as we did in the case $N = 0$ of Theorem 1. ◻

Lemma 4. *If $\operatorname{Re} a > 0$ and $\chi + \gamma / \operatorname{Re} a < 1$, where*

$$\gamma = \int_0^1 \frac{1}{1-q} |K_0(q)\, d\nu_0(q) + K_1(q)\, d\nu_1(q)| < \infty,$$

then the solution $y(t)$ of problem (4.3) *satisfies*

$$\lim_{t \to \infty} y(t) e^{-at} = y^*$$

for some (*possibly zero*) *constant y^*.*

Proof. From the variation of constants formula, we have

$$y(t) = e^{at} y(0) + \int_0^t e^{a(t-\tau)} \sum_{n=0}^{\infty} T^n (T_0 + T_1) y(\tau)\, d\tau. \tag{4.4}$$

Noting that

$$\left|\int_0^t e^{-\alpha\tau} y(q_1\cdots q_n q\tau)\,d\tau\right| \le \frac{1}{\operatorname{Re} a(1-q)} \max_{\tau\in[0,t]} |y(\tau)e^{-a\tau}|,$$

where $q \in [0,1), q_1, \ldots, q_n \in [0,1]$, we obtain

$$\left|\int_0^1 \int_0^t e^{-a\tau} y(q_1\cdots q_n q\tau)(K_0(q)\,d\nu_0(q) + K_1(q)\,d\nu_1(q))\,d\tau\right| \le \frac{\gamma}{\operatorname{Re} a} \max_{\tau\in[0,t]} |y(\tau)e^{-a\tau}|.$$

Hence,

$$\begin{aligned}\left|\int_0^t e^{-a\tau} \sum_{n=0}^{\infty} T^n(T_0+T_1)y(\tau)\,d\tau\right| &\le \sum_{n=0}^{\infty} \int_0^t \int_{[0,1]^{n+1}} |e^{-a\tau} y(q_1\cdots q_n q\tau)(K_0(q)\,d\nu_0(q) \\ &\qquad + K_1(q)\,d\nu_1(q))\,d\nu(q_1)\cdots d\nu(q_n)\,d\tau| \\ &\le \sum_{n=0}^{\infty} \frac{\gamma}{\operatorname{Re} a} \int_{[0,1]^n} |K(q_1)\cdots K(q_n)\,d\nu(q_1)\cdots \\ &\qquad d\nu(q_n)| \max_{\tau\in[0,t]} |y(\tau)e^{-a\tau}| \\ &\le \frac{\gamma}{\operatorname{Re} a(1-\chi)} \max_{\tau\in[0,t]} |y(\tau)e^{-a\tau}|.\end{aligned}$$

From (4.4) and the preceding estimate, we obtain

$$|y(t)e^{-at}| \le |y_0| + \frac{\gamma}{\operatorname{Re} a(1-\chi)} \max_{\tau\in[0,t]} |y(\tau)e^{-a\tau}|.$$

Hence,

$$|y(t)e^{-at}| \le \frac{\operatorname{Re} a(1-\chi)}{\operatorname{Re} a(1-\chi)-\gamma}|y_0|, \quad t \ge 0,$$

which means that $y(t)e^{-at}$ is uniformly bounded. Denote this bound by M. Again, from (4.4), we obtain for $t_2 > t_1 > 0$ that

$$y(t_2)e^{-at_2} - y(t_1)e^{-at_1} = \int_{t_1}^{t_2} e^{-a\tau} \sum_{n=0}^{\infty} T^n(T_0+T_1)y(\tau)\,d\tau.$$

Noting that

$$\int_{t_1}^{t_2} |e^{-a\tau} y(q_1 \cdots q_n q\tau)|\, d\tau \le \frac{Me^{-\operatorname{Re} a(1-q)t_1}}{\operatorname{Re} a(1-q)},$$

where $q \in [0, 1)$, $q_1, \ldots, q_n \in [0, 1]$, we obtain similarly that

$$\begin{aligned} &|y(t_2)e^{-at_2} - y(t_1)e^{-at_1}| \\ &\qquad \le \frac{M}{\operatorname{Re} a(1-\chi)} \int_0^1 \frac{e^{-\operatorname{Re} a(1-q)t_1}}{1-q} |K_0(q)\, d\nu_0(q) + K_1(q)\, d\nu_1(q)|. \end{aligned}$$

For any given $\varepsilon > 0$, it is evident that there exist $h \in (0, 1)$ and $T > 0$ such that

$$\int_h^1 \frac{1}{1-q} |K_0(q)\, d\nu_0(q) + K_1(q)\, d\nu_1(q)| < \frac{\operatorname{Re} a(1-\chi)}{2M} \varepsilon,$$

and

$$e^{-\operatorname{Re} a(1-h)T} < \frac{\operatorname{Re} a(1-\chi)}{2M\gamma} \varepsilon.$$

Hence we have for $t_2 > t_1 > T$ that

$$\begin{aligned} &|y(t_2)e^{-at_2} - y(t_1)e^{-at_1}| \\ &\qquad \le \frac{1}{1-\chi} \bigg\{ \frac{Me^{-\operatorname{Re} a(1-h)T}}{\operatorname{Re} a} \int_0^h \frac{1}{1-q} |K_0(q)\, d\nu_0(q) + K_1(q)\, d\nu_1(q)| \\ &\qquad\qquad + \frac{M}{\operatorname{Re} a} \int_h^1 \frac{1}{1-q} |K_0(q)\, d\nu_0(q) + K_1(q)\, d\nu_1(q)| \bigg\} \\ &\qquad \le \varepsilon, \end{aligned}$$

which means that $\lim_{t\to\infty} y(t)e^{-at}$ exists. □

Lemma 5. *If* $\chi < 1$ *and* $\operatorname{Re} a + (\chi_0 + \chi_1)/(1-\chi) \le 0$, *then the solution* $y(t)$ *of problem* (4.3) *is uniformly bounded by* $|y_0|$.

Proof. The case $\operatorname{Re} a = 0$ is trivial. Consider the case $\operatorname{Re} a < 0$. For any fixed positive constant L, we derive from formula (4.4) that

$$\begin{aligned} |y(t)| &\le |e^{at} y_0| + \frac{\chi_0 + \chi_1}{1-\chi} ||y||_{[0,L]} \int_0^t e^{\operatorname{Re} a(t-\tau)}\, d\tau \\ &\le (|y_0| - ||y||_{[0,L]}) e^{\operatorname{Re} at} + ||y||_{[0,L]}, \quad t \in [0, L]. \end{aligned}$$

Hence, $||y||_{[0,L]} \leq |y_0|$. The arbitrariness of L implies that the solution is uniformly bounded by $|y_0|$. □

Lemma 6. *If* $\chi < 1$, $\operatorname{Re} a + (\chi_0 + \chi_1)/(1-\chi) < 0$, *and*

$$\lim_{h\to 0} \int_0^h |K_i(q)\, d\nu_i(q)| = 0, \quad i = 0, 1,$$

then the solution $y(t)$ *of problem* (4.3) *satisfies* $\lim_{t\to\infty} y(t) = 0$.

Proof. It follows from Lemma 5 that the solution $y(t)$ of problem (4.3) is uniformly bounded by $|y(0)|$. Hence, $\phi = \limsup_{t\to+\infty} |f(t)|$ exists. We shall prove that $\phi > 0$ leads to a contradiction. Let

$$\delta = \left\{1 + |y(0)| + \frac{2|y(0)|(\chi_0 + \chi_1 + 1) + \chi_0 + \chi_1}{|\operatorname{Re} a|(1-\chi)}\right\}^{-1} \left\{1 - \frac{\chi_0 + \chi_1}{|\operatorname{Re} a|(1-\chi)}\right\}\phi > 0.$$

By our assumption, there exists $t_0 > 0$ such that

$$|y(t)| < \phi + \delta, \quad t > t_0,$$

$h \in (0,1)$ such that

$$\int_0^h |K_i(q)\, d\nu_i(q)| < \delta, \quad i = 0, 1,$$

integer $m > 0$ such that $\chi^m < \delta$ and $t_1 > t_0/h^{m+1}$ such that

$$e^{\operatorname{Re} a(t - t_0/h^{m+1})} < \delta, \quad t > t_1.$$

From

$$\left|\sum_{n=0}^{m-1} T^n(T_0 + T_1)y(\tau)\right| \leq \frac{|y(0)|(\chi_0 + \chi_1)}{1-\chi}, \quad \tau > 0,$$

we obtain

$$\left|\int_0^{t_0/h^{m+1}} e^{a(t-\tau)} \sum_{n=0}^{m-1} T^n(T + T_1)y(\tau)\, d\tau\right| \leq \frac{|y(0)|(\chi_0 + \chi_1)}{|\operatorname{Re} a|(1-\chi)}\delta, \quad t > t_1.$$

From

$$\sum_{n=0}^{m-1}\int_{[0,h]^{n+1}}|y(q_1\cdots q_n q\tau)K(q_1)\cdots K(q_n)$$
$$(K_0(q)\,d\nu_0(q)+K_1(q)\,d\nu_1(q))\,d\nu(q_1)\cdots d\nu(q_n)|$$
$$\leq \sum_{n=0}^{m-1}|y(0)|\bigg(\int_{[0,h]}|K(q)\,d\nu(q)|)^n|\int_0^h|K_0(q)\,d\nu_0(q)+K_1(q)\,d\nu_1(q)\bigg)$$
$$< \frac{2|y(0)|}{1-\chi}\delta, \quad \tau\geq 0$$

and

$$\sum_{n=0}^{m-1}\int_{[0,1]^{n=1}-[0,h]^{n+1}}|y(q_1\cdots q_n q\tau)K(q_1)\cdots K(q_n)$$
$$(K_0(q)\,d\nu_0(q)+K_1(q)\,d\nu_1(q))\,d\nu(q_1)\cdots d\nu(q_n)|$$
$$< \frac{\chi_0+\chi_1}{1-\chi}(\phi+\delta), \quad \tau\geq t_0/h^{m+1}$$

we obtain

$$\left|\int_{t_0/h^{m+1}}^t e^{a(t-\tau)}\sum_{n=0}^{m-1}T^n(T_0+T_1)y(\tau)\,d\tau\right|$$
$$< \frac{2|y(0)|\delta+(\chi_0+\chi_1)(\phi+\delta)}{|\mathrm{Re}\,a|(1-\chi)}, \quad t>t_1.$$

Hence,

$$\left|\int_0^t e^{a(t-\tau)}\sum_{n=0}^{m-1}T^n(T_0+T_1)y(\tau)\,d\tau\right|$$
$$< \frac{|y(0)|(\chi_0+\chi_1+2)\delta+(\chi_0+\chi_1)(\phi+\delta)}{|\mathrm{Re}\,a|(1-\chi)}, \quad t>t_1.$$

Moreover,

$$\left|\sum_{n=m}^{\infty}T^n(T_0+T_1)y(\tau)\right|\leq \frac{|y(0)|(\chi_0+\chi_1)}{1-\chi}\chi^m, \quad \tau\geq 0,$$

implies

$$\left|\int_0^t e^{a(t-\tau)} \sum_{n=m}^{\infty} T^n (T_0 + T_1) y(\tau)\, d\tau\right| < \frac{|y(0)|(\chi_0+\chi_1)}{|\mathrm{Re}\, a|(1-\chi)}\delta, \quad t > 0.$$

Exploiting the above bounds, we obtain from formula (4.4) that

$$\begin{aligned} |y(t)| &< |y(0)|\delta + \frac{2|y(0)|(\chi_0 + \chi_1 + 1)\delta + (\chi_0 + \chi_1)(\phi + \delta)}{|\mathrm{Re}\, a|(1 - \chi)} \\ &= \phi - \delta, \quad t > t_1, \end{aligned}$$

which contradicts the definition of ϕ. Hence, $\phi = 0$. □

5. Unboundedness of solutions in the case $\mathrm{Re}\, a \geq 0$**.** In this section we first use a very old result from the theory of complex functions to derive the unboundedness of solutions in some particular cases.

The order of an entire function

$$f(z) = \sum_{n=0}^{\infty} \frac{a_n}{n!} z^n,$$

plays an important part in the first half of this section. Recall that the order ρ_f can be evaluated through the following formula (see Titchmarch [**18**])

$$\rho_f = \limsup_{n\to\infty} \frac{n \ln n}{n \ln n - \ln |a_n|}.$$

According to Ahlfors theorem [**5**], an entire function f of order ρ_f has at most $[2\rho_f]$ finite asymptotes at ∞. Thus, $\rho_f < 1/2$ means that f has no finite asymptotes at all. In other words, given any continuous curve $\gamma(s)$, $s \in [0, 1)$, in the complex plane such that $\gamma(s) \to \infty$ as $s \to 1$, it is true that $\limsup_{s\to 1-} |f(\gamma(s))| = \infty$. It should be noted that the same result holds for all the derivatives $f^{(k)}$ since they have the same order as f.

Theorem 4. *If $a = 0$ and $d\mu(q) = 0$ for $q > q_0$, where $q_0 \in (0, 1)$, then the solutions of problem* (1.1) *and all of their derivatives are unbounded.*

Proof. From $|\mu_k| \le q_0^k \mu^*$, we obtain

$$\sum_{k=N}^{n-1} \ln|\mu_k| \le \frac{1}{2}(n-N)(n-1+N)\ln q_0 + (n-N)\ln \mu^*.$$

From $|\nu_k| \le \nu_N^* < 1$, $k \ge N$, we obtain

$$\left| \sum_{k=N}^{n-1} \ln|1-\nu_k| \right| \le -(n-N)\ln(1-\nu_N^*). \tag{5.1}$$

Hence, the order of the solution $y(t)$ is zero. It follows from Ahlfors theorem that the solution is unbounded. □

Theorem 5. *If $a = 0$ and $\mu(q) = \sum_{j=1}^{\infty} a_j q^{\alpha_j}$, where $\operatorname{Re}\alpha_j > 0$ for all $j \ge 1$, the series $\sum_{j=1}^{\infty} a_j$ and $\sum_{j=1}^{\infty} a_j \alpha_j^2$ converge absolutely, and $\sum_{j=1}^{\infty} a_j \alpha_j = 0$, then the solutions of the initial-value problem* (1.1) *and all of their derivatives are unbounded.*

Proof. Noting that

$$\mu_k = \sum_{j=1}^{\infty} \frac{a_j \alpha_j}{k+\alpha_j} = -\sum_{j=1}^{\infty} \frac{a_j \alpha_j^2}{k(k+\alpha_j)}, \quad k \ge 1,$$

we have for $k \ge 1$,

$$|\mu_k| \le k^{-2} \sum_{j=1}^{\infty} |a_j \alpha_j^2|,$$

$$\sum_{k=N}^{n-1} \ln|\mu_k| \le (n-N)\ln \sum_{j=1}^{\infty} |a_j\alpha_j^2| - 2\ln(n-1)! + 2\ln(N-1)!, \quad n > N.$$

Together with (5.1), we see that the order of the solution is $\rho_y \le 1/3$. It follows from Ahlfors theorem that the solution is unbounded. □

Example 3. If $\mu(q) = q - q^2/2$, then $\rho_y = 1/3$.

Suppose next that $\sum_{j=1}^{\infty} a_j \alpha_j \ne 0$. From

$$\mu_k = \frac{1}{k}\mu'(1) - \sum_{j=1}^{\infty} \frac{a_j \alpha_j^2}{k(k+\alpha_j)}, \quad k > 0,$$

we obtain

$$\sum_{k=N}^{n-1} \ln|\mu_k| = -\ln(n-1)! + O(n).$$

Hence, $\rho_y = 1/2$. The Ahlfors theorem is no longer of use in this case.

Example 4. Consider the case that $a = 0$, $\mu(\tau) = a_0\tau^\gamma/\gamma$, $\nu(\tau) \equiv 0$, where $\operatorname{Re}\gamma > 0$, $a_0 \neq 0$. The corresponding solution of problem (1.1) with $y_0 = 1$ is

$$y(t) = {}_0F_1(-;\gamma;a_0 t), \tag{5.2}$$

where ${}_0F_1$ is the hypergeometric function. From the following formulas (see Abramowitz and Stegun [**1**, pp. 362, 364])

$$J_\nu(z) = \frac{(z/2)^\nu}{\Gamma(\nu+1)}{}_0F_1(-;\nu+1;-z^2/4),$$

and

$$J_\nu(z) = \left(\frac{2}{\pi z}\right)^{1/2}\left\{\cos\left(z - \frac{1}{2}\nu\pi - \frac{1}{4}\pi\right) + e^{|\operatorname{Im} z|}O(|z|^{-1})\right\}, \quad |\arg(z)| < \pi,$$

where $J_\nu(z)$ is a Bessel function, we obtain

$$\begin{aligned}{}_0F_1(-;\beta,z) &= \Gamma(\beta)\pi^{-1/2}(-z)^{(1/2-\beta)/2}\{\cos(2(-z)^{1/2} - ((\beta-1)/2)\pi - \pi/4) \\ &\quad + e^{2|\operatorname{Im}(-z)^{1/2}|}O(|z|^{-1/2})\}, \qquad |\arg((-z)^{1/2})| < \pi.\end{aligned}$$

If $a_0 < 0$, then the solution (5.2) oscillates unboundedly if $\gamma < 1/2$, oscillates boundedly if $\gamma = 1/2$, and tends to zero as $t \to \infty$ if $\gamma > 1/2$. This implies that the solution (5.2) displays entirely different asymptotic properties as the parameter γ changes, though its order remains constant.

Theorem 6. *If* $a > 0$ *and*

$$\int_0^1 \frac{1}{1-q}|d(\mu(q) + a\nu(q))| < \infty, \tag{5.3}$$

then the solution $y(t)$ of problem (1.1) satisfies

$$\lim_{t\to\infty} y^{(m)}(t)e^{-at} = a^{m-N}y_N^*, \tag{5.4}$$

for all $m \in \mathbf{Z}^+$, where

$$y_N^* = y_N \prod_{k=N}^{\infty} \frac{1+\mu_k/a}{1-\nu_k} < \infty.$$

Proof. The convergence of the product $\prod_{k=N}^{\infty}(1+\mu_k/a)/(1-\nu_k)$ follows from the inequality

$$\sum_{k=N}^{\infty}\left|\frac{1+\mu_k/a}{1-\nu_k}-1\right| \le \frac{1}{a(1-\nu_N^*)}\int_0^1 \frac{1}{1-q}|d(\mu(q)+a\nu(q))|.$$

For any given $\varepsilon > 0$, there exist an integer $M > N$ and a real number $T > 0$ such that

$$\left|y_N \prod_{k=N}^{n+m-1}\frac{1+\mu_k/a}{1-\nu_k} - y_N^*\right| < \frac{a^{N-m}\varepsilon}{2}, \quad n \ge M,$$

$$\left|e^{-at}\sum_{n=0}^{M-1}(y_{n+m}-a^{m-N}y_N^*)\frac{t^n}{n!}\right| < \frac{\varepsilon}{2}, \quad t > T.$$

From the preceding estimates and the expression

$$\begin{aligned} y^{(m)}(t) - a^{m-N}y_N^*e^{at} &= \sum_{n=0}^{M-1}(y_{n+m}-a^{m-N}y_N^*)\frac{t^n}{n!} \\ &\quad + a^{m-N}\sum_{n=M}^{\infty}\left(y_N\prod_{k=N}^{n+m-1}\frac{1+\mu_k/a}{1-\nu_k} - y_N^*\right)\frac{(at)^n}{n!}, \end{aligned}$$

obtain

$$|y^{(m)}(t)e^{-at} - a^{m-N}y_N^*| < \varepsilon, \quad t > T,$$

which implies (5.4). □

It is easy to see from (5.4) that the solution $y(t)$ of problem (1.1) increases (in modulus) exponentially if and only if $y_N^* \neq 0$ and $\mu_k \neq -a$ for all $k \geq N$. If $y_N^* = 0$ or $\mu_k = -a$ for some $k \geq N$, then the solution is a polynomial, as we can see from Theorem 1.

Example 5. Fox, Mayers, Ockendon and Tayler [**4**] presented some interesting numerical results obtained by the method of deferred correction for the problem (1.2) with $a = 0.95$, $b = -1$, $p = 0.99$ and $y_0 = 1$. Their results show that the numerical solution seemingly tends to zero (less than 5×10^{-5} in modulus) for $t \in (85, 125)$, but increases rapidly to positive infinity soon after $t_n > 145$. This is in conformity with the estimate (5.4), since $y_0^* \simeq 2.1 \times 10^{-81}$, which is a surprisingly small number compared with $e^{-125a} \simeq 2.68 \times 10^{-52}$.

Theorem 7. *If* $\operatorname{Re} a > 0$ *and* (5.3) *holds, then there exists a* (*possibly zero*) *constant* y^* *such that the solution* $y(t)$ *of problem* (1.1) *satisfies*

$$\lim_{t\to\infty} y^{(m)}(t)e^{-at} = a^m y^* \tag{5.5}$$

for all $m \in \mathbf{Z}^+$.

Proof. Let $y(t)$ be a solution of problem (1.1), and denote $y_n(t) = y^{(n)}(t)$, $n \geq 0$. We have

$$y_n'(t) = ay_n(t) + \int_0^1 y_n(qt)q^n\,d\mu(q) + \int_0^1 y_n'(qt)q^n\,d\nu(q), \qquad t > 0, \quad n \geq 0. \tag{5.6}$$

According to our assumption and because of the identity

$$\lim_{n\to\infty} \mu_n^* = \lim_{n\to\infty} \nu_n^* = \lim_{n\to\infty} \int_0^1 \frac{q^n}{1-q} |d(\mu(q) + a\nu(q))| = 0,$$

there exists an integer $M \geq 0$ such that $\nu_m^* < 1$ and

$$\nu_m^* + \frac{1}{\operatorname{Re} a} \int_0^1 \frac{q^m}{1-q} |d(\mu(q) + a\nu(q))| < 1$$

for all $m \geq M$. Applying Lemma 1 to equation (5.6) in the case $n = m \geq M$, we see that $y_m(t)$ satisfies

$$y_m'(t) = a y_m(t) + \sum_{n=0}^{\infty} T^n (aT + T_1) y_m(t), \tag{5.7}$$

where

$$Ty(t) = \int_0^1 y(qt) q^m \, d\nu(q), \qquad T_1 y(t) = \int_0^1 y(qt) q^m \, d\mu(q).$$

Applying Lemma 4 to equation (5.7), we see that there exists a (possibly zero) constant x_m, such that

$$\lim_{t\to\infty} y^{(m)}(t) e^{-at} = x_m.$$

For $0 \leq n < m$, we obtain from

$$y^{(n)}(t) = \sum_{k=0}^{m-n-1} \frac{1}{k!} y^{(k+n)}(0) t^k + \frac{1}{(m-n-1)!} \int_0^t (t-\tau)^{m-n-1} y^{(m)}(\tau) \, d\tau$$

that

(5.8)

$$\begin{aligned}\lim_{t\to\infty} y^{(n)}(t) e^{-at} &= \lim_{t\to\infty} \frac{1}{(m-n-1)!} \int_0^t (t-\tau)^{m-n-1} e^{-a(t-\tau) y^{(m)}} (\tau) e^{-a\tau} \, d\tau \\ &= \frac{x_m}{(m-n-1)!} \int_0^\infty \tau^{m-n-1} e^{-a\tau} \, d\tau \\ &= a^{n-m} x_m.\end{aligned}$$

The integer $m \geq M$ being arbitrary, we deduce that $\lim_{t\to\infty} y^{(n)}(t)e^{-at}$ exists for all $n \geq 0$. Let $y^* = x_0$. It is easy to see from (5.8) that (5.5) holds for all $m \in \mathbf{Z}^+$. □

Comparing (5.5) with (5.4), we conjecture that the identity

$$y^* = a^{-N} y_N \prod_{k=N}^{\infty} \frac{1 + \mu_k/a}{1 - \nu_k}$$

can be extended from $a > 0$ to $\mathrm{Re}\, a > 0$. A special case of this conjecture will be published in the "problems and solutions" section of SIAM Review [**13**].

Example 6. Consider the case where $a \neq 0$, $\mu(\tau) = a_0\tau^{\gamma}/\gamma$, $\nu(\tau) \equiv 0$, where $\mathrm{Re}\,\gamma > 0$, $a_0 \neq 0$. The corresponding solution of problem (1.1) with $y_0 = 1$ is

$$y(t) = {}_1F_1(\gamma + a_0/a; \gamma; at), \tag{5.9}$$

where ${}_1F_1$ is the confluent hypergeometric function. From Abramowitz and Stegun [**1**, p. 508], we have

$$ {}_1F_1(\alpha;\beta;z) = \Gamma(\beta)\left\{\frac{e^{\pm i\pi\alpha}z^{-\alpha}}{\Gamma(\beta - a)} + \frac{e^z z^{\alpha-\beta}}{\Gamma(\alpha)}\right\}(1 + \mathcal{O}(|z|^{-1})), \tag{5.10}$$

where the upper sign being taken if $-\pi/2 < \arg(zx) < 3\pi/2$, the lower sign if $-3\pi/2 < \arg(z) \leq -\pi/2$. If $\mathrm{Re}\, a > 0$, we obtain from (5.10) that the solution (5.9) satisfies

$$y(t)e^{-at} = \frac{\Gamma(\gamma)}{\Gamma(\gamma + a_0/a)}(at)^{a_0/a}(1 + \mathcal{O}(t^{-1})).$$

Hence, $y(t)e^{-at} \to \infty$ as $t \to \infty$ if $\mathrm{Re}\,(a_0/a) > 0$, $y(t)e^{-at} \to \Gamma(\gamma)/\Gamma(\gamma + a_0/a)$ as $t \to \infty$ if $\mathrm{Re}\,(a_0/a) = 0$, and $y(t)e^{-at} \to 0$ as $t \to \infty$ if $\mathrm{Re}\,(a_0/a) < 0$. This shows that condition (5.3) in Theorem 6 and Theorem 7 cannot be removed.

6. Asymptotic stability in the case $\operatorname{Re} a < 0$.

Theorem 8. *If* $\operatorname{Re} a < 0$, $\mu^* < |a|$, *and*

$$\lim_{h\to 0}\int_0^h |d\mu(q)| = \lim_{h\to 0}\int_0^h |d\nu(q)| = 0, \tag{6.1}$$

then the zero solution of the initial-value problem (1.1) *is asymptotically stable.*

Proof. Let $y(t)$ be a solution of the initial-value problem (1.1) and denote $y_n(t) = y^{(n)}(t)$, $n \geq 0$. We observe that $y_n(t)$ satisfies equation (5.6). According to our assumption and because of the identity (2.5), there exists an integer $M \geq 0$ such that $\nu_M^* < 1$ and

$$\operatorname{Re} a + \frac{|a|\nu_M^* + \mu_M^*}{1-\nu_M^*} < 0.$$

Applying Lemma 1 to equation (5.6) in the case $n = M$, we see that $y_M(t)$ satisfies (5.7) with $m = M$. Applying Lemma 6 to this equation, we see that $\lim_{t\to\infty} y_M(t) = 0$. From equation (5.6), we have

$$\begin{gathered} y_n(t) = -\frac{1}{a}\int_0^1 y_n(qt)q^n\, d\mu(q) + f_n(t), \\ t > 0, \quad n = 0, 1, \dots, M-1, \end{gathered} \tag{6.2}$$

where

$$f_n(t) = -\frac{1}{a}\left(y_{n+1}(t) - \int_0^1 y_{n+1}(qt)q^n\, d\nu(q)\right).$$

For $n = M-1$, observing that

$$\left| -\frac{1}{a}\int_0^1 q^{M-1}\, d\mu(q)\right| < 1$$

and that $\lim_{t\to\infty} f_{M-1}(t) = 0$, we derive from Lemma 2 that $\lim_{t\to\infty} y_{M-1}(t) = 0$. Continuing the above procedure for descending n, we finally obtain $\lim_{t\to\infty} y(t) = 0$. Hence, the zero solution is asymptotically stable. □

Next we discuss the necessity of the condition (6.1) made in Theorem 8. The purpose of this condition is to exclude the case where the functions $\mu(q)$ or $\nu(q)$ have a jump at $q = 0$. If this is the case, we can replace $d\mu(q)$ and $d\nu(q)$ by $d\mu^*(q) + b\delta(0)\,dq$ and $d\nu^*(q) + c\delta(0)\,dq$, respectively, where $\mu^*(q)$ and $\nu^*(q)$ are continuous at $q = 0$. Then (1.1) can be written as

(6.3)
$$y'(t) = ay(t) + \int_0^1 y(qt)\,d\mu^*(q) + \int_0^1 y'(qt), d\nu^*(q) + by(0) + cy'(0),$$
$$t > 0, \quad y(0) = y_0.$$

Assume that $y_0 \neq 0$, $c + \beta \neq 1$, $a + \alpha \neq 0$ and $b(1-\beta) + c(a+\alpha) \neq 0$, where $\alpha = \int_0^1 d\mu^*(q)$, $\beta = \int_0^1 d\nu^*(q)$. Substituting $t = 0$ into the preceding equation yields

$$y'(0) = \frac{a + \alpha + b}{1 - \beta - c} y_0.$$

Let

$$z(t) = y(t) + \frac{b(1-\beta) + c(a+\alpha)}{(a+\alpha)(1-\beta-c)} y_0,$$

we see that $z(t)$ satisfies

$$z'(t) = az(t) + \int_0^1 z(qt)\,d\mu(q) + \int_0^1 z'(qt)\,d\nu(q), \quad t > 0,$$
$$z(0) = \left(1 + \frac{b(1-\beta) + c(a+\alpha)}{(a+\alpha)(1-\beta-c)}\right) y_0.$$

Note that $\lim_{t\to\infty} z(t) = 0$ implies

$$\lim_{t\to\infty} y(t) = -\frac{b(1-\beta) + c(a+\alpha)}{(a+\alpha)(1-\beta-c)} y_0 \neq 0.$$

Hence, the zero solution is stable but not asymptotically stable.

Example 7. Consider the case where $a \neq 0$, $\mu(\tau) = a_0 \tau^\gamma/\gamma$, $\nu(\tau) \equiv 0$, where $\operatorname{Re}\gamma > 0$, $a_0 \neq 0$. The corresponding solution of problem (1.1) with $y_0 = 1$ is given by (5.9). Applying Theorem 8 to this case, we see that the zero solution is asymptotically stable if

$\mathrm{Re}\, a < 0$ and $\mathrm{Re}\, \gamma > |a_0/a|$. However, we can obtain a better result from (5.10), namely, that the zero solution is asymptotically stable if $\mathrm{Re}\, a < 0$ and $\mathrm{Re}\, \gamma > \mathrm{Re}\, (-a_0/a)$.

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