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WEAK ALMOST PERIODICITY OF CONVOLUTIONS

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ABSTRACT. It is shown that, for an Eberlein-weakly almost periodic (w.a.p.) function f with values in a Banach space X, the convolution $dK*f$ is again w.a.p. in Z, whenever the kernel $K(t) \in \mathcal{B}(X,Z)$ is of uniform bounded variation. This result is applied to abstract Cauchy problems as well as to abstract Volterra equations.

1. Introduction. Let X be a Banach space, A the generator of a C_0-semigroup $(S(t))_{t\geq 0}$ on X of type $\omega(A) < 0$, and consider the abstract Cauchy problem

$$\dot{u}(t) = Au(t) + f(t), \qquad u(0) = u_0, \quad t \geq 0, \tag{1.1}$$

where $f \in C_b(\mathbf{R}_+; X)$, $u_0 \in X$, as well as the evolution equation

$$\dot{v}(t) = Av(t) + g(t), \quad t \in \mathbf{R}, \tag{1.2}$$

on the line, where $g \in C_b(\mathbf{R}; X)$. The unique mild solutions of (1.1) and (1.2) are then given by

$$u(t) = S(t)u_0 + \int_0^t S(\tau)f(t-\tau)\,d\tau, \quad t \geq 0, \tag{1.3}$$

and

$$v(t) = \int_0^\infty S(\tau)g(t-\tau)\,d\tau, \quad t \in \mathbf{R}, \tag{1.4}$$

respectively. Such variation of parameters formulae also arise in the context of abstract linear Volterra equations, where $S(t)$ then denotes the so-called resolvent family; see Section 4 for details.

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The study of almost periodicity properties of the solutions to equations of this kind naturally leads to the question whether convolutions of the form

$$v(t) = \int_{-\infty}^{\infty} dK(\tau)g(t-\tau), \quad t \in \mathbf{R}, \tag{1.5}$$

and

$$u(t) = \int_{-\infty}^{t} dK(\tau)f(t-\tau), \quad t \in \mathbf{R}, \tag{1.6}$$

preserve such properties; here $\{K(t)\}_{t\in\mathbf{R}}$ is a family of bounded linear operators from X to another Banach space Z which is of uniform bounded variation, i.e.,

$$\begin{aligned} \operatorname{Var} K|_{-\infty}^{\infty} = \sup\Bigg\{ \sum_{j=1}^{n} |K(t_j) - K(t_{j-1})|_{\mathcal{B}(X,Z)} :& \\ -\infty < t_0 < t_1 < \cdots < t_n < \infty \Bigg\} &< \infty, \end{aligned}$$

$K \in BV(\mathbf{R};\mathcal{B}(X,Z))$ for short. To draw the connection between the semigroup $S(t)$ and $K(t)$, observe that if $K(t) = 0$ for $t < 0$, and $K(t) = \int_0^t S(\tau)\,d\tau$ for $t \geq 0$, then $K \in BV(\mathbf{R};\mathcal{B}(X))$.

It is well known (and easy to prove) that $g \in AP(X)$ implies $v \in AP(X)$, where $AP(X)$ denotes the space of all almost periodic (a.p.) functions on $\mathbf{R}$ with values in X in the sense of Bochner. Similarly, if f is asymptotically almost periodic (a.a.p.), i.e., $f \in AAP(X)$ then $u \in AAP(X)$.

In this note we address the problem of whether corresponding invariance properties of the convolution hold for weak almost periodicity in the sense of Eberlein. This latter kind of asymptotic behavior has recently been shown to arise naturally for solutions of both linear and nonlinear evolution equations, see Ruess/Summers [**13, 15, 16**].

Section 2 contains some background material about a.p. and w.a.p. functions. The formulation of our main result, which answers the above question in the affirmative, and its proof are presented in Section 3. For the sake of completeness and easy reference, we also include the corresponding results for a.p. and a.a.p. functions. Section 4 is devoted to the discussion of some consequences of the main result for abstract Cauchy problems as well as for abstract Volterra equations.

2. Preliminaries. In the sequel, the spaces of bounded, continuous and of bounded, uniformly continuous X-valued functions on an interval $J \subset \mathbf{R}$, equipped with the sup-norm $|\cdot|_\infty$, will be denoted by $C_b(J;X)$ and $C_{ub}(J;X)$, respectively. We abbreviate $C_b(X) = C_b(\mathbf{R};X)$ and $C_{ub}(X) = C_{ub}(\mathbf{R};X)$. The group of translations $\{T_t\}_{t\in\mathbf{R}}$, defined by

$$(T_t f)(\tau) = f(t+\tau), \quad t,\tau \in \mathbf{R},$$

consists of isometries and is strongly continuous in $C_{ub}(\mathbf{R};X)$, but not in $C_b(\mathbf{R};X)$. Recall that a function $f \in C_b(X)$ is called *almost periodic* (a.p.), respectively, *weakly almost periodic* in the sense of Eberlein (w.a.p.), if $T_{\mathbf{R}} f = \{T_\tau f : \tau \in \mathbf{R}\}$ is relatively compact, respectively, weakly relatively compact, in $C_b(X)$ (Bohl/Bohr/Bochner, cf. [**3**], and Eberlein [**7**] and Ruess/Summers [**14**]). Both a.p. and w.a.p. functions are uniformly continuous (Bochner [**3**] and Ruess/Summers [**17**]), so that the spaces $AP(X)$ and $W(X)$ of all a.p. and of all w.a.p. functions, respectively, are closed translation invariant subspaces of $C_{ub}(X)$. Note that periodic functions p are characterized among a.p. functions by the stronger property that $T_{\mathbf{R}} p$ is compact in $C_b(X)$.

We further recall that, according to the Jacobs-DeLeeuw/Glicksberg decomposition theorem (cf. Krengel [**10**, Section 2.4]), we have the topological direct sum decomposition

$$W(X) = AP(X) \oplus W_0(X),$$

where the space $W_0(X) \subset W(X)$ is given by

$$W_0(X) = \{f \in W(X) : w - \lim_{n\to\infty} T_{\tau_n} f = w - \lim_{n\to\infty} T_{\sigma_n} f = 0, \\ \text{for some sequences } \tau_n \to \infty, \sigma_n \to -\infty\}.$$

The *Bohr transform*, given by

$$\alpha(\rho, f) := \lim_{N\to\infty} N^{-1} \int_0^N e^{-i\rho t} f(t)\, dt, \quad \rho \in \mathbf{R},\ f \in AP(X),$$

is well defined and continuous from $AP(X)$ to X. For $f \in AP(X)$ its exponent set

$$\exp(f) = \{\rho \in \mathbf{R} : \alpha(\rho, f) \neq 0\}$$

is at most countable. Bochner's approximation theorem states that, given a fixed countable subset $\{\rho_j\}_1^\infty$, there are convergence factors $\gamma_{nj} \in \mathbf{R}$ with $\gamma_{nj} \to 1$ as $n \to \infty$ such that the trigonometric polynomials

$$B_n f = \sum_{j=1}^{n} \gamma_{nj} \alpha(\rho_j, f) e^{i\rho_j t}, \quad n \in \mathbf{N},$$

converge to f uniformly on $\mathbf{R}$, provided $\exp(f) \subset \{\rho_j\}_1^\infty$. By virtue of this result, it is also clear that $f \in AP(X)$ is uniquely determined by its Bohr transform. (See Amerio and Prouse [**1**] for proofs.)

If $f \in C_b(X)$ is ω-periodic, then $\exp(f) \subset (2\pi/\omega)\mathbf{Z}$ and

$$\alpha(2\pi n/\omega, f) = f_n = \frac{1}{\omega} \int_0^\omega e^{-2\pi i n t/\omega} f(t)\, dt, \quad n \in \mathbf{Z},$$

are the Fourier coefficients of f. The space of all continuous ω-periodic X-valued functions will be denoted by $P_\omega(X)$.

Turning to the halfline $\mathbf{R}^+ = [0, \infty)$, we first recall that a function $f \in C_{ub}(X)$ is called *asymptotically almost periodic* (a.a.p.) (to the right) if $T_{\mathbf{R}_+} f|_{\mathbf{R}+} \subset C_b(\mathbf{R}_+; X)$ is relatively compact. Similarly, a function $f \in C_{ub}(X)$ is called *weakly asymptotically almost periodic* in the sense of Eberlein [**7**] (w.a.a.p.) if $T_{\mathbf{R}_+} f|_{\mathbf{R}_+} \subset C_b(\mathbf{R}_+; X)$ is weakly relatively compact. The spaces $AAP^+(X)$ and $W^+(X)$ of all a.a.p. and of all w.a.a.p. functions, respectively, are closed translation invariant subspaces of $C_{ub}(X)$. Once again, the Jacobs-DeLeeuw/Glicksberg theorem (cf. Krengel [**10**, Section 2.4]) yields topological direct sum decompositions of these spaces of the form

$$AAP^+(X) = AP(X) \oplus C_0^+(X), \quad \text{and} \quad W^+(X) = AP(X) \oplus W_0^+(X),$$

respectively, where $C_0^+(X)$ and $W_0^+(X)$ are defined by

$$C_0^+(X) = \{f \in C_{ub}(X) : f(t) \to 0 \text{ as } t \to \infty\}$$

and

$$W_0^+(X) = \{f \in W^+(X) : w - \lim_{n\to\infty} T_{\tau_n} f|_{\mathbf{R}_+} = 0 \text{ in } C_b(\mathbf{R}_+; X), \\ \text{for some sequence } \tau_n \to \infty\},$$

respectively.

Note that the space

$$C_l^+(X) = \{f \in C_{ub}(X) : f(\infty) = \lim_{t\to\infty} f(t) \text{ exists}\}$$

is a closed subspace of $AAP^+(X)$.

Finally, once again, both for $f \in AAP^+(X)$ and $f \in W^+(X)$, the Bohr transform is well defined and, in either case, we have

$$\alpha(\rho, f) = \lim_{N\to\infty} N^{-1} \int_0^N e^{-i\rho t} f(t)\, dt = \alpha(\rho, f_a), \quad \rho \in \mathbf{R},$$

where f_a denotes the respective a.p. part of f in the Jacobs-DeLeeuw and Glicksberg decomposition of f. This is implied by the general mean ergodic theorem for bounded C_0-semigroups; see, e.g., Hille/Phillips [**9**, Chapter XVIII] and Ruess/Summers [**17**].

3. The main result. Let $K \in BV(\mathbf{R}; \mathcal{B}(X, Z))$, where Z denotes another Banach space and, for a given $f \in C_b(X)$, define $v \in C_b(Z)$ as the convolution

$$v(t) = \int_{-\infty}^{\infty} dK(\tau) f(t - \tau), \quad t \in \mathbf{R}, \tag{3.1}$$

where the integral is well defined in the Riemann-Stieltjes sense. The map $G : f \mapsto v$ obviously is bounded linear from $C_b(X)$ to $C_b(Z)$ with norm $|G| \le \operatorname{Var} K|_{-\infty}^{\infty}$ and commutes with the group of translations. For this reason, and since G is also weakly continuous, it is evident that G maps the subspaces $C_{ub}(X)$, $AP(X)$, $P_\omega(X)$, $W(X)$, and $W_0(X)$ into the corresponding spaces of Z-valued functions. The same is easily shown to be true also for $C_0^+(X)$ and for $C_l^+(X)$, hence also for $AAP^+(X)$. It is also valid for $W^+(X)$ and $W_0^+(X)$ which, however, is less obvious. For such kernels K the Fourier transform of dK is denoted by $\widetilde{dK}$, i.e.,

$$\widetilde{dK}(\rho) = \int_{-\infty}^{\infty} e^{-i\rho t} dK(t), \quad t \in \mathbf{R};$$

then $\widetilde{dK} \in C_{ub}(\mathcal{B}(X, Z))$ is easily proved. The main result of this paper reads as follows.

Theorem 1. *Suppose $K \in BV(\mathbf{R}; \mathcal{B}(X, Z))$, and let G be defined according to $(Gf)(t) = v(t)$, $t \in \mathbf{R}$, where $v(t)$ is given by* (3.1), *for $f \in C_b(X)$. Then G belongs to $\mathcal{B}(C_b(X), C_b(Z))$ and commutes with the group of translations $\{T_t\}_{t\in\mathbf{R}}$. Furthermore, we have*:

(i) *$f \in C_{ub}(X)$ implies $Gf \in C_{ub}(Z)$;*

(ii) *$f \in P_\omega(X)$ implies $Gf \in P_\omega(Z)$, and*

$$(Gf)_n = \widetilde{dK}(2\pi n/\omega) f_n, \quad n \in \mathbf{Z}; \tag{3.2}$$

(iii) *$f \in AP(X)$ implies $Gf \in AP(Z)$,* $\exp(Gf) \subset \exp(f)$, *and*

$$\alpha(\rho, Gf) = \widetilde{dK}(\rho)\alpha(\rho, f), \quad \rho \in \exp(f); \tag{3.3}$$

(iv) *$f \in AAP^+(X)$ implies $Gf \in AAP^+(Z)$, $(Gf)_0 = Gf_0$, $(Gf)_a = Gf_a$ and also* (3.3) *holds*;

(v) *$f \in C_l^+(X)$ implies $Gf \in C_l^+(Z)$ and*

$$(Gf)(\infty) = \widetilde{dK}(0) f(\infty); \tag{3.4}$$

in particular, $f \in C_0^+(X)$ implies $Gf \in C_0^+(Z)$;

(vi) *$f \in W^+(X)$ (respectively, $W(X)$) implies $Gf \in W^+(Z)$ (respectively, $W(Z)$), $(Gf)_a = Gf_a$, $(Gf)_0 = Gf_0$, and* (3.3) *holds*; *in particular, G maps $W_0^+(X)$ (respectively, $W_0(X)$) into $W_0^+(Z)$ (respectively, $W_0(Z)$).*

Proof. Let $\mathcal{H}$ denote any of the symbols C_{ub}, AP, P_ω, C_0^+, C_l^+, AAP^+, W, W_0, W^+, W_0^+; then $\mathcal{H}(X)$ is a closed subspace of $C_{ub}(X)$ which is translation invariant, the group of translations $\{T_t\}_{t\in\mathbf{R}}$ is strongly continuous and bounded in $\mathcal{H}(X)$. Furthermore, if $K \in$

$\mathcal{B}(X,Z)$ and $f \in \mathcal{H}(X)$, then Kf, defined by $(Kf)(t) = Kf(t)$, $t \in \mathbf{R}$, belongs to $\mathcal{H}(Z)$. This way K is extended to an operator $K \in \mathcal{B}(\mathcal{H}(X),\mathcal{H}(Z))$, and similarly, $K \in BV(\mathbf{R};\mathcal{B}(X,Z))$ extends to $K \in BV(\mathbf{R};\mathcal{B}(\mathcal{H}(X),\mathcal{H}(Z)))$. For such $K(t)$ and $f \in H(X)$, we have

$$(dK * f)(t) = \int_{-\infty}^{\infty} dK(\tau)(T_{-\tau}f)(t) = \left(\int_{-\infty}^{\infty} dK(\tau)T_{-\tau}f\right)(t), \quad t \in \mathbf{R};$$

since $T_\tau f$ is uniformly continuous on $\mathbf{R}$ in $\mathcal{H}(X)$, $\int_{-\infty}^{\infty} dK(\tau)T_{-\tau}f$ exists as a Riemann-Stieltjes integral in $\mathcal{H}(Z)$. Therefore, $G \in \mathcal{B}(\mathcal{H}(X),\mathcal{H}(Z))$, which proves the first part of each of the statements (i)–(vi) of Theorem 1.

If $f \in W^+(X)$, then there is the unique decomposition $f = f_0 + f_a$, where $f_0 \in W_0^+(X)$ and $f_a \in AP(X)$; this implies $Gf = Gf_0 + Gf_a$ and $Gf_0 \in W_0^+(Z)$, $Gf_a \in AP(Z)$, hence $(Gf)_a = Gf_a$ and $(Gf)_0 = Gf_0$, by uniqueness of the decomposition. This proves the second statement of (vi), and replacing W^+ by AAP^+, W_0^+ by C_0^+ in the argument just given, the second assertion of (iv) follows as well.

To prove (3.3), let $f(t) = \sum_{k=1}^n e^{i\rho_k t} f_k$ be a trigonometric polynomial; then

$$(Gf)(t) = \sum_{k=1}^n \int_{-\infty}^{\infty} e^{i\rho_k(t-\tau)} dK(\tau) f_k = \sum_{k=1}^n e^{i\rho_k t}\widetilde{dK}(\rho_k) f_k$$

is again a trigonometric polynomial with $\exp(Gf) \subset \exp(f)$ and (3.3) holds. The continuity of the Bohr transform and Bochner's approximation theorem for a.p. functions then imply (iii), in particular (3.3). Equations (3.2) and (3.4) are direct consequences of (3.3); hence, the proof is complete. $\square$

As an easy consequence of Theorem 1, we also obtain the mapping behavior of G_+, defined by

$$(G_+f)(t) = \int_{-\infty}^{t} dK(\tau) f(t-\tau), \quad t \in \mathbf{R}, \tag{3.5}$$

where $f \in C_b(X)$. G_+ is also bounded linear; however, it does not commute with the group of translations. But the estimate

$$|(Gf)(t) - (G_+f)(t)| = \left| \int_t^\infty dK(\tau)f(t-\tau) \right| \leq \operatorname{Var} K|_t^\infty |f|_\infty \to 0$$

as $t \to \infty$ shows $Gf - G_+f \in C_0^+(Z)$ for any $f \in C_b(X)$. Therefore we obtain:

Corollary 1. *Suppose $K \in BV(\mathbf{R}; \mathcal{B}(X, Z))$, let G be defined as in Theorem* 1, *and G_+ by* (3.5); *let $\mathcal{H}$ denote any of the symbols C_b, C_{ub}, C_0^+, C_l^+, AAP^+, W^+, W_0^+. Then $G - G_+$ belongs to $\mathcal{B}(C_b(X), C_0^+(Z))$; consequently, G_+ maps $\mathcal{H}(X)$ into $\mathcal{H}(Z)$ and* (3.3) *and* (3.4) *remain valid for G_+f.*

Observe that in case $K(t) \equiv K(-\infty)$ on $t < 0$, Corollary 1 yields an analogous result for the convolution

$$w(t) = \int_0^t dK(\tau)f(t-\tau), \quad t \geq 0,$$

on the halfline.

4. Applications to evolutionary integral equations. Let X and Y be Banach spaces such that Y is densely embedded into X, let $A \in L^1_{\mathrm{loc}}(\mathbf{R}_+; \mathcal{B}(Y, X))$, and $h \in L^1_{\mathrm{loc}}(\mathbf{R}_+; X)$. In this section we consider the application of our main result to the linear evolutionary integral equation

$$u(t) = \int_0^t A(t-s)u(s)\,ds + h(t), \quad t \in \mathbf{R}_+. \tag{4.1}$$

Clearly, the abstract Cauchy problem (1.1) is a special case of (4.1); choose $A(t) \equiv A$, $Y = D(A)$, the domain of A equipped with the graph norm of A, and

$$h(t) = u_0 + \int_0^t f(s)\,ds, \quad t \geq 0. \tag{4.2}$$

It is well known that many problems from mathematical physics can be formulated as (4.1), in particular, linear viscoelasticity, the theory of heat conduction with memory, and electrodynamics with memory lead to such problems; see Dafermos [**6**], Miller [**11**], Carr/Hannsgen [**4**], Clément/Nohel [**5**], Bloom [**2**], and many others. A recent account of the theory is presented in the first author's monograph [**12**]. In these applications one should think of X as a space of functions defined on a region $\Omega \subset \mathbf{R}^n$, of $A(t)$ as a partial differential operator, and of Y as the domain of $A(t)$, including the boundary conditions involved.

The most important concept for (4.1) is that of the resolvent $S(t)$, which is defined as follows. A family $\{S(t)\}_{t\in\mathbf{R}_+} \subset \mathcal{B}(X)$ is called a *resolvent family* for (4.1) if $S(\cdot)x$ is continuous in X, for each $x \in X$, $S(t)$ leaves Y invariant, $S(\cdot)y$ is continuous in Y, and the *resolvent equations* hold.

(4.3)
$$S(t)y = y + \int_0^t A(t-\tau)S(\tau)y\,d\tau, \quad \text{for all } y \in Y, t \geq 0;$$

(4.4)
$$S(t)y = y + \int_0^t S(\tau)A(t-\tau)y\,d\tau, \quad \text{for all } y \in Y, t \geq 0;$$

there can be at most one resolvent family for (4.1) and, if this equation admits such a family, then every solution of (4.1) is represented by the *variation of parameters formula*

$$u(t) = \frac{d}{dt}\int_0^t S(t-\tau)h(\tau)\,d\tau, \quad t \geq 0. \tag{4.5}$$

If $h(t)$ is of the form (4.2), then (4.5) becomes (1.3) and, in the special case of the Cauchy problem (1.1) we have $S(t) = e^{At}$, the semigroup generated by A.

The resolvent $S(t)$ for (4.1) is called *integrable* if there is a function $\varphi \in L^1(\mathbf{R}_+)$ such that $|S(t)| \leq \varphi(t)$ for all $t > 0$. In the setting of the Cauchy problem, this is fulfilled if and only if the semigroup is of type $\omega(A) < 0$.

If the resolvent $S(t)$ is integrable, then the function K, defined by $K(t) = 0$ for $t < 0$, $K(t) = \int_0^t S(\tau)\,d\tau$ for $t \geq 0$, belongs to

$BV(\mathbf{R};\mathcal{B}(X))$; hence, Theorem 1 and Corollary 1 apply. The resolvent equations show that $\widetilde{dK}(\rho)$ is then given by $\widetilde{dK}(\rho) = H(i\rho)$, where

$$H(\lambda) = \hat{S}(\lambda), \quad \operatorname{Re}\lambda \geq 0. \tag{4.6}$$

As a result, we obtain

Theorem 2. *Let X and Y be Banach spaces with Y densely embedded into X, and assume $A \in L^1_{\mathrm{loc}}(\mathbf{R}_+;\mathcal{B}(Y,X))$. Suppose* (4.1) *admits an integrable resolvent $S(t)$, let $H(\lambda)$ be defined by* (4.6) *and let $u(t)$, respectively $v(t)$, be defined according to* (1.3), *respectively* (1.4), *with $g = f$, where h is of the form* (4.2), *with $u_0 = 0$ and $f \in C_b(X)$.*

Then $u, v \in C_b(X)$ and $u(t) - v(t) \to 0$ as $t \to \infty$. Furthermore, the following assertions hold.

(a) *$f \in C_{ub}(X)$ implies $u, v \in C_{ub}(X)$;*

(b) *$f \in AAP^+(X)$ implies $u, v \in AAP^+(X)$, and*

$$\alpha(\rho, u) = \alpha(\rho, v) = H(i\rho)\alpha(\rho, f), \quad \rho \in \mathbf{R}; \tag{4.7}$$

(c) *$f \in W^+(X)$ implies $u, v \in W^+(X)$, and* (4.7) *holds*; *in particular, if $f \in W_0^+(X)$ then $u, v \in W_0^+(X)$;*

(d) *$f \in C_l^+(X)$ implies $u, v \in C_l^+(X)$ and $u(\infty) = v(\infty) = H(0)f(\infty)$*; *in particular, if $f \in C_0^+(X)$, then $u, v \in C_0^+(X)$;*

(e) *$F \in P_\omega$ implies $v \in P_\omega$ and*

$$v_n = H(2\pi ni/\omega)f_n, \quad n \in \mathbf{Z}; \tag{4.8}$$

(f) *$f \in AP(X)$ implies $v \in AP(X)$ and* (4.7) *holds.*

If $u_0 \in X$ is nonzero and, in addition, $S(t)x \to 0$ as $t \to \infty$ for each $x \in X$, the same results are valid except that $u(t)$ has a jump discontinuity of magnitude u_o at $t = 0$.

If, in addition to the assumptions of Theorem 2, we have $A(t) = A_1(t) + A_2(t)$, $t > 0$, where $A_1 \in L^1(\mathbf{R}_+;\mathcal{B}(Y,X))$ and $A_2 \in BV(\mathbf{R}_+;\mathcal{B}(Y,X))$, then by the resolvent equations,

$$H(\lambda) = (\lambda - \lambda\hat{A}_1(\lambda) - \widehat{dA_2}(\lambda))^{-1}, \quad \operatorname{Re}\lambda > 0; \tag{4.9}$$

moreover, the function $v(t)$ can then be considered as the solution of (4.10)

$$\dot{v}(t) = \int_0^\infty A_1(\tau)\dot{v}(t-\tau)\,d\tau + \int_0^\infty dA_2(\tau)v(t-\tau) + f(t), \quad t \in \mathbf{R}.$$

Since $u(t)$ behaves asymptotically as $v(t)$, (4.10) can be termed *limiting equation* for (4.1). Note that the special case $A_1 = 0$ and $A_2 \equiv A$ in (4.10) corresponds to the Cauchy problem (1.2). For more details and further discussion, we refer to Prüss [**12**].

REFERENCES

1. L. Amerio and G. Prouse, *Almost-periodic functions and functional equations*, Van Nostrand Reinhold Co., New York, 1971.

2. F. Bloom, *Ill-posed problems for integrodifferential equations in mechanics and electromagnetic theory*, Volume 3 of Studies in Applied Mathematics, SIAM, Philadelphia, 1981.

3. S. Bochner, *Abstrakte fastperiodische Funktionen*, Acta Math. **61** (1933), 149–184.

4. R.W. Carr and K.B. Hannsgen, *A nonhomogeneous integrodifferential equation in Hilbert space*, SIAM J. Math. Anal. **10** (1979), 961–984.

5. Ph. Clément and J.A. Nohel, *Abstract linear and nonlinear Volterra equations preserving positivity*, SIAM J. Math. Anal. **10** (1979), 365–388.

6. C.M. Dafermos, *An abstract Volterra equation with applications to linear viscoelasticity*, J. Differential Equations **7** (1970), 554–569.

7. W.F. Eberlein, *Abstract ergodic theorems and weak almost periodicity*, Trans. Amer. Math. Soc. **67** (1949), 217–240.

8. M. Fréchet, *Les fonctions asymptotiquement presque-périodiques*, Rev. Sci. **79** (1941), 341–354.

9. E. Hille and R.S. Phillips, *Functional analysis and semigroups*, Volume 31 of Amer. Math. Soc. Colloq. Publ., Amer. Math. Soc., Providence, Rhode Island, 1957.

10. U. Krengel, *Ergodic theorems*, volume 6 of de Gruyter Studies in Mathematics, de Gruyter, 1985.

11. R. Miller, *An integrodifferential equation for rigid heat conductors with memory*, J. Math. Anal. Appl. **66** (1978), 313–332.

12. J. Prüss, *Evolutionary integral equations and applications*, Monographs in Mathematics **87**, Birkhäuser, Basel, 1993.

13. W.M. Ruess and W.H. Summers, *Weak almost periodicity and the strong ergodic limit theorem for contraction semigroups*, Israel J. Math. **64** (1988), 139–157.

14. ———, *Integration of asymptotically almost periodic functions and weak asymptotic almost periodicity*, Dissertationes Math. **279** (1989), 1–38.

15. ———, *Weakly almost periodic semigroups of operators*, Pacific J. Math. **143** (1990), 175–193.

16. ———, *Weak almost periodicity and the strong ergodic limit theorem for periodic evolution systems*, J. Funct. Anal. **94** (1990), 177–195.

17. ———, *Ergodic theorems for semigroups of operators*, Proc. Amer. Math. Soc. **114** (1992), 423–432.

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