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GENERALIZED SOLUTIONS FOR ONE DIMENSIONAL FLOWS OF K – BKZ FLUIDS

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ABSTRACT. A boundary traction problem for a viscoelastic liquid is studied. The problem is posed in one-space dimension and is described by a third order integrodifferential equation of Volterra type. Existence, uniqueness and asymptotic behavior are proved for a class of generalized solutions. The results obtained correspond to what is already known for smooth solutions to these types of problems.

1. Introduction. In this work we consider the motion of one-dimensional shearing flows of a K-BKZ fluid with a Newtonian viscosity. Our main interest is to investigate a class of generalized solutions for certain initial-boundary value problems associated with these flows. We assume that the fluid occupies a reference configuration given by the unit interval $0 \leq x \leq 1$ and a Newtonian viscosity given by a positive constant $\eta > 0$. If $u(x,t)$ denotes the displacement, then the motion is governed by a third order partial integrodifferential equation of the form

$$\text{(IDE)} \quad u_{tt}(x,t) - \eta u_{xxt}(x,t) = \int_0^\infty g(s, u_x(x,t) - u_x(x,t-s))_x \, dx + f(x,t),$$

where $0 \leq x \leq 1$, $0 \leq t < \infty$. The function g is called the memory function and f represents the body force. Equation (IDE) is based on a constitutive law proposed by Kaye [**7**] and Bernstein, Kearsley, Zapas [**2**] for viscoelastic liquids. A discussion of the derivation of (IDE) from the principles of continuum mechanics is given in [**10**].

There are several possible initial-boundary value problems that are naturally associated with equation (IDE). In this work we wish to discuss one particular type of problem called the boundary traction problem. This means that, in addition to prescribing the initial history

$\tilde{u}(x,t)$ of the displacement $u(x,t)$ for $t < 0$, and prescribing initial conditions $u_0(x)$, $u_1(x)$ for the one-sided limits $u(x,0+)$, $u_t(x,0+)$, respectively, we also assume that traction forces act on the fluid at the boundary. Thus, we are considering equation (IDE) with the following initial and boundary conditions:

$$\text{(BC)} \qquad u_{xt}(j,t) + \int_0^\infty g(s, u_x(j,t) - u_x(j,t-s))\,ds = h_j(t), \qquad t > 0,\ j = 0,1,$$

$$\text{(IC)} \qquad u(x,0+) = u_0(x),\ u_t(x,0+) - u_1(x), \quad 0 < x < 1,$$

$$\text{(IH)} \qquad u(x,t) = \tilde{u}(x,t), \quad 0 < x < 1,\ t < 0.$$

Problem (IDE), (BC), (IC), (IH) has been solved by H. Engler [**5**] in the context of smooth solutions. Along with the related boundary displacement problem for (IDE), Engler has shown that, under general smoothness and compatibility assumptions, local smooth solutions of the boundary traction problem exist and are unique. Furthermore, if g, f, h_0, h_1 satisfy certain growth and integrability conditions, then the smooth solutions exist globally in time, and their asymptotic behavior can be described with rates of convergence depending on the material parameters of the problem.

In this work we take advantage of the "divergence structure" of equation (IDE) and extend the concept of a solution to a wider class of functions not necessarily having derivatives up to order three. By doing this, we are able to require less regularity on $g, f, \tilde{u}, u_0, u_1, h_0, h_1$ than found in [**5**]. Our method of approach is to first replace the nonlocal boundary conditions (BC) by a pair of Neumann conditions

$$\text{(NBC)} \qquad u_x(j,t) = \varphi_j(t), \quad t > 0,\ j = 0,1.$$

This intermediate step allows us to derive a simple integral equation for $u(x,t)$. The bulk of the paper is devoted to the study of this integral equation which is derived by using some ideas from G. Andrews [**1**]. The integral equation can be easily solved locally in time by the Contraction Mapping Principle. We do not require the Krasnoselskii Fixed Point Theorem that was used in [**1**]. Under the assumptions that we make on the data, we show that the solution $u(x,t)$ has the property that

$$u \in C([0,T]; L^\infty(0,1)), \quad u_x \in C([0,T], L^\infty(0,1))$$
$$u_t \in C((0,1) \times (0,T)), \quad u_{xt} \in L^2((0,1) \times (0,T))$$

for a suitable constant $T > 0$. Also, we show that, under some mild conditions on the data, the solution $u(x,t)$ satisfies the traction boundary condition (BC) almost everywhere on $0 \le t \le T$.

Under suitable assumptions (same as in [**5**]), we show that generalized solutions exist globally in time, even if the data is large. They also share the same asymptotic behavior as smooth solutions. The proofs of these statements are based on energy methods and use an integral identity satisfied by all generalized solutions of (IDE), (BC), (IC), (IH).

The organization of the paper is as follows. In Section 2 we introduce notation and present background material on parabolic equations that will be needed in the sequel. Section 3 is devoted to the proofs of existence and uniqueness of generalized solutions. In Section 4 we use energy methods to obtain the necessary estimates for the proof of global existence. In Section 5 we discuss the asymptotic behavior of generalized solutions.

For references on the type of problem considered in this work, we refer to the book [**10**]. We especially mention Chapter IV for a synopsis of the boundary displacement problem for (IDE). This problem can also be treated by the methods used here and corresponding results can be proved under appropriate assumptions.

2. Preliminaries. In this section we introduce notation and discuss some results that will be used in subsequent sections. For a given set S, let $L^p(S)$ and $W^{m,p}(S)$ denote the standard L^p and Sobolev spaces with norms $\|\cdot\|_p$ and $\|\cdot\|_{m,p}$, respectively, where $1 \le p \le \infty$, $m = 1, 2, \dots$. Given a positive number $T > 0$, we let $Q_T = (0,1) \times (0,T)$. Let $W_2^{1,0}(QT)$ denote the space of all measurable functions $u : Q_T \to \mathbf{R}$ such that the distributional derivatives $\partial_x^r u \in L^2(Q_T)$ for $r = 0, 1$. When equipped with the norm

$$\|u\|^2_{W_2^{1,0}(Q_t)} = \int_{Q_T} (u^2 + u_x^2)\, dx\, dt,$$

the space $W_2^{1,0}(Q_T)$ is a Hilbert space. Since Q_T is a rectangle, we can reverse the roles of x and t and define the Hilbert space $W_2^{0,1}(Q_T)$ in a similar manner with norm

$$\|u\|^2_{W_2^{0,1}(Q_T)} = \int_{Q_T} (u^2 + u_t^2)\, dx\, dt.$$

Let $W_p^{2,1}(Q_T)$ denote the space of all measurable functions $u : Q_T \to \mathbf{R}$ whose distributional derivatives $\partial_x^s \partial_t^r u \in L^p(Q_T)$ for all $2r + s \le 2$. When equipped with the norm

$$\|u\|_{W_p^{2,1}(Q_T)} = \sum_{2r+s\le 2} \|\partial_x^s \partial_t^r u\|_p,$$

the space $W_p^{2,1}(Q_T)$ is a Banach space.

If X is any Banach space and J is an interval of real numbers, let $C_b(J, X)$ denote the Banach space of all bounded continuous functions $u : J \to X$ with norm

$$\|u\|_{C_b(J;X)} = \sup_{t\in J} \|u(t)\|_X.$$

When J Is a compact interval, we simply write $C(J; X)$ instead of $C_b(J; X)$.

In the sequel, we will consider various Green's functions associated with the linear heat equation. We let

$$K(x,t) = \frac{1}{\sqrt{4\pi t}} \exp\left(-\frac{x^2}{4\eta t}\right), \quad x \in \mathbf{R},\ t > 0, \tag{2.1}$$

and define the theta-function by

$$\theta(x,t) = \sum_{n=-\infty}^{+\infty} K(x + 2n, t), \quad x \in \mathbf{R},\ t > 0. \tag{2.2}$$

We define

$$G(x,\xi,t) = \theta(x-\xi,t) - \theta(x+\xi,t)$$
$$H(x,\xi,t) = \theta(x-\xi,t) + \theta(x+\xi,t)$$

for $x, \xi \in \mathbf{R}$ and $0 < t < \infty$. Then $G(x, \xi, t)$ and $H(x, \xi, t)$ represent Green's functions for the linear heat equation with Dirichlet and Neumann boundary conditions, respectively, in the strip $(0,1)\times(0,\infty)$ (cf. [**4**]).

From results in (**8**; Chapter IV, Section 9]), if $\psi \in L^p(Q_T)$ with $p > 3/2$, then the function

$$w(x,t) = \int_0^t \int_0^1 G(x,\xi,t-\tau)\psi(\xi,\tau)\, d\xi\, d\tau \tag{2.3}$$

belongs to $W_p^{2,1}(Q_T)$ and is the unique solution almost everywhere on Q_T of the problem

$$\begin{aligned} w_t(x,t) - \eta w_{xx}(x,t) = \psi(x,t), &\quad (x,t) \in Q_T, \\ w(0,t) = w(1,t) = 0, &\quad t > 0, \\ w(x,0) = 0, &\quad 0 < x < 1. \end{aligned}$$

Furthermore, there is a constant $C(T) > 0$ such that

$$\int_0^t \int_0^1 |G(x,\xi,t-\tau)\psi(\xi,\tau)|\, d\xi\, d\tau \le C(T)\|\psi\|_p, \quad (x,t) \in Q_T, \tag{2.4}$$

and $w(x,t)$ is Hölder continuous in x and t. If $p > 3$, then we also have

$$\int_0^t \int_0^1 |G_x(x,\xi,t-\tau)\psi(\xi,\tau)|\, d\xi\, d\tau \le C(T)\|\psi\|_p, \quad (x,t) \in Q_T, \tag{2.5}$$

and $w_x(x,t)$ is Hölder continuous in x and t. Using methods similar to [**1**], we can also show that

$$\sup_{0 \le x \le 1} \int_0^1 |G(x,\xi,t)\varphi(\xi)|\, d\xi \le \sqrt{\eta}\|\varphi\|_\infty, \qquad 0 < t < \infty, \tag{2.6}$$

$$\sup_{0 \le x \le 1} \int_0^1 |G_x(x,\xi,t)\varphi(\xi)|\, d\xi \le \frac{1}{\sqrt{\pi t}}\|\varphi\|_\infty, \quad 0 < t < \infty, \tag{2.7}$$

for all $\varphi \in L^\infty(0,1)$.

Next, consider the linear heat equation with Neumann boundary conditions:

$$v_t(x,t) - \eta v_{xx}(x,t) = \psi(x,t), \quad (x,t) \in Q_T \tag{2.8}$$

$$v_x(0,t) = h_0(t),\ v_x(1,t) = h_1(t), \quad t \in (0,T) \tag{2.9}$$

$$v(x,0) = v_0(x), \quad 0 < x < 1. \tag{2.10}$$

A formal representation of a solution $v(x,t)$ of (2.8), (2.9), (2.10) is given by

$$
\begin{aligned}
v(x,t) &= \int_0^t \int_0^1 H(x,\xi,t-\tau)\psi(\xi,\tau)\,d\xi\,d\tau \\
&\quad + \int_0^1 H(x,\xi,t)v_0(\xi)\,d\xi \\
&\quad + 2\int_0^t \theta(x-1,t-\tau)h_1(\tau)h\tau \\
&\quad - 2\int_0^t \theta(x,t-\tau)h_0(\tau)\,d\tau.
\end{aligned}
\tag{2.11}
$$

Without giving the most optimal result possible, it will be sufficient for our purpose to state

LEMMA 1. *Suppose that $T > 0$ is fixed, and assume that the following conditions are satisfied*:

(i) $\psi \in L^p(Q_T)$ *for some* $p \in (3,\infty)$,

(ii) $v_0 \in L^\infty(0,1)$,

(iii) $h_0, h_1 \in L^1(0,T)$.

Then the function $v(x,t)$ *given by* (2.11) *satisfies* (2.8) *almost everywhere in* Q_T *and also satisfies the initial-boundary conditions* (2.9), (2.10) *in the sense that*

$$
\begin{aligned}
&\lim_{x\to j} v_x(j,t) = h_j(t) \quad a.e.\ t \in (0,T),\ j = 0,1, \\
&\lim_{t\to 0+} v(x,t) = v(x,t) \quad a.e.\ x \in (0,1).
\end{aligned}
$$

We also have

$$
\sup_{(x,t)\in Q_T} |v(x,t)| < \infty, \qquad \sup_{(x,t)\in Q_T} \int_0^t |v_x(x,s)|\,ds < \infty. \tag{2.12}
$$

In addition, if we replace (ii), (iii) *by*

(ii)′ $v_0 \in W^{1,\infty}(0,1)$,

(iii)′ $h_0, h_1 \in L^\infty(0,T)$,

then we have

$$\sup_{(x,t)\in Q_T} |v_x(x,t)| < \infty. \tag{2.13}$$

PROOF. With the possible exception of the second inequality in (2.12), the results of Lemma 1 are quite well known. Their validity may be inferred from standard sources [**6, 8**], or the results may be proven directly from the basic definitions (2.1), (2.2). So we shall only comment on the estimation of the integral

$$\int_0^t v_x(x,s)\,ds. \tag{2.14}$$

It suffices to consider each term in (2.11) separately. Since (2.4)–(2.7) are also valid for the Green's function $H(x,\xi,t)$, the first two terms are easy to estimate. We concentrate on the last term, which we define by

$$\tilde{v}(x,t) = (-2)\int_0^t \theta(x,t-\tau)h_0(\tau)\,d\tau, \tag{2.15}$$

since the third term is handled in a similar way. From (2.2), one may differentiate $\tilde{v}(x,t)$ with respect to x and obtain

$$\tilde{v}_x(x,t) = (-2)\int_0^t K_x(x,t-\tau)h_0(\tau)h\tau - 2\int_0^t J(x,t-\tau)h_0(\tau)\,d\tau,$$

where $J(x,t)$ is a function in $L^\infty(Q_T)$. Therefore

$$\begin{aligned}\int_0^t &|\tilde{v}_x(x,s)|\,ds \\ &\le 2\int_0^t\int_0^s |K_x(x,s-\tau)h_0(\tau)|\,d\tau\,ds + T\|J\|_\infty\int_0^t |h_0(\tau)|\,d\tau.\end{aligned} \tag{2.16}$$

Assuming $x>0$, the double integral in (2.16) is absolutely convergent and then, by Fubini's Theorem,

$$\begin{aligned}\int_s^t\int_0^s &|K_x(x,s-\tau)h_0(\tau)|\,d\tau\,ds \\ &\le \frac{x}{4\eta\sqrt{\pi}}\int_0^t |h_0(\tau)|\int_0^{t-\tau}\frac{1}{s^{3/2}}\exp\left(-\frac{x^2}{4\eta s}\right)ds\,d\tau \\ &\le \frac{1}{\eta\sqrt{\pi}}\int_0^t |h_0(\tau)|\,d\tau\cdot\int_0^\infty \frac{1}{y^{1/2}}e^{-y}\,dy.\end{aligned}$$

Since this expression is finite and independent of x, the estimate is proven. $\square$

Later on in the sequel, we shall use the approximation arguments that are based on properties of the spaces $W_2^{1,0}(Q_T)$ and $W_2^{0,1}(Q_T)$. We state the results that we need in the next two lemmas.

LEMMA 2. *Let $u \in W_2^{1,0}(Q_T)$ be given. Then there is a sequence $\varphi_n \in C^\infty(\overline{Q}_T)$ such that*

(i) *$\varphi_n \to u$ in $W_2^{1,0}(Q_T)$ as $n \to \infty$,*

(ii) *$\varphi_n(x, \cdot) \to u(x, \cdot)$ in $L^2(0, T)$ as $n \to \infty$ for each $0 \le x \le 1$.*

PROOF. From [**9**, p. 159] the space $C^\infty(\overline{Q}_T)$ is dense in $W_2^{1,0}(Q_T)$. This gives (i) and the Sobolev Imbedding Theorem gives (ii). $\square$

Using similar arguments, we can also prove the following result.

LEMMA 3. *Let $u \in W_2^{0,1}(Q_T) \cap L^\infty(Q_T)$ be given. Then there is a sequence $\varphi_n \in C^\infty(\overline{Q}_T)$ such that*

(i) *$\varphi_n \to u$ in $W_2^{0,1}(Q_T)$ as $n \to \infty$,*

(ii) *$\varphi_n(\cdot, t) \to u(\cdot, t)$ in $L^2(0, 1)$ as $n \to \infty$ for each $0 \le t \le T$,*

(iii) *$\varphi_n(x, t) \to u(x, t)$ a.e. $(x, t) \in Q_T$,*

(iv) *$\{\varphi_n\}$ is uniformly bounded on Q_T.*

3. Existence and uniqueness. In this section we define generalized solutions of (IDE), (BC), (IC), (IH) be means of an integral equation. Under suitable assumptions on the data, we show that generalized solutions exist and are unique and have certain regularity properties. The derivation of the integral equation is facilitated by first considering the following related initial-boundary value problem:

(3.1)
$$u_{tt}(x,t) - \eta u_{xxt}(x,t) = \int_0^\infty g(s, u_x(x,t) - u_x(x,t-s))_x \, ds + f(x,t), \quad 0 < x < 1, \ t > 0,$$

$$u_x(0,t) = \varphi_0(t), \quad u_x(1,t) = \varphi_1(t), \quad t > 0, \tag{3.2}$$

$$u(x,0+) = u_0(x), \quad u_t(x,0+) = u_1(x), \quad 0 < x < 1, \tag{3.3}$$

$$u(x,t) = \tilde{u}(x,t), \quad 0 < x < 1, \quad t < 0. \tag{3.4}$$

Suppose $u(x,t)$ is a smooth solution of (3.1)–(3.4) with smooth data. We introduce the operator Γ defined by

$$\Gamma(v)(x,t) = \int_0^t g(t-s, v(x,t) - v(x,s)) \, ds + \int_{-\infty}^0 g(t-s, v(x,t) - \tilde{u}_x(x,s)) \, ds \tag{3.5}$$

and combine (3.1)–(3.4) into a single equation,

$$\left(\frac{\partial}{\partial t} - \eta \frac{\partial^2}{\partial x^2}\right) u_t(x,t) = \frac{\partial}{\partial x} \Gamma(u_x)(x,t) + f(x,t).$$

Using the Green's function $H(x,\xi,t)$, one can write $u_t(x,t)$ in the form

$$u_t(x,t) = \Phi(x,t) - \int_0^t \int_0^1 H_\xi(x,\xi,t-\tau) \Gamma(u_x)(\xi,\tau) \, d\xi \, d\tau, \tag{3.6}$$

where

$$\begin{aligned} \Phi(x,t) = & \int_0^t \int_0^1 H(x,\xi,t-\tau) f(\xi,\tau) \, d\xi \, d\tau \\ & + \int_0^1 H(x,\xi,t) u_1(\xi) \, d\xi \\ & + 2 \int_0^t \theta(x-1, t-\tau) h_1(\tau) \, d\tau \\ & - 2 \int_0^t \theta(x, t-\tau) h_0(\tau) \, d\tau. \end{aligned} \tag{3.7}$$

The functions h_0, h_1 are defined by

$$
\begin{aligned}
h_j(t) &= \varphi_j'(y) \\
&\quad + \int_0^t g(t-s, \varphi_j(t) - \varphi_j(s))\, ds \\
&\quad + \int_{-\infty}^0 g(t-s, \varphi_j(t) - \tilde{u}_x(j,s))\, ds, \quad t > 0,\ j = 0, 1.
\end{aligned} \tag{3.8}
$$

Integrate (3.6) with respect to t to obtain the following integral equation satisfied by all smooth solutions of (3.1)–(3.4):

$$
\begin{aligned}
u(x,t) &= u_0(x) + \int_0^t \Phi(x,s)\, ds \\
&\quad - \int_0^t \int_0^s \int_0^1 H_\xi(x,\xi,s-\tau)\Gamma(u_x)(\xi,\tau)\, d\xi\, d\tau,\ ds
\end{aligned} \tag{3.9}
$$

for $0 \le x \le 1$, $0 \le t < \infty$.

Now relax the smoothness assumptions on (3.1)–(3.4) and consider solutions $u(x,t)$ of (3.9) from a larger class of functions. We temporarily ignore the formal connection between h_0, h_1 and φ_0, φ_1 given by (3.8) and make the following assumptions:

(A1) Let $p \in (3, \infty)$ be given. Assume that the data $(f, h_0, u_0, u_1, \tilde{u})$ satisfies the following conditions:

(i) $f \in L^p_{\mathrm{loc}}((0,1) \times (0,\infty))$;

(ii) $h_0, h_1 \in L^1_{\mathrm{loc}}(0,\infty)$, where $1 \le q < \infty$;

(iii) $u_0, u_1 \in W^{1,\infty}(0,1)$;

(iv) $\tilde{u} \in C_b(-\infty, 0; W^{1,\infty}(0,1))$.

(G1) The function $g : [0,\infty] \times \mathbf{R} \to \mathbf{R}$ is continuous and, for each $R > 0$, there is a positive function $a_R \in L^1(0,\infty)$ such that if $|u| \le R$, $|v| \le R$, then

$$|g(t,u) - g(t,v)| \le a_R(t)|u - v|, \quad 0 \le t < \infty.$$

Furthermore, the mapping $t \to g(t,0)$ belongs to $L^1(0,\infty)$.

From Lemma 1, observe that assumption (A1) implies that $\Phi(x,t)$ satisfies problems (2.8)–(2.10) with $\psi = f$, $v_0 = u_1$ and has the estimates (2.12) if $q = 1$.

DEFINITION 1. Let assumptions (A1), (G1) be satisfied. A function $u(x,t)$ will be called a *generalized solution* of (IDE), (BC), (IC), (IH) on a rectangle Q_T if (i) $u \in C([0,T]; W^{1,\infty}(0,1))$; (ii) $u = \tilde{u}$ on $[0,1] \times (-\infty, 0)$; and (iii) u satisfies (3.9) on Q_T.

The next result gives the local existence and uniqueness of generalized solutions.

THEOREM 1. *Let $T_0 > 0$ be given, and let $(f, h_0, h_1, u_0, u_1, \tilde{u})$ satisfy assumption* (A1) *with $q = 1$. Let*

$$M_0 = \text{ess sup} \left\{ |u_0'(x)| + \int_0^t |\Phi_x(x,s)|\, ds : (x,t) \in Q_{T_0} \right\},$$

and define constants R, C by

$$R = 2 \max\{M_0, \|\tilde{u}\|_{C_b(-\infty,0;W^{1,\infty}(0,1))}\},$$
$$C = \max\{\sqrt{\eta}, \pi^{-1/2}\}.$$

Assume that g satisfies (G1)*, and let T be a positive number such that*

$$\begin{cases} 0 < T \le T_0, \\ \frac{T}{\eta}(1+C)\|a_{2R}\|_1 < 1, \\ \frac{T}{\eta}(1+C)(2R\|a_{2R}\|_1 + \int_0^\infty |g(s,0)|\, ds) \le R - M_0. \end{cases} \tag{3.10}$$

Then there exists a generalized solution u on Q_T having the following regularity properties:

$$\begin{cases} u \in C([0,T]; L^\infty(0,1)), & u_x \in C([0,T]; L^\infty(0,1)) \\ u_t \in C(\overline{Q}_T), & u_{xt} \in L^\infty(Q_T), \|u_x\|_{L^\infty(Q_T)} \le R \end{cases} \tag{3.11}$$

PROOF. Assume that (3.10) holds, and let $B_R = \{v \in L^\infty(Q_T) : \|v\|_\infty \le R\}$. Define an operator $\mathcal{F}$ on $L^\infty(Q_T)$ by

$$\begin{aligned} \mathcal{F}(v)(x,t) = u_0'(x) &+ \int_0^t \Phi_x(x,s)\, ds \\ &+ \frac{1}{\eta} \int_0^t \int_0^1 G(x,\xi,t-\tau)\Gamma(v)(\xi,\tau)\, d\xi\, d\tau \\ &- \frac{1}{\eta} \int_0^t \Gamma(v)(x,s)\, ds, \quad (x,t) \in Q_T. \end{aligned} \tag{3.12}$$

We first show that $\mathcal{F}$ has a fixed point in B_R. Let $v \in B_R$, then, by (G1),

$$|g(t-s, v(x,t) - v(x,s))| \leq 2Ra_{2R}(t-s) + |g(t-s,0)|,$$
$$|g(t-s, v(x,t) - \tilde{u}(x,s))| \leq 2Ra_{2R}(t-s) + |g(t-s,0)|.$$

From (3.5) it follows that

$$|\Gamma(v)(x,t)| \leq 2R\|a_{2R}\|_1 + \int_0^\infty |g(s,0)\,ds, \quad (x,t) \in Q_T, \tag{3.13}$$

and, therefore,

$$|\mathcal{F}(v)(x,t)| \leq M_0 + \frac{T}{\eta}(1+C)\bigg(2R\|a_{2R}\|_1 + \int_0^\infty |g(s,0)|\,ds\bigg) \leq R$$

for almost all $(x,t) \in Q_T$. This shows that $\mathcal{F}$ maps B_R into itself.

Next, let $u^1, u^2 \in B_R$; then, by (3.12),

$$\begin{aligned}\mathcal{F}(u^1)(x,t) &- \mathcal{F}(u^2)(x,t)\\ &= \frac{1}{\eta}\int_0^t\int_0^1 G(x,\xi,t-\tau)[\Gamma(u^1)(\xi,\tau) - \Gamma(u^2)(\xi,\tau)]\,d\xi\,d\tau\\ &\quad - \frac{1}{\eta}\int_0^t [\Gamma(u^1)(x,s) - \Gamma(u^2)(x,s)]\,ds.\end{aligned}$$

From (2.4) and (G1),

$$\|\mathcal{F}(u^1) - \mathcal{F}(u^2)\|_\infty \leq \frac{2T}{\eta}(1+C)\|a_{2R}\|_1\|u^1 - u^2\|_\infty.$$

So, by (3.10), it follows that $\mathcal{F}$ is a contraction mapping on B_R, and there is a unique function $v \in B_R$ such that $v = \mathcal{F}(v)$.

Using the fixed point $v \in B_R$, define a function

$$w(x,t) = \int_0^t\int_0^1 G(x,\xi,t-\tau)\Gamma(v)(\xi,\tau)\,d\xi\,d\tau, \quad (x,t) \in Q_T.$$

From (3.13) we have that $\Gamma(v) \in L^\infty(Q_T)$, and so, from our remarks in Section 2 concerning (2.3),

$$\begin{aligned} v(x,t) &= u_0'(x) + \int_0^t \Phi_x(x,s)\,ds + \frac{1}{\eta} w(x,t) - \frac{1}{\eta}\int_0^t \Gamma(v)(x,s)\,ds \\ &= \frac{\partial}{\partial x}\left\{ u_0(x) + \int_0^t \Phi(x,s)\,ds + \int_0^t w_x(x,s)\,ds \right\} \end{aligned}$$

for almost all $(x,t) \in Q_T$. Define

$$u(x,t) = u_0(x) + \int_0^t \Phi(x,s)\,ds + \int_0^t w_x(x,s)\,ds; \tag{3.14}$$

then $v = u_x$ a.e. on Q_T and, by (2.5)

$$\begin{aligned} w_x(x,t) &= \int_0^t \int_0^1 G_x(x,\xi,t-\tau)\Gamma(v)(\xi,\tau)\,d\xi\,d\tau \\ &= -\int_0^t \int_0^1 H_\xi(x,\xi,t-\tau)\Gamma(u_x)(\xi,\tau)\,d\xi\,d\tau, \quad (x,t) \in Q_T. \end{aligned}$$

It follows that $u(x,t)$ satisfies (3.9) on Q_T.

Since $u_x \in B_R$, it follows from (3.9) that $u \in W^{1,\infty}(0,T;L^\infty(0,1))$ with $u_x \in C([0,T];L^\infty(0,1))$. From (3.6) we get $u_t \in C(\overline{Q}_T)$, and, from (3.14), it follows that

$$u_{xt}(x,t) = \Phi_x(x,t) + \frac{1}{\eta} w_t(x,t) - \frac{1}{\eta}\Gamma(u_x)(x,t), \text{ a.e. } (x,t) \in Q_T.$$

Therefore, $u_{xt} \in L^2(Q_T)$ and this proves (3.11). □

Corollary 1. *Let $u(x,t)$ be a generalized solution given by Theorem 1. Then*

$$\lim_{t\to 0+} u(x,t) = u_0(x), \text{ uniformly on } 0 \le x \le 1, \tag{3.15}$$

$$\lim_{t\to 0+} u_t(x,t) = u_1(x), \text{ uniformly on } 0 \le x \le 1. \tag{3.16}$$

In addition, suppose there is a point $x_0 \in [0,1]$ *such that*

$$\lim_{x\to x_0} u_0'(x) = u_0'(x_0), \tag{3.17}$$

$$\lim_{x\to x_0} \tilde{u}_x(x,t) = \widetilde{u_x}(x_0,t), \text{ uniformly for } -\infty < t \le 0. \tag{3.18}$$

Then

$$\lim_{x\to x_0} u_x(x,t) = u_x(x_0,t), \text{ uniformly for } 0 \le t \le T. \tag{3.19}$$

In particular, if (3.17), (3.18) *occur at the endpoints* $x_0 = 0, 1$, *then* $u(x,t)$ *satisfies the traction boundary condition* (BC) *a.e.* $t \in [0,T]$.

Proof. By (3.14),

$$\begin{aligned} u(x,t) &= u_0(x) + \int_0^t \Phi(x,s)\,ds \\ &\quad + \eta \int_0^t \int_0^s \int_0^1 G_x(x,\xi,s-\tau)\cdot\Gamma(u_x)(\xi,\tau)\,d\xi\,d\tau\,ds \end{aligned} \tag{3.20}$$

so that, by (2.5),

$$|u(x,t) - u_0(x)| \le \int_0^t |\Phi(x,s)|\,ds + t\eta C(T)\|\Gamma(u_x)\|_p, \quad (x,t) \in Q_T.$$

This proves (3.15). From (3.20) we have

$$u_t(x,t) = \Phi(x,t) + \eta \int_0^t \int_0^1 G_x(x,\xi,t-\tau)\Gamma(u_x)(\xi,\tau)\,d\xi\,d\tau,$$

so that

$$\begin{aligned} &|u_t(x,t) - u_1(x)| \\ &\qquad \le |\Phi(x,t) - u_1(x)| + \eta C(T)\left\{\int_0^t \int_0^1 |\Gamma(u_x)(\xi,\tau)|^p\,d\xi\,d\tau\right\}^{1/p} \end{aligned}$$

for all $(x,t) \in Q_T$. This proves (3.16).

Now suppose (3.17), (3.18) hold and consider (3.14). We have

$$
\begin{aligned}
u_x(x,t) = u_0'(x) + \int_0^t \Phi_x(x,s)\,ds + w(x,t)& \\
- \int_0^t \Gamma(u_x)(x,s)\,ds, \quad \text{a.e. } (x,t) \in Q_T.&
\end{aligned} \tag{3.21}
$$

Let $S \subset Q_T$ be the set where (3.21) makes sense for all $(x,t) \in S$. For any pair $(\lambda, t) \in S$, define

$$
\begin{aligned}
z_\lambda(t) &= u_x(\lambda,t), \\
h_\lambda(t) &= u_0'(\lambda) + \int_0^t \Phi_x(\lambda;s)\,ds + w(\lambda,t),
\end{aligned}
$$

for $t \in [0,T]$. Then $z_\lambda, h_\lambda \in C([0,T])$ by (3.11) and

$$
\begin{aligned}
z_\lambda(t) + \int_0^t \bigg\{ \int_0^s g(s-\tau, z_\lambda(t) - z_\lambda(\tau))\,d\tau& \\
+ \int_{-\infty}^0 g(s-\tau, z_\lambda(s) - \tilde{u}_x(\lambda,\tau))\,d\tau \bigg\}\,ds& \\
= h_\lambda(t), \quad 0 \le t \le T.&
\end{aligned}
$$

Since $|z_\lambda(t)| \le R$ for $t \in [0,T]$, it follows from assumption (G1) that

$$
\begin{aligned}
|z_\lambda(t) - z_\mu(t)| \le \int_0^t (\|a_{2R}\|_1 + A(t-\tau))|z_\lambda(\tau) - z_\mu(\tau)|\,d\tau& \\
+ T\|a_{2R}\|_1 \sup_{-\infty<\tau\le 0} |\tilde{u}_x(\mu,\tau) - \tilde{u}_x(\mu,\tau)|& \\
+ |h_\lambda(t) - h_\mu(t)|, \quad 0 \le t \le T,&
\end{aligned}
$$

where

$$
A(t) = \int_0^t a_{2R}(s)\,ds.
$$

Now let $\mu = x_0$ and put

$$
\begin{aligned}
\omega(t,\lambda) &= |z_\lambda(t) - z_{x_0}(t)|, \\
H(t,\lambda) &= T\|a_{2R}\|_1 \sup_{-\infty<\tau\le 0} |\tilde{u}_x(\lambda,\tau) - \tilde{u}_x(x_0,\tau)| + |h_\lambda(t) - h_{x_0}(t)|, \\
K(t) &= \|a_{2R}\|_1 + A(t).
\end{aligned}
$$

Then

$$\omega(t,\lambda) \le H(t,\lambda) + \int_0^t K(t-\tau)\omega(\tau,\lambda)\,d\tau, \quad 0 \le t \le T. \tag{3.22}$$

If $R(t)$ denotes the resolvent corresponding to $K(t)$, then (3.22) implies that

$$\omega(t,\lambda) \le H(t,\lambda) + \int_0^t |R(t-\tau)| H(\tau,\lambda)\,d\tau, \quad 0 \le t \le T.$$

It follows that $\omega(t,\lambda) \to 0$ as $\lambda \to x_0$, uniformly on $[0,T]$, and this proves (3.19).

Next, we suppose that (3.17), (3.18) hold for $x_0 = 0$. Then, by (3.5),

$$\lim_{x\to 0} \Gamma(u_x)(x,t) = \Gamma(u_x)(0,t), \quad 0 \le t \le T.$$

So, (3.13) obtains

$$\lim_{x\to 0} \int_0^t \Gamma(u_x)(x,s)\,ds = \int_0^t \Gamma(u_x)(0,s)\,ds, \quad 0 \le t \le T.$$

Letting $x \to 0$ in (3.21) gives

$$u_x(0,t) = u_0'(0) + \int_0^t h_0(s)\,ds - \int_0^t \Gamma(u_x)(0,s)\,ds,$$

and so, for almost all $t \in [0,T]$,

$$u_{xt}(0,t) = h_0(t) - \Gamma(u_x)(0,t).$$

Thus, $u_x(0,t)$ satisfies (BC) for almost all $t \in [0,T]$. A similar argument works for the case $x_0 = 1$. □

Corollary 2. *Let the assumptions of Theorem* 1 *be satisfied. Then there exists only one generalized solution satisfying* (3.11).

Proof. Suppose $u(x,t)$ and $u^1(x,t)$ are two solutions of (3.9) which satisfy (3.11). Then $u_x = u_x^1$ a.e. on Q_T so that there is a function $\chi \in C^1([0,T])$ such that

$$u(x,t) - u^1(x,t) = \chi(t), \quad (x,t) \in Q_T.$$

By (3.6) this implies that $\chi'(t) = 0$ for all $0 \leq t \leq T$. So $\chi(t) \equiv \chi_0$ is constant on $[0, T]$ and, by (3.15),

$$\chi_0 = \lim_{t \to 0+} (u(x,t) - u^1(x,t)) = 0.$$

Hence, $u = u^1$ on Q_T and this proves uniqueness. ◻

4. A priori estimates. In order to obtain a global generalized solution $u(x,t)$ of (IDE), (BC), (IC), (IH), it is clear from Theorem 1 that we must control the size of the constant R in (3.10). This implies that we must obtain an a priori estimate in $L^\infty(Q_T)$ for the gradient $u_x(x,t)$. The main effort in this section is to prove such an estimate. We first begin by establishing a variational property shared by all generalized solutions.

LEMMA 4. *Let the data* $(f, h_0, h_1, u_0, u_1, \tilde{u})$ *satisfy assumption* (A1) *with* $q = 1$, *and let* g *satisfy* (G1). *Let* u *be the corresponding generalized solution on* Q_T. *Then, for all* $\psi \in C^\infty(\overline{Q}_T)$, *we have*

$$\begin{aligned}
&- \int_0^T \int_0^1 u_t(x,t)\psi_t(x,t)\,dx\,dt \\
&+ \int_0^T \int_0^1 u_{xt}(x,t)\psi_x(x,t)\,dx\,dt \\
&+ \int_0^T \int_0^1 \Gamma(u_x(x,t)\psi_x(x,t)\,dx\,dt \\
&\quad = \int_0^1 u_1(x)\psi(x,0)\,dx - \int_0^1 u_t(x,T)\psi(x,T)\,dx \\
&\qquad + \int_0^T h_1(t)\psi(1,t)\,dt - \int_0^T h_0(t)\psi(0,t)\,dt \\
&\qquad + \int_0^T \int_0^1 f(x,t)\psi(x,t)\,dx\,dt.
\end{aligned} \tag{4.1}$$

PROOF. The proof of (4.1) is long and tedious, but only involves integration by parts, which can be justified using (3.11). The boundary terms in the integrals are easily evaluated if on the left side of (4.1) we use the representations (see (3.14))

$$u_t(x,t) = \Phi(x,t) + w_x(x,t), \tag{4.2}$$

$$u_{xt}(x,t) = \Phi_x(x,t) + w_{xx}(x,t) \tag{4.3}$$

for almost all $(x,t) \in Q_T$. We omit the details. $\square$

Using the variational equation (4.1), we next show that all generalized solutions satisfy an energy equation.

THEOREM 2. *Let the data* $(f, h_0, h_1, u_0, u_1, \tilde{u})$ *satisfy assumption* (A1) *with* $q = 2$, *and let* g *satisfy* (G1). *Let* u *be the corresponding generalized solution on* Q_T. *Then, for almost all* $t \in [0,T]$,

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int_0^1 u_t^2(x,t)\,dx + \int_0^1 u_{xt}^2(x,t)\,dx \int_0^1 \Gamma(u_x)(x,t)u_{xt}(x,t)\,dx \\ &\quad = h_1(t)u_t(1,t) - h_0(t)u_t(0,t) + \int_0^t f(x,t)u_t(x,t)\,dx \end{aligned} \tag{4.4}$$

PROOF. Because of a lack of regularity, the proof of (4.4) is accomplished through approximations. The method of [**8**, pp. 141–143] can be used, which is based on the *Steklov average*

$$v_h(x,t) = \frac{1}{h}\int_t^{t+h} v(x,s)\,ds, \quad h > 0,$$

of a function $v(x,t)$. The basic property of Steklov averages is that if $v \in L^2(Q_T)$, then, for each $0 < \delta < T$,

$$v_h,\ \partial_t v_h \in L^2(Q_{T-\delta}), \quad \text{for all } 0 < h < \delta,$$

and

$$v_h \to v \quad \text{in } L^2(Q_{T-\delta}) \quad \text{as } h \to 0.$$

We will not give the details of the proof of (4.4) since the arguments require very little change from those used in [**8**]. We only remark that it is convenient to use Lemma 2 to approximate the Steklov average u_{th} of u_t. $\square$

Now consider the primitive

$$F(t,u) = \int_0^u g(t,\xi)\,d\xi, \quad (t,u) \in [0,\infty) \times \mathbf{R}. \tag{4.5}$$

If g satisfies (G1), then, for each $R > 0$, there is a function $a_R \in L^1(0,\infty)$ such that if $|u| \le R$, $|v| \le R$, then

$$|F(t,u) - F(t,v)| \le b_R(t)|u-v|, \quad 0 < t < \infty, \tag{4.6}$$

where $b_R(t) = Ra_R(t) + |g(t,0)|$. We make the following additional hypotheses:

(G2) The mapping $t \to g(t,u)$ is continuously differentiable on $(0,\infty)$ for each $u \in \mathbf{R}$.

(F1) For each $R > 0$, there is a function $d_R \in L^1(0,\infty)$ such that, for all $0 < t < \infty$, $|u| \le R$, we have

$$|F_{,1}(t,u)| \min(1,t) \le d_R(t),$$

where $F_{,1} = \frac{\partial F}{\partial t}$.

(F2) There is a constant $C_0 \ge 0$ and a function $C_1 \in L^1(0,\infty)$ such that, for all $0 < t < \infty$, $u \in \mathbf{R}$, we have

$$\begin{aligned} F(t,u) &\ge -C_1(t)(1+|u|^2), \\ F_{,1}(t,u) &\le C_0\{F(t,u) + C_1(t)(1+|u|^2)\}. \end{aligned}$$

REMARK . Let $a(t)$ be a nonnegative, nonincreasing function on $[0,\infty)$ with both a, a' integrable on $(0,\infty)$. In the separated kernel case, assumptions (G2), (F1), (F2) are broad enough to allow memory functions of the form

$$g(t,\xi) = a(t)(g_0(\xi) - a_1\xi - a_1), \quad (t,\xi) \in [0,\infty) \times \mathbf{R},$$

where a_1, a_2 are constants and g_0 is nondecreasing and locally Lipschitz continuous on $\mathbf{R}$ with $g_0(0) = 0$

LEMMA 5. *Let u be a generalized solution of* (IDE), (BC), (IC), (IH) *on Q_T given by Theorem* 1. *Suppose that assumptions* (G2) *and* (F1) *are satisfied. Then, for each $0 < t < T$, we have*

$$
\begin{aligned}
&\int_0^t \int_0^1 \Gamma(u_x)(x,\tau)u_{xt}(x,\tau)\,dx\,d\tau \\
&\quad = \int_0^\infty \int_0^1 F(s, u_x(x,t) - u_x(x,t-s))\,dx\,ds \\
&\qquad - \int_0^\infty \int_0^1 F(s, u_x(x,0+) - u_x(x,-s))\,dx\,ds \\
&\qquad - \int_0^t \int_0^\infty \int_0^1 F_{,1}(s, u_x(x,\tau) - u_x(x,\tau - s))\,dx\,ds\,d\tau.
\end{aligned} \tag{4.7}
$$

PROOF. We first note that, for any function $\varphi \in C^\infty(\overline{Q}_T)$ such that $\varphi(x,t) = \tilde{u}_x(x,t)$ a.e. $(x,t) \in (0,1) \times (-\infty, 0)$, we have

$$
\begin{aligned}
\Gamma(\varphi)(x,t)\varphi_t(x,t) &= \frac{\partial}{\partial t}\int_{-\infty}^t F(t-s, \varphi(x,t) - \varphi(x,s))\,ds \\
&\quad - \int_{-\infty}^t F_{,1}(t-s, \varphi(x,t) - \varphi(x,s))\,ds
\end{aligned}
$$

for $0 < t \le T$, $0 \le x \le 1$. So, if $0 < \epsilon \le t \le T$, then

$$
\begin{aligned}
&\int_\epsilon^t \int_0^1 \Gamma(\varphi)(x,\tau)\varphi_t(x,\tau)\,dx\,ds \\
&\quad = \int_0^\infty \int_0^1 F(s, \varphi(x,t) - \varphi(x,t-s)\,dx\,ds \\
&\qquad - \int_0^\infty \int_0^1 F(s, \varphi(x,\epsilon) - \varphi(x,\epsilon - s))\,dx\,ds \\
&\qquad - \int_\epsilon^t \int_0^\infty \int_0^1 F_{,1}(s, \varphi(x,\tau) - \varphi(x,\tau - s))\,dx\,ds\,d\tau.
\end{aligned} \tag{4.8}
$$

Since $u_x \in W_2^{0,1}(Q_T) \cap L^\infty(Q_T)$, we may apply Lemma 2 and obtain an approximating sequence $\{\varphi_n\}$ for u_x such that (4.8) holds for each function $\varphi = \varphi_n$. Passing to the limit as $n \to \infty$ gives

$$\begin{aligned}(4.9)\qquad &\int_\epsilon^t \int_0^1 \Gamma(u_x)(x,\tau)u_{xt}(x,\tau)\,dx\,d\tau \\ &\quad= \int_0^\infty \int_0^1 (s, u_x(x,t) - u_x(x,t-s))\,dx\,ds \\ &\qquad - \int_0^\infty \int_0^1 F(s, u_x(x,\epsilon) - u_x(x,\epsilon - s))\,dx\,ds \\ &\qquad - \int_\epsilon^t \int_0^\infty \int_0^1 F_{,1}(s, u_x(x,\tau) - u_x(x,\tau - s))\,dx\,ds\,d\tau.\end{aligned}$$

We wish to pass to the limit in (4.9) as $\epsilon \to 0^+$. The only troublesome term is the second integral on the right-hand side. We write

$$\begin{aligned}&\int_0^\infty \int_0^1 F(s, u_x(x,\epsilon) - u_x(x,\epsilon - s))\,dx\,ds \\ &\quad= \int_0^\epsilon \int_0^1 F(\epsilon - s, u_1(x,\epsilon) - u_x(x,s))\,dx\,ds \\ &\qquad + \int_{-\infty}^0 \int_0^1 F(\epsilon - s, u_x(x,\epsilon) - u_x(x,s))\,dx\,ds.\end{aligned}$$

Let $R = \sup\{|u_x(x,s)| : (x,s) \in Q_T\}$; then, by (4.6),

$$\int_0^\epsilon \int_0^1 |F(\epsilon - s, u_x(x,\epsilon) - u_x(x,s))|\,dx\,ds \le 2R \int_0^\epsilon b_R(t)\,dr \to 0 \text{ as } \epsilon \to 0.$$

By (3.11) and Lebesque's Dominated Convergence Theorem,

$$\begin{aligned}&\lim_{\epsilon\to 0^+} \int_{-\infty}^0 \int_0^1 F(\epsilon - s, u_x(x,\epsilon) - u_x(x,s))\,dx\,ds \\ &\qquad = \int_\infty^0 \int_0^1 F(-s, u_x(x,0+) - u_x(x,s))\,dx\,ds.\end{aligned}$$

So one may let $\epsilon \to 0+$ in (4.9) to obtain (4.7). □

THEOREM 3. *Let the data* $(f, h_0, h_1, u_0, u_1, \tilde{u})$ *satisfy* (A1) *with* $q = 2$, *and let* g *satisfy* (G1), (G2). *Assume hypotheses* (F1), (F2) *are satisfied, and let u be the corresponding generalized solution on Q_T. Define*

$$\begin{aligned} E(t) &= \frac{1}{2}\int_0^1 u_t^2(x,t)\,dx \\ &\quad + \int_0^\infty \int_0^1 \{F(s, u_x(x,t) - u_x(x,t-s)) \\ &\quad + C(s)(1 + |u_x(x,t) - u_x(x,t-s)|^2)\}\,dx\,ds \end{aligned} \tag{4.10}$$

Then there is a constant $\Lambda > 0$ depending only on the data and T such that

$$E(t) + \int_0^t \int_0^1 |u_{xt}(x,s)|^2\,dx\,ds \le \Lambda, \quad 0 \le t \le T. \tag{4.11}$$

PROOF. From Theorem 2 and Lemma 2, we see that u satisfies the energy equation

$$\begin{aligned} &\frac{1}{2}\int_0^1 u_t^2(x,t)\,dx + \int_0^t \int_0^1 u_{xt}^2(x,\tau)\,dx\,d\tau \\ &\qquad + \int_0^\infty \int_0^1 F(s, u_x(x,t) - u_x(x,t-s))\,dx\,ds \\ &\quad = \frac{1}{2}\int_0^1 u_1^2(x)\,dx + \int_0^\infty \int_0^1 F(s, u_0'(x) - \tilde{u}_x(x,-s))\,dx\,ds \\ &\qquad + \int_0^t h_1(\tau)u_t(1,\tau)\,d\tau - \int_0^t h_0(\tau)u_t(0,\tau)\,d\tau \\ &\qquad + \int_0^t \int_0^\infty \int_0^1 F_{,1}(s, u_x(x,\tau) - xu_x(x,\tau-s))\,dx\,ds\,d\tau \\ &\qquad + \int_0^t \int_0^1 f(x,\tau)u_t(x,\tau)\,dx\,d\tau. \end{aligned} \tag{4.12}$$

For any $\epsilon > 0$, we have the estimates

$$\begin{aligned} &\int_0^t |h_j(\tau)u_t(j,\tau)|\,d\tau \\ &\qquad \le \frac{1}{\epsilon}\int_0^t |h_j(\tau)|^2\,d\tau + 2\epsilon \int_0^t \int_0^1 (u_t^2(x,\tau) + u_{xt}^2(x,\tau))\,dx\,d\tau \end{aligned}$$

for $j = 0, 1$. Substituting this into (4.12), taking $0 < \epsilon < 1/4$, and using (F2) gives

$$\begin{aligned} E(t) &+ \frac{1}{2}\int_0^t \int_0^1 u_{xt}^2(x,\tau)\,dx\,d\tau \\ &\le C(T) + C_2 \int_0^t E(\tau)\,d\tau \\ &\quad + \int_0^\infty \int_0^1 C_1(s)(1 + |u_x(x,t) - u_x(x,t-s)|^2)\,dx\,ds, \end{aligned}$$

where

$$\begin{aligned} C(T) =& \frac{1}{2}\int_0^1 u_1^2(x)\,dx + \int_0^\infty \int_0^1 F(s, u_0'(x) - \tilde{u}_x(x,-s))\,dx\,ds \\ &+ \frac{1}{\epsilon}\int_0^T (|h_0(\tau)|^2 + |h_1(\tau)|^2)\,d\tau + \int_0^T \int_0^1 |f(x,\tau)|^2\,dx\,d\tau \end{aligned}$$

and $C_2 = \max(3/2, C_0)$. Following the arguments of [**5**, Lemma 2.1], leads to the inequality

$$\begin{aligned} E(t) + \frac{1}{4}\int_0^t \|u_{x\tau}(\cdot,\tau)\|_2^2\,d\tau \le C\biggl\{1 + \int_0^t (1 + C_1(t-s))(E(s) \\ + \int_0^s \|u_{x\tau}(\cdot,\tau)\|_2^2\,d\tau)\,ds\biggr\}, \quad 0 \le t \le T. \end{aligned}$$

By Gronwall's inequality, we obtain (4.11). $\square$

Having established the energy estimate (4.11), we can proceed to obtain an L^∞ estimate on the gradient $u_x(x,t)$. We shall use the integral equation method.

THEOREM 4. *Let $T > 0$ be given. Assume* (A1) *with $q = 2$,* (G1), (G2), (F1), (F2) *are satisfied. In addition, suppose that*

$$g(t,u) = g_0(t,u) + L(t)u, \tag{4.13}$$

where $L \in L^1(0,\infty) \cap C([0,\infty])$ and $g_0(t,u)$ is nondecreasing in u. Let $u(x,t)$ be the corresponding generalized solution on $Q_{t_0} = (0,1) \times (0,t_0)$

for some $t_0 \in [0, T)$. Then there is a constant $M > 0$, depending on the data and T but not on t_0, such that

$$|u_x(x,t)| \le M \quad a.e.\ (x,t) \in Q_{t_0}. \tag{4.14}$$

PROOF. Let $u(x,t)$ be the generalized solution on Q_{t_0} corresponding to the data $(f, h_0, h_1, u_0, u_1, \tilde{u})$. Define

$$q(x,t) = \int_0^x u_t(y,t)\,dy, \quad (x,t) \in Q_{t_0}.$$

Then, from (3.14), (4.11), we have

$$q(x,t) = \int_0^x \Phi(y,t)\,dy + w(x,t),$$
$$|q(x,t)| \le \|u_t(\cdot,t)\|_2 \le \sqrt{2\Lambda}$$

for all $(x,t) \in Q_{t_0}$. It follows that

$$\begin{aligned} \eta u_x(x,t) - q(x,t) = \eta u_0'(x) + \eta \int_0^t \Phi_x(x,s)\,ds - \int_0^x \Phi(y,t)\,dy \\ - \int_0^t \Gamma(u_x)(x,s)\,ds. \end{aligned} \tag{4.15}$$

From the initial-boundary value problem satisfied by $\Phi(x,t)$, we have

$$\begin{aligned} \int_0^x \Phi(y,t)\,dy - \eta \int_0^t \Phi_x(x,s)\,ds = \int_0^x u_1(y)\,dy - \eta \int_0^t h_0(s)\,ds \\ + \int_0^t \int_0^x f(y,s)\,dy\,ds. \end{aligned}$$

Also, by (4.13),

$$\begin{aligned} \Gamma(u_x)(x,t) = &\int_0^t g_0(t-s, u_x(x,t) - u_x(x,s))\,ds \\ &+ \int_{-\infty}^0 g_0(t-s, u_x(x,t) - \tilde{u}_x(x,s))\,ds \\ &+ L_0 u_x(x,t) - \int_0^t L(t-s) u_x(x,s)\,ds, \end{aligned}$$

where $L_0 = \int_0^\infty L(t)\,dt$. So (4.15) can be written as

$$\begin{aligned}(4.16)\qquad &\eta\, u_x(x,t) - q(x,t) + \int_0^t K(t-s)u_x(x,s)\,ds \\ &+ \int_0^t \int_0^s g_0(s-\tau, u_x(x,s) - u_x(x,\tau))\,d\tau\,ds \\ &+ \int_0^t \int_{-\infty}^0 g_0(s-\tau, u_x(x,s) - \tilde{u}_x(x,\tau))\,d\tau\,ds = k(x,t),\end{aligned}$$

where

$$K(t) = L_0 - \int_0^t L(s)\,ds, \quad t \geq 0,$$

$$\begin{aligned}k(x,t) = \eta u_0'(x) - \int_0^x u_1(y)\,dy + \eta \int_0^t h_0(s)\,ds - \int_0^t \int_0^x f(y,s)\,dy\,ds \\ + \int_0^t \int_{-\infty}^0 L(s-\tau)\tilde{u}_x(x,\tau)\,d\tau\,ds, \quad (x,t) \in Q_T.\end{aligned}$$

It is clear that the functions $k(x,t)$ and $q(x,t)$ are bounded on Q_T. So, in order to prove (4.14), it suffices to show that the function $\psi(x,t)$, defined by

$$(4.17)\qquad \psi(x,t) = \eta u_x(x,t) - q(x,t) - k(x,t), \quad \text{a.e. } (x,t) \in Q_{t_0},$$

is bounded on Q_{t_0}.

We fix $x \in [0,1]$ where $u_0'(x)$ is defined and $u_x(x,\cdot), \tilde{u}_x(x,\cdot)$ are continuous functions of t. We use the notation

$$\psi(t) = \psi(x,t), \quad q(t) = q(x,t)$$
$$k(t) = k(x,t), \quad \tilde{u}_x(t) = \tilde{u}_x(x,t).$$

From equation (4.16), we see that $\psi \in C^1([0,t_0])$ and satisfies

(4.18)

$$\begin{aligned}\dot{\psi}(t) &+ \int_0^t g_0\left(t-s, \frac{1}{\eta}(\psi(t) - \psi(s) + k(t) - k(s) + q(t) - q(s))\right) ds \\ &+ \int_{-\infty}^0 g_0\left(t-s, \frac{1}{\eta}(\psi(t) + k(t) + q(t)) - \tilde{u}_x(s)\right) ds \\ &+ \frac{L_0}{\eta}(\psi(t) + k(t) + q(t)) \\ &= \frac{1}{\eta}\int_0^t L(t-s)(\psi(s) + k(s) + q(s))\,ds, \quad 0 \leq t \leq t_0.\end{aligned}$$

We shall show that there is constant $M_0 > 0$ such that

$$|\psi(t)| \le e^{M_0 t}, \text{ for all } 0 \le t \le t_0, \tag{4.19}$$

and M_0 does not depend on x or t_0. Indeed, let

$$\begin{aligned} C'_T = \sup_{0\le t\le T} \bigg\{ \int_0^t \left| g_0\left(t-s, \frac{1}{\eta}(k(t)+q(t)-k(s)-q(s))\right)\right| ds \\ + \int_{-\infty}^0 \left| g_0\left(t-s, \frac{1}{\eta}(k(t)+q(t))-\tilde{u}_x(s)\right)\right| ds \bigg\}, \\ C''_T = \sup_{0\le t\le T} \int_0^t |L(t-s)(k(s)+q(s))|\, ds + \sup_{0\le t\le T} |L_0||k(t)+q(t)|. \end{aligned}$$

Choose M_0 such that

$$M_0 > \frac{|L_0|}{\eta} + \frac{1}{\eta}C''_T + C'_T + \frac{1}{\eta}\int_0^\infty |L(s)|\, ds. \tag{4.20}$$

We prove that (4.19) holds for this choice of M_0. Let

$$t_1 = \sup\{0 \le t < t_0 : |\psi(s)| \le e^{M_0 s},\ 0 \le s \le t\}.$$

Since $\psi(0) = 0$, the number t_1 is well defined, and it suffices to show that $t_1 = t_0$. If this is false, then $t_1 < t_0$ and either we have

$$\dot\psi(t_1) \ge M_0 e^{M_0 t_1} \quad \text{if } \psi(t_1) = e^{M_0 t_1}, \tag{4.21}$$

or else we have

$$\dot\psi(t_1) \le -M_0 e^{M_0 t_1} \quad \text{if } \psi(t_1) = -e^{M_0 t_1}. \tag{4.22}$$

We only consider the case (4.21) since the other case is handled in an identical manner.

Assuming that (4.21) holds, it follows that $\psi(t_1) \ge \psi(t)$ for all $0 \le t \le t_1$. Using the fact that $g_0(t,u)$ is nondecreasing in u, it follows from (4.18) that

$$M_0 e^{M_0 t_1} - C'_T \le \frac{|L_0|}{\eta} e^{M_0 t_1} + \frac{e^{M_0 t_1}}{\eta}\int_0^{t_1} |L(s)|\, ds + \frac{1}{\eta}C''_T.$$

Therefore,

$$M_0 \le \frac{|L_0|}{\eta} + \left(\frac{1}{\eta} C''_T + C'_T\right) e^{-M_0 t_1} + \frac{1}{\eta}\int_0^\infty |L(s)|\, ds,$$

and this contradicts the definition of M_0. So (4.19) holds and the Theorem is proved. ◻

COROLLARY 3. *Let the data* $(f, h_0, h_1, u_0, u_1, \tilde{u})$ *satisfy assumption* (A1) *with* $q = 2$. *Assume also that* (G1), (G2), (F1), (F2) *are satisfied and that* $g(t, u)$ *can be written in the form of equation* (4.13). *Then there exists a unique global generalized solution* $u(x, t)$ *of* (IDE), (BC), (IC), (IH) *define on* $Q_\infty = (0, 1) \times (0, \infty)$ *and having the regularity given by* (3.11) *for each* $T > 0$.

PROOF. This is an immediate consequence of Theorem 1, especially (3.10) and Theorem 4. ◻

5. Asymptotic behavior. In this section we assume that the hypotheses of Corollary 3 are satisfied and $u(x, t)$ is the corresponding globally defined generalized solution of (IDE), (BC), (IC), (IH). We let $U(t)$ denote the mean displacement function

$$U(t) = \int_0^1 u(x, t)\, dx, \quad t \ge 0, \tag{5.1}$$

and we introduce a new function

$$u^*(x, t) = u(x, t) - U(t). \tag{5.2}$$

Then

$$\int_0^1 u^*(x, t)\, dx = 0, \quad t \ge 0,$$

and we have the inequalities

$$|u_t^*(x, t)| \le \|u_{xt}^*(\cdot, t)\|_2, \quad 0 \le x \le 1,\ t \ge 0, \tag{5.3}$$

$$\|u_t^*(\cdot, t)\|_2 \le \|u_{xt}^*(\cdot, t)\|_2, \quad t \ge 0. \tag{5.4}$$

From (3.14),

$$\text{(5.5)} \qquad \begin{cases} u_t^*(x,t) = u_t(x,t) - \int_0^1 \Phi(x,t)\,dx \\ U''(t) = \int_0^1 \Phi_t(x,t)\,dx = h_0(t) - h_1(t) + \int_0^1 f(x,t)\,dx \end{cases}$$

for almost all (x,t). We substitute (5.2) into (4.2) (noting that $u_x^* = u_x$) and use (5.5) to obtain

(5.6)

$$\frac{1}{2}\frac{d}{dt}|u_t^*(x,t)|^2\,dx + \int_0^1 |u_{xt}^*(x,t)|^2\,dx + \int_0^1 \Gamma(u_x^*)(x,t)u_{xt}^*(x,t)\,dx$$
$$= h_1(t)u_t^*(1,t) - h_0(t)u_t^*(0,t) + \int_0^1 \tilde{f}(x,t)u_t^*(x,t)\,dx, \quad \text{a.e. } t \geq 0,$$

where

$$\tilde{f}(x,t) = f(x,t) - h_1(t) + h_0(t) - \int_0^1 f(x,t)\,dx.$$

We use the energy equation (5.6) to obtain weighted estimates on $u^*(x,t)$ needed to examine asymptotic behavior. We need an additional assumption on the primitive $F(t,u)$.

(F3) $F(t,u) \geq 0$ and there is a constant $\delta \in (0,1/2)$ such that $F_{,1}(t,u) + \delta F(t,u) \leq 0$ for all $y \in \mathbf{R}$, $t > 0$.

THEOREM 5. *Assume hypotheses* (A1) *with* $q = 2$, (G1), (G2), (F1), (F2), (F3) *are satisfied. Suppose* g *satisfies* (4.13), *and let* u *be the corresponding globally defined generalized solution. Let* $b \in C^1([0,\infty))$ *be a function such that*

$$\text{(5.8)} \qquad b(0) = 1 \quad \text{and} \quad 0 \leq b'(t) \leq \delta b(t) \quad \text{for all } t \geq 0.$$

Define $\epsilon_0 = 1/4 - \delta/2$ *and*

(5.9)

$$K(t) = \left\{ \|u_1\|_2^2 + \|u_0\|_1^2 + 2\int_0^\infty \int_0^1 F(s, u_0'(x) - \tilde{u}_x(x,-s))\,dx\,ds \right.$$
$$\left. + \frac{1}{\epsilon_0}\int_0^t b(\tau)(|h_0(\tau)|^2 + |h_1(\tau)|^2)\,d\tau \right\}^{1/2}, \quad t \geq 0.$$

Then we have the estimates

$$b^{1/2}(t)\|u_t^*(\cdot,t)\|_2 \le K(t) + \int_0^t b^{1/2}(\tau)\|\tilde{f}(\cdot,\tau)\|_2\,d\tau, \tag{5.10}$$

$$\begin{aligned}\frac{1}{2}\int_0^t b^{1/2}(\tau)\|u_{xt}^*(\cdot,\tau)\|_2^2\,d\tau \\ \le K^2(t) + \left\{\int_0^t b^{1/2}(\tau)\|\tilde{f}(\cdot,\tau)\|_2\,d\tau\right\}^2\end{aligned} \tag{5.11}$$

for all $t \ge 0$.

Proof. We multiply (5.6) by $b(t)$ and integrate the resulting equation from 0 to t:

$$\begin{aligned}&\frac{1}{2}b(t)\|u_t^*(\cdot,t)\|_2^2 - \int_0^t b'(\tau)\|u_t^*(\cdot,\tau)\|_2^2\,d\tau\\ &\quad + \int_0^t b(\tau)\|u_{xt}^*(\cdot,\tau)\|_2^2\,d\tau\\ &\quad + \int_0^t b(\tau)\int_0^1 \Gamma(u_x^*)(x,\tau)u_{xt}^*(x,\tau)\,dx\,d\tau\\ &= \frac{1}{2}\|u_t^*(\cdot,0)\|_2^2 + \int_0^t b(\tau)(h_1(\tau)u_t^*(1,t) - h_0(\tau)u_t^*(0,\tau))\,d\tau\\ &\quad + \int_0^t b(\tau)\int_0^1 \tilde{f}(x,\tau)u_t^*(x,\tau)\,dx\,d\tau.\end{aligned} \tag{5.12}$$

If we multiply both sides of (4.4) by $b(t)$ and differentiate with respect to t, we have, a.e. $t \ge 0$,

$$\begin{aligned}b(t)&\int_0^1 \Gamma(u_x)(x,t)u_{xt}(x,t)\,dx\\ &= \frac{d}{dt}\int_0^\infty b(t)\int_0^1 F(s,u_x(x,t)-u_x(x,t-s))\,dx\,ds\\ &\quad - b'(t)\int_0^\infty\int_0^1 F(s,u_x(x,t)-\eta(x)(s,t-s))\,dx\,ds\\ &\quad - b(t)\int_0^\infty\int_0^1 F_{,1}(s,u_x,(x,t)-u_x(x,t-s))\,dx\,ds.\end{aligned}$$

So, by (5.8) and (F3), it follows that for almost all $t \geq 0$,

$$\begin{aligned} &b(t) \int_0^1 \Gamma(u_x)(x,t) u_{xt}(x,t)\, dx \\ &\quad \geq \frac{d}{dt} \int_0^\infty b(t) \int_0^1 F(s, u_x(x,t) - u_x(x,t-s))\, dx\, ds. \end{aligned} \tag{5.13}$$

From (5.4), (5.8), we also have

$$\begin{aligned} &- b'(t)\|u_t^*(\cdot,t)\|_2^2 + b(t)\|u_{xt}^*(\cdot,t)\|_2^2 \\ &\quad \geq (1-\delta) b(t) \|u_{xt}^*(\cdot,t)\|_2^2 \geq 0, \end{aligned} \tag{5.14}$$

$$\begin{aligned} &|b(t) h_j(t) u_t^*(j,t)| \\ &\quad \leq \frac{1}{\epsilon} b(t) |h_j(t)|^2 + \epsilon b(t) \|u_{tx}^*(\cdot,t)\|_2^2, \end{aligned} \tag{5.15}$$

for $\epsilon > 0$ constant. Choosing $\epsilon = \epsilon_0$ and substituting (5.13), (5.14), (5.15) into (5.12) gives

$$\begin{aligned} &\frac{1}{2} b(t) \|u_t^*(\cdot,t)\|_2^2 + \frac{1}{2}\int_0^t b(\tau)\|u_{xt}^*(\cdot,\tau)\|_2^2\, d\tau \\ &\quad + \int_0^\infty b(t) \int_0^1 F(s, u_x(x,t) - u_x(x,t-s))\, dx\, ds \\ &\qquad \leq \frac{1}{2} K^2(t) + \int_0^t b(\tau) \|\tilde{f}(\cdot,\tau)\|_2 \|u_t^*(\cdot,\tau)\|_2\, d\tau, \quad t \geq 0. \end{aligned} \tag{5.16}$$

Since $F \geq 0$ by assumption (F3), one may apply Gronwall's inequality [**3**, Lemma A.5] to obtain (5.10). Then write

$$\begin{aligned} &\frac{1}{2}\int_0^t b(\tau) \|u_{xt}^*(\cdot,\tau)\|_2^2\, d\tau \\ &\quad \leq \frac{1}{2} K^2(t) + \int_0^t b^{1/2}(\tau) \|\tilde{f}(\cdot,\tau)\|_2 b^{1/2}(\tau) \|u_t^*(\cdot,\tau)\|_2\, d\tau \end{aligned}$$

and use (5.10) to obtain the inequality (5.9). □

The next Theorem contains the principle result on asymptotic behavior. It is stated somewhat differently from [**5**, Corollary 4.4] because of assumption (5.21). Since in most cases we choose $b(t) = e^{\delta t}$, this integrability condition does not seem to be too restrictive.

THEOREM 6. *Assume hypotheses* (A1) *with* $q = 2$, (G1), (G2), (F1), (F2), (F3) *are satisfied. Suppose* g *satisfies* (4.13), *and let* u *be the corresponding globally defined generalized solution. Let* $U(t)$ *be the mean displacement function given by* (5.1). *Suppose also that*

$$\int_0^\infty \frac{1}{b(t)}\,dt < \infty, \tag{5.17}$$

$$\int_0^\infty b(t)|h_j(t)|^2\,dt < \infty, \quad \text{for } j = 0, 1, \tag{5.18}$$

$$\int_0^\infty b^{1/2}(t)\|\tilde{f}(\cdot,t)\|_2\,dt < \infty. \tag{5.19}$$

Then there is a function $u_{\infty,x} \in L^2(0,1)$ *such that*

$$\|u_x(\cdot,t) - u_{\infty,x}\|_2^2 = O\left(\int_t^\infty \frac{1}{b(\sigma)}\,d\sigma\right) \quad \text{as } t \to \infty. \tag{5.20}$$

If we also assume that

$$\int_0^\infty \frac{1}{b^{1/2}(t)}\,dt < \infty \tag{5.21}$$

then there is a function $u_\infty \in W^{1,2}(0,1)$ *such that* $u'_\infty(x) = u_{\infty,x}(x)$ *a.e.* $0 \le x \le 1$ *and*

$$\|u(\cdot,t) - U(t) - u_\infty\|_2 = O\left(\int_t^\infty \frac{1}{b^{1/2}(\sigma)}\,d\sigma\right) \quad \text{as } t \to \infty. \tag{5.22}$$

PROOF. Let

$$M_\infty = K(\infty) + \int_0^\infty b^{1/2}(\tau)\|\tilde{f}(\cdot,\tau)\|_2\,d\tau.$$

By (3.11), we have $u_x \in W^{1/2}(0,T;L^2(0,1))$, and so $u_x(x,t)$ is absolutely continuous in t. So if $0 \le s < t < \infty$, then, by (5.11),

$$\begin{aligned}\int_0^1 |u_x(x,t) - u_x(x,s)|^2\,dx &\le \left\{\int_s^t b(\sigma)\int_0^1 |u_{xt}(x,\sigma)|^2\,dx\,d\sigma\right\}\\ &\qquad \left\{\int_s^t \frac{1}{b(\sigma)}\,d\sigma\right\}\\ &\le 2M_\infty^2 \int_s^t \frac{1}{b(\sigma)}\,d\sigma.\end{aligned}$$

So $u_x(x,t)$ is Cauchy in the space $C(0,\infty;L^2(0,1))$, and there is an element $u_{\infty,x}\in L^2(0,1)$ such that

$$\|u_x(\cdot,t)-u_{\infty,x}\|_2^2 \le 2M_\infty^2\int_t^\infty \frac{1}{b(\sigma)}\,d\sigma, \quad 0\le t<\infty.$$

This proves (5.20).

Let u^* be given by (5.2), then $u^*\in W^{1/2}(0,T;L^\infty(0,1))$. So, by absolute continuity and (5.10),

$$\begin{aligned}\left(\int_0^1 |u^*(x,t)-u^*(x,s)|^2\,dx\right)^{1/2} &\le \int_s^t \|u_t^*(\cdot,\sigma)\|_2\,d\sigma\\ &\le M_\infty\int_s^t \frac{1}{b^{1/2}(\sigma)}\,d\sigma\end{aligned}$$

for all $0\le s<t<\infty$. Hence u^* is also Cauchy in the space $C(0,\infty;L^2(0,1))$, and there is an element $u_\infty\in L^2(0,1)$ such that

$$\|u^*(\cdot,t)-u_\infty\|_2 \le M_\infty\int_t^\infty \frac{1}{b^{1/2}(\sigma)}\,d\sigma, \quad 0\le t<\infty.$$

This proves (5.21). Finally, if $\varphi\in C_0^\infty(0,1)$, then, using the limits (5.20), (5.21), it follows that

$$\int_0^1 u_\infty(x)\varphi'(x)\,dx = -\int_0^1 u_{\infty,x}(x)\varphi(x)\,dx.$$

Hence, $u'_\infty(x)=u_{\infty,x}$ a.e. $0\le x\le 1$. □

REFERENCES

1. G. Andrews, *On the existence of solutions to the equation* $u_{tt}=u_{xxt}+\sigma(u_x)_x$, J. Differential Equations **35** (1980), 220–231.

2. B. Bernstein, E.A. Kearsley, and L.J. Zapas, *A study of stress relaxation with finite strain*, Trans. Soc. Rheology **7** (1963), 391–410.

3. H. Brezis, *Opérateurs maximaux monotones*, North-Holland Math. Stud. **5**, North-Holland, Amsterdam, 1973.

4. J.R. Canon, *The one-dimensional heat equation*, Encyclopedia Math. Appl. (Gian-Carlo Rota, ed.), Vol. 18, Cambridge Univ. Press, Cambridge, 1984.

5. H. Engler, *On the dynamic shear flow problem for viscoelastic liquids*, SIAM J. Math. Anal. **18** (1987), 972–990.

6. A. Friedman, *Partial differential equations of parabolic type*, Krieger, Florida, 1983.

7. A. Kaye, *Non-Newtonian flow in incompressible fluids*, College of Aeronautics, Cranfield, England, Tech. Note 134, 1962.

8. O.A. Ladyženskaja, V.A. Solonnikov and N.N. Ural'ceva, *Linear and quasilinear equations of parabolic type*, Transl. Math. Monographs, Amer. Math. Soc., Providence, 1968.

9. V.P. Mikhailov, *Partial differential equations*, MIR Publishers, Moscow, 1978.

10. M. Renardy, W.J. Hrusa, and J.A. Nohel, *Mathematical problems in viscoelasticity*, Pitman Monographs, vol. **35**, Longman Sci. Tech., Harlow, 1987.

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