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REGULARITY PROPERTIES OF SOLUTIONS OF LINEAR INTEGRODIFFERENTIAL EQUATIONS WITH SINGULAR KERNELS

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Dedicated to John A. Nohel for his sixty-fifth birthday

1. Introduction. Let $\mathbf{H}$ be a Hilbert space with norm $|\cdot|$ and inner product $\langle\cdot,\cdot\rangle$. Let A be a possibly unbounded linear operator in $\mathbf{H}$, and let $a : [0,\infty) \to \mathbf{R}$ be a given locally integrable kernel function. In this note we study properties of solutions of the abstract linear integrodifferential equation

$$u'(t) + a * Au(t) = 0, \quad 0 < t < T, \tag{1}$$

in $\mathbf{H}$. Here the usual convolution notation is employed: $f * g(t) = \int_0^t f(t-s)g(s)\,ds$, if one of the two functions is scalar-valued and the other one is vector-valued.

We want to consider *mild solutions* of (1), i.e., continuous functions $u(\cdot)$ for which $1 * a * u(t) \in D(A)$ for all $0 \le t \le T$ and for which the integrated version

$$u(t) + A(1 * a * u(t)) = u(0) \tag{2}$$

of (1) holds for all $t \in [0,T]$. The goal of this note is to give conditions on the kernel function a under which such mild solutions satisfy an estimate of the form

$$|Au(t)| \le \frac{C}{t^M}|u(0)|$$

for a suitable power M. It will be proved that such an estimate is always true if the derivative a' is integrable and behaves like $-t^{-2\alpha}$ near zero, and that in this case $M = 1/\alpha$ is a suitable exponent.

The following assumptions will be used. The assumptions for the kernel function $a(\cdot)$ are to hold on any finite interval $[0,T]$. A subscript T denotes that the corresponding quantity depends on T.

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(A) The operator A is densely defined, self adjoint, and there exists $\Lambda > 0$ such that $\langle Ax, x\rangle \geq \Lambda|x|^2$ for all $x \in D(A)$.

(a1) The kernel $a(\cdot)$ satisfies

$$\begin{gathered} a(0) = 1 \\ a' \in L^1(0, T; \mathbf{R}), \qquad \int_0^T t|a''(t)|\, dt < \infty \\ a''(\cdot) \geq -c_T(\cdot) \quad \text{with } c_T \in L^1(0, T; \mathbf{R}). \end{gathered}$$

(a2) There exist $\delta > 0$ and $\alpha \in (0, 1/2)$ such that

$$a''(t) \geq \delta t^{-1-2\alpha} - c_T(t) \quad \text{with } c_T \in L^1(0, T; \mathbf{R}) \tag{3}$$

$$\int_0^T t^3|a^{(4)}(t)|\, dt < \infty. \tag{4}$$

It is known that operators satisfying (A) have a spectral decomposition

$$Ax = \int_\Lambda^\infty \lambda\, dE_\lambda x$$

for all $x \in D(A)$ [**8**]. Then fractional powers of A can be defined in the usual way,

$$A^\beta x = \int_\Lambda^\infty \lambda^\beta\, dE_\lambda x,$$

for $-\infty < \beta < \infty$ and for x from a suitable maximal domain of definition. Also, property (3) forces a' to have an integrable singularity at the origin that is at least algebraic: $a'(t) \leq -\delta_1 t^{-2\alpha} + C_T$ for some $\delta_1 > 0$.

The two main results of this paper are the following:

THEOREM 1. *Let assumptions* (A) *and* (a1) *hold. Then, for any* $u_0 \in \mathbf{H}$, *there exists a unique mild solution of* (2) *with* $u(0) = u_0$. *It can be obtained as* $u(t) = R(t)u_0$, *where* $(R(t))_{0\leq t\leq T}$ *is a strongly continuous operator family in* $L(\mathbf{H}, \mathbf{H})$. *Moreover, the operator families* $A^{-1}R''(\cdot)$, $A^{-\frac{1}{2}}R'(\cdot)$, $A^{\frac{1}{2}}(1 * R)(\cdot)$ *and* $A(1 * 1 * R)(\cdot)$ *are also strongly continuous on* $[0, T]$.

THEOREM 2. *Let assumptions* (A), (a1) *and* (a2) *hold, and let* $0 \le \beta \le 1$, $0 \le \gamma \le 1/2$. *Then* $A^\beta R(\cdot)$ *and* $A^{\gamma+1/2}(1 * R)(\cdot)$ *are strongly continuous families of bounded operators on* $(0, T]$, *and the estimates*

$$\|A^\beta R(t)\| \le C_T t^{-\frac{\beta}{\alpha}}, \tag{5}$$

$$\|A^{\frac{1}{2}+\gamma}(1 * R)(t)\| \le C_T t^{-\frac{\gamma}{\alpha}} \tag{6}$$

hold.

Theorem 1 has a natural and well-known consequence for the inhomogeneous equation

$$u'(t) + a * Au(t) = f(t). \tag{7}$$

In analogy to the definition above, we call a continuous function $u : [0, T] \to \mathbf{H}$ a mild solution of (7) if $1 * a * u(t) \in D(A)$ for all $t \in [0, T]$ and if the integrated equation

$$u(t) + A(1 * a * u(t)) = u(0) + \int_0^t f(s)\, ds \tag{8}$$

holds for all t.

COROLLARY 1. *For any* $f \in L^1(0, T; \mathbf{H})$, $u_0 \in \mathbf{H}$ *there exists a unique solution of* (8) *with* $u(0) = u_0$. *It is given by*

$$u(t) = R(t)u_0 = \int_0^t R(t-s)f(s)\, ds, \quad 0 \le t \le T.$$

Theorem 2 implies estimates in various intermediate regularity classes which we give next.

COROLLARY 2. *Under the assumptions of Theorem* 2, *let* $0 < \beta < 1$, $1 < p < \infty$, $r = \beta/\alpha$. *Then*

$$\int_0^T t^{rp} \|A^\beta R(t)\|^p \frac{dt}{t} \le C(T, p, r), \tag{9}$$

and, if $0 \leq \beta < 1/2$*, then*

$$\int_0^T t^{rp} \|A^{\beta+\frac{1}{2}}(1 * R)(t)\|^p \frac{dt}{t} \leq C(T, p, r). \tag{10}$$

There exist several results like Theorem 1 in the literature, see, e.g., [**2, 3, 4, 12**]. The new features in the result given here are the weakened assumptions on the kernel function. The main reason for stating Theorem 1 is to give a framework for Theorem 2, which seems to be new in this form. Other results concerning regularizing effects of the resolvent family $R(\cdot)$ have been given for classes of kernels with either stronger or weaker singularities. In [**5**] and [**9**], the kernel a itself is assumed to have a singularity at the origin, and the resolvent family is shown to possess an analytic extension into some sector in the complex plane that contains the positive real axis. These arguments use the Laplace transform and a suitable deformation of the integration path in the inversion formula, much as in the classical argument for the construction of analytic semigroups [**11**]. In [**6, 7, 10, 13**] kernels are considered for which (3) holds with $\alpha = 0$, i.e., $a'(\cdot)$ has a logarithmic singularity at the origin. In this case, $A^\beta R(t)$ becomes a bounded operator for $t > \beta t_0$, where $t_0 > 0$. These results can be proved with a variety of techniques, e.g., again by deforming an integration path in the complex inversion formula [**13**], by casting the problem in an abstract semigroup framework and appealing to a general result on differentiability of semigroups [**6**], or by explicitly computing the resolvent in a model problem [**7, 10**]. Some explicit examples in [**10**] show that this "delayed" regularization property in the case $\alpha = 0$ is sharp.

We shall construct the solution of Theorem 1 as

$$u(t) = \int_\Lambda^\infty u_\lambda(t) \, dE_\lambda u_0,$$

where the scalar functions u_λ are solutions of the equations

$$u'_\lambda(t) + \lambda a * u_\lambda(t) = 0, \quad u_\lambda(0) = 1. \tag{11}$$

To do this, suitable *a priori* estimates have to be derived, which are listed in Lemma 1 in the next section. The remainder of the next section is concerned with deriving additional estimates from which Theorem 2 follows. The main tool then is simply the variation-of-constants formula for solutions of inhomogeneous equations associated with (11) which is employed in a "bootstrap" fashion. One advantage of this approach is that the sign conditions and regularity assumptions for the kernel can be kept to a minimum. The two main results together with the Corollaries are then proved in Section 3.

We use the usual notation of Sobolev spaces and refer the reader to [**1**] for the necessary background material. In particular, fractional order Sobolev spaces $W^{s,2}(I)$ of scalar functions on intervals I will be used. Their definition and properties are also listed in [**1**]. Constants that may change from line to line are denoted by the same letter, C; they are allowed to depend on $T, a(\cdot)$ and its properties, and other parameters, but not on the parameter $\lambda > \Lambda$ that is used in (11). We also write

$$d_j^k(t) = t^j d^{(k)}(t)$$

whenever $d(\cdot)$ is a function for which the k-th derivative is defined.

2. A priori estimates for scalar equations. In this section we study solutions of the scalar equations (11) in more detail. The goal is to collect estimates that display the dependence on λ in detail. A standard contraction argument shows that solutions u_λ exist and are unique on any time interval $[0,T]$ and that they will be in $C^2([0,T])$. From now on, T will be arbitrary and fixed. We set $v_\lambda = 1 * u_\lambda$.

LEMMA 1. *Let a satisfy assumptions* (a1). *Then, for any solution u_λ of* (11) *and for all $0 \le t \le T < \infty$,*

$$\lambda^{-1}|u_\lambda''(t)| + \lambda^{-1/2}|u_\lambda'(t)| + |u_\lambda(t)| + \lambda^{1/2}|v_\lambda(t)| + \lambda|1 * v_\lambda(t)| \le C_T. \tag{12}$$

If a also satisfies (a2)*, then, additionally,*

$$\int_0^T \int_0^T \left(\frac{|u_\lambda(t) - u_\lambda(s)|^2}{|t-s|^{1+2\alpha}} + \lambda \frac{|v_\lambda(t) - v_\lambda(s)|^2}{|t-s|^{1+2\alpha}} \right) ds\, dt \le C_T. \tag{13}$$

PROOF. Equation (11) can be written in the equivalent form

$$u'_\lambda(t) + \lambda\biggl(\int_0^t a'(t-s)(v_\lambda(s) - v_\lambda(t))\,ds + a(t)v_\lambda(t)\biggr) = 0.$$

Multiply with $u_\lambda(t)$ and integrate. After some manipulations, the result is

$$\begin{aligned}(14)\qquad &|u_\lambda(t)|^2 - 1 + \lambda|v_\lambda(t)|^2\\ &\qquad - \lambda\int_0^t a'(t-s)(|v_\lambda(t-s)|^2 + |v_\lambda(s)|^2 - 2v_\lambda(t)v_\lambda(s))\,ds\\ &\qquad + \lambda\int_0^t\int_0^s a''(s-\tau)|v_\lambda(s) - v_\lambda(\tau)|^2\,d\tau\,ds = 0.\end{aligned}$$

Note that, due to the smoothness of v_λ, the integral involving a'' is convergent. Since a'' is bounded below by an integrable function, (14) implies that for some $m \in L^1(0,T;\mathbf{R})$

$$(15)\qquad |u_\lambda(t)|^2 + \lambda|v_\lambda(t)|^2 \le C + \lambda\int_0^t (m(t-s) + m(s))|v_\lambda(s)|^2\,ds.$$

Gronwall's lemma now implies the bound for the third and fourth term on the left-hand side of estimate (12). The last term on the left-hand side of (12) can be estimated since, by (11), $1 * v_\lambda + a' * 1 * v_\lambda = \lambda^{-1}(1 - u_\lambda)$. Similarly, using equation (11) and its derivative, the other two terms can be estimated.

To prove (13), we use (14) again and note that, by the preceding argument, all terms in (14) except the double integral involving a'' are now estimated independent of λ. Now use the assumption that, up to an integrable function, $a''(t) \ge \delta t^{-1-2\alpha}$. This implies that the second term on the left-hand side of (13) can be estimated as stated. To estimate also the other term, we differentiate (11), multiply the result with $\lambda^{-1}u_\lambda(t)$ and integrate once. The result is the identity

(16)
$$\begin{aligned}&\frac{1}{\lambda}|u'_\lambda(t)|^2 + \int_0^t\int_0^s a''(t-s)|u_\lambda(s) - u_\lambda(\tau)|^2\,d\tau\,ds\\ &\quad = 1 + a(t)|u_\lambda(t)|^2 + \int_0^t (a'(t-s)|u_\lambda(t) - u_\lambda(s)|^2 + a'(s)|u_\lambda(s)|^2)\,ds.\end{aligned}$$

Since the right-hand side can be estimated uniformly in λ, the same argument as before shows that also the first term in (13) can be estimated as claimed. □

LEMMA 2. *If assumptions* (a1) *and* (a2) *hold, then, for any solution of* (11),

$$\lambda^{2\alpha}\int_0^T |u_\lambda(t)|^2\,dt \le C_T. \tag{17}$$

PROOF. Estimate (13) implies that

$$\begin{aligned} \|v_\lambda\|_\alpha &\le C_T\lambda^{-1/2}\\ \|v_\lambda\|_{1+\alpha} &\le C_T, \end{aligned} \tag{18}$$

where $\|\cdot\|_s$ is a norm on the fractional order Sobolev space $W^{s,2}([0,T],\mathbf{R})$ [**1**]. Since these are complex interpolation spaces, a standard interpolation inequality now implies that

$$\|u_\lambda\|_0 \le \|v_\lambda\|_1 \le C\|v_\lambda\|_\alpha^\alpha \|v_\lambda\|_{1+\alpha}^{1-\alpha} \le C\lambda^{-\alpha}.$$

This is the desired estimate. □

Lemmata 1 and 2 provide all *a priori* estimates that will be needed for the proofs. The rest of this section will be concerned with deriving related estimates for functions of the form $t^l u_\lambda(t)$, using the variation-of-constants formula. We therefore write

$$u_{\lambda,l}(t) = t^l u_\lambda(t), \quad v_{\lambda,l}(t) = t^l v_\lambda(t)$$

for $0 \le t \le T$, $\lambda \ge \Lambda$, $l = 0, 1, 2, \dots$. We also set

$$w_{\lambda,l}(t) = t^l (u_\lambda * u_\lambda)(t)$$

and set $w_{\lambda,-1}(t) = 0$ for all t by convention.

LEMMA 3. *Suppose a satisfies* (a1) *and* (a2) *and u_λ solves* (11). *Then, for all $l = 0, 1, 2, \dots$, and $0 \le t \le T$,*

$$w_{\lambda,l}(t) = \sum_{i=0}^{l} \binom{l}{i} u_{\lambda,i} * u_{\lambda,l-i}(t) \tag{19}$$

$$\left| \int_0^t (t-s) u_{\lambda,l}(s)\, ds \right| \le C(l,T)\lambda^{-1} \tag{20}$$

$$|(1 * u_{\lambda,l})(t) - v_{\lambda,l}(t)| \le C(l,T)\lambda^{-1}. \tag{21}$$

PROOF. Equation (19) (which reduces to the Leibniz rule for derivatives of products after taking the Fourier transform) is proved by a standard induction argument.

Estimate (20) also follows by induction: For $l = 0$, the estimate is contained in Lemma 1. For the step from l to $l+1$ we write

$$\begin{aligned} \int_0^t (t-s) u_{\lambda,l+1}(s)\, ds = & -2 \int_0^t \int_0^s (s-\tau) u_{\lambda,l}(\tau)\, d\tau\, ds \\ & + t \cdot \int_0^t (t-s) u_{\lambda,l}(s)\, ds, \end{aligned}$$

and this term is bounded by $C\lambda^{-1}$ by the induction assumption.

The proof of (21) also uses induction. For $l = 0$, nothing has to be shown. For the step from l to $l+1$, we write

$$\begin{aligned} (1 * u_{\lambda,l+1})(t) - v_{\lambda,l+1}(t) = & -\int_0^t (t-s) u_{\lambda,l}(s)\, ds \\ & + t\Big(\int_0^t u_{\lambda,l}(s)\, ds - v_{\lambda,l}(t)\Big), \end{aligned}$$

and by (20) and the induction assumption, both terms on the right-hand side are bounded by $C\lambda^{-1}$. □

Next an auxiliary kernel is introduced which can be shown to have the same regularity properties as a_1^1. Let $k : [0,T] \to \mathbf{R}$ be the *resolvent* *kernel* associated with a', i.e., the L^1-function with the equivalent properties

$$a'(t) + k * a'(t) + k(t) = 0, \quad \text{for a.e. } t \in [0,T], \tag{22}$$
$$a(t) + k * a(t) = 1, \quad \text{for all } t \in [0,T]. \tag{23}$$

We set

$$b(t) = 1 + a_1^1(t) + k * a_1^1(t) = 1 + ta'(t) + \int_0^t k(t-s)sa'(s)\,ds. \tag{24}$$

LEMMA 4. *The kernel b has the properties*

$$t \cdot a(t) = b * a(t), \quad \textit{for all } t \in [0,T], \tag{25}$$
$$a_2^1 \in W^{2,1}([0,T],\mathbf{R}) \Longrightarrow b_1 \in W^{2,1}([0,T],\mathbf{R}) \tag{26}$$
$$a_3^1 \in W^{3,1}([0,T],\mathbf{R}) \Longrightarrow b_2 \in W^{3,1}([0,T],\mathbf{R}). \tag{27}$$

PROOF. We have

$$\begin{aligned} b = 1 + a_1^1 + k * a_1^1 &= a + a_1^1 + k * (a + a_1^1) \\ &\Longrightarrow b + a' * b = a + a_1^1 \\ &\Longrightarrow \frac{d}{dt}(b * a) = \frac{d}{dt} a_1^1. \end{aligned}$$

This proves (25). To prove (26) and (27), we first compute $k_1(t) = tk(t)$ and $k_2(t) = t^2 k(t)$. Multiplying (22) with t implies

$$a_1^1 + k_1 * a' + k * a_1^1 + k_1 = 0,$$

and thus

$$k_1 = -\,a_1^1 - k * a_1^1 - k * (a_1^1 + k * a_1^1) = -\,(a_1^1 + 2k * a_1^1 + k * k * a_1^1).$$

Multiplying again with t one obtains, after some manipulations,

$$k_2 = -\,a_2^1 - 2k * a_2^1 - k * k * a_2^1 + 2(a_1^1 + 2k * a_1^1 + k * k * a_1^1) * (a_1^1 + k * a_1^1).$$

Thus, k_i is as regular as a_i^1 for $i = 1, 2$. Now

$$b_1^1(t) = t + a_2^1(t) + k_1 * a_1^1(t) + k * a_2^1(t) \tag{28}$$

and

$$b_2(t) = t^2 + a_3^1(t) + \sum_{i=0}^{2} \binom{2}{i} k_i * a_{i+1}^1(t). \tag{29}$$

From (28) we read off that $b_1 \in W^{2,1}([0,T], \mathbf{R})$ if a_2^1 is in this class, which proves (26). To prove (27), we note that the convolution products involving k_1 and k_2 will be in $W^{3,1}([0,T], \mathbf{R})$ as soon as $a_2^1 \in W^{2,1}([0,T], \mathbf{R})$ and that the other terms will be as regular as a_3^1. Since assumption (a2) implies in particular that $a_3^1 \in W^{3,1}([0,T], \mathbf{R})$, property (27) follows. $\square$

Lemma 5. *If assumptions* (a1) *and* (a2) *hold, then, for all* $t \in [0,T]$, $\lambda > \Lambda$, $l = 1, 2, 3, \ldots,$

$$\begin{aligned} &u_{\lambda,l}(t) + b * u_{\lambda,l-1}(t) \\ &\quad = 2w_{\lambda,l-1}(t) + b' * w_{\lambda,l-1}(t) + (l-1)b_1^1 * w_{\lambda,l-2}(t) + f_{\lambda,l}(t), \end{aligned} \tag{30}$$

where $|f_{\lambda,l}(t)| \le C(l,T)\lambda^{-1}$.

Proof. The proof uses induction. For $l = 1$, one computes

$$\begin{aligned} u'_{\lambda,1}(t) &= u_\lambda(t) + tu'_\lambda(t) \\ &= u_\lambda(t) - \lambda a_1 * u_\lambda(t) - \lambda a * u_{\lambda,1}(t). \end{aligned}$$

By Lemma 4,

$$\begin{aligned} \lambda a_1 * u_\lambda &= \lambda b * a * u_\lambda \\ &= - b * u'_\lambda \\ &= - u_\lambda + b - b' * u_\lambda. \end{aligned}$$

Thus, $u_{\lambda,1}$ solves the equation

$$u'_{\lambda,1} + \lambda a * u_{\lambda,1} = 2u_\lambda + b' * u_\lambda - b.$$

Applying the variation-of-constants formula gives (30) for $l = 1$.

The induction step from l to $l+1$ is a straightforward calculation. One only has to show that the remainder term $f_{\lambda,l+1}$ can be estimated as stated.

$$\begin{aligned} u_{\lambda,l+1}(t) &+ b * u_{\lambda,l}(t) \\ &= t(u_{\lambda,l}(t) + b * u_{\lambda,l-1}(t)) - b_1 * u_{\lambda,l}(t) \\ &= t(2w_{\lambda,l-1}(t) + b' * w_{\lambda,l-1}(t) + (l-1)b_1^1 * w_{\lambda,l-2}(t) \\ &\quad + f_{\lambda,l}(t)) - b_1 * u_{\lambda,l}(t) \\ &= 2w_{\lambda,l}(t) + b' * w_{\lambda,l}(t) + l b_1^1 * w_{\lambda,l-1}(t) + f_{\lambda,l+1}(t) \end{aligned}$$

with $f_{\lambda,l+1}(t) = (l-1)b_2^1 * w_{\lambda,l-1}(t) + t f_{\lambda,l}(t) - b_1 * u_{\lambda,l}(t)$. By Lemma 4, b_2^1 and b_1 are in $W^{2,1}([0,T],\mathbf{R})$ and vanish for $t=0$. Therefore, by Lemma 3, the two convolution integrals, and thus also $f_{\lambda,l+1}(t)$ can be estimated by $C\lambda^{-1}$. The lemma follows. □

LEMMA 6. *If assumptions* (a1) *and* (a2) *hold, then for* $0 < t \le T$, $0 \le r \le 1/\alpha$, $\lambda \ge \Lambda$,

$$|u_\lambda(t)| \le C t^{-r} \lambda^{-r\alpha} \tag{31}$$

and for $0 \le r \le 1/(2\alpha)$,

$$|v_\lambda(t)| \le C t^{-r} \lambda^{-r\alpha - \frac{1}{2}}. \tag{32}$$

Here C *depends on* r, a, T, *but not on* λ.

PROOF. We use induction on n to prove first (31) and then (32) for all $r \le n \le 1/\alpha$, respectively, for all $r \le n \le 1/(2\alpha)$, and then give a concluding argument in the case where $1/\alpha$ is not an integer. We start with the estimate for u_λ for $r = n = 1$. By Lemma 5,

$$t u_\lambda(t) = -\, b * u_\lambda(t) + 2u_\lambda * u_\lambda(t) + b' * u_\lambda * u_\lambda(t).$$

The terms on the right-hand side can be estimated for each t as follows:

$$|b * u_\lambda(t)| = |v_\lambda(t) + b' * v_\lambda(t)| \le C\lambda^{-1/2} \le C\lambda^{-\alpha}$$

by Lemma 1;

$$|u_\lambda * u_\lambda(t)| \le \|u_\lambda\|_{L^2}^2 \le C\lambda^{-\alpha} \tag{33}$$

by Hölder's inequality and Lemma 2; and

$$|b' * u_\lambda * u_\lambda(t)| \le C\lambda^{-\alpha}$$

by (33). These estimates prove (31) for $r = 1$. Since (31) is also true for $r = 0$, we obtain (31) for all $0 \le r \le 1$.

We next show (32), first for $r = 1/2$ and then for $r = 1$. One computes

$$tv_\lambda(t) = 1 * 1 * u_\lambda(t) + 1 * b * u_\lambda(t) - 2v_\lambda * u_\lambda(t) - 2b' * v_\lambda * u_\lambda(t)$$

due to Lemma 5. The first two terms can be estimated by $C\lambda^{-1}$. The other two terms can be estimated by

$$|2v_\lambda * u_\lambda(t) + 2b' * v_\lambda * u_\lambda(t)| \le 2\, t^{\frac{1}{2}} \|v_\lambda\|_{L^\infty} \|u_\lambda\|_{L^2} \le Ct^{\frac{1}{2}} \lambda^{-\frac{\alpha+1}{2}}, \tag{34}$$

by Lemma 3 and Hölder's inequality. Thus $|v_\lambda(t)| \le Ct^{-1/2}\lambda^{-(\alpha+1)/2}$. Repeating the estimate (34) and using (31) with $r = 1/2$ now also gives

$$|2v_\lambda * u_\lambda(t) + 2b' * v_\lambda * u_\lambda(t)| \le C\lambda^{-\alpha-\frac{1}{2}}.$$

This proves (32) also for $r = 1$. As before, since the estimate is also true for $r = 0$, we then obtain it for all $r \in [0, 1]$.

For the induction step, suppose that (31) and (32) are true for $0 \le r \le n \le 1/(2\alpha) - 1$. By Lemma 5 and Lemma 3,

$$\begin{aligned} t^{n+1}u_\lambda(t) = &- v_{\lambda,n}(t) - b' * v_{\lambda,n}(t) + 2w_{\lambda,n}(t) \\ &+ b' * w_{\lambda,n}(t) + nb_1^1 * w_{\lambda,n-1}(t) + g_{\lambda,n}(t), \end{aligned} \tag{35}$$

where $|g_{\lambda,n}(t)| \le C\lambda^{-1}$. We first show (31) for $r = n + 1/2$ and then for $r = n+1$. The terms on the left-hand side of (35) can be estimated as follows:

$$|v_{\lambda,n}(t) + b' * v_{\lambda,n}(t)| \le Ct^{-n}\lambda^{-n\alpha-\frac{1}{2}} \le Ct^{-n}\lambda^{-(n+1)\alpha}$$

by induction assumption;

$$\begin{aligned} |w_{\lambda,n}(t)| &\le \sum_{i=0}^{n} \binom{n}{i} u_{\lambda,i} * u_{\lambda,n-i}(t)| \\ &\le Ct^{\frac{1}{2}}\lambda^{-(n+\frac{1}{2})\alpha} \end{aligned} \tag{36}$$

due to the estimates

$$|u_{\lambda,j}(t)| \le Ct^{-\frac{1}{2}}\lambda^{-(j+\frac{1}{2})\alpha}, \tag{37}$$

for $j = 0, \dots, n-1$, and

$$|u_{\lambda,n}(t)| \le C\lambda^{-n\alpha}. \tag{38}$$

Thus, also

$$|b' * w_{\lambda,n}(t)| \le Ct^{\frac{1}{2}}\lambda^{-(n+\frac{1}{2})\alpha}. \tag{39}$$

The terms involving $w_{\lambda,n-1}$ can be written as

$$b_1^1 * w_{\lambda,n-1} = (b' + b_1^2) * \sum_{i=0}^{n-1} v_{\lambda,i} * u_{\lambda,n-1-i} + g_{\lambda,n-1}$$

by Lemma 3, where $|g_{\lambda,n-1}(t)| \le C\lambda^{-1}$. Since $b', b_1^2 \in L^1(0,T;\mathbf{R})$, we only have to estimate each term in the sum pointwise:

$$|v_{\lambda,i} * u_{\lambda,n-1-i}(t)| \le C\lambda^{-n\alpha-\frac{1}{2}} \le C\lambda^{-(n+1)\alpha},$$

due to (37) and

$$|v_{\lambda,j}(t)| \le Ct^{-\frac{1}{2}}\lambda^{-(j+\frac{1}{2})\alpha-\frac{1}{2}}, \tag{40}$$

for $j = 0, \dots, n-1$, by induction assumption. Put together, all these estimates prove (31) for $r = n + 1/2$. This means that

$$|u_{\lambda,n}(t)| \le Ct^{-\frac{1}{2}}\lambda^{-(n+\frac{1}{2})\alpha}. \tag{41}$$

Using (41) instead of (38) now implies that

$$|w_{\lambda,n}(t)| \le C\lambda^{-(n+1)\alpha},$$

and this improved estimate implies (31) also for $r = n + 1$. As before, estimate (31) then follows also for all $r \in [n, n+1]$.

We next show (32) for $r = n + 1/2$ and $r = n + 1$, still assuming that $n \le 1/(2\alpha) - 1$. The argument is essentially the same as in the case $n = 1$. First, according to Lemma 3, we can replace $v_{\lambda,n+1}(t)$ by $1 * u_{\lambda,n+1}$ with an error that is bounded by $C\lambda^{-1}$. Next, by Lemma 5,

$$\begin{aligned} 1 * u_{\lambda,n+1} = & - b * v_{\lambda,n} + 2 * w_{\lambda,n} \\ & + b' * 1 * w_{\lambda,n} + n \cdot 1 * b_1^1 * w_{\lambda,n-1} + 1 * f_{\lambda,n+1}. \end{aligned} \tag{42}$$

Lemmata 1 and 4 imply that the first and the last two terms can be estimated by $C\lambda^{-1}$. The second term can be estimated by

$$\begin{aligned} |2 * w_{\lambda,n}(t)| & \leq C \sum_{j=0}^{n} |v_{\lambda,j} * u_{\lambda,n-j}|(t) + C\lambda^{-1} \\ & \leq Ct^{\frac{1}{2}} \lambda^{-\frac{1}{2}-(n+\frac{1}{2})\alpha}. \end{aligned} \tag{43}$$

Here (37) and (40) have been used for $j = 0, \ldots, n$, except for the term involving $v_{\lambda,n}$, where we use the induction assumption. The same estimate holds for the third term in (42). This proves (32) for $r = n + 1/2$ and implies in particular that

$$|v_{\lambda,n}(t)| \leq Ct^{-\frac{1}{2}} \lambda^{-\frac{1}{2}-(n+\frac{1}{2})\alpha}. \tag{44}$$

Using (44) instead of the induction assumption allows us to improve (43) to

$$|2 * w_{\lambda,n}(t)| \leq C\lambda^{-\frac{1}{2}-(n+1)\alpha}$$

and to estimate the third term in (42) in the same way. Thus (32) is proved for $r = n + 1$ and consequently for all $r \in [n, n+1]$.

There remain some additional cases, all of which can be handled by the same arguments. Estimate (32) is shown for the case $[1/(2\alpha)] \leq r \leq 1/(2\alpha)$ by an argument that is similar to the one in the induction step. If $1/(2\alpha) - 1 < n < 1/\alpha$, then only (31) has to be shown by induction, and the argument is identical to the one given above. Finally, (31) follows for the remaining range $[1/\alpha] < r \leq 1/\alpha$ as in the induction step. The Lemma is completely proved. □

3. Proofs of the main results.

Proof of Theorem 1 and of Corollary 1. Let u be a solution of (1) with $u(0) = 0$. For $N \geq \Lambda$, $t \geq 0$, set $u^N(t) = E_N u(t)$, where

E_N is the spectral measure of the set $[\Lambda, N]$. Then u^N solves the same equation. Since the operator A is bounded on the space $E_N\mathbf{H}$ in which this equation holds, we obtain $u^N = 0$, due to a standard uniqueness argument for ordinary integral equations. Since N is arbitrary, $u = 0$ follows, which established uniqueness of mild solutions of (1) and (7).

To construct a solution of (1), suppose that $u_0 \in \mathbf{H}$ is given. Set

$$u^N(t) = \int_\Lambda^N u_\lambda(t)\, dE_\lambda u_0 \tag{45}$$

for $N \geq \Lambda$, $0 \leq t \leq T$. The u^N are (classical) solutions of (1) with initial data $u^N(0) = E_N u_0$. Then the estimates in Lemma 1 show that the functions $A^{-1}(u^N)''$, $A^{-\frac{1}{2}}(u^N)'$, u^N, $A^{1/2}(1*u^N)$, and $A(1*1*u^N)$ all converge uniformly in $C([0,T],\mathbf{H})$ to continuous limit functions $A^{-1}u''$, $A^{-\frac{1}{2}}u'$, u, $A^{1/2}(1*u)$, and $A(1*1*u)$, and u is a mild solution of (1). This proves Theorem 1. Corollary 1 and the variation-of-constants formula follow by using the same construction. □

Proof of Theorem 2. The estimates of Theorem 2 follow by using Lemma 6 in (45) and passing to the limit as $N \to \infty$. □

For the proof of Corollary 2, we need a standard Marcinkiewicz-type interpolation argument.

Lemma 7. *Let $f : [0,T] \to \mathbf{R}$ be measurable such that $\|f\|_{L^\infty} \leq C_1$ and $|f(t)| \leq C_2 t^{-\gamma}$ for almost all t, for some $\gamma > 0$. Then, for all p with $1 \leq p < \infty$ and $1/\gamma < p$,*

$$\int_0^T |f(t)|^p\, dt \leq \frac{C_1^{p-\frac{1}{\gamma}} C_2^{\frac{1}{\gamma}}}{p - \frac{1}{\gamma}}.$$

PROOF. Let $\mu(s) = |\{t \mid |f(t)| > s\}|$ be the distribution function of f. The assumptions imply

$$\mu(s) = 0, \quad \text{for } s \geq C_1,$$

and

$$s^{\frac{1}{\gamma}} \cdot \mu(s) \leq C_2^{\frac{1}{\gamma}}, \quad \text{for all } s.$$

Then

$$\begin{aligned} \int_0^T |f(t)|^p \, dt &= \int_0^{C_1} s^{p-1} \mu(s) \, ds \\ &\leq \int_0^{C_1} s^{p-1-\frac{1}{\gamma}} C_2^{\frac{1}{\gamma}} \, ds \\ &= \frac{C_2^{p-\frac{1}{\gamma}} C_2^{\frac{1}{\gamma}}}{p - \frac{1}{\gamma}}. \ \square \end{aligned}$$

PROOF OF COROLLARY 2. Let p, β, r be given as in the assumptions. For given λ, we set $f(t) = t^{r-1/p} u_\lambda(t)$ and $\gamma = 1/\alpha + 1/p - r$. By Lemma 6,

$$|f(t)| \leq C\lambda^{-r\alpha} \text{ and } |f(t)| \leq C\lambda^{-1} t^{-\gamma}.$$

Applying Lemma 7 then leads to the estimate

$$\int_0^T |f(t)|^p \, dt \leq C\lambda^{-\beta}.$$

Using this estimate in (45) and passing to the limit as $N \to \infty$ implies the first half of the Corollary. The second half follows analogously. $\square$

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