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PARAMETER IDENTIFICATION IN A VOLTERRA EQUATION WITH WEAKLY SINGULAR KERNEL

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This paper is dedicated with gratitude to my thesis advisor,
John A. Nohel, on the occasion of his sixty-fifth birthday.

ABSTRACT. We consider identification of parameters in a Volterra integrodifferential system with a weakly singular kernel. Such kernels arise in fractional derivative damping models of viscoelastic materials. The Volterra equation is cast in a semigroup setting to establish results on the differentiability of the solution with respect to a parameter. These results are needed for convergence of the identification algorithm. Numerical results are presented.

1. Introduction. In this paper we consider the identification of parameters in a Volterra integrodifferential equation with a singular kernel. The equation of interest has the form

$$(1.1)\quad \begin{cases} \dot{w}(t) = Mw(T) + \int_{-\infty}^{t} K(t-s,p)w(s)\,ds + F(t), & t \geq 0, \\ w(0) = \eta,\ w(s) = \phi(s), & s < 0, \end{cases}$$

where M is an $n \times n$ constant matrix, $\eta \in \mathbf{R}^n$, $\phi \in L^1(-\infty, 0; \mathbf{R}^n)$ and $K(\cdot, p)$ is an $n \times n$ singular kernel depending on a parameter p contained in an admissible parameter set. We are particularly interested in a kernel function of the form

$$g(s,p) = \frac{\gamma e^{-\beta s}}{\Gamma(1-\alpha)s^{\alpha}}, \quad s > 0,$$

where $\Gamma(\cdot)$ denotes the gamma function, $p = (\alpha, \beta, \gamma) \in \mathbf{R}^3$ with $0 \leq \alpha < 1$ and $\beta, \gamma > 0$. Such kernels arise in the study of fractional derivative models of viscoelastic structures. For a more complete discussion of the origins of this kernel and the viscoelastic models, we

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refer the reader to [**12, 18, 13, 15**], and, in particular, to [**17**] and the extensive bibliography therein.

Banks, et al. [**2**] have identified parameters corresponding to β and γ in a similar (but different) model, but assumed that α was known. Torvik and Bagley [**1, 18**] have estimated the parameter α, but in the Laplace transform domain. In this paper we restrict ourselves to identifying α only, though the theory may be modified to include β and γ as well.

In order to relate equation (1.1) to a (idealized) physical model, consider the longitudinal motions of a uniform bar fixed at both ends with Boltzmann type damping. The governing equation is [**9, 14**]

$$\begin{aligned}\rho u_{tt}(x,t) = \frac{\partial}{\partial x}\bigg\{Eu_x(x,t) + \frac{\partial}{\partial t}\int_0^t g(t-s)u_x(x,s)\,ds\bigg\}\\ + f(x,t), \quad 0 < x < 1,\ t > 0,\end{aligned} \tag{1.2}$$

with boundary conditions $\quad u(0,t) = 0, \quad u(1,t) = 0,$
and initial conditions $\quad u(x,0) = d(x), \quad u_t(x,0) = v(x).$

Here, $u(x,t)$ represents the axial displacement of position x at time t, ρ is the density of the material, E a stiffness parameter, $f(x,t)$ a forcing function, and

$$g(s) = \frac{\gamma e^{-\beta s}}{\Gamma(1-\alpha)s^{\alpha}}$$

represents a fractional derivative damping term modified to have exponential decay [**12**].

A common approach to the parameter identification problem [**4**] is to apply a Galerkin-type approximation scheme to the beam equation and then incorporate some type of identification algorithm to the approximating system of integrodifferential equations. If one applies a Galerkin scheme to equation (1.2) (e.g., using linear splines), one obtains a system of equations of the form

$$\hat{A}\frac{d^2v}{dt^2}(t) = \hat{B}v(t) + \hat{C}\frac{d}{dt}\int_0^t g(t-s)v(s)\,ds + \hat{F}(t). \tag{1.3}$$

In this equation $\hat{A}, \hat{B}$, and $\hat{C}$ represent constant matrices and $v(t)$ and $\hat{F}(t)$ are vectors of appropriate dimension.

In order to retain the salient features but simplify the analysis in the following sections, we shall consider the following scalar version of (1.3):

$$\begin{cases} \frac{d^2x}{dt^2}(t) = ax(t) + \frac{d}{dt}\int_0^t g(t-s)x(s)\,ds + \hat{f}(t), \\ x(0) = x_0, \quad \dot{x}(0) = x_1. \end{cases} \tag{1.4}$$

Integrating (1.4) we obtain

$$\begin{cases} \dot{x}(t) = a\int_0^t x(s)\,ds + \int_0^t g(t-s)x(s)\,ds + f(t), \\ x(0) = x_0, \end{cases} \tag{1.5}$$

where

$$f(t) = x_1 + \int_0^t \hat{f}(s)\,ds.$$

Define $z(t) = \int_0^t x(s)\,ds$; then $\dot{z}(t) = x(t)$ and we obtain the system of integrodifferential equations

$$\begin{cases} \dot{x}(t) = az(t) + \int_0^t g(t-s)x(s)\,ds + f(t) \\ \dot{z}(t) = x(t), \end{cases} \tag{1.6}$$

with $x(0) = x_0$, $z(0) = 0$.

A standard assumption in viscoelasticity [**13**] is that the material is in an unstrained state for time $t < 0$. This would correspond to $u(x, s) = 0$ for $s < 0$ in equation (1.2). It follows then that $x(s) = 0$ and $z(s) = 0$ for $s < 0$ in (1.6). If we define $w(t) = \operatorname{col}(x(t), z(t))$, then $w(t)$ satisfies

$$\begin{cases} \dot{w}(t) = Mw(t) + \int_0^t K(t-s)w(s)\,ds + F(t), \\ w(0) = \begin{pmatrix} x_0 \\ 0 \end{pmatrix}, \end{cases} \tag{1.7}$$

where $M = \begin{pmatrix} 0 & a \\ 1 & 0 \end{pmatrix}$, $K(s) = \begin{pmatrix} g(s) & 0 \\ 0 & 0 \end{pmatrix}$, and $F(t) = \begin{pmatrix} f(t) \\ 0 \end{pmatrix}$.

Since $w(s) = \operatorname{col}(0, 0)$, for $s < 0$, we may rewrite (1.7) as

$$\dot{w}(t) = Mw(t) + \int_{-\infty}^t K(t-s)w(s)\,ds + F(t),$$
$$w(0) = \begin{pmatrix} x_0 \\ 0 \end{pmatrix}, \quad w(s) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad s < 0,$$

which is in the form of equation (1.1).

The remainder of the paper is outlined as follows. In Section 2 we review previous results that place equation (1.1) in a semigroup setting in order to establish existence of solutions. Differentiability results needed for the parameter estimation algorithm are then proved. In Section 3 the quasilinearization algorithm used for the identification procedure is discussed along with convergence results. Numerical examples are given in Section 4.

2. The abstract setting. In this section we develop an abstract framework for the Volterra integral equation discussed in the previous section. Namely, we will consider equation (1.1) in the form

$$\begin{cases} \dot{w}(t) = Mw(t) + \int_{-\infty}^{0} K(-s,\alpha)w(t+s)\,ds + F(t), & t > 0, \\ w(0) = \eta, \; w(s) = \varphi(s), & s < 0, \end{cases} \tag{2.1}$$

where $\eta = \binom{x_0}{0} \in \mathbf{R}^2$, $M = \begin{pmatrix} 0 & a \\ 1 & 0 \end{pmatrix}$, $F(t) = \begin{pmatrix} f(t) \\ 0 \end{pmatrix}$, and

$$K(s,\alpha) = \frac{\gamma e^{-\beta s}}{\Gamma(1-\alpha)s^{\alpha}} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad s > 0. \tag{2.2}$$

We assume β and γ are positive constants and $0 \leq \alpha < 1$. By a solution of (2.1) we mean a function $w : (-\infty,\infty) \to \mathbf{R}^2$ such that w is absolutely continuous (A.C.) on $[0,\infty)$ and satisfies the integral equation a.e. on $[0,\infty)$, $w(0) = \eta$, and $w(s) = \varphi(s)$ a.e. on $(-\infty, 0]$.

Our semigroup formulation follows the construction in [**5**] as further developed in [**10**] and [**11**]. Define the product space $X = \mathbf{R}^2 \times L^1(-\infty,0)$ with norm $\|(\eta,\varphi)\|_X = |\eta| + \|\varphi\|_{L^1(-\infty,0)}$. Consider the homogeneous equation

$$\begin{cases} \dot{y}(t) = My(t) + \int_{-\infty}^{0} K(-s,\alpha)y(t+s)\,ds, & t > 0 \\ y(0) = \eta, \; y(s) = \varphi(s), & s < 0. \end{cases} \tag{2.3}$$

Then, for each pair $(\eta,\varphi) \in X$, (2.3) has a unique solution, and, moreover, the mapping $S(t,\alpha)(\eta,\varphi) = (y(t), y_t(\cdot))$ defines a C_0-semigroup on X. Here we have used the notation $y_t(s) = y(t+s)$, $t \geq 0$, $s < 0$.

Fix $\epsilon \in (0,1)$ and define the parameter set $P = [0, 1-\epsilon]$. Then it is readily seen from (2.2) that there is a constant C, independent of α, such that

$$\int_0^{\infty} |K(s,\alpha)|\,ds \leq C \quad \text{for all } \alpha \in P. \tag{2.4}$$

Under this condition it is shown in [**10**] and [**11**] that the semigroup $S(t,\alpha)$ is generated by a closed and densely-defined operator $\mathcal{A}(\alpha)$ defined by

$$\begin{aligned}\mathrm{Dom}\,(\mathcal{A}(\alpha)) \equiv D = \{(\eta,\varphi)\in X : \varphi \text{ is A.C. on compact subsets}\\ \text{of } (-\infty,0],\ \dot{\varphi}\in L^1(-\infty,0),\ \varphi(0)=\eta\}\end{aligned}$$

and

$$\mathcal{A}(\alpha)(\eta,\varphi) = \left(M\eta + \int_{-\infty}^{0} K(-s,\alpha)\varphi(s)\,ds, \dot{\varphi}\right) \quad \text{for } (\eta,\varphi)\in D.$$

Our task is to show that the solution $w(t,\alpha) \equiv w(t)$ of (2.1) is differentiable with respect to α and that this derivative is sufficiently smooth to establish the local convergence of the algorithm defined in Section 3. This involves verifying the conditions in the semigroup setting established in [**6**] and [**8**]. We, therefore, assume in what follows that the reader has these papers in hand.

Since we are interested in dependence on α, we write $\mathcal{A}(\alpha) = \mathcal{A} + \mathcal{B}(\alpha)$, where $\mathcal{A}$ is independent of α and

$$\mathcal{B}(\alpha)(\eta,\varphi) = \left\langle \int_{-\infty}^{0} K(-s,\alpha)\varphi(s)\,ds,\ 0\right\rangle, \quad (\eta,\varphi)\in D. \tag{2.5}$$

Note that the range of $\mathcal{B}(\alpha)$ is the finite-dimensional space $Y = \mathbf{R}^2 \times \{0\}$.

Fix $y_0 \in X$, $\alpha_0 \in P$, and $T > 0$. Then the differentiability with respect to α at α_0 of the solution $y(t,\alpha)$ of (2.3) is a consequence of the following theorem.

THEOREM 2.1. *For every $t \in [0,T]$, $S(t,\alpha)y_0$ as defined above is Frechét differentiable with respect to α at α_0, and its derivative is given by*

$$D_\alpha S(t,\alpha_0)y_0 = \int_0^t S(t-s,\alpha_0)[D_\alpha \mathcal{F}(\alpha_0)y_0](s)\,ds, \quad 0\le t\le T,$$

where $\mathcal{F}(\alpha)$ is, for each $\alpha\in P$, a mapping from X into $L^1(0,T;X)$ defined by

$$[\mathcal{F}(\alpha)y_0](t) = \left(\int_{-\infty}^{0} K(-s,\alpha)y(t+s)\,ds,\ 0\right) \tag{2.6}$$

for $y_0 \in X$, $0 \le t \le T$. Recall that y is the solution of (2.3) with $(\eta, \varphi) = y_0$.

PROOF. This result is proved in [**7**] for a general Volterra kernel $K(s,\alpha)$ satisfying condition (2.4) under the following hypothesis:

(2.7) the mapping $\alpha \to K(\cdot,\alpha)$ from P into $L^1(0,\infty)$ is Frechét differentiable with respect to α at α_0.

Recall that $K(s,\alpha) = g(s,\alpha)\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, where

$$g(s,\alpha) = \frac{\gamma e^{-\beta s}}{\Gamma(1-\alpha)s^{\alpha}}, \quad s > 0,\ \alpha \in P.$$

Let $'$ denote differentiation with respect to α. Then computation shows that g' and g'' are of the form

$$g'(s,\alpha) = g_1(\alpha)(\ln s)e^{-\beta s}s^{-\alpha} + g_2(\alpha)e^{-\beta s}s^{-\alpha}$$

and

$$\begin{aligned} g''(s,\alpha) = g_3(\alpha)(\ln s)^2 e^{-\beta s}s^{-\alpha} &+ g_4(\alpha)(\ln s)e^{-\beta s}s^{-\alpha} \\ &+ g_5(\alpha)e^{-\beta s}s^{-\alpha}, \end{aligned}$$

where $g_1, \dots, g_5$ are continuous functions of α on P which can be explicitly calculated in terms of the gamma function and its derivatives. Important properties of g' and g'' for our purposes are that there are functions $\psi_1, \psi_2 \in L^1(0,\infty)$ such that

$$|g'(s,\alpha)| \le \psi_1(s) \quad \text{for } s > 0,\ \alpha \in P, \tag{2.8}$$

and

$$|g''(s,\alpha)| \le \psi_2(s) \quad \text{for } s > 0,\ \alpha \in P. \tag{2.9}$$

For example, one can take

$$\psi_1(s) = \begin{cases} (M_1|\ln s|e^{-\beta s} + M_2 e^{-\beta s})/s^{1-\epsilon}, & \text{for } 0 < s < 1 \\ M_1|\ln s|e^{-\beta s} + M_2 e^{-\beta s}, & \text{for } s \ge 1, \end{cases}$$

where M_1 and M_2 are upper bounds on P of $|g_1(\alpha)|$ and $|g_2(\alpha)|$, respectively. There is an analogous expression for $\psi_2(s)$. Therefore, by Taylor's theorem with remainder, we obtain

$$\begin{aligned}&|K(s,\alpha+h)-K(s,\alpha)-K'(s,\alpha)h|\\ &\quad=|g(s,\alpha+h)-g(s,\alpha)-g'(s,\alpha)h|\\ &\quad=|g''(s,\xi_1(s))h^2/2|\\ &\quad\le \psi_2(s)|h|^2/2\end{aligned}$$

for $s>0$, α, $\alpha+h\in P$, and $\xi_1(s)$ between α and $\alpha+h$. Integrating this inequality over $(0,\infty)$ and using $\psi_2\in L^1(0,\infty)$ yields (2.7) and completes the proof of Theorem 2.1.

A sufficient smoothness property for the local convergence of the parameter estimation algorithm is established in the following theorem.

THEOREM 2.2. *For every $t\in[0,T]$, $\alpha^*\in P$ and $y_0\in X$, $D_\alpha S(t,\alpha)y_0$ is strongly locally Lipschitz continuous with respect to α at α^*.*

PROOF. The proof relies on Lemma 3.3 of [**8**]. We must show that hypotheses (H11) and (H12) of that paper hold in this application. By definition (2.6), (H11) requires that there exist constants K_1, $\delta_1>0$ such that

$$\int_0^T\left|\int_{-\infty}^0(K(-s,\alpha+h)-K(-s,\alpha))y(t+s)\,ds\right|dt\le K_1|h||\eta| \tag{2.10}$$

for $|h|\le\delta_1$, where y is the solution of (2.3) with $\alpha=\alpha^*$ and $\varphi\equiv 0$. Note that, since $(y(t),y_t)=S(t,\alpha_0)(\eta,0)$, we have $|y(t)|\le M_1e^{\omega t}|\eta|$ for $t\ge 0$, and $y(t)=\varphi(t)=0$ for $t<0$. It is shown in [**7**] that the constants M_1 and ω may be taken independently of $\alpha\in P$. Therefore, by Fubini's theorem and the mean value theorem, we obtain

$$\begin{aligned}\int_0^T \int_{-\infty}^0 &|(K(-s,\alpha+h)-K(-s,\alpha))y(t+s)|\,ds\,dt\\ &= \int_{-\infty}^0 |K(-s,\alpha+h)-K(-s,\alpha)| \int_s^{T+s} |y(t)|\,dt\,ds\\ &\le \int_{-\infty}^0 |K(-s,\alpha+h)-K(-s,\alpha)| \int_0^T |y(t)|\,dt\,ds\\ &\le M_1Te^{\omega T}|\eta| \int_{-\infty}^0 |K(-s,\alpha+h)-K(-s,\alpha)|\,ds\\ &\le M_1Te^{\omega T}|\eta||h| \int_{-\infty}^0 \psi_1(-s)\,ds\end{aligned}$$

for α, $\alpha+h\in P$. Here we have used (2.8) in the last inequality. Since $\psi_1 \in L^1(0,\infty)$, this establishes (2.10).

Hypothesis (H12) of [**8**] requires the Lipschitz continuity of the derivative with respect to α of the mapping $\mathcal{F}(\alpha)$ defined by (2.6). For brevity, we denote the value of this derivative at $\alpha\in P$ by $D\mathcal{F}(\alpha)$. The existence of $D\mathcal{F}(\alpha)$ was shown in Theorem 2.1, and from the proof of that theorem we have the formula

$$[D\mathcal{F}(\alpha)h](t) = \left(\int_{-\infty}^0 [K'(-s,\alpha)h]y(t+s)\,ds,\ 0\right)$$

for $0\le t\le T$, $h\in\mathbf{R}$, $\alpha\in P$, where y is the solution of (2.3) with $\alpha=\alpha_0$ and $(\eta,\varphi)=y_0$. Recall that $'$ denotes differentiation with respect to α. The local Lipschitz continuity of $D\mathcal{F}(\alpha)$ at a point $\alpha=\alpha^*\in P$ now follows from estimates similar to those used to establish (H11) but with ψ_2 in place of ψ_1. This completes the proof of Theorem 2.2.

We now turn our attention to solutions of the nonhomogeneous equation (2.1). It is well known that a mild solution to this equation is given by the variation of constants formula

$$(w(t),w_t) = S(t,\alpha)(\eta,\varphi) + Q(t,\alpha),$$

where

$$Q(t,\alpha) = \int_0^t S(t-s,\alpha)(F(s),0)\,ds. \tag{2.11}$$

It remains, therefore, to consider the existence and smoothness of the derivative of $Q(t,\alpha)$ with respect to α. We again appeal to [**8**] where these properties of $Q(t,\alpha)$ are demonstrated by considering similar properties of the mapping $\mathcal{G}(\alpha) : L^1(0,T;X) \to L^1(0,T;X)$ defined by

$$[\mathcal{G}(\alpha)v](t) = \int_0^t \mathcal{B}(\alpha)S(t-s,\alpha_0)v(s)\,ds$$

for $v \in L^1(0,T;X)$, $\alpha \in P$, α_0 fixed. Note that if $v(t) = (F(t),0)$ for some $F \in L^1(0,T)$, then

$$(w(t),w_t) = \int_0^t S(t-s,\alpha_0)v(s)\,ds,$$

where w is a mild solution of (2.1) with $(\eta,\varphi) = (0,0)$. Since $\mathcal{B}(\alpha)$ is a difference of closed operators,

$$[\mathcal{G}(\alpha)v](t) = \mathcal{B}(\alpha)(w(t),w_t).$$

Therefore, using definition (2.5), we obtain in this setting that

$$[\mathcal{G}(\alpha)v](t) = \left(\int_{-\infty}^0 K(-s,\alpha)w(t+s)\,ds, 0\right) \tag{2.12}$$

where $v(t) = (F(t),0)$ and w is the solution of (2.1) with $\alpha = \alpha_0$ and $(\eta,\varphi) = (0,0)$. Here we assume F is sufficiently smooth for the solution w to exist in the strong sense defined earlier.

Comparing definitions (2.6) and (2.12), we see that properties of $\mathcal{G}(\alpha)$ with respect to α can be proven in the same way as the corresponding properties of $\mathcal{F}(\alpha)$ using the solution of (2.1) in the place of the solution of (2.3) at a fixed $\alpha_0 \in P$. For this reason the following theorems, which are consequences of Lemmas 3.2 and 3.4 of [**8**], are stated without proof.

THEOREM 2.3. *For $F \in L^1(0,\infty)$, let $Q(t,\alpha)$ be defined by* (2.11). *Then, for every $t \in [0,T]$ and $\alpha_0 \in P$, $Q(t,\alpha)$ is Frechét differentiable with respect to α at α_0, and this derivative is given by the formula*

$$D_\alpha Q(t,\alpha_0) = \int_0^t S(t-s,\alpha_0)[D_\alpha \mathcal{G}(\alpha_0)v](s)\,ds,$$

where $v(t) = (F(t),0)$ and $\mathcal{G}(\alpha)$ is defined by (2.12).

THEOREM 2.4. *Suppose the hypotheses of Theorem* 2.3 *hold. Then the mapping* $D_\alpha Q(t,\alpha)$ *is locally Lipschitz continuous with respect to* α *at every* $\alpha^* \in P$.

3. The algorithm. In this section we define a parameter estimation algorithm based on quasilinearization and state some local convergence results. For later adaptation we develop the algorithm for the case $\alpha \in P \subset \mathbf{R}^n$ with canonical basis e_i, $i = 1, 2, \dots, n$. In Section 4 the algorithm is applied in the case $n = 1$. The definitions and theorems stated here may also be found in [**8**] but are included here for completeness.

Using the notation of the previous section, let $y_0 = (\eta, \varphi) \in X$ and $\alpha \in P$. Let C be a bounded linear mapping from X into a finite-dimensional space $\mathbf{R}^l$, and define

$$w(t,\alpha) = C[S(t,\alpha)y_0 + Q(t,\alpha)].$$

The parameter estimation algorithm is related to the following optimization problem.

PROBLEM 3.1. Let $w_j \in \mathbf{R}^l$, $j = 1, 2, \dots, m$, be data values taken at times $t_j \in [0, T]$, $j = 1, 2, \dots, m$, respectively. For $\alpha \in P$, define the quadratic cost function

$$J(\alpha) = \sum_{j=1}^{m} |w(t_j,\alpha) - w_j|^2.$$

Find $\alpha^* \in P$ such that $J(\alpha^*) \le J(\alpha)$ for all $\alpha \in P$.

The quasilinearization algorithm method defines a recursive algorithm whose fixed point is a local solution of Problem 3.1. A more complete exposition is given in [**3**]. Given an initial guess $\alpha_0 \in P$, define

$$\alpha_{k+1} = f(\alpha_k), \quad k = 0, 1, 2, \dots,$$

where

$$f(\alpha) = \alpha - [D(\alpha)]^{-1} b(\alpha)$$
$$D(\alpha) = \sum_{j=1}^{m} M^T(t_j, \alpha) M(t_j, \alpha)$$
$$b(\alpha) = \sum_{j=1}^{m} M^T(t_j, \alpha)[w(t_j, \alpha) - w_j]$$

and the matrix $M(t, \alpha)$ has its ith column $M^i(t, \alpha)$ given by

$$M^i(t, \alpha) = C D_\alpha [S(t, \alpha) y_0 + Q(t, \alpha)] e_i, \quad i = 1, 2, \ldots, n.$$

The following theorems are typical of quasilinearization methods. Their proofs may be found in [**8**]. We obtain superlinear convergence when there is an exact fit to data (Theorem 3.1) and linear convergence in the presence of error (Theorem 3.2).

THEOREM 3.1. *Suppose the conditions of the previous section are satisfied. Moreover, assume* $[D(\alpha)]^{-1}$ *exists,* $f(\alpha^*) = \alpha^*$*, and* $J(\alpha^*) = 0$*. Then, for every* $\epsilon > 0$*, there exists* $\delta > 0$ *such that*

$$|f(\alpha) - f(\alpha^*)| \le \epsilon |\alpha - \alpha^*|$$

for $|\alpha - \alpha^*| \le \delta$*. In particular, there is a neighborhood* $\mathcal{U}$ *of* α^* *such that* $\alpha_k \to \alpha^*$ *as* $k \to \infty$ *whenever* $\alpha_0 \in \mathcal{U}$*.*

The following theorem does not require an exact fit to data but does place some technical restrictions on the behavior of the matrix $M(t, \alpha)$ near α^*. Note that, under the conditions of Theorem 2.2, there exists a number $\bar{\delta} > 0$ such that, for $0 < \delta < \bar{\delta}$, there exists a constant $K(\delta)$ such that

$$\sum_{j=1}^{m} |M^T(t_j, \alpha) - M^T(t_j, \alpha^*)| \le K(\delta) |\alpha - \alpha^*|$$

for $|\alpha - \alpha^*| \le \delta$. Let $K^* = \limsup_{\delta \downarrow 0} K(\delta)$, and define

$$\lambda^* = K^* |D(\alpha^*)^{-1}| \max_j |w(t_j, \alpha^*) - w_j|.$$

THEOREM 3.2. *Suppose the conditions of the previous section are satisfied. Moreover, assume* $[D(\alpha^*)]^{-1}$ *exists and* $f(\alpha^*) = \alpha^*$. *Let* λ^* *be defined as above and assume* $\lambda^* < 1$. *Then there exists* $\delta^* > 0$ *such that*

$$|f(\alpha) - f(\alpha^*)| \le \lambda^* |\alpha - \alpha^*|$$

for $|\alpha - \alpha^*| \le \delta^*$. *In particular,* $\alpha_k \to \alpha^*$ *as* $k \to \infty$ *whenever* $|\alpha_0 - \alpha^*| \le \delta^*$.

4. Numerical results. In this section we present several examples that illustrate the ideas discussed in the previous sections. Recall the identification problem: given observations w_j at times t_j, $j = 1, 2, \dots, m$, determine $\alpha \in [0, 1)$ that minimizes the cost functional

$$J(\alpha) = \sum_{j=1}^{m} (x(t_j) - w_j)^2,$$

where $x(t)$ satisfies

$$\begin{cases} \dot{x}(t) = a \int_0^t x(s)\, ds + \frac{\gamma}{\Gamma(1-\alpha)} \int_0^t \frac{e^{-\beta(t-s)}}{(t-s)^\alpha} x(s)\, ds + f(t), \\ x(0) = x_0. \end{cases} \tag{4.1}$$

The quasilinearization algorithm requires that we solve (4.1) along with its sensitivity equation which has the form

$$\begin{cases} \dot{x}_\alpha(t) = a \displaystyle\int_0^t x_\alpha(s)\, ds + \frac{\gamma}{\Gamma(1-\alpha)} \int_0^t \frac{e^{-\beta(t-s)}}{(t-s)^\alpha} x_\alpha(s)\, ds \\ \qquad\qquad + \dfrac{\gamma \Gamma'(1-\alpha)}{\Gamma(1-\alpha)^2} \displaystyle\int_0^t \frac{e^{-\beta(t-s)}}{(t-s)^\alpha} x(s)\, ds \\ \qquad\qquad - \dfrac{\gamma}{\Gamma(1-\alpha)} \displaystyle\int_0^t \frac{\ln(t-s) e^{-\beta(t-s)}}{(t-s)^\alpha} x(s)\, ds, \\ x_\alpha(0) = 0, \end{cases} \tag{4.2}$$

where $x_\alpha(t) = (\partial x / \partial \alpha)(t)$. The zero initial condition reflects the fact that the value $x(0) = x_0$ is independent of the parameter α.

The implementation of the identification scheme begins with an initial guess for α. Equations (4.1) and (4.2) are integrated using this initial value, then $x(t)$ and $x_\alpha(t)$ are used to give an updated estimate of the parameter. For this particular problem the quasilinearization algorithm updates the current estimate α_k according to

$$\alpha_{k+1} = \alpha_k - \frac{\sum_{j=1}^{m}(x(t_j) - w_j)x_\alpha(t_j)}{\sum_{j=1}^{m}(x_\alpha(t_j))^2},$$

where $x(t)$ and $x_\alpha(t)$ are the solutions of (4.1)–(4.2) computed for $\alpha = \alpha_k$. In order to numerically integrate the state and sensitivity equations, we first convert (4.1)–(4.2) to integral equations via the substitution $z(t) = \dot{x}(t)$ and $z_\alpha(t) = \dot{x}_\alpha(t)$. Then one has the 4×4 system of integral equations consisting of (4.1)–(4.2) with the above substitutions coupled with

$$\begin{cases} x(t) = x(t_p) + \int_{t_p}^{t} z(s)\,ds, \\ x_\alpha(t) = x_\alpha(t_p) + \int_{t_p}^{t} z_\alpha(s)\,ds, \end{cases} \tag{4.3}$$

where $t_p \in [0, t)$ is determined by the approximation scheme. The solution of the system of the integral equations is then approximated by applying a product integration method based on Simpson's rule to the singular integral terms, and Simpson's rule to all other integrals. For a description of product integration methods, we refer the interested reader to [**16**].

In each of the following examples, we numerically solve (4.1)–(4.3) in time on the interval $[0, 1]$. Define $t_j = j/N$, $j = 0, 1, \dots, N$. The numerical integration scheme then computes values for $x(t_j)$. Examples (4.1) and (4.2) presented here are computed using a value of $N = 50$, and Example (4.3) is computed using $N = 200$. In each case, 5 data points located at $t = .2, .4, .6, .8$, and 1.0 are used in the identification procedure. The true values of $x(t)$ in all of the Figures (4.1)–(4.5) are denoted by X.

Example 4.1. In this example we set the values of a, β, and γ to 1., 1., and 5., respectively. The parameter value to be identified is $\alpha = 1/2$, and the nonhomogeneous term $f(t)$ is

$$\begin{aligned} f(t) &= e^{-t}(1024t^3 + 2176t + 1792) - 1825 \\ &\quad - \frac{e^{-t}t^{.5}}{\Gamma(.5)}\left(\frac{32768}{315}t^4 - \frac{8192}{35}t^3 + \frac{512}{3}t^2 - \frac{128}{3}t + 2\right) \end{aligned}$$

The true solution is $x(t) = e^{-t}T_4(2t-1)$, where $T_4(s)$ is the Chebyshev polynomial of degree 4 on $-1 \leq s \leq 1$. Tables (4.1) and (4.2) contain the results for two computer runs, one for an initial α of $\alpha_0 = .9$, and the other for $\alpha_0 = .2$. The sequence of α_k values, their corresponding costs $J(\alpha_k)$, and the values of the state $x(t)$ at time $t = 1$ are included to illustrate the convergence. The true value of $x(1)$ is .3678794. Note that, in each case, once an estimate of α is greater than .5, then the sequence of iterates converges monotonically down to the true value. This is a characteristic of all of our computer simulations and seems to indicate that it is better to choose an initial value of α that is high instead of low. In fact, for all simulations, another characteristic is that if the initial choice is excessively low, then the next estimate of α is greater than 1, and the integral becomes undefined. For this particular example, $\alpha_0 = .1$ resulted in a value greater than 1 for the next iterate and the integration scheme broke down. However, though not shown here, some examples ran successfully even with a negative initial value for α. Figures (4.1)–(4.2) show the convergence of the state $x(t)$ for the initial values of $\alpha_0 = .9$, and .2, respectively.

TABLE 4.1.

$a = 1$, $\gamma = 5$, $\beta = 1$, true $\alpha = .5$, initial $\alpha = .9$			
Iteration	α	$J(\alpha)$	$x(1)$
0	.9	78.9130445	-7.7572184
1	.8061489	11.8804011	-2.7175508
2	.7036760	1.6122215	-.7432785
3	.6046577	.1689001	.0158974
4	.5327346	.0090401	.2877378
5	.5034419	.0000834	.3602260
6	.5000500	.0000000	.3677829

TABLE 4.2.

$a = 1$, $\gamma = 5$, $\beta = 1$, true $\alpha = .5$, initial $\alpha = .2$			
Iteration	α	$J(\alpha)$	$x(1)$
0	.2	.1205106	.637832
1	.8835656	55.8792983	-6.443195
2	.7876713	8.3050417	-2.201167
3	.6845883	1.0919149	-.542773
4	.5884893	.1055430	.090622
5	.5236488	.0045683	.311016
6	.5018820	.0000246	.363724
7	.5000214	.0000000	.367845

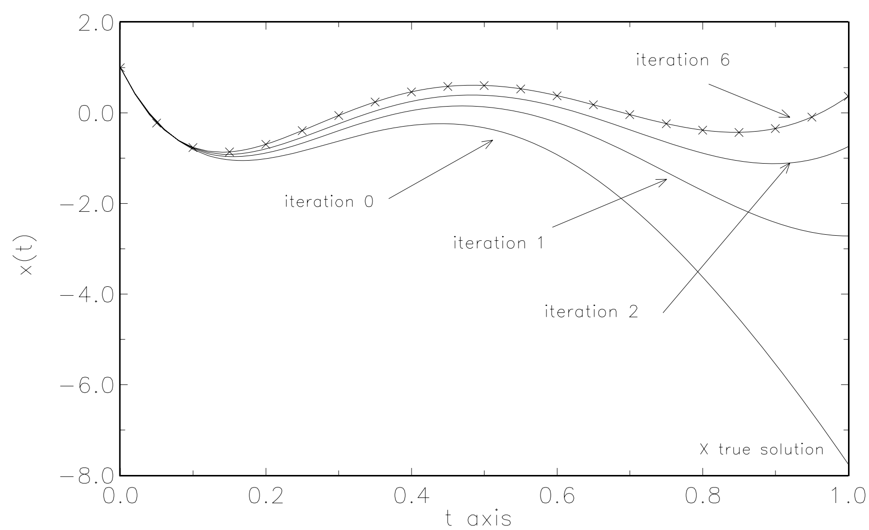

FIGURE 4.1

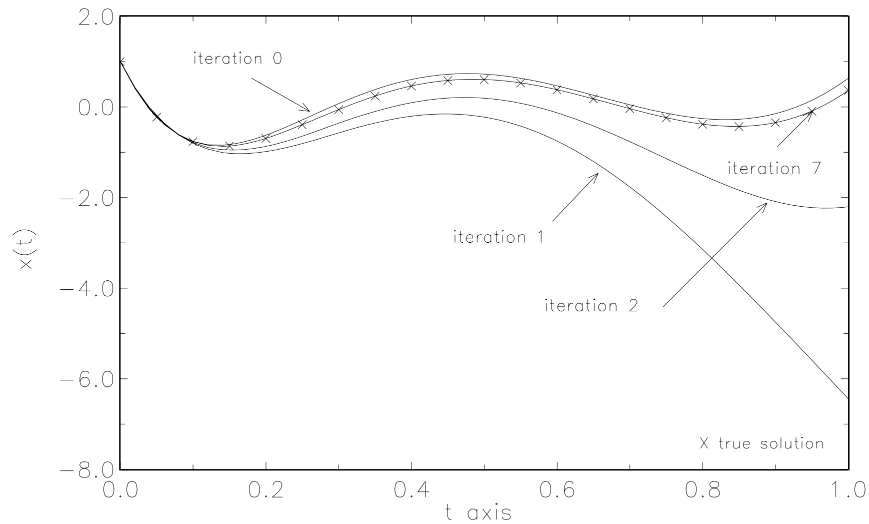

FIGURE 4.2

EXAMPLE 4.2. Here we set $a = 1$, $\beta = 1$, $\gamma = 4$, and $\alpha = .9$. The nonhomogeneous term is

$$f(t) = -1 - \gamma \frac{10e^{-t}}{\Gamma(.1)} t^{.1}.$$

In this case, the true solution is e^{-t}.

This example contains a kernel that is more singular than that of Example 1. The results for initial values of $\alpha_0 = .999$ and $\alpha_0 = .8$ are given in Tables (4.3) and (4.4), respectively. For comparison, the true value of $x(1)$ is $x(1) = .3678794$. Note, again, it appears that a high initial guess of α is preferable to a low one. Moreover, for an initial guess of $\alpha_0 = .75$, the algorithm updates the parameter to a value greater than 1 and the program stops. The convergence of the states is shown in Figures (4.3) and (4.4).

TABLE 4.3.

$a = 1$, $\gamma = 4$, $\beta = 1$, true $\alpha = .9$, initial $\alpha = .999$			
Iteration	α	$J(\alpha)$	$x(1)$
0	.999	56.1016515	7.127846
1	.9400160	4.37204423	2.223429
2	.9075174	.1077531	.656523
3	.9002876	.0001462	.378490
4	.9000004	.0000000	.367895

TABLE 4.4.

$a = 1$, $\gamma = 4$, $\beta = 1$, true $\alpha = .9$, initial $\alpha = .8$			
Iteration	α	$J(\alpha)$	$x(1)$
0	.8	7.0279032	-1.895701
1	.9676194	17.3817062	4.096880
2	.9200856	.8805865	1.195951
3	.9019895	.0071213	.441968
4	.9000204	.0000007	.368632
5	.9000000	.0000000	.367879

Example 4.3. This example has two features that are different than the previous examples. In Section 2 we assumed that $\beta > 0$, ensuring that the integral in equation (2.4) exists. This example illustrates that it may be possible to lift this restriction to include $\beta = 0$, which results in a true fractional derivative model. Also, for this example, $x(t) = t^{1.5}$. Thus, the true solution has an unbounded second derivative at $t = 0$. Because integration methods based on Simpson's rule converge slowly for functions that do not have four continuous derivatives, it was necessary to increase N to 200 for this example to gain accuracy.

Here we set the values $a = 1$, $\beta = 0$, $\gamma = 4$, and $\alpha = .5$. The function $f(t)$ is, in this case,

$$f(t) = \frac{3t^{.5}}{2} - \frac{2t^{2.5}}{5} - \Gamma(.5)\gamma\frac{3t^2}{8}.$$

The results of the quasilinearization algorithm are given in Table (4.5)

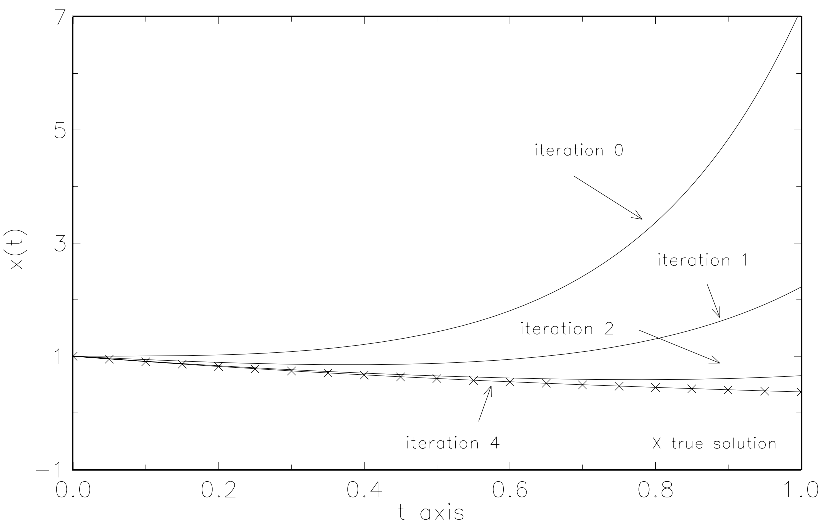

FIGURE 4.3

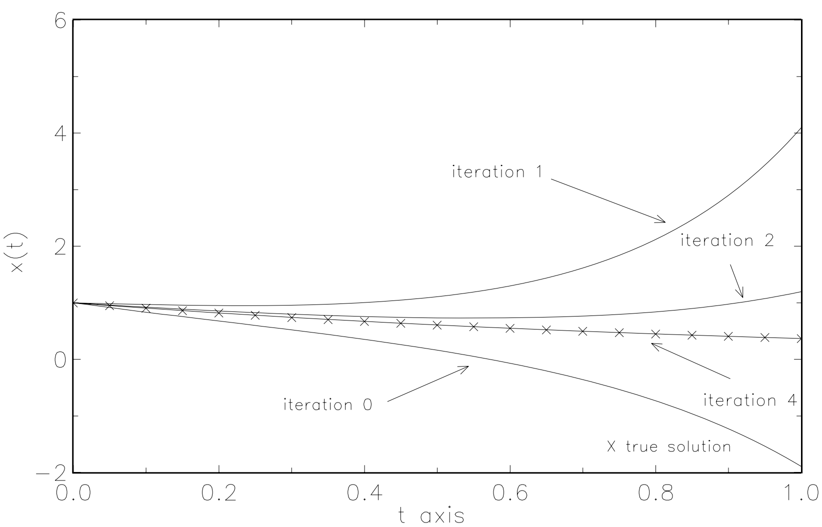

FIGURE 4.4

for the initial value of $\alpha_0 = .9$. The fact that we could only obtain α to 3 correct digits is due to the inaccuracy of the integration scheme used. Figure (4.5) illustrates the convergence of the states to the true solution.

TABLE 4.5.

$a = 1$, $\gamma = 4$, $\beta = 0$, true $\alpha = .5$, initial $\alpha = .9$			
Iteration	α	$J(\alpha)$	$x(1)$
0	.9	24.1608906	5.563632
1	.7641130	3.5153401	2.711820
2	.6246967	.3481690	1.531838
3	.5298167	.0127381	1.101062
4	.5019498	.0000392	1.005624
5	.5002170	.0000000	1.000050
6	.5002110	.0000000	1.000030

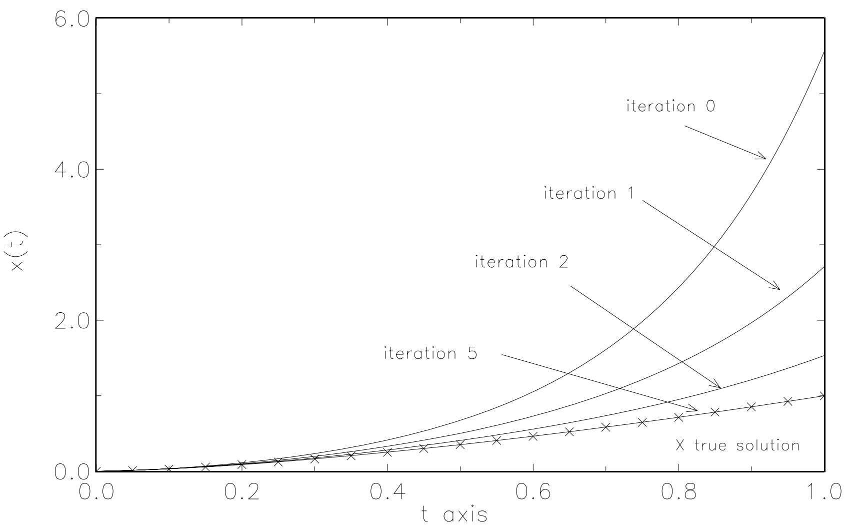

FIGURE 4.5

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